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A solution of the problem of inverse approximation for the sampling Kantorovich operators in case of Lipschitz functions

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Abstract

The study of inverse results of approximation for the family of sampling Kantorovich operators in case of α-Hölder function, $0 < \alpha < 1$, has been solved in a recent paper of some of the authors. However, the limit case of Lipschitz functions, i.e., when $\alpha = 1$, in which standard methods fail, remained unsolved. In this paper, a solution of the above open problem of inverse approximation has been proposed.

1 Introduction

The sampling Kantorovich operators K_w^χ, based upon the kernel function χ, have been widely studied by many authors in the recent years (see, e.g, [25, 4, 8, 20, 15, 12]). The above operators have been introduced in [3] as an L^p version of the generalized sampling series

$$\sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi\left(wx-k\right) \qquad \text{(I)}$$

where $w > 0$ and $\chi : \mathbb{R} \to \mathbb{R}$ is a suitable kernel function (see, e.g., [7, 23, 22]), in the same spirit of the works of Kantorovich [14] and Lorentz [19], i.e., by replacing sample values $f(k/w)$ by mean values of the form $w\int_{k/w}^{(k+1)/w} f(u)\,du$. We recall that Kantorovich-type operators allow to approximate not necessarily continuous functions (see, e.g., [17, 18, 21, 24]).

The latter peculiarity revealed to be very suitable in order to study a wide rage of applications in various fields related to applied mathematics, like image [2, 9] and signal processing [16].

One of the most complicated tasks in approximation theory is the study of inverse results of approximation. The so-called saturation problem belongs to the class of inverse results, which substantially deals with the determination of the best possible order of approximation that can be achieved by a family of operators in a certain class of functions. For what concerns the sampling Kantorovich operators, the latter problem has been solved in [10] in the space $C(\mathbb{R})$ of the uniformly continuous and bounded functions.

Another important question that can be classified among the inverse results is of course the study of the regularity properties of a function f, when its order of approximation, by means of $K_w^\chi f$, is known.

A partial solution to the above question has been given always in [10], where, under suitable assumptions of the kernel χ, it has been proved that:

$$\left\|K_w^\chi f - f\right\|_\infty = O\left(w^{-\alpha}\right), \quad as \quad w\to+\infty \quad \Longrightarrow \quad f\in \mathrm{Lip}_\alpha(\mathbb{R}),$$

for every $0 < \alpha < 1$, where:

$$\mathrm{Lip}_\alpha(\mathbb{R}) := \left\{f\in C(\mathbb{R}) : \|f(\cdot)-f(\cdot+t)\|_\infty = O(|t|^\alpha),\ \text{as}\ t\to 0\right\}.$$

The above proof was inspired by the paper of Becker [5] in case of the Bernstein polynomials; however, the above approach fails for the limit case $\alpha = 1$, how usually happens also for other well-known families of approximation operators.

Other possible standard methods for the proof of inverse results, such as that one based on the telescopic sums [6], show the same limit, namely, they fail for the case $\alpha = 1$.

The fact that, in general, the techniques for the proof of inverse results of approximation do not work in the limit case, was remarked and highlighted many years ago by DeVore in his book [13], and this has been confirmed also in our case. This fact happens when the approximation operators do not satisfy a *Bernstein inequality*, i.e., when it is not possible to prove that the norm of the first derivative of the operator is less or equal to a quantity depending only from the norm of the operator itself.

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In this paper, we focus our attention to the above open problem, providing a proof of the inverse result in the limit case $\alpha = 1$.

In order to reach our main theorem (Theorem 7) we need to recall some preliminary results (see Section 2) mainly based on the notion of averaged-kernel originally introduced in [1], and we need a necessary and sufficient condition (Theorem 6) involving the first derivatives of our operators.

2 Notations and preliminary results

In this section we recall some definitions and preliminary results which will be useful in the paper.

Let $\Pi = \{t_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that $-\infty < t_k < t_{k+1} < +\infty$ for every $k \in \mathbb{Z}$, $\lim_{k\to\pm\infty} t_k = \pm\infty$ and such that there are two positive constants Δ, δ with $\delta \le t_{k+1} - t_k \le \Delta$. We also define $\Delta_k := t_{k+1} - t_k$, for every $k \in \mathbb{Z}$.

Definition 1. A function $\chi : \mathbb{R} \to \mathbb{R}$ is called a kernel if it belongs to $L^1(\mathbb{R})$, is bounded in a neighbourhood of the origin, and satisfies the following conditions:

(χ_1) for every $x \in \mathbb{R}$

$$\sum_{k\in\mathbb{Z}} \chi(u - t_k) = 1,$$

where the series $\sum_{k\in\mathbb{Z}} \chi(u - t_k)$ converges uniformly on the compact subsets of $\mathbb{R}$;

(χ_2) for some $\beta > 0$,

$$m_{\beta,\Pi}(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u - t_k)| \, |u - t_k|^{\beta} < +\infty.$$

Now, we recall the definition of the operators studied in this paper.

Definition 2. We define by $\left(K_w^{\chi}\right)_{w>0}$ the family of operators defined by

$$\left(K_w^{\chi} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi(wx - t_k) \left[\frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du\right], \; x \in \mathbb{R}$$

where $f : \mathbb{R} \to \mathbb{R}$ is locally integrable function such that the above series is a convergent for every $x \in \mathbb{R}$.

We recall the following lemma.

Lemma 3. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be any function which satisfies (χ_2) and that is bounded in a neighborhood of the origin. Then:*

$$m_{\nu,\Pi}(\chi) < +\infty \tag{1}$$

for every $0 \le \nu \le \beta$.

For a proof of Lemma 3, see, e. g., [3].

Now, we denote by $M(\mathbb{R})$ the linear space of all Lebesgue measurable real functions defined on $\mathbb{R}$, by $C(\mathbb{R})$ the space of bounded and uniformly continuous functions, and by $C^1(\mathbb{R})$ the space of bounded and uniformly continuous functions, with first derivative $f' \in C(\mathbb{R})$.

Proposition 4. *Let χ be a kernel belonging to $C^1(\mathbb{R})$, such that:*

(χ_3) *the series $\sum_{k\in\mathbb{Z}} |\chi'(wx - t_k)|$, $x \in \mathbb{R}$, $w > 0$ is uniformly convergent on the compact subsets of $\mathbb{R}$, with respect to the variable x.*

Then, for any bounded and locally integrable $f \in M(\mathbb{R})$ we have that $K_w^{\chi} f \in C^1(\mathbb{R})$ and:

$$\frac{d}{dx}\left(K_w^{\chi} f\right)(x) = w\left(K_w^{\chi'} f\right)(x)$$

for every $x \in \mathbb{R}$.

Proof. The proof is an immediate consequence of the classical theorem of term by term differentiation of series (see also [10]). □

We also need the following property:

Proposition 5. *For any bounded kernel χ, it turns out that the function*

$$\chi_s(\cdot) := \chi(\cdot + s), \; \forall s \in \mathbb{R}$$

is a kernel.

Proof. The proof follows immediately observing that

$$\int_{\mathbb{R}} |\chi_s(u)|\, du = \int_{\mathbb{R}} |\chi(u + s)|\, du = \|\chi\|_1$$

so $\chi_s \in L^1(\mathbb{R})$. Furthermore

$$\sum_{k\in\mathbb{Z}} \chi_s(u - t_k) = \sum_{k\in\mathbb{Z}} \chi(u + s - t_k) = 1$$

for every $u \in \mathbb{R}$, in view of (χ_1) since χ is a kernel. Finally, if χ satisfies condition (χ_2) for some $\beta \geq 1$, we have

$$\begin{aligned}
\sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u-t_k|^\beta &= \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u+s-t_k-s|^\beta \\
&\leq \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, (|u+s-t_k|+|s|)^\beta \\
&= 2^\beta \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)| \left(\frac{|u+s-t_k|}{2} + \frac{|s|}{2}\right)^\beta \\
&\leq 2^{\beta-1} \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u+s-t_k|^\beta \\
&+ 2^{\beta-1} |s|^\beta \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|,
\end{aligned}$$

where the last inequality follows by the convexity of the function $|\cdot|^\beta$, $\beta \geq 1$. Then we just have to notice that

$$\begin{aligned}
\sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u+s-t_k|^\beta \ &= \ \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u+s-t_k)|\, |u+s-t_k|^\beta \\
&= m_{\beta,\Pi}(\chi) \ < \ +\infty,
\end{aligned}$$

in view of assumption $(\chi 2)$, and similarly,

$$\sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)| \ = m_{0,\Pi}(\chi) \ < \ +\infty,$$

by Lemma 3.

If $0 < \beta < 1$ we have that the function $|\cdot|^\beta$ is concave and then subadditive, so we can write

$$\begin{aligned}
\sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u-t_k|^\beta &= \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u+s-t_k-s|^\beta \\
&\leq \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, (|u+s-t_k|+|s|)^\beta \\
&\leq \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|\, |u+s-t_k|^\beta \\
&+ |s|^\beta \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi_s(u-t_k)|
\end{aligned}$$

and, arguing as in the previous part, we get the statement. □

Following the calculation as in [1], it is quite simple to prove that, if χ is a continuous kernel, then the averaged type kernel

$$\overline{\chi}_s(t) := \frac{1}{s} \int_0^s \chi(u+t)\, du, \ s > 0 \tag{2}$$

turns out to be a differentiable kernel with:

$$\overline{\chi}'_s(t) = \chi(t+s) - \chi(t). \tag{3}$$

3 Main Theorems

In order to establish our main result, we first recall the following definition:

$$\mathrm{Lip}_1(\mathbb{R}) \ := \ \left\{ f \in C(\mathbb{R}) : \|f(\cdot) - f(\cdot + t)\|_\infty = O(|t|), \text{ as } t \to 0 \right\}.$$

Now, we can prove the following necessary and sufficient condition.

Theorem 6. *Let $f \in C(\mathbb{R})$ and $\chi \in C(\mathbb{R})$ be fixed. Assume that*

$$\left\| K_w^\chi f - f \right\|_\infty = O\left(w^{-1}\right) \tag{4}$$

as $w \to +\infty$. Then, if we consider the kernel $\overline{\chi}_s(t)$, for some fixed $s \geq 1$, we have:

$$\left\| w K_w^{\overline{\chi}'_s} f \right\|_\infty = O(1), \ \text{as } w \to +\infty \qquad \Longleftrightarrow \qquad f \in \mathrm{Lip}_1(\mathbb{R}).$$

Proof. Let $s \geq 1$ be fixed. We first prove the implication $(\Rightarrow)$. By (4) and condition $\left\| w K_w^{\overline{\chi}'_s} f \right\|_\infty = O(1)$, as $w \to +\infty$, there exist $M, C > 0$ and some $\overline{w} > 0$ such that for all $w \geq \overline{w}$ we have

$$\left\| K_w^\chi f - f \right\|_\infty \leq M w^{-1},$$

and,

$$\left\| wK_w^{\overline{\chi}_s'} f \right\|_\infty \leq C. \tag{5}$$

Let now $\overline{\gamma} := \overline{w}^{-1}$. Then for all $x, y \in \mathbb{R}$, with $x < y$ (if $x = y$ is trivial) such that $|x - y| \leq \overline{\gamma}$ there exists some $w \geq \overline{w}$ such that, with the above $s \geq 1$, the equality

$$y - x = sw^{-1},$$

holds. Then we have

$$\begin{aligned} f(y) - f(x) = &\left[f(y) - \left(K_w^\chi f \right)(y) \right] + \left[\left(K_w^\chi f \right)(x) - f(x) \right] \\ &+ \left(K_w^\chi f \right)(y) - \left(K_w^\chi f \right)(x) = I_1 + I_2 + I_3. \end{aligned} \tag{6}$$

We immediately obtain

$$|I_1| \leq Mw^{-1} = \frac{M}{s}(y - x), \ |I_2| \leq Mw^{-1} = \frac{M}{s}(y - x).$$

Let us analyze I_3. By (2) and (3) we can observe that

$$\begin{aligned} \left(K_w^\chi f \right)(y) - \left(K_w^\chi f \right)(x) = &\sum_{k \in \mathbb{Z}} \left[\chi(wy - t_k) - \chi(wx - t_k) \right] \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \\ = &\sum_{k \in \mathbb{Z}} \left[\chi(wx - t_k + s) - \chi(wx - t_k) \right] \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \\ = &\sum_{k \in \mathbb{Z}} \overline{\chi}_s'(wx - t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \\ = &w^{-1} \left(wK_w^{\overline{\chi}_s'} f \right)(x). \end{aligned}$$

Thus we can conclude that

$$|f(y) - f(x)| \leq 2\frac{M}{s}(y - x) + w^{-1} \left\| wK_w^{\overline{\chi}_s'} f \right\|_\infty$$

but, since $sw^{-1} = y - x$, we finally get:

$$|f(y) - f(x)| \leq \left(2\frac{M}{s} + \frac{C}{s} \right)(y - x)$$

and so the thesis.

Now we prove ($\Leftarrow$). Then, for any $x \in \mathbb{R}$ and sufficiently large w we have:

$$\begin{aligned} \left(wK_w^{\overline{\chi}_s'} f \right)(x) = &w \sum_{k \in \mathbb{Z}} \left[\chi(wx - t_k + s) - \chi(wx - t_k) \right] \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\, du \\ = &w \left[\left(K_w^\chi f \right)\left(x + \frac{s}{w} \right) - f\left(x + \frac{s}{w} \right) \right] - w \left[\left(K_w^\chi f \right)(x) - f(x) \right] \\ &+ w \left[f\left(x + \frac{s}{w} \right) - f(x) \right] \end{aligned}$$

and so, by (4) and the lipschitzianity of f we get, for suitable positive constants M_1, M_2, and $w > 0$ sufficiently large, that

$$\left\| wK_w^{\overline{\chi}_s'} f \right\|_\infty \leq 2M_1 + sM_2,$$

and so the thesis. □

Now we can prove our main theorem, i.e., the inverse result of approximation for the sampling Kantorovich operators.

Theorem 7. *Let $f \in C(\mathbb{R})$ and $\chi \in C(\mathbb{R})$ be a kernel. Assume that*

$$\left\| K_w^\chi f - f \right\|_\infty = O\left(w^{-1} \right) \tag{7}$$

and there exists $s \geq 1$ such that

$$\left\| K_w^{\chi_s} f - f \right\|_\infty = O\left(w^{-1} \right), \ w \to +\infty. \tag{8}$$

Then $f \in \mathrm{Lip}_1(\mathbb{R})$.

Proof. By Theorem 6, it is sufficient to show that for all $x \in \mathbb{R}$ we have

$$w \left(K_w^{\overline{\chi}_s'} f \right)(x) = O(1),$$

as $w \to +\infty$ for the averaged kernel $\overline{\chi}_s$. We have that:

$$\begin{aligned} w\left(K_w^{\overline{\chi}_s'} f\right)(x) =& w \sum_{k\in\mathbb{Z}} [\chi(wx-t_k+s)-\chi(wx-t_k)] \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du \\ =& w\left[\sum_{k\in\mathbb{Z}} \chi(wx-t_k+s) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du - f(x)\right] \\ &- w\left[\sum_{k\in\mathbb{Z}} \chi(wx-t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du - f(x)\right] \end{aligned}$$

and, for every sufficiently large w, we obtain:

$$\begin{aligned} \left| w\left(K_w^{\overline{\chi}_s'} f\right)(x)\right| \leq& w \left| \sum_{k\in\mathbb{Z}} \chi_s(wx-t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du - f(x)\right| \\ &+ w \left| \sum_{k\in\mathbb{Z}} \chi(wx-t_k) \frac{w}{\Delta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du - f(x)\right| \\ =& w\left|\left(K_w^{\chi_s} f\right)(x) - f(x)\right| + w\left|\left(K_w^{\chi} f\right)(x) - f(x)\right| \\ \leq& M_1 + M_2, \end{aligned}$$

where M_1, $M_2 > 0$ are the absolute constants arising from assumptions (7) and (8). This concludes the proof. □

Remark 8. *We stress that assumption (8) is not restrictive. Indeed, it is well-known that (see [4]) if χ is a given kernel satisfying assumption $(\chi 2)$ with $\beta \geq 1$, the following quantitative estimate holds:*

$$\left\|K_w^{\chi} f - f\right\|_\infty \leq \left[\frac{3}{2} m_{0,\Pi}(\chi) + m_{1,\Pi}(\chi)\right] \omega\left(f, \frac{1}{w}\right),$$

for every sufficiently large w, where $\omega\left(f, \frac{1}{w}\right)$ is the modulus of continuity of f. In view of the above estimate, and recalling the computation performed in the proof of Proposition 5 we can deduce that, for any fixed $s \geq 1$:

$$\begin{aligned} \left\|K_w^{\chi_s} f - f\right\|_\infty \;\leq&\; \left[\frac{3}{2} m_{0,\Pi}(\chi_s) + m_{1,\Pi}(\chi_s)\right] \omega\left(f, \frac{1}{w}\right) \\ \leq&\; \left[\left(\frac{3}{2} + s\right) m_{0,\Pi}(\chi) + m_{1,\Pi}(\chi)\right] \omega\left(f, \frac{1}{w}\right), \end{aligned}$$

as $w \to +\infty$, i.e., the order of approximation of f by means of the sampling Kantorovich operators based upon χ and χ_s depends only from f. Now, since in assumption (7) we assume that the order of approximation in case of the kernel χ for a fixed f is $1/w$, we expect that in case of χ_s the order is at least the same. Obviously, from the above inequality, it turns out that the converse of Theorem 7 holds.

In conclusion we recall that, several examples of kernels satisfying assumption $(\chi 2)$ with $\beta \geq 1$ can be found, e.g., in [2, 11].

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