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On Optimal Points for Interpolation by Univariate Exponential Functions

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Abstract

We discuss the aysmptotics of the points that maximize the determinant of the interpolation matrix for interpolants of the form $I_1(x) = \sum_{i=1}^{n} a_i e^{\alpha x t_i}$ and $I_2(x) = \sum_{i=1}^{n} a_i e^{-\beta(x-t_i)^2}$.

Suppose that we are given a set of *nodes* $t_1 < t_2 < \cdots < t_n \in [a,b]$ and a set of interpolation *sites* $s_1 < s_2 < \cdots < s_n \in [a,b]$ and a kernel

$$K : [a,b]^2 \to \mathbb{R}.$$

For values $y_1, y_2, \cdots, y_n \in \mathbb{R}$ we may attempt to interpolate these values y at the sites s using the basis function

$$K_j(x) := K(x,t_j),\ 1 \le j \le n,$$

i.e., find

$$I_K(x) := \sum_{j=1}^{n} a_j K_j(x) \tag{1}$$

with the property that

$$I_K(s_i) = y_i,\ 1 \le i \le n. \tag{2}$$

In this note we will consider the two kernels

$$K_1(x,y) := e^{\alpha xy},\ \alpha > 0, \tag{3}$$

which results in an interpolation by *exponential ridge* functions and

$$K_2(x,y) := e^{-\beta(x-y)^2},\ \beta > 0, \tag{4}$$

which gives an interpolation by a gaussian *radial basis function*.

Of course the interpolants (1) and (2) will exist and be unique if and only if the interpolation matrix

$$M_K(s,t) := [K(s_i,t_j)] \in \mathbb{R}^{n\times n} \tag{5}$$

is non-singular. Of particular interest, from the computational point of view, would be to know for which nodes and sites the matrix $M_K(s,t)$ is as well-conditioned as possible. However, this is likely a forbiddingly difficult problem and hence it is reasonable to ask for which sites and nodes

$$\det(M_K(s,t))$$

is as large as possible, giving an analogue of the classical Fekete points for polynomial interpolation. Note that the choice of $K(x,y) = (x-y)^{n-1}$ results in classical polynomial interpolation, in which case $M_K(s,t)$ is equivalent to the classical Vandermonde matrix and

$$\det(M_K(s,t)) = a_n V(s) V(t) \tag{6}$$

where

$$a_n = \prod_{j=0}^{n-1} \binom{n-1}{j}$$

and

$$V(x) := \prod_{1\le i<j\le n} (x_j - x_i) \tag{7}$$

is the classical Vandermonde determinant. To see this, just note that by the binomial theorem we may write the matrix

$$[(s_i - t_j)^{n-1}] = S_n \times T_n$$

where

$$S_n = \begin{bmatrix} 1 & s_1 & s_1^2 & \cdot & \cdot & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \cdot & \cdot & s_2^{n-1} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 1 & s_n & s_n^2 & \cdot & \cdot & s_n^{n-1} \end{bmatrix}$$

and

$$T_n = \begin{bmatrix} \binom{n-1}{n-1}(-t_1)^{n-1} & \binom{n-1}{n-1}(-t_2)^{n-1} & \cdot & \cdot & \cdot & \binom{n-1}{n-1}(-t_n)^{n-1} \\ \binom{n-1}{n-2}(-t_1)^{n-2} & \binom{n-1}{n-2}(-t_2)^{n-2} & \cdot & \cdot & \cdot & \binom{n-1}{n-2}(-t_n)^{n-2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \binom{n-1}{1}(-t_1)^1 & \binom{n-1}{1}(-t_2)^1 & \cdot & \cdot & \cdot & \binom{n-1}{1}(-t_n)^1 \\ 1 & 1 & & & & 1 \end{bmatrix}.$$

S_n is the classical Vandermonde matrix and hence $\det(S_n) = V(s)$. Further, factoring the common factors from each of the rows, we have

$$\begin{aligned} \det(T_n) &= (-1)^{1+2+\cdots+(n-1)} \prod_{j=0}^{n-1} \binom{n-1}{j} \begin{vmatrix} t_1^{n-1} & t_2^{n-1} & \cdot & \cdot & \cdot & t_n^{n-1} \\ t_1^{n-2} & t_2^{n-2} & \cdot & \cdot & \cdot & t_n^{n-2} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ t_1^1 & t_2^1 & \cdot & \cdot & \cdot & t_n^1 \\ 1 & 1 & & & & 1 \end{vmatrix} \\ &= (-1)^{n(n-1)} \left\{ \prod_{j=0}^{n-1} \binom{n-1}{j} \right\} V(t), \end{aligned}$$

after suitably reordering the columns. The formula (6) follows by noting that $n(n-1)$ is always even.

The classical Fekete points for polynomial interpolation are those points $f_1 \le f_2 \le \cdots \le f_n \in [a,b]$ which maximize $V(x)$, $x \in [a,b]^n$. As is well known (see e.g. [1]), they tend weak-* to the arcsine measure for the interval $[a,b]$, i.e., the discrete measures

$$\mu_f^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \delta_{f_i} \tag{8}$$

have the property that, for every $g \in C[a,b]$,

$$\lim_{n\to\infty} \int_a^b g(x) d\mu_f^{(n)} = \int_a^b g(x) d\mu^*$$

where

$$d\mu^* = \frac{1}{\pi} \frac{1}{\sqrt{(b-x)(x-a)}} dx \tag{9}$$

is the arcsine measure for the interval $[a,b]$.

In this note we will prove the following theorem.

Theorem 1. *Suppose that the kernels $K_1(s,t)$ and $K_2(s,t)$ are given by* (3) *and* (4)*, respectively. Suppose further that $\widehat{s}_1 < \widehat{s}_2 < \cdots < \widehat{s}_n \in [a,b]$ are points which maximize either* *(a)* $\det(M_{K_1}(s,t))$, $s \in [a,b]^n$, *where* $t \in [a,b]^n$ *are fixed but distinct*

(b) $\det(M_{K_1}(s,s))$, $s \in [a,b]^n$

(c) $\det(M_{K_2}(s,t))$, $s \in [a,b]^n$, *where* $t \in [a,b]^n$ *are fixed but distinct*

(d) $\det(M_{K_2}(s,s))$, $s \in [a,b]^n$.

Then the discrete measures $\mu_{\widehat{s}}^{(n)}$ *(cf.* (8)*) tend weak-* to the arcsine measure* μ^* *given by* (9).

We remark that, in contrast, for radial basis interpolation by basis functions of the form $g(|x|)$ with $g'(0) \neq 0$, the optimal points are asymptotically *uniformly* distributed; see [3] or [2].

Proof. We first consider the exponential ridge kernel $K_1(x,y) = e^{\alpha xy}$ with $\alpha > 0$. Note that we write

$$K_1(x,y) = e^{x'y'}$$

where $x' := \sqrt{\alpha}x$ and $y' = \sqrt{\alpha}y$.

Then, by the remarkable formula (3.15) of Gross and Richards [5], we have

$$\begin{aligned}\det(M_{K_1}(s,t)) &= \det([e^{s_i't_j'}]) \\ &= \beta_n^{-1} V(s')V(t') \int_{U(n)} e^{\mathrm{tr}(s'ut'u^*)} du\end{aligned}$$

where

$$\beta_n := \prod_{j=1}^{n} (j-1)!$$

and the integral is over $U(n)$ the group of complex unitary matrices with Haar measure normalized to have volume 1. Here u^* denotes the conjugate transpose of the matrix $u \in U(n)$. By an abuse of notation, in the integrand, s' and t' are the $n \times n$ diagonal matrices with the elements s_i' and t_j' on the diagonal, respectively.

Now, note that

$$\begin{aligned}V(s') &= \prod_{1 \le i < j \le n} (s_j' - s_i') \\ &= \prod_{1 \le i < j \le n} \sqrt{\alpha}(s_j - s_i) \\ &= (\sqrt{\alpha})^{n(n-1)/2} \prod_{1 \le i < j \le n} (s_j - s_i) \\ &= (\sqrt{\alpha})^{n(n-1)/2} V(s).\end{aligned}$$

Similarly,

$$V(t') = (\sqrt{\alpha})^{n(n-1)/2} V(t).$$

Further,

$$\mathrm{tr}(s'ut'u^*) = \alpha \mathrm{tr}(sutu^*)$$

and thus we have

$$\det(M_{K_1}(s,t)) = \beta_n^{-1} \alpha^{n(n-1)/2} V(s)V(t) \int_{U(n)} e^{\alpha \mathrm{tr}(sutu^*)} du. \tag{10}$$

Now, as in condition (a), let $t_1 < t_2 < \cdots < t_n \in [a,b]$ be fixed, and $\widehat{s}_1 < \widehat{s}_2 < \cdots < \widehat{s}_n \in [a,b]$ be a set of points which maximizes $\det(K_1(s,t))$ for $s \in [a,b]^n$ (we do not claim that they are unique). We will use the Gross-Richards formula (10) to show that

$$\lim_{n\to\infty} V(\widehat{s})^{1/\binom{n}{2}} = \delta([a,b]), \tag{11}$$

the *transfinite diameter* of the interval $[a,b]$. It is known (see e.g. [1]) that this is sufficient for the claim of the theorem.

First consider the integral term of (10). Coope and Rinaud ([4, Thm. 4.1]) have shown that

$$\mathrm{tr}(sutu^*) \le \sum_{i=1}^{n} s_i t_i \tag{12}$$

for $u \in U(n)$. It follows that $\mathrm{tr}(sutu^*) \le n\max\{a^2,b^2\}$. From their Cor. 4.2 it follows that $\mathrm{tr}(sutu^*) \ge n\min\{a^2,b^2,ab\}$, i.e.,

$$n\min\{a^2,b^2,ab\} \le \mathrm{tr}(sutu^*) \le n\max\{a^2,b^2\}. \tag{13}$$

Setting $F_n(s) := \int_{U(n)} e^{\alpha \mathrm{tr}(sutu^*)} du$, it follows that

$$e^{\alpha n \min\{a^2,b^2,ab\}} \le F_n(s) \le e^{\alpha n \max\{a^2,b^2\}}.$$

In particular

$$\lim_{n\to\infty} F_n(s)^{1/\binom{n}{2}} = 1 \tag{14}$$

for *any* set of points $s \in [a,b]^n$.

Now, rewrite (10) as

$$V(s) = \frac{c_n}{F_n(s)} \det(M_{K_1}(s,t)),$$

where $c_n := \beta_n \alpha^{-n(n-1)/2}/V(t)$, and let $f_1 < f_2 < \cdots < f_n \in [a,b]$ be the classical Fekete points for the interval $[a,b]$, i.e., those such that $V(s) \le V(f)$, $\forall s \in [a,b]^n$. Then,

$$\begin{aligned} V(s^*) &\le V(f) \\ &= \frac{c_n}{F_n(f)} \det(M_{K_1}(f,t)) \\ &\le \frac{c_n}{F_n(f)} \det(M_{K_1}(s^*,t)) \\ &= \frac{F_n(s^*)}{F_n(f)} \frac{c_n}{F_n(s^*)} \det(M_{K_1}(s^*,t)) \\ &= \frac{F_n(s^*)}{F_n(f)} V(s^*). \end{aligned}$$

In other words,

$$\frac{F_n(f)}{F_n(s^*)} V(f) \le V(s^*) \le V(f). \tag{15}$$

Since

$$\lim_{n\to\infty} V(f)^{1/\binom{n}{2}} = \delta([a,b]),$$

it follows from (14) that we have (11) and the result follows for case (a).

The proof of (b) is very similar. In this case we re-write (10)

$$V^2(s) = \frac{\beta_n}{\alpha^{n(n-1)/2} F_n(s)} \det(M_{K_1}(s,s))$$

and by the same manipulations as above, we obtain

$$\frac{F_n(f)}{F_n(s^*)} V^2(f) \le V^2(s^*) \le V^2(f). \tag{16}$$

Taking $1/(2\binom{n}{2})$th roots gives the result.

To see (c) and (d), notice that

$$e^{-\beta(x-y)^2} = e^{-\beta x^2} e^{2\alpha xy} e^{-\beta y^2}$$

so that

$$M_{K_2}(s,t) = \mathrm{diag}(e^{-\beta s_i^2}) M_{K_1}(s,t) \mathrm{diag}(e^{-\beta t_j^2})$$

where the kernel $K_1(x,y) = e^{\alpha xy}$ with $\alpha := 2\beta$. It follows that

$$a_n \det(M_{K_1}(s,t)) \leq \det(M_{K_2}(s,t)) \leq b_n \det(M_{K_1}(s,t))$$

where

$$a_n = \det(\mathrm{diag}(e^{-\beta s_i^2})) \geq e^{-n\beta \max\{a^2,b^2\}}$$

and

$$b_n = \det(\mathrm{diag}(e^{-\beta t_j^2})) \leq e^{-n\beta \min\{a^2,b^2\}}.$$

Consequently, the inequalities (15) and (16) allow us to reduce the cases of (c) and (d) to (a) and (b) respectively, and we are done. □

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