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L_p Markov exponent of certain domains with cusps

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Abstract

In this paper we give sharp L_p Markov type inequality for derivatives of polynomials for some family of domains with cusps.

1 Introduction

Let $\mathcal{P}_n(\mathbb{R}^m)$ be the class of all algebraic polynomials in m variables with real coefficients of degree n. Further, let $C(\Omega)$ be the real space of all real valued continuous functions f defined on a compact set $\Omega \subset \mathbb{R}^m$ with the norm $\|f\|_{C(\Omega)} := \sup_{x\in\Omega} |f(x)|$, and let $L_{p,W}(\Omega)$, $1 \le p \le \infty$, be the space of all Lebesgue-measurable functions f on $\Omega \subset \mathbb{R}^m$ such that $\|f\|_{L_{p,W}(\Omega)} := (\int_\Omega |f(x)|^p W(x)\,dx)^{1/p} < \infty$ if $1 \le p < \infty$, and $L_{\infty,W} := C(\Omega)$. Set $L_p(\Omega) := L_{p,1}(\Omega)$, $1 \le p \le \infty$. Here W denotes an integrable weight defined on a set $\Omega \subset \mathbb{R}^m$ with the property that the set $\{x \in \Omega : W(x) = 0\}$ has m-dimensional Lebesgue measure 0. Moreover, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Definition 1.1. We say that a compact set $\emptyset \ne E \subset \mathbb{R}^m$ satisfies L_p Markov type inequality (or: is a L_p Markov set) if there exist $\kappa, C > 0$ such that, for each polynomial $P \in \mathcal{P}(\mathbb{R}^m)$ and each $\alpha \in \mathbb{N}_0^m$,

$$\|D^\alpha P\|_{L_p(E)} \le (C(\deg P)^\kappa)^{|\alpha|}\|P\|_{L_p(E)}, \tag{1}$$

where $D^\alpha P = \frac{\partial^{|\alpha|} P}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Clearly, by iteration, it is enough to consider in the above definition multi-indices α with $|\alpha| = 1$. The inequality (1) is a generalization of the classical Markov inequality:

$$\|P'\|_{C([-1,1])} \le (\deg P)^2 \|P\|_{C([-1,1])}.$$

Markov-type inequalities play an important role in Approximation Theory since they are widely used for verifying inverse theorems of approximation. These inequalities and its various generalizations (restricted not only to nonpluripolar subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ but also their versions for pieces of semialgebraic sets or other "small" subsets of $\mathbb{R}^n$ ($\mathbb{C}^n$)) found many applications in approximation theory, analysis, constructive function theory, but also in other branches of science (for example, in physics or chemistry).

In this paper we shall consider the following problem:

For a given L_p Markov set E determine $\mu_p(E) := \inf\{\kappa : E$ satisfies (1)$\}$.

The quantity $\mu_p(E)$ is called L_p Markov exponent and was first considered by Baran and Pleśniak in [2] for $p = \infty$. This is related to the linear extension operator for C^∞ functions with restricted growth of derivatives (see [8, 9]). For any compact set E in $\mathbb{R}^m$ we have $\mu_p(E) \ge 2$. If E is a fat convex subset of $\mathbb{R}^m$, then $\mu_p(E) = 2$. It is known that L_∞ Markov exponent, for $Lip\gamma$, $0 < \gamma < 1$ cuspidal domains in $\mathbb{R}^m$ is equal to $\frac{2}{\gamma}$ (see for instance, [4], [1], [6]). If $K \subset \mathbb{R}^m$ is a $Lip\gamma$, $0 < \gamma < 1$ cuspidal piecewise graph domain such that it is imbedded in an affine image of the l_γ ball having one of its vertices on the boundary ∂K of K, then $\mu_p(E) = \frac{2}{\gamma}$ for $1 \le p < \infty$ (see [7]). Our goal is to establish L_p Markov exponent of the following domains $\Psi_k := \{(x, y) \in \mathbb{R}^2 : |x| \le 1, 0 \le y \le x^{2k}\}$, $\Upsilon_k := \{(x, y) \in \mathbb{R}^2 : |x| \le 1, 0 \le y \operatorname{sgn} x \le |x|^{2k+1}\}$ and $\Lambda_k := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, ax^k \le y \le bx^k\}$, $k \in \mathbb{N}$, $0 < a < b$. More precisely, we show that $\mu_p(\Psi_k) = \mu_p(\Lambda_k) = 2k$ and $\mu_p(\Upsilon_k) = 2k + 1$ for every $k \in \mathbb{N}$, $1 \le p < \infty$. Since none of the domains Ψ_k, Υ_k and Λ_k is cuspidal piecewise graph domain, the above results cannot be obtained using the methods of [7]. In particular, Λ_k has a cusp at the origin that cannot be connected to the interior of Λ_k by a straight line. However, these results are known in case of supremum norm (see [6]).

2 Auxiliary results

In order to verify our main results we shall need some auxiliary statements. The following inequalities play a central role in our considerations.

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Lemma 2.1. *Let $S := \{(t,y) \in \mathbb{R}^2 : 0 \le t \le 1, 0 \le y \le t\}$. For each $\alpha > -1$ and $\beta, \mu, \nu \ge 0$ there exists a positive constant C such that*

$$\int_S t^\alpha y^\beta |\frac{\partial Q}{\partial t}(t,y)|^p\,dtdy \le Cn^{2p}\int_S t^\alpha y^\beta |Q(t,y)|^p\,dtdy, \quad \int_S t^\alpha y^\beta |\frac{\partial Q}{\partial y}(t,y)|^p\,dtdy \le Cn^{2p}\int_S t^\alpha y^\beta |Q(t,y)|^p\,dtdy \tag{2}$$

$$\int_S t^\alpha y^\beta |Q(t,y)|^p\,dtdy \le Cn^{2\mu}\int_S t^{\alpha+\mu} y^\beta |Q(t,y)|^p\,dtdy, \quad \int_S t^\alpha y^\beta |Q(t,y)|^p\,dtdy \le Cn^{2\nu}\int_S t^\alpha y^{\beta+\nu} |Q(t,y)|^p\,dtdy \tag{3}$$

for every $Q \in \mathcal{P}_n(\mathbb{R}^2)$.

Proof. Let $L := \{(t,y) \in \mathbb{R}^2 : 0 \le y \le \frac{1}{2}, y \le t \le y + \frac{1}{2}\}$ and let $T := \{(t,y) \in \mathbb{R}^2 : \frac{1}{2} \le t \le 1, \frac{1}{2} \le y \le t\}$. Then

$$\int_S t^\alpha y^\beta |Q(t,y)|^p\,dtdy \le \int_L t^\alpha y^\beta |Q(t,y)|^p\,dtdy + 2^\alpha\int_T |Q(t,y)|^p\,dtdy + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} t^\alpha y^\beta |Q(t,y)|^p\,dtdy \tag{4}$$

and

$$\int_T |Q(t,y)|^p\,dtdy \le 2^{\alpha+\beta+1}\int_T t^\alpha y^\beta |Q(t,y)|^p\,dtdy. \tag{5}$$

Since T is a fat convex set, there exists $D > 0$ such that

$$\int_T |\frac{\partial Q}{\partial t}(t,y)|^p\,dtdy \le Dn^{2p}\int_T |Q(t,y)|^p\,dtdy, \quad \int_T |\frac{\partial Q}{\partial y}(t,y)|^p\,dtdy \le Dn^{2p}\int_T |Q(t,y)|^p\,dtdy. \tag{6}$$

Now consider the integral over L. Using the change of variables $t - y = z$, we have

$$\int_L t^\alpha y^\beta |Q(t,y)|^p\,dtdy = \int_{[0,1/2]^2} (y+z)^\alpha y^\beta |Q(y+z,y)|^p\,dzdy. \tag{7}$$

Set $R(z,y) := Q(y+z,y)$. By Theorem 7.4 of [5] there exists a positive constant A such that

$$\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |\frac{\partial R}{\partial z}(z,y)|^p\,dz \le An^{2p}\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |R(z,y)|^p\,dz, \tag{8}$$

$$\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |\frac{\partial R}{\partial y}(z,y)|^p\,dy \le An^{2p}\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |R(z,y)|^p\,dy. \tag{9}$$

Thus, by (4)-(9), and by the fact that $\frac{\partial R}{\partial z} = \frac{\partial Q}{\partial t}$, $\frac{\partial Q}{\partial y} = \frac{\partial R}{\partial y} - \frac{\partial R}{\partial z}$, we obtain the inequalities (2).

In order to prove the inequalities (3), it suffices to prove that there exists a positive constant A' such that

$$\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |G(z,y)|^p\,dz \le A'n^{2\mu}\int_0^{\frac{1}{2}} (y+z)^{\alpha+\mu} y^\beta |G(z,y)|^p\,dz,$$

$$\int_0^{\frac{1}{2}} (y+z)^\alpha y^\beta |G(z,y)|^p\,dy \le A'n^{2\nu}\int_0^{\frac{1}{2}} (y+z)^\alpha y^{\beta+\nu} |G(z,y)|^p\,dy.$$

for every $G \in \mathcal{P}_n(\mathbb{R}^2)$. This follows from the inequality (7.22) of [5]. □

In a similar way one can derive the following lemma.

Lemma 2.2. *Fix $0 \le c < d$, and let $V := V_{c,d} := \{(x,\eta) \in \mathbb{R}^2 : 0 \le t \le 1, ct \le \eta \le dt\}$. For each $\alpha > -1$ and $\beta, \mu, \nu \ge 0$ there exists a positive constant C such that*

$$\int_V t^\alpha \eta^\beta |\frac{\partial Q}{\partial t}(t,\eta)|^p\,dtd\eta \le Cn^{2p}\int_V t^\alpha \eta^\beta |Q(t,\eta)|^p\,dtd\eta, \quad \int_V t^\alpha \eta^\beta |\frac{\partial Q}{\partial \eta}(t,\eta)|^p\,dtd\eta \le Cn^{2p}\int_V t^\alpha \eta^\beta |Q(t,\eta)|^p\,dtd\eta \tag{10}$$

$$\int_V t^\alpha \eta^\beta |Q(t,\eta)|^p\,dtd\eta \le Cn^{2\mu}\int_V t^{\alpha+\mu} \eta^\beta |Q(t,\eta)|^p\,dtd\eta, \quad \int_V t^\alpha \eta^\beta |Q(t,\eta)|^p\,dtd\eta \le Cn^{2\nu}\int_V t^\alpha \eta^{\beta+\nu} |Q(t,\eta)|^p\,dtd\eta \tag{11}$$

for every $Q \in \mathcal{P}_n(\mathbb{R}^2)$.

3 Main results

This section addresses main theorems.

Theorem 3.1. *Let k be a natural number, and let $\Psi_k = \{(x,y)\in\mathbb{R}^2 : |x|\le 1, 0\le y\le x^{2k}\}$, $\Upsilon_k = \{(x,y)\in\mathbb{R}^2 : |x|\le 1, 0\le y\operatorname{sgn} x\le |x|^{2k+1}\}$. Then for $1\le p\le\infty$ we have $\mu_p(\Psi_k)=\mu_p(\Upsilon_k)-1=2k$.*

Proof. First we prove that $\mu_p(\Psi_k)=2k$. It is clear that for each $P\in\mathcal{P}_n(\mathbb{R}^2)$ there exist $P_0,P_1\in\mathcal{P}_n(\mathbb{R}^2)$ such that $P(x,y)=P_0(x^2,y)+xP_1(x^2,y)$. Hence

$$\int_{\Psi_k}|P(x,y)|^p\,dxdy=\int_S\frac{kr^{k-1}|P_0(t,r^k)+\sqrt{t}P_1(t,r^k)|^p}{2\sqrt{t}}\,dtdr+\int_S\frac{kr^{k-1}|P_0(t,r^k)-\sqrt{t}P_1(t,r^k)|^p}{2\sqrt{t}}\,dtdr, \tag{12}$$

where $S=\{(t,r)\in\mathbb{R}^2 : 0\le t\le 1, 0\le r\le t\}$. In (2) of Lemma 2.1, let $\alpha=-\frac{1}{2}$, $\beta=k-1$ to conclude that

$$\int_S\frac{kr^{k-1}|kr^{k-1}\frac{\partial P_0}{\partial y}(t,r^k)|^p}{2\sqrt{t}}\,dtdr\le C(kn)^{2p}\int_S\frac{kr^{k-1}|P_0(t,r^k)|^p}{2\sqrt{t}}\,dtdr.$$

Again by Lemma 2.1, for $\mu=(k-1)p$,

$$\int_S\frac{kr^{k-1}|\frac{\partial P_0}{\partial y}(t,r^k)|^p}{2\sqrt{t}}\,dtdr\le C(kn)^{2(k-1)p}\int_S\frac{kr^{k-1}|r^{k-1}\frac{\partial P_0}{\partial y}(t,r^k)|^p}{2\sqrt{t}}\,dtdr.$$

Therefore

$$\int_S\frac{kr^{k-1}|\frac{\partial P_0}{\partial y}(t,r^k)|^p}{2\sqrt{t}}\,dtdr\le C^2(kn)^{2kp}\int_S\frac{kr^{k-1}|P_0(t,r^k)|^p}{2\sqrt{t}}\,dtdr. \tag{13}$$

Similarly,

$$\int_S\frac{kr^{k-1}|\sqrt{t}\frac{\partial P_1}{\partial y}(t,r^k)|^p}{2\sqrt{t}}\,dtdr\le C^2(kn)^{2kp}\int_S\frac{kr^{k-1}|\sqrt{t}P_1(t,r^k)|^p}{2\sqrt{t}}\,dtdr. \tag{14}$$

Thus, by (12)-(14), and by the fact that $\frac{\partial P}{\partial y}(x,y)=\frac{\partial P_0}{\partial y}(x^2,y)+x\frac{\partial P_1}{\partial y}(x^2,y)$, we have

$$\int_{\Psi_k}|\frac{\partial P}{\partial y}(x,y)|^p\,dxdy\le 2^pC^2(kn)^{2kp}\int_S\frac{kr^{k-1}|P_0(t,r^k)|^p}{2\sqrt{t}}\,dtdr+2^pC^2(kn)^{2kp}\int_S\frac{kr^{k-1}|\sqrt{t}P_1(t,r^k)|^p}{2\sqrt{t}}\,dtdr. \tag{15}$$

From (12) and the inequality

$$\|2f\|^p_{L_p(S)}\le 2^{p-1}(\|f-g\|^p_{L_p(S)}+\|f+g\|^p_{L_p(S)})$$

we see that

$$\int_S\frac{kr^{k-1}|P_0(t,r^k)|^p}{\sqrt{t}}\,dtdr\le\int_{\Psi_k}|P(x,y)|^p\,dxdy, \tag{16}$$

$$\int_S\frac{kr^{k-1}|\sqrt{t}P_1(t,r^k)|^p}{\sqrt{t}}\,dtdr\le\int_{\Psi_k}|P(x,y)|^p\,dxdy. \tag{17}$$

By using inequalities (15)-(17), we obtain

$$\int_{\Psi_k}|\frac{\partial P}{\partial y}(x,y)|^p\,dxdy\le 2^{p+1}C^2(kn)^{2kp}\int_{\Psi_k}|P(x,y)|^p\,dxdy. \tag{18}$$

In a similar way we can prove that

$$\int_{\Psi_k}|\frac{\partial P}{\partial y}(x,y)|^p|x|\,dxdy\le 2^{p+1}C^2(kn)^{2kp}\int_{\Psi_k}|P(x,y)|^p|x|\,dxdy. \tag{19}$$

We need now to consider $\frac{\partial P}{\partial x}$. Using the change of variables $y=\eta^{2k}$, we have

$$\int_{\Psi_k}|\frac{\partial P}{\partial x}(x,y)|^p\,dxdy=\int_S 2k\eta^{2k-1}|\frac{\partial P}{\partial x}(x,\eta^{2k})|^p\,dxd\eta+\int_S 2k\eta^{2k-1}|\frac{\partial P}{\partial x}(-x,\eta^{2k})|^p\,dxd\eta,$$

$$\int_{\Psi_k}|P(x,y)|^p\,dxdy=\int_S 2k\eta^{2k-1}|P(x,\eta^{2k})|^p\,dxd\eta+\int_S 2k\eta^{2k-1}|P(-x,\eta^{2k})|^p\,dxd\eta.$$

By Lemma 2.1, using $\alpha=0$, $\beta=2k-1$,

$$\int_S \eta^{2k-1}|\frac{\partial P}{\partial x}(x,\eta^{2k})|^p\,dx\,d\eta \le C(2kn)^{2p}\int_S \eta^{2k-1}|P(x,\eta^{2k})|^p\,dx\,d\eta,$$
$$\int_S \eta^{2k-1}|\frac{\partial P}{\partial x}(-x,\eta^{2k})|^p\,dx\,d\eta \le C(2kn)^{2p}\int_S \eta^{2k-1}|P(-x,\eta^{2k})|^p\,dx\,d\eta.$$

Hence

$$\int_{\Psi_k}|\frac{\partial P}{\partial x}(x,y)|^p\,dx\,dy \le C(2kn)^{2p}\int_{\Psi_k}|P(x,y)|^p\,dx\,dy. \tag{20}$$

Similarly,

$$\int_{\Psi_k}|\frac{\partial P}{\partial x}(x,y)|^p|x|\,dx\,dy \le C(2kn)^{2p}\int_{\Psi_k}|P(x,y)|^p|x|\,dx\,dy. \tag{21}$$

By (18) and (20) we know that $\mu_p(\Psi_k)\le 2k$ for $1\le p<\infty$. To prove the reverse inequality, define $\Xi_n(x,y)=yP_n^{(\omega,\omega)}(1-x^2)$. Here $P_n^{(\omega,\sigma)}$ denotes the Jacobi polynomial of degree n associated with parameters ω, σ. Then

$$\int_{\Psi_k}\left|\frac{\partial\Xi_n}{\partial y}(x,y)\right|^p\,dx\,dy=\int_0^1\left|P_n^{(\omega,\omega)}(t)\right|^p(1-t)^{k-1/2}\,dt,$$
$$\int_{\Psi_k}|\Xi_n(x,y)|^p\,dx\,dy=\frac{1}{p+1}\int_0^1\left|P_n^{(\omega,\omega)}(t)\right|^p(1-t)^{pk+k-1/2}\,dt.$$

It is known (see [10], Chap. VII) that

$$\int_0^1\left|P_n^{(\omega,\omega)}(x)\right|^p(1-x)^\gamma\,dx\sim n^{\omega p-2\gamma-2}\quad\text{whenever}\quad 2\gamma<\omega p-2+p/2. \tag{22}$$

If $2(p+1)k<\omega p-1+p/2$, then by (22),

$$\frac{\left\|\frac{\partial\Xi_n}{\partial y}\right\|_{L^p(\Lambda_k)}}{\|\Xi_n\|_{L^p(\Lambda_k)}}\sim n^{2k}.$$

Hence $\mu_p(\Psi_k)\ge 2k$.

Now we wish to prove that $\mu_p(\Upsilon_k)=2k+1$. Since $T(x,y)=(x,yx)$ maps Υ_k onto Ψ_k,

$$\int_{\Upsilon_k}|f(x,y)|^p\,dx\,dy=\int_{\Psi_k}|f(x,yx)|^p|x|\,dx\,dy. \tag{23}$$

Applying (19) and (21) to $Q(x,y):=P(x,yx)$, we find that

$$\int_{\Psi_k}|x\frac{\partial P}{\partial y}(x,yx)|^p|x|\,dx\,dy\le 2^{p+1}C^2(2kn)^{2kp}\int_{\Psi_k}|P(x,yx)|^p|x|\,dx\,dy, \tag{24}$$
$$\int_{\Psi_k}|\frac{\partial P}{\partial x}(x,yx)+y\frac{\partial P}{\partial y}(x,yx)|^p|x|\,dx\,dy\le 2^{p+1}C^2(2kn)^{2p}\int_{\Psi_k}|P(x,yx)|^p|x|\,dx\,dy. \tag{25}$$

In order to establish $\mu_p(\Upsilon_k)\le 2k+1$ it will be enough to prove that there exists a positive constant B such that

$$\int_{\Psi_k}|R(x,y)|^p|x|\,dx\,dy\le Bn^p\int_{\Psi_k}|xR(x,y)|^p|x|\,dx\,dy \tag{26}$$

for every $R\in\mathcal{P}_n(\mathbb{R}^2)$. If we write $R(x,y)=R_0(x^2,y)+xR_1(x^2,y)$, then

$$2\int_{\Psi_k}|R(x,y)|^p|x|\,dx\,dy=\int_S kr^{k-1}|R_0(t,r^k)+\sqrt{t}R_1(t,r^k)|^p\,dt\,dr+\int_S kr^{k-1}|R_0(t,r^k)-\sqrt{t}R_1(t,r^k)|^p\,dt\,dr.$$

By Lemma 2.1, we conclude that

$$\int_S r^{k-1}|R_0(t,r^k)|^p\,dt\,dr\le Cn^p\int_S r^{k-1}|\sqrt{t}R_0(t,r^k)|^p\,dt\,dr,$$
$$\int_S r^{k-1}|\sqrt{t}R_1(t,r^k)|^p\,dt\,dr\le Cn^p\int_S r^{k-1}|tR_1(t,r^k)|^p\,dt\,dr.$$

From this it follows that

$$2\int_{\Psi_k} |R(x,y)|^p |x|\,dxdy \le 2^p C n^p \left(\int_S r^{k-1} |\sqrt{t} R_0(t,r^k)|^p \,dtdr + \int_S r^{k-1} |t R_1(t,r^k)|^p \,dtdr \right).$$

Therefore

$$\int_{\Psi_k} |R(x,y)|^p |x|\,dxdy \le 2^p C n^p \int_{\Psi_k} |xR(x,y)|^p |x|\,dxdy.$$

From (23)-(26), it follows that $\mu_p(\Upsilon_k) \le 2k+1$. Let Ξ_n be given as above. If $2(p+1)k < \omega p - 2 - p/2$, then

$$\int_{\Upsilon_k} \left| \frac{\partial \Xi_n}{\partial y}(x,y) \right|^p \,dxdy = \int_0^1 \left| P_n^{(\omega,\omega)}(t) \right|^p (1-t)^k \,dt \sim n^{\omega p - 2k - 2},$$

$$\int_{\Upsilon_k} |\Xi_n(x,y)|^p \,dxdy = \frac{1}{p+1} \int_0^1 \left| P_n^{(\omega,\omega)}(t) \right|^p (1-t)^{pk+k+p/2} \,dt \sim n^{\omega p - 2(pk+k+p/2)-2}.$$

Hence $\mu_p(\Upsilon_k) \ge 2k+1$. Thus $\mu_p(\Upsilon_k) = 2k+1$. □

The second main theorem is as follows.

Theorem 3.2. *Let k be a natural number. Fix $0 \le a < b$, and let $\Lambda_k := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, ax^k \le y \le bx^k\}$. Then $\mu_p(\Lambda_k) = 2k$ for every $1 \le p \le \infty$.*

Proof. We note first that the L_∞ Markov exponent of Λ_k is known (see [6]). For $1 \le p < \infty$, using the change of variables $y = z^k$, we have

$$\int_{\Lambda_k} |P(x,y)|^p \,dxdy = \int_{V_{c,d}} kz^{k-1} |P(x,z^k)|^p \,dxdz, \tag{27}$$

where $c = \sqrt[k]{a}$ and $d = \sqrt[k]{b}$. In (10) of Lemma 2.2, let $\alpha = 0$, $\beta = k-1$ to conclude that

$$\int_{V_{c,d}} z^{k-1} |\frac{\partial P}{\partial x}(x,z^k)|^p \,dxdz \le C(kn)^{2p} \int_{V_{c,d}} z^{k-1} |P(x,z^k)|^p \,dxdz, \tag{28}$$

$$\int_{V_{c,d}} z^{k-1} |kz^{k-1} \frac{\partial P}{\partial y}(x,z^k)|^p \,dxdz \le C(kn)^{2p} \int_{V_{c,d}} z^{k-1} |P(x,z^k)|^p \,dxdz \tag{29}$$

for every $P \in \mathcal{P}_n(\mathbb{R}^2)$. Another application of Lemma 2.2 shows that

$$\int_{V_{c,d}} z^{k-1} |\frac{\partial P}{\partial y}(x,z^k)|^p \,dxdz \le C(kn)^{2kp-2p} \int_{V_{c,d}} z^{k-1} |z^{k-1} \frac{\partial P}{\partial y}(x,z^k)|^p \,dxdz. \tag{30}$$

Thus, by (27)-(30), we have $\mu_p(\Lambda_k) \le 2k$.

To prove $\mu_p(\Lambda_k) \ge 2k$, define $U_n(x,y) = y P_n^{(\alpha,\alpha)}(1-x)$. Then

$$\int_{\Lambda_k} \left| \frac{\partial U_n}{\partial y}(x,y) \right|^p \,dxdy = (b-a) \int_0^1 \left| P_n^{(\alpha,\alpha)}(t) \right|^p (1-t)^k \,dt,$$

$$\int_{\Lambda_k} |U_n(x,y)|^p \,dxdy = \frac{b^{p+1} - a^{p+1}}{p+1} \int_0^1 \left| P_n^{(\alpha,\alpha)}(t) \right|^p (1-t)^{(p+1)k} \,dt.$$

Proceeding as before, whenever $2(p+1)k < \alpha p - 2 + p/2$ we have

$$\frac{\left\| \frac{\partial U_n}{\partial y} \right\|_{L^p(\Lambda_k)}}{\|U_n\|_{L^p(\Lambda_k)}} \sim n^{2k}.$$

This completes the proof. □

Similarly, one can prove the following theorem.

Theorem 3.3. *Let k, l be natural numbers such that $l < k$. Fix $0 \le a < b$, and let $\Lambda_{k,l} := \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, ax^k \le y^l \le bx^k\}$. Then $\frac{2k}{l} \le \mu_p(\Lambda_{k,l}) \le 2k$ for every $1 \le p < \infty$.*

Note that the above result is valid for $p = \infty$ (see [6]).

In the last statement we turn our attention to more general types of cuspidal domains. Specifically, we replace x^k (in Ψ_k) by any convex function f such that $f(0) = f'(0) = 0$.

Theorem 3.4. *Let f be a real-valued convex function on the interval $[0,1]$. Suppose that $f(0)=f'(0)=0$, $f'(1)<\infty$ and $f(t)>0$ for $t\in(0,1]$. Let $K=\{(x,y)\in\mathbb{R}^2 : |x|\le 1,\ 0\le y\le f(x^2)\}$. Then for $1\le p<\infty$,*

$$\mu_p(K)\le \inf\{\tau>0 : \exists_{C>0}\,\forall_{n\in\mathbb{N}}\ n^2\le Cf'(1/n^2)n^{\tau}\}. \tag{31}$$

Moreover, if there exists a constant $I>0$ such that $I\cdot f(x)\ge xf'(x)$ then the above inequality becomes an equality.

Proof. If $P_0,P_1\in\mathcal{P}_n(\mathbb{R}^2)$, $P(x,y)=P_0(x^2,y)+xP_1(x^2,y)$, then

$$\int_K |P(x,y)|^p\,dxdy=\int_{K'}\frac{|P_0(t,y)+\sqrt{t}P_1(t,y)|^p}{2\sqrt{t}}\,dtdy+\int_{K'}\frac{|P_0(t,y)-\sqrt{t}P_1(t,y)|^p}{2\sqrt{t}}\,dtdy, \tag{32}$$

where $K'=\{(t,y)\in\mathbb{R}^2 : 0\le t\le 1,\ 0\le y\le f(t)\}$. For $n\in\mathbb{N}$, let

$$K'_n=\{(t,y)\in\mathbb{R}^2 : \frac{1}{n^2}\le t\le 1,\ 0\le y\le f(t)\}.$$

It follows from (7.17) of [5] that for every $\alpha>-1$ there exists a positive constant B such that if H is a polynomial in one variable of degree at most n, then

$$\int_0^1 t^{\alpha}|H(t)|^p\,dt\le B\int_{1/n^2}^1 t^{\alpha}|H(t)|^p\,dt.$$

Hence

$$\int_{K'} t^{\alpha}|Q(t,y)|^p\,dtdy\le B\int_{K'_n} t^{\alpha}|Q(t,y)|^p\,dtdy \tag{33}$$

for every Q in $\mathcal{P}_n(\mathbb{R}^2)$. Let $\eta_n=f(1/n^2)$, $\eta'_n=f'(1/n^2)$. Our assumptions guarantee that

$$T_n=:\{(t,y)\in\mathbb{R}^2 : \frac{1}{n^2}\le t\le 1,\ 0\le y\le \eta'_n t-\frac{\eta'_n}{n^2}\}\subset K'_n. \tag{34}$$

For $n\ge 2$, define

$$L_n:=\{(t,y)\in\mathbb{R}^2 : \frac{1}{n^2}\le t\le 1,\ \eta'_n t-\frac{\eta'_n}{n^2}\le y\le \eta'_n t-\frac{\eta'_n}{n^2}+f(1/4)\}\cap K'_n. \tag{35}$$

If we define $V:=\{(t,y)\in\mathbb{R}^2 : 1/4\le t\le 1,\ f(1/4)\le y\le f(t)\}$, then $K'_n\subset L_n\cup T_n\cup V$ for $n\ge 2$. Hence

$$\int_{K'_n} t^{\alpha}|Q(t,y)|^p\,dtdy\le \int_{L_n} t^{\alpha}|Q(t,y)|^p\,dtdy+\int_{T_n} t^{\alpha}|Q(t,y)|^p\,dtdy+\int_V t^{\alpha}|Q(t,y)|^p\,dtdy. \tag{36}$$

Since V is a locally Lipschitzian compact subset of $\mathbb{R}^2$, $\mu_p(V)=2$ (see [3]). Hence there exists a positive constant B_1 such that

$$\max\left\{\int_V |\frac{\partial Q}{\partial t}(t,y)|^p\,dtdy,\int_V |\frac{\partial Q}{\partial y}(t,y)|^p\,dtdy\right\}\le B_1(\deg Q)^{2p}\int_V |Q(t,y)|^p\,dtdy. \tag{37}$$

By the definition of V, we can write

$$\frac{1}{4}\int_V t^{\alpha}|Q(t,y)|^p\,dtdy\le \int_V |Q(t,y)|^p\,dtdy\le 4^{\alpha+1}\int_V t^{\alpha}|Q(t,y)|^p\,dtdy. \tag{38}$$

By (37) and (38),

$$\max\left\{\int_V t^{\alpha}|\frac{\partial Q}{\partial t}(t,y)|^p\,dtdy,\int_V t^{\alpha}|\frac{\partial Q}{\partial y}(t,y)|^p\,dtdy\right\}\le 4^{\alpha+2}B_1(\deg Q)^{2p}\int_V t^{\alpha}|Q(t,y)|^p\,dtdy. \tag{39}$$

Now consider the integral over T_n. Using similar ideas to those applied in the proof of Lemma 2.1, we can establish that there is a constant B_2, depending only on α, such that

$$\int_{T_n} t^{\alpha}|\frac{\partial Q}{\partial t}(t,y)|^p\,dtdy\le B_2(1-1/n^2)^{-p}(\deg Q)^{2p}\int_{T_n} t^{\alpha}|Q(t,y)|^p\,dtdy, \tag{40}$$

$$\int_{T_n} t^{\alpha}|\frac{\partial Q}{\partial y}(t,y)|^p\,dtdy\le B_2(\eta'_n-\frac{\eta'_n}{n^2})^{-p}(\deg Q)^{2p}\int_{T_n} t^{\alpha}|Q(t,y)|^p\,dtdy. \tag{41}$$

In order to deal with L_n, we define

$$L'_n:=\{(t,y)\in\mathbb{R}^2 : \frac{1}{n^2}\le t\le 1,\ 0\le y\le \eta'_2-\frac{\eta'_2}{4}+f(1/4)\}\cap K'_n,$$
$$l(\theta):=\{(t,y)\in\mathbb{R}^2 : y=\theta\},\quad \theta\in\mathbb{R}.$$

Then $l(\theta)$ intersects L_n' along a single line segment of lengths not smaller then some positive constant depending only on f for every $\theta \in [0, \eta_2' - \frac{\eta_2'}{4} + f(1/4)]$. Thus by using Theorem 7.4 of [5] along each of these segments implies that there exists a positive constant B_3 such that

$$\int_{L_n'} t^\alpha |\frac{\partial Q}{\partial t}(t,y)|^p \, dt dy \le B_3 (\deg Q)^{2p} \int_{L_n'} t^\alpha |Q(t,y)|^p \, dt dy.$$

By our assumptions, $L_n \subset L_n' \subset K_n'$. Therefore

$$\int_{L_n} t^\alpha |\frac{\partial Q}{\partial t}(t,y)|^p \, dt dy \le B_3 (\deg Q)^{2p} \int_{K_n'} t^\alpha |Q(t,y)|^p \, dt dy. \tag{42}$$

An illustration of L_n, T_n, L_n', with $f(x) = x^2$, $n = 3$ is shown in Figure 1.

Using the change of variables $s = y - \eta_n' t + \frac{\eta_n'}{n^2}$ and proceeding as before, one can verify that

$$\int_{L_n} t^\alpha |\frac{\partial Q}{\partial t}(t,y) + \eta_n' \frac{\partial Q}{\partial y}(t,y)|^p \, dt dy \le B_3 (\deg Q)^{2p} \int_{K_n'} t^\alpha |Q(t,y)|^p \, dt dy. \tag{43}$$

From (42), (43) and the convexity of $x \mapsto x^p$ for $p \ge 1$,

$$\int_{L_n} t^\alpha |\frac{\partial Q}{\partial y}(t,y)|^p \, dt dy \le B_3 \left(\frac{2}{\eta_n'}\right)^p (\deg Q)^{2p} \int_{K_n'} t^\alpha |Q(t,y)|^p \, dt dy. \tag{44}$$

By (32), (33), (39), (40), (41), (42) and (44), there exists a positive constant C such that

$$\max\left\{\int_K |\frac{\partial P}{\partial x}(x,y)|^p \, dx dy, \int_K |\frac{\partial P}{\partial y}(x,y)|^p \, dx dy\right\} \le C \left(\frac{n^2}{\eta_n'}\right)^p \int_K |P(x,y)|^p \, dx dy$$

for every P in $\mathcal{P}_n(\mathbb{R}^2)$. Therefore,

$$\mu_p(K) \le \inf\{\tau > 0 : \exists_{C>0} \forall_{n \in \mathbb{N}} \; n^2 \le C f'(1/n^2) n^\tau\}.$$

To prove the reverse inequality, we shall use Jacobi polynomials $P_n^{(\omega,\sigma)}$. An easy computation leads to

$$\int_{K_n'} t^\alpha |y P_n^{(\omega,\sigma)}(1-t)|^p \, dt dy = \int_{\frac{1}{n^2}}^1 (f(t))^{p+1} \frac{t^\alpha}{p+1} |P_n^{(\omega,\sigma)}(1-t)|^p \, dt = \int_0^{1-\frac{1}{n^2}} (f(1-t))^{p+1} \frac{(1-t)^\alpha}{p+1} |P_n^{(\omega,\sigma)}(t)|^p \, dt. \tag{45}$$

Using the change of variables $t = \cos\theta$, we have

$$\int_{K_n'} t^\alpha |y P_n^{(\omega,\sigma)}(1-t)|^p \, dt dy = \int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} \frac{(1-\cos\theta)^\alpha}{p+1} |P_n^{(\omega,\sigma)}(\cos\theta)|^p \sin\theta \, d\theta, \tag{46}$$

where $u_n = \arccos(1 - 1/n^2)$. Applying certain properties of Jacobi polynomials $P_n^{(\omega,\sigma)}$ verified in [10], (7.32.5), p. 169, we conclude that there exists a natural number n_0 so that

$$\int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} \frac{(1-\cos\theta)^\alpha}{p+1} |P_n^{(\omega,\sigma)}(\cos\theta)|^p \sin\theta \, d\theta \le \frac{\Lambda n^{-p/2}}{p+1} \int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} (1-\cos\theta)^\alpha \theta^{-\omega p - p/2} \sin\theta \, d\theta \tag{47}$$

for $n \ge n_0$ and appropriately adjusted constant Λ. The fact that $1 - \cos x \le \sin^2 x$ for $-\pi/2 \le x \le \pi/2$ allows us to conclude that

$$\int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} (1-\cos\theta)^\alpha \theta^{-\omega p - p/2} \sin\theta \, d\theta \le \int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} (\sin\theta)^{2\alpha+1} \theta^{-\omega p - p/2} \, d\theta. \tag{48}$$

Since $\sin x \le x$ for $x \ge 0$, we have

$$\int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} (\sin\theta)^{2\alpha+1} \theta^{-\omega p - p/2} \, d\theta \le \int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} \theta^{-\omega p - p/2 + 2\alpha + 1} \, d\theta. \tag{49}$$

Integration by parts gives us, for $-\omega p - p/2 + 2\alpha + 1 \ne -1$,

$$\int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1} \theta^{-\omega p - p/2 + 2\alpha + 1} \, d\theta = \left[\frac{(f(1-\cos\theta))^{p+1} \theta^{-\omega p - p/2 + 2\alpha + 2}}{-\omega p - p/2 + 2\alpha + 2}\right]_{u_n}^{\pi/2} + \int_{u_n}^{\frac{\pi}{2}} \frac{(p+1)(f(1-\cos\theta))^p \theta^{-\omega p - p/2 + 2\alpha + 2}}{\omega p + p/2 - 2\alpha - 2} f'(1-\cos\theta) \sin\theta \, d\theta. \tag{50}$$

If $-1 \le x \le 1$, then $\sqrt{1-x^2}\arccos x \le 2(1-x)$. Therefore,

$$\theta \sin\theta f'(1-\cos\theta) \le 2(1-\cos\theta) f'(1-\cos\theta).$$

Hence, by our assumption on f,

$$\theta \sin\theta f'(1-\cos\theta) \le 2I f(1-\cos\theta). \tag{51}$$

If $\omega p + p/2 - 2\alpha - 2 > 0$, then by (51),

$$\int_{u_n}^{\frac{\pi}{2}} \frac{(p+1)(f(1-\cos\theta))^p \theta^{-\omega p - p/2+2\alpha+2}}{\omega p + p/2 - 2\alpha - 2} f'(1-\cos\theta)\sin\theta \, d\theta \le \int_{u_n}^{\frac{\pi}{2}} \frac{2I(p+1)(f(1-\cos\theta))^{p+1}\theta^{-\omega p - p/2+2\alpha+1}}{\omega p + p/2 - 2\alpha - 2} \, d\theta. \tag{52}$$

Thus, by (50) and (52),

$$\int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1}\theta^{-\omega p - p/2+2\alpha+1} \, d\theta \le \frac{(\eta_n)^{p+1} u_n^{-\omega p - p/2+2\alpha+2}}{\omega p + p/2 - 2\alpha - 2 - 2I(p+1)} \tag{53}$$

whenever $\omega p + p/2 - 2\alpha - 2 > 2I(p+1)$. Hence, there exists Λ_1 such that

$$\int_{u_n}^{\frac{\pi}{2}} (f(1-\cos\theta))^{p+1}\theta^{-\omega p - p/2+2\alpha+1} \, d\theta \le \Lambda_1 (\eta_n)^{p+1} n^{\omega p + p/2 - 2\alpha - 2}. \tag{54}$$

Putting inequalities (45)-(54) and (33) together, we have

$$\int_{K'} t^\alpha |y P_n^{(\omega,\sigma)}(1-t)|^p \, dt\, dy \le \frac{B\Lambda\Lambda_1}{p+1}(\eta_n)^{p+1} n^{\omega p - 2\alpha - 2} \le \frac{B\Lambda\Lambda_1}{p+1}(\eta_n')^{p+1} n^{\omega p - 2\alpha - 4 - 2p}. \tag{55}$$

By our assumption,

$$\int_{K'} t^\alpha |P_n^{(\omega,\sigma)}(1-t)|^p \, dt\, dy \ge \int_0^{\frac{1}{n^2}} \int_0^{f(t)} t^\alpha |P_n^{(\omega,\sigma)}(1-t)|^p \, dy\, dt = \int_0^{\frac{1}{n^2}} f(t) t^\alpha |P_n^{(\omega,\sigma)}(1-t)|^p \, dt. \tag{56}$$

By making the change of variable $t = \frac{z}{2n^2}$, we obtain

$$\int_0^{\frac{1}{n^2}} f(t) t^\alpha |P_n^{(\omega,\sigma)}(1-t)|^p \, dt = \frac{1}{2n^2}\int_0^2 f(g_n(z))(g_n(z))^\alpha |P_n^{(\omega,\sigma)}(1-g_n(z))|^p \, dz, \tag{57}$$

where $g_n(z) = \frac{z}{2n^2}$. Again certain properties of Jacobi polynomials $P_n^{(\omega,\sigma)}(x)$ play a role. By the formula of Mehler-Heine type (see [10], Theorem 8.1.1.)

$$\frac{1}{2n^2}\int_0^2 f(g_n(z))(g_n(z))^\alpha |P_n^{(\omega,\sigma)}(1-g_n(z))|^p \, dz \ge \frac{n^{\omega p}}{4^p n^2}\int_0^2 f(g_n(z))(g_n(z))^\alpha (4(z/2)^{-\omega} J_\omega(z) - 1/\Gamma(\omega+2))^p \, dz$$

for $\omega > 0$ all sufficiently large n. Here $J_\omega(z)$ is the Bessel functions of the first kind. Since

$$\min_{z\in[0,2]}\{(z/2)^{-\omega} J_\omega(z)\} \ge \min_{z\in[0,2]}\left\{\frac{1}{\Gamma(\omega+1)} - \frac{z^2}{4\Gamma(\omega+2)}\right\} = \frac{1}{\Gamma(\omega+1)} - \frac{1}{\Gamma(\omega+2)} = \frac{\omega}{\Gamma(\omega+2)},$$

we have

$$\frac{1}{2n^2}\int_0^2 f(g_n(z))(g_n(z))^\alpha |P_n^{(\omega,\sigma)}(1-g_n(z))|^p \, dz \ge \left(\frac{4\omega-1}{4\Gamma(\omega+2)}\right)^p n^{\omega p - 2}\int_0^2 f(g_n(z))(g_n(z))^\alpha \, dz. \tag{58}$$

Then integration by parts shows that

$$\int_0^2 f(g_n(z))(g_n(z))^\alpha \, dz = \frac{2\eta_n}{n^{2\alpha}(\alpha+1)} - \frac{1}{2n^2}\int_0^2 \frac{z}{\alpha+1} f'(g_n(z))(g_n(z))^\alpha \, dz.$$

Since $If(x) \ge x f'(x)$, this leads to

$$\int_0^2 f(g_n(z))(g_n(z))^\alpha \, dz \ge \frac{2\eta_n}{n^{2\alpha}(I+\alpha+1)} \ge \frac{2\eta_n'}{n^{2\alpha+2} I(I+\alpha+1)}. \tag{59}$$

From (56), (57), (58) and (59) we see that

$$\int_{K'} t^\alpha |P_n^{(\omega,\sigma)}(1-t)|^p \, dt\, dy \ge \left(\frac{4\omega-1}{4\Gamma(\omega+2)}\right)^p \frac{2n^{\omega p - 2}\eta_n'}{n^{2\alpha+2} I(I+\alpha+1)}$$

This last inequality together with (55) imply that there exists a positive constant λ depending only on f, α and ω such that, for each n,

$$\int_{K'} t^{\alpha}|P_n^{(\omega,\sigma)}(1-t)|^p \, dt dy \geq \lambda \frac{n^{2p}}{(\eta_n')^p} \int_{K'} t^{\alpha}|yP_n^{(\omega,\sigma)}(1-t)|^p \, dt dy.$$

Since

$$\int_K |P_n^{(\omega,\sigma)}(1-x^2)|^p \, dx dy = \int_{K'} t^{-1/2}|P_n^{(\omega,\sigma)}(1-t)|^p \, dt dy$$

$$\int_K |yP_n^{(\omega,\sigma)}(1-x^2)|^p \, dx dy = \int_{K'} t^{-1/2}|yP_n^{(\omega,\sigma)}(1-t)|^p \, dt dy,$$

it follows that $\mu_p(K) \geq \inf\{\tau > 0 : \exists_{C>0} \forall_{n\in\mathbb{N}} \; n^2 \leq Cf'(1/n^2)n^{\tau}\}$. □

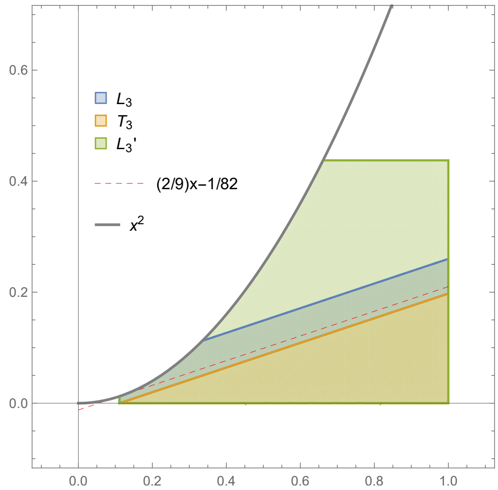

Figure 1: L_3, T_3, L_3', with $f(x) = x^2$ and the tangent line to the curve $y = x^2$ at the point $\frac{1}{9}$.

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