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Computation of the Bell-Laplace transforms

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Abstract

An extension of the Laplace transform by using Bell polynomials was recently introduced. In the present paper computational techniques for approximating the transformed functions are derived. The theoretical approach exploits the generating function method, but for the numerical experiments the matrix pencil method has been used, since it proved to be more effective.

1 Introduction

Laplace Transform (shortly LT) [9, 25] is one of the most useful tools in the applications of mathematics, ranging form signals analysis to imagine processing. Its aplications, jointly with the Fourier transform in ordinary and partial differential equations, is widely exploited in literature [2]. The classical form of this operator writes:

$$\mathcal{L}(f) := \int_0^\infty \exp^{-1}(s\,t) f(t)\,dt = F(s),$$

since it converts a function of a real variable t (often representing the time) to a function of a complex variable s (complex frequency). This transform is used for solving differential equations, since it transforms differential into algebraic equations and convolution into multiplication. It can be applied to local integrable functions on $[0, +\infty)$ and it converges in each half plane $Re(s) > a$, the constant a, also known as the convergece abscissa, depends on the growth behavior of $f(t)$.
A large number of LT can be found in the literature and, together with the respective antitransforms, are usually used in the solution of the most diverse differential problems. The numerical computation of the link between transforms and antitransforms was considered, for example, by F.G. Tricomi [22, 23], which highlighted the link with the series expansions in Laguerre polynomials. These results have been extended to more general expansions [17], however the results of Tricomi have been proven numerically more convenient by the point of view of numerical complexity.
Recently, extensions of the LT have been considered in [16] and in [6] the numerical computation was carried out by approximating the respective kernels by means of expansions in a general Dirichlet series. Extensions of LT (called Laguerre-Laplace transforms) have been obtained in the first place [6] by replacing the exponential with the Laguerre-type exponentials, introduced in [8] and previously studied in [10], [11], [13]. Subsequently the kernel was replaced by an expansion whose coefficients are combination of Bell polynomials, exploiting a transformation, introduced in [16], which uses the Blissard formula, a typical tool of the umbral calculus [19, 20].
In order to validate the computational methods, some examples were taken into consideration, by the first author, with the aid of the computer algebra program Mathematica©.
The rapid decay of the considered kernels allows to extend the integration interval to a right neighborhood of the origin. However, the higher computational complexity using the theoretical approach based on the generating function of the generalized Lucas polynomials suggested to approximate the original kernel by a truncation of a general Dirichlet's series, and to use the matrix pencil method for evaluating the best coefficients.
The results obtained confirm the correctness of the procedure introduced.

2 Recalling the Bell polynomials

Considering the n-times differentiable functions $x = g(t)$ and $y = f(x)$, defined in given intervals of the real axis, the composite function $\Phi(t) := f(g(t))$, can be differentiated with respect to t, up to the n-th order, by using the chain rule.
We use the notations:

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t).$$

Then the n-th derivative of $\Phi(t)$ is represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n),$$

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where Y_n denotes the nth Bell polynomial [3, 18].

The traditional form of the Bell polynomials [7] is given by:

$$Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) f_k \,, \tag{1}$$

where $\forall\, n$, the $B_{n,k}$ satisfy the recursion:

$$\begin{gathered} B_{n+1,1} = g_{n+1}\,, \qquad B_{n+1,n+1} = g_1^{n+1}\,, \\ B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \dots, , g_{n-h-k+1})\, g_{h+1}\,. \end{gathered} \tag{2}$$

The $B_{n,k}$ functions, for any $k = 1, 2, \dots, n$, are polynomials in the $g_1, g_2, \dots, g_n$ variables - actually depending only on $g_1, g_2, \dots, g_{n-k+1}$ - homogeneous of degree k and *isobaric* of weight n (i.e. they are linear combinations of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$), so that

$$B_{n,k}(\alpha\beta\, g_1, \alpha\beta^2 g_2, \dots, \alpha\beta^{n-k+1} g_{n-k+1}) = \alpha^k \beta^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1})\,, \tag{3}$$

and

$$Y_n(f_1, \beta\, g_1; f_2, \beta^2 g_2; \dots; f_n, \beta^n g_n) = \beta^n\, Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n)\,.$$

3 Generalized second kind Lucas polynomials

Putting for $r \geq 1$ $\mathbf{u} \equiv (u_1, u_2, \dots, u_r)$ we write the linear homogeneous difference equation

$$X_n(\mathbf{u}) = u_1 X_{n-1} - u_2 X_{n-2} + \cdots + (-1)^{r-1} u_r X_{n-r}\,, \tag{4}$$

whose characteristic polynomial writes

$$P(\mathbf{u}; \lambda) := \lambda^r - u_1 \lambda^{r-1} + u_2 \lambda^{r-2} + \cdots + (-1)^r u_r\,. \tag{5}$$

The generalized second kind Lucas polynomials [14], that are the solution of (4), satisfy the following recurrence relation

$$\Phi_{-1} = 0,\ \Phi_0 = 0,\ \dots,\ \Phi_{r-3} = 0,\ \Phi_{r-2} = 1\,,$$

$$\Phi_n(\mathbf{u}) = u_1 \Phi_{n-1} - u_2 \Phi_{n-2} + \cdots + (-1)^{r-1} u_r \Phi_{n-r}\,,$$

In a former article [5] it has been recalled that the generating function of these polynomials is

$$\sum_{n=0}^{\infty} \Phi_{n+r-2}(\mathbf{u})\, \lambda^n = \frac{1}{P(\mathbf{u}; \lambda)}\,. \tag{6}$$

4 Bell-Laplace Transforms

In the framework of the umbral calculus [20, 19], in [16] it was introduced a class of generalized LTs. Here we refer to them as Bell-Laplace transforms, since their definition is based on Bell polynomials.
More precisely, given the umbral symbol $a := \{a_k\} = (1, a_1, a_2, a_3, \dots)$, we consider the function:

$$\Lambda_a(t) := \frac{1}{1 + \sum_{k=1}^{\infty} a_k \frac{t^k}{k!}}\,, \quad (t \geq 0). \tag{7}$$

According to the Blissard problem [4] (see e.g. Sect. 3 in [14]), and recalling (1), it results:

$$\begin{aligned} \frac{1}{1 + a_1(st) + a_2\frac{(st)^2}{2!} + a_3\frac{(st)^3}{3!} + \ldots} &= 1 + \sum_{k=1}^{\infty} Y_k(-1!, a_1; 2!, a_2; \dots; (-1)^k k!, a_k) \frac{(s\,t)^k}{k!} = \\ &= 1 + \sum_{k=1}^{\infty}\sum_{h=1}^{k} (-1)^h h!\, B_{k,h}(a_1, a_2, \dots, a_{k-h+1}) \frac{(s\,t)^k}{k!}\,. \end{aligned} \tag{8}$$

By setting

$$\mathcal{L}_a(f) = \int_0^{\infty} f(t) \left[1 + \sum_{k=1}^{\infty}\sum_{h=1}^{k} (-1)^h h!\, B_{k,h}(a_1, a_2, \dots, a_{k-h+1}) \frac{(s\,t)^k}{k!}\right] dt = F_a(s)\,, \tag{9}$$

putting, for shortness:

$$C_k(a) := \sum_{h=1}^{k} (-1)^h h!\, B_{k,h}(a_1, a_2, \ldots, a_{k-h+1})\,, \quad C_0(a) := 1\,, \tag{10}$$

equation (9) writes

$$\mathcal{L}_a(f) = \int_0^\infty f(t) \sum_{k=0}^{\infty} C_k(a) \frac{(s\,t)^k}{k!}\, dt = F_a(s)\,. \tag{11}$$

Note that the ordinary Laplace transform corresponds to the sequence: $a = \{a_k\}$, where $a_k \equiv 1\,, \forall k$. In fact [15]:

$$B_{k,h}(1, 1, \ldots, 1) = S(k, h)\,,$$

where $S(k, h)$ are the Stirling numbers of the second kind [7].
Then, using a known identity, we find:

$$C_k(1, 1, \ldots, 1) = \sum_{h=1}^{k} (-1)^h h!\, S(k, h) = (-1)^k\,, \quad \forall k \geq 0\,,$$

so that the ordinary expression of the Laplace transform is recovered, that is it results:

$$\mathcal{L}_{(1,1,\ldots,1,\ldots)}(f) \equiv \mathcal{L}(f)\,. \tag{12}$$

In [15] it is possible to find several sums of the type (10), corresponding to different sequences $a = \{a_k\}$. However, in order to preserve the main properties of the LT, we assume, the following hypothesis:

HP. *For every fixed s in the region of convergence, the power series $\sum_{k=0}^{\infty} C_k(a)(s\,t)^k/k!$, in equation (11) has an exponential decay to zero when $t \to \infty$, i.e.*

$$C_k(a) = O(1)\,, \quad (k \to \infty)\,.$$

$C_k(a)$ is obviously bounded if it contains only a finite number of terms.

Possible kernels, considered in numerical applications, are obtained assuming $a_k = k!$, (Figure 1), that is:

$$\frac{1}{1 + t + t^2 + t^3 + \ldots} \quad (t \geq 0)\,. \tag{13}$$

The truncation of the series in (13) allows to functions whose behavior is similar to that depicted in Figure 1.

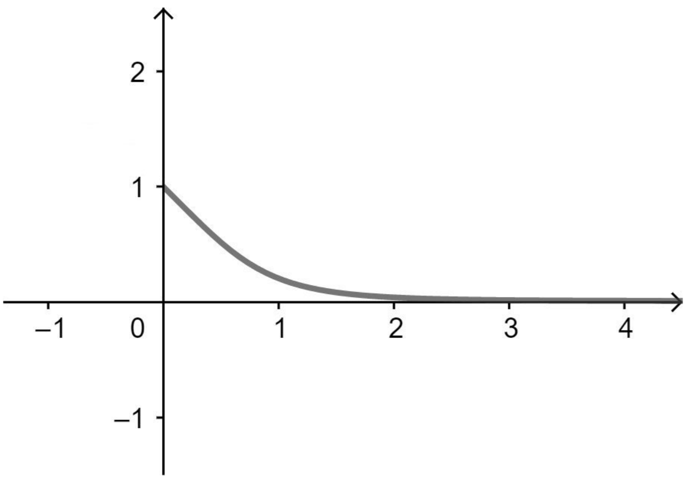

Figure 1: Graph of the function: $\frac{1}{1+t+t^2+t^3+t^4}$

For $\{a_k\} = (k!)^2$, i.e.:

$$\frac{1}{1 + t + 2!t^2 + 3!t^3 + \ldots} \quad (t \geq 0)\,, \tag{14}$$

the truncation of the series in (14) gives graphs similar to that depicted if Figure 2.

Remark 1. Note that, increasing the values of the sequence $\{a_k\}$ in equation *(7)*, the corresponding graphs exhibit a more fast decreasing character. Then it is possible to limit the integration to an interval of the type $[0, L]$, as the kernel becomes negligible outside it. For example, in Figure 1 we can assume $L = 4$, and in Figure 2 $L = 2$.

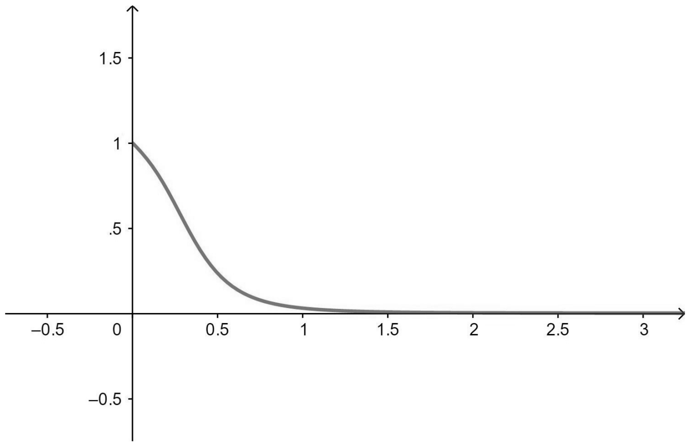

Figure 2: Graph of the function: $\frac{1}{1+t+2!t^2+3!t^3+4!t^4}$

4.1 The matrix pencil method

For a better understanding of the numerical technique used, we premise a concise exposition of the matrix pencil method, a computational tool usually exploited in signal theory [21].
Suppose we have N samples $(y_0, y_1, \ldots, y_{N-1})$ of a given function $y(t)$ taken at equal intervals of length T. Let us model the function as a sum of M complex exponentials:

$$y(t) = \sum_{j=1}^{M} R_j e^{s_j t}, \tag{15}$$

so that:

$$y_k = y(kT) = \sum_{j=1}^{M} R_j z_j^k, \tag{16}$$

with $z_j = e^{s_j T}$. The matrix pencil method allows evaluating the best estimates of the parameters M and R_j, z_j $(j = 1, 2, \ldots, M)$ in (16). To this end, let us introduce the matrices $[Y_1]$ and $[Y_2]$ as follows:

$$[Y_1] = \begin{bmatrix} y_1 & y_2 & \cdots & y_L \\ y_2 & y_3 & \cdots & y_{L+1} \\ \vdots & \vdots & \cdots & \vdots \\ y_{N-L} & y_{N-L+1} & \cdots & y_{N-1} \end{bmatrix}_{(N-L)\times L}, \tag{17}$$

$$[Y_2] = \begin{bmatrix} y_0 & y_1 & \cdots & y_{L-1} \\ y_1 & y_2 & \cdots & y_L \\ \vdots & \vdots & \cdots & \vdots \\ y_{N-L-1} & y_{N-L} & \cdots & y_{N-2} \end{bmatrix}_{(N-L)\times L}, \tag{18}$$

where $1 \le L \le N-1$. Using simple algebra, it can be readily verified that:

$$[Y_1] = [Z_1][R][Z_0][Z_2], \tag{19}$$

$$[Y_2] = [Z_1][R][Z_2], \tag{20}$$

with:

$$[Z_1] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & \cdots & \vdots \\ z_1^{N-L-1} & z_2^{N-L-1} & \cdots & z_M^{N-L-1} \end{bmatrix}_{(N-L)\times M}, \tag{21}$$

$$[Z_2] = \begin{bmatrix} 1 & z_1 & \cdots & z_1^{L-1} \\ 1 & z_2 & \cdots & z_2^{L-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & z_M & \cdots & z_M^{L-1} \end{bmatrix}_{M\times L}, \tag{22}$$

$$[Z_0] = \mathrm{diag}(z_1, z_2, \ldots, z_M)\,, \tag{23}$$

$$[R] = \mathrm{diag}(R_1, R_2, \ldots, R_M)\,. \tag{24}$$

Using (19) and (20), the following matrix pencil can be defined:

$$[Y_1] - \lambda [Y_2] = [Z_1][R]\{[Z_0] - \lambda [I]\}[Z_2]\,, \tag{25}$$

where $[I]$ is the identity matrix of order M.
The rank of the left-hand side of (25) is M if $M \leq L \leq N - M$ [21]. However, if $\lambda = z_j$ $(j = 1, 2, \ldots, M)$, the rank is reduced to $M - 1$ since the j–th row and column of $[Z_0] - \lambda [I]$ become zero. This implies that the quantities z_j are the generalized eigenvalues of the matrix pair $\{[Y_1], [Y_2]\}$. Hence:

$$[Y_1]\left[r_j\right] = z_j\,[Y_2]\left[r_j\right]\,, \tag{26}$$

with $\left[r_j\right]$ denoting the generalized eigenvectors corresponding to z_j so that:

$$\left\{[Y_2]^{\dagger}[Y_1] - z_j\,[I]\right\}\left[r_j\right] = [0]\,, \tag{27}$$

where $[Y_2]^{\dagger}$ is the Moore-Penrose pseudo-inverse of $[Y_2]$, that is:

$$[Y_2]^{\dagger} = \left\{[Y_1]^H [Y_1]\right\}^{-1}[Y_1]^H\,, \tag{28}$$

denoting the superscript ^{H} the complex conjugate matrix transposition. It is straightforward to conclude that the parameters z_j for $j = 1, 2, \ldots, M$ can be evaluated as the eigenvalues of $[Y_2]^{\dagger}[Y_1]$.
Once M and z_j $(j = 1, 2, \ldots, M)$ are known, the complex amplitudes R_j can be easily determined by solving the least-square problem:

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & \cdots & \vdots \\ z_2^{N-1} & z_2^{N-1} & \cdots & z_M^{N-1} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_M \end{bmatrix}. \tag{29}$$

The matrix equation (29) can be solved using the QR decomposition with Householder transformations [21]. In the following, we assume $L = \lfloor N/2 \rfloor$.

4.2 Approximation of Bell-Laplace transforms

By (6), for $\mathbf{u} = (-t, t^2, \ldots, (-1)^r t^r)$ consider the following generating function

$$\frac{1}{\lambda^r + t\lambda^{r-1} + \cdots + t^{r-1}\lambda + t^r} = \sum_{n=0}^{\infty} \Phi_{n+r-2}[-t, t^2, \ldots, (-1)^r t^r]\lambda^n\,.$$

For $\lambda = 1$ we have:

$$\frac{1}{1 + t + \cdots + t^{r-1} + t^r} = \sum_{n=0}^{\infty} \Phi_{n+r-2}[-t, t^2, \ldots, (-1)^r t^r]\,.$$

Comparing the above equation with the expansion (8), corresponding to the sequence $a := (1, 1, 2!, \ldots, r!, 0, 0, \ldots)$:

$$\frac{1}{1 + t + \cdots + t^{r-1} + t^r} = \sum_{k=0}^{\infty} \left[\frac{1}{k!} \sum_{h=1}^{k} (-1)^h h! B_{k,h}(1, 2!, \ldots, r!, 0, 0, \ldots)\right] t^k\,.$$

Then it must be:

$$\sum_{n=0}^{\infty} \Phi_{n+r-2}[-t, t^2, \ldots, (-1)^r t^r] = \sum_{k=0}^{\infty} \left[\frac{1}{k!} \sum_{h=1}^{k} (-1)^h h! B_{k,h}(1, 2!, \ldots, r!, 0, 0, \ldots)\right] t^k\,.$$

Considering the generalized LT associated to the sequence $a = (1, 1, 2!, \ldots, r!, 0, 0, \ldots)$, and supposing the kernel negligible outside the interval $[0, L]$, we can write:

$$\mathcal{L}_a(f) = \int_0^L f(t) \sum_{k=0}^{\infty} C_k(a) \frac{(st)^k}{k!}\,dt =$$

$$= \int_0^L f(t) \sum_{n=0}^{\infty} \Phi_{n+r-2}[-st, (st)^2, \ldots, (-1)^r (st)^r]\,dt =$$

$$= \sum_{n=0}^{\infty} \int_0^L f(t)\,\Phi_{n+r-2}[-st, (st)^2, \ldots, (-1)^r (st)^r]\,dt = F_a(s)\,.$$

In what follows the preceding results are extended in a straightforward way to the generalized LT associated to a sequence $a=(1,a_1,2!a_2,\dots,r!a_r,0,0,\dots)$, where $\forall k, a_k \geq 0$.
Consider the generating function:

$$\frac{1}{\lambda^r+a_1 t\lambda^{r-1}+\cdots+a_r t^r}=\sum_{n=0}^{\infty}\Phi_{n+r-2}[-a_1 t, a_2 t^2,\dots,(-1)^r a_r t^r]\lambda^n .$$

For $\lambda=1$ we have:

$$\frac{1}{\lambda^r+a_1 t\lambda^{r-1}+\cdots+a_r t^r}=\sum_{n=0}^{\infty}\Phi_{n+r-2}[-a_1 t, a_2 t^2,\dots,(-1)^r a_r t^r].$$

Comparing the above equation with the expansion (8) corresponding to the sequence $a:=(1,a_1,2!a_2,\dots,r!a_r,0,0,\dots)$:

$$\frac{1}{\lambda^r+a_1 t\lambda^{r-1}+\cdots+a_r t^r}=\sum_{k=0}^{\infty}\left[\frac{1}{k!}\sum_{h=1}^{k}(-1)^h h! B_{k,h}(a_1,2!a_2,\dots,r!a_r,0,0,\dots)\right]t^k ,$$

assuming equal to 1 the addend of the internal summation when $k=0$. Then it must be:

$$\sum_{n=0}^{\infty}\Phi_{n+r-2}[-a_1 t, a_2 t^2,\dots,(-1)^r a_r t^r]=\sum_{k=0}^{\infty}\left[\frac{1}{k!}\sum_{h=1}^{k}(-1)^h h! B_{k,h}(a_1,2!a_2,\dots,r!a_r,0,0,\dots)\right]t^k .$$

Considering the generalized LT associated to the sequence $a:=(1,a_1,2!a_2,\dots,r!a_r,0,0,\dots)$, supposing the kernel negligible outside the interval $[0,L]$ we can write:

$$\mathcal{L}_a(f)=\int_0^L f(t)\sum_{k=0}^{\infty}C_k(a)\,\frac{(s\,t)^k}{k!}\,dt=$$

$$=\int_0^L f(t)\sum_{n=0}^{\infty}\Phi_{n+r-2}[-a_1 st, a_2(st)^2,\dots,(-1)^r a_r(st)^r]\,dt=$$

$$=\sum_{n=0}^{\infty}\int_0^L f(t)\,\Phi_{n+r-2}[-a_1 st, a_2(st)^2,\dots,(-1)^r a_r(st)^r]\,dt=F_a(s).$$

Remark 2. Note that increasing the number r of positive terms in the considered sequence a, the function in (7) faster decreases towards zero and consequently the approximation gradually becomes more accurate.
We have checked that the theoretical appoximation given in this Section is not convenient by the numerical point of view, being computationally more complex. Therefore, for the numerical results, we have used the matrix pencil method passing from the original kernels as in (7) to those given in equations (30) and (39).

5 Numerical Results

In what follows we will show some numerical examples concerning the computation of Bell-Laplace transforms.

5.1 Example: Bell-Laplace Transform of $e^t\,\Gamma(t)$

The kernel of the Bell-Laplace transform defined by the umbral symbol $a=\{a_k\}$ with $a_k=\frac{1}{k!}\left|\cos\frac{k\pi}{2}\right|$ ($\forall k\geq 1$) is given by:

$$\Lambda_a(x)=\frac{1}{1+\sum_{k=1}^{\infty}a_k\frac{x^k}{k!}}=\frac{2}{I_0\left(2\sqrt{x}\right)+J_0\left(2\sqrt{x}\right)}, \tag{30}$$

for $x\geq 0$ (see Figure 3), where I_0 denote the modified Bessel functions of the first kind, and J_0 the Bessel functions of the second kind [24].
Upon defining the threshold $\Lambda_{min}=10^{-7}$, the parameter L such that:

$$\Lambda_a(x)\leq\Lambda_{min}, \quad \forall x\geq L, \tag{31}$$

can be numerically evaluated using the Newton's method. In this way, one can readily find that $L\simeq 92.16933300080161$.
Let us now consider the function:

$$f(t)=e^t\,\Gamma(t), \tag{32}$$

whose graph is displayed in Figure 4.
Thence, using a suitable quadrature formula on the interval $[0,L]$ where $\Lambda_a(x)$ is not negligible, the Bell-Laplace transform of $f(t)$ can be evaluated numerically as $\tilde{F}_a(s)$ by the procedure discussed in the previous section.

The distribution of $\tilde{F}_a(s)$ in the complex plane $s = \sigma + \mathrm{i}\,\omega$ is shown in Figure 5, and here compared to the one obtained by application of the rigorous theoretical expression:

$$\mathcal{L}_a(f) = \int_0^\infty f(t)\Lambda_a(s\,t)\,dt = F_a(s). \tag{33}$$

As it can be noticed, the agreement between $\tilde{F}_a(s)$ and $F_a(s)$ is very good, both in magnitude and argument.
It is important to mention that the Gauss-Kronrod quadrature formula [12] has been adopted in the computations presented in this section. Obviously, alternative quadrature formulas can be utilized for the same purpose, while preserving a good numerical accuracy. For the sake of completeness, in order to demonstrate the robustness of the proposed procedure from a computational standpoint, the considered Bell-Laplace transform of $f(t)$ has been, also, evaluated by using the trapezoidal quadrature formula [1]. As it can be noticed in Figure 6, the pseudo relative deviation between $\tilde{F}_a(s)$ as calculated by the two aforementioned quadrature algorithms, namely:

$$\varepsilon_r(s) = \frac{\tilde{F}_a^{(GK)}(s) - \tilde{F}_a^{(T)}(s)}{F_a(s)}, \tag{34}$$

is very small, typically below 0.01% in magnitude.
Finally, for further insight, the distribution of $\tilde{F}_a(s)$ along the cut sections $\omega = \mathrm{Im}\{s\} = -1$ and $\sigma = \mathrm{Re}\{s\} = 5$ is reported in Figures 7 and 8, respectively.

5.2 Example: Bell-Laplace Transform of $e^{\mathrm{i}\pi t}$

The kernel of the Bell-Laplace Laplace transform defined by the sequence $a = (1, 1+\sqrt{5}, 2(3+\sqrt{5}), 6(3+\sqrt{5}), 24(1+\sqrt{5}), 120, 0, 0, 0, \dots)$ is given by:

$$\begin{aligned}\Lambda_a(x) &= \frac{1}{1 + \sum_{k=1}^{\infty} a_k \frac{x^k}{k!}} \\ &= \frac{1}{1 + (1+\sqrt{5})x + (3+\sqrt{5})x^2 + (3+\sqrt{5})x^3 + (1+\sqrt{5})x^4 + x^5},\end{aligned} \tag{35}$$

for $x \geq 0$ (see Figure 9).
Upon defining the threshold $\Lambda_{min} = 10^{-7}$, the parameter L such that:

$$\Lambda_a(x) \leq \Lambda_{min}, \quad \forall x \geq L, \tag{36}$$

can be numerically evaluated using the Newton's method. In this way, one can readily find that $L \simeq 24.46315443022993$.
Let us now consider the function:

$$f(t) = e^{\mathrm{i}\pi t}, \tag{37}$$

whose graph is displayed in Figure 10.
Thence, using a suitable quadrature formula on the interval $[0, L]$ where $\Lambda_a(x)$ is not negligible, the Bell-Laplace transform of $f(t)$ can be evaluated numerically as $\tilde{F}_a(s)$ by the procedure discussed in the previous section.
The distribution of $\tilde{F}_a(s)$ in the complex plane $s = \sigma + \mathrm{i}\,\omega$ is shown in Figure 11, and here compared to the one obtained by application of the rigorous theoretical expression:

$$\mathcal{L}_a(f) = \int_0^\infty f(t)\Lambda_a(s\,t)\,dt = F_a(s). \tag{38}$$

As it can be noticed, the agreement between $\tilde{F}_a(s)$ and $F_a(s)$ is very good, both in magnitude and argument.
It is important to mention that the Gauss-Kronrod quadrature formula [12] has been adopted in the computations presented in this section. Obviously, alternative quadrature formulas can be utilized for the same purpose, while preserving a good numerical accuracy. For the sake of completeness, in order to demonstrate the robustness of the proposed procedure from a computational standpoint, the considered Bell-Laplace transform of $f(t)$ has been, also, evaluated by using the trapezoidal quadrature formula [1]. As it can be noticed in Figure 12, the pseudo relative deviation between $\tilde{F}_a(s)$ as calculated by the two aforementioned quadrature algorithms, namely:

$$\varepsilon_r(s) = \frac{\tilde{F}_a^{(GK)}(s) - \tilde{F}_a^{(T)}(s)}{F_a(s)}, \tag{39}$$

is very small, always below 0.00001% in magnitude.
Finally, for further insight, the distribution of $\tilde{F}_a(s)$ along the cut sections $\omega = \mathrm{Im}\{s\} = 1$ and $\sigma = \mathrm{Re}\{s\} = 1$ is reported in Figures 13 and 14, respectively.

6 Conclusion

In preceding articles [16, 6] we have shown that generalized forms of the Laplace transform can be defined thorough the Laguerre-type exponentials. Furthermore, a wider extension can be obtained exploiting the Blissard problem, which is connected with Bell's polynomials. Then it was possible to associate, to every sequence of numbers, satisfying a simple hypothesis, a formal transform extending that of Laplace.
A number of open problems arise:
1. Is the extended transform associated to the Laguerre-type exponentials [6] useful to solve differential equations making use of the Laguerre derivative instead of the ordinary one?
2. What kind of differential operators are involved with the Bell-Laplace transforms introduced in this way?
3. Is it possible to define the inverse for these transformations?
In this article we have shown that a standard procedure, based on the matrix pencil method, approximating the kernel of the relevant transform as the sum of suitable dumped exponential terms, can be used to the numerical evaluation of the Laguerre-Laplace and Bell-Laplace transforms. In this way, the problem have been reduced to the computation of the standard Laplace transform, which have been treated by a judicious use of Tricomi's method.
The effectiveness of the proposed methodology has been shown by several test cases involving complex functions typically encountered in Mathematical Physics.

Compliance with ethical standards

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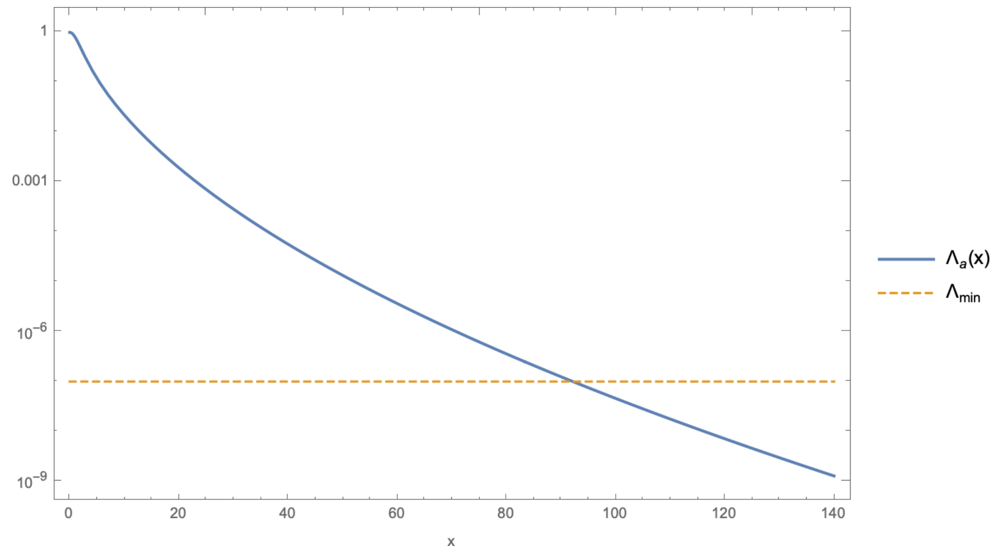


Figure 3: Distribution of the kernel $\Lambda_a(x)$ relevant to the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\cos\frac{k\pi}{2}\right|$.

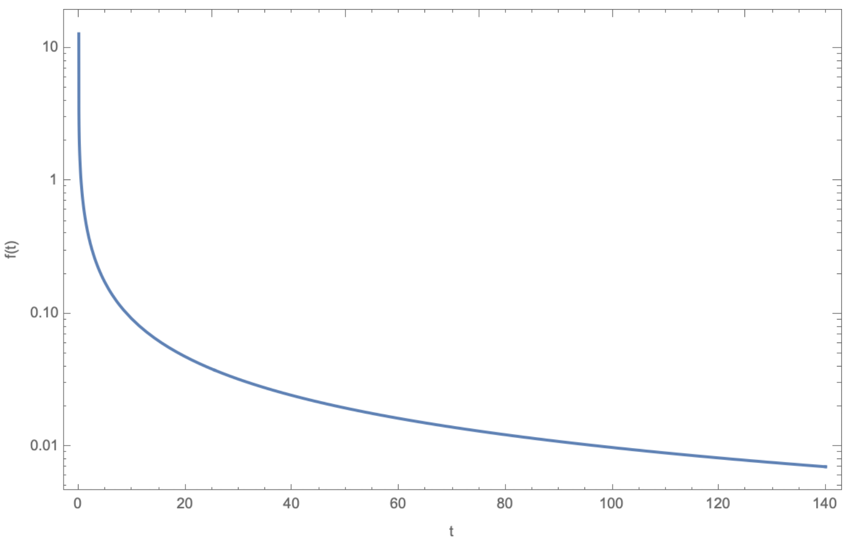


Figure 4: Distribution of the function $f(t) = e^t\,\Gamma(t)$.

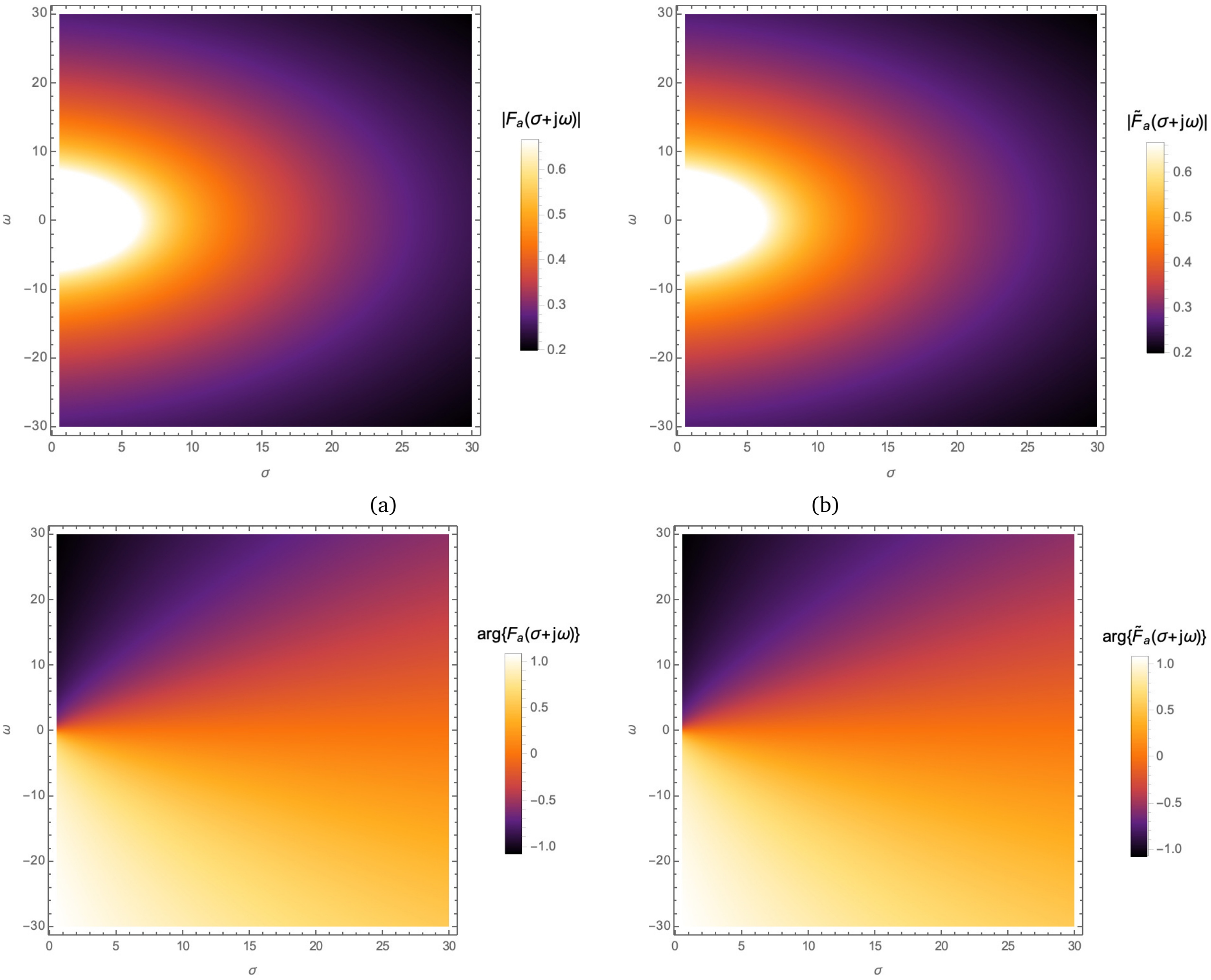


Figure 5: Magnitude and argument of the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\cos\frac{k\pi}{2}\right|$ relevant to $f(t) = e^t\,\Gamma(t)$ as evaluated, through the approximant $\tilde{F}_a(s)$ and by means of the rigorous theoretical expression $F_a(s)$, in the domain of the complex variable $s = \sigma + \mathrm{i}\,\omega$.

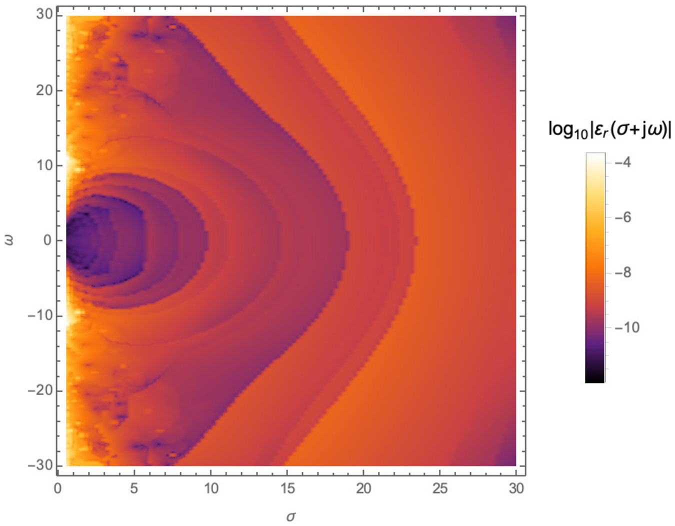
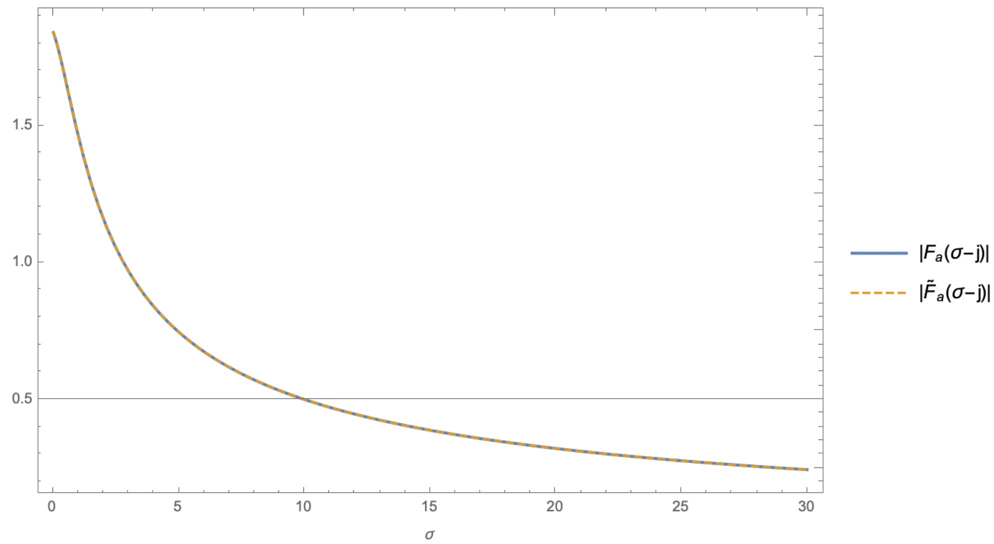
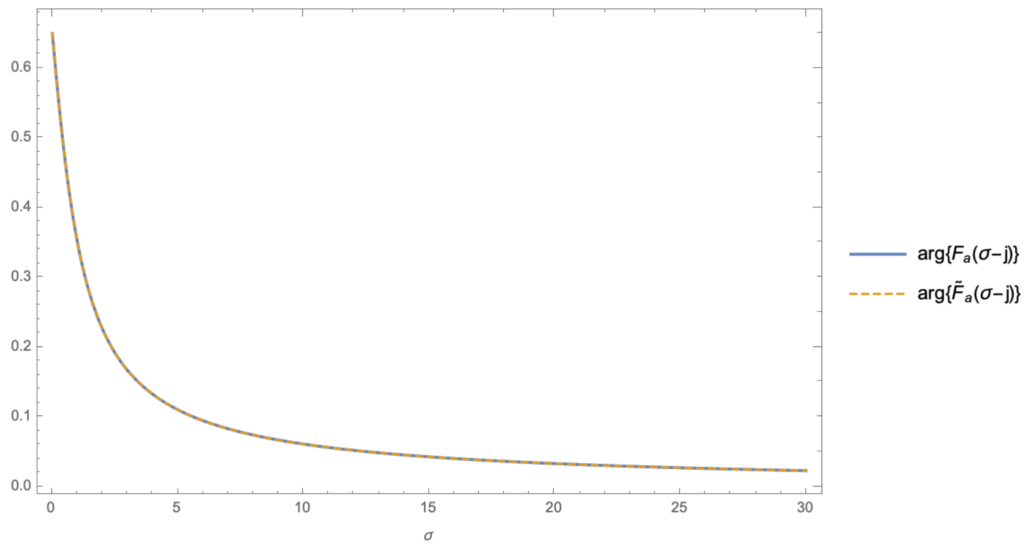


Figure 6: Magnitude in logarithmic scale of the deviation $\varepsilon_r(s)$ between the approximants $\tilde{F}_a(s)$ of the Bell-Laplace transform of $f(t) = e^t\,\Gamma(t)$ as calculated by the Gauss-Kronrod and trapezoidal quadrature formulas, respectively, relative to the rigorous theoretical expression $F_a(s)$.



(a)



(b)

Figure 7: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\cos\frac{k\pi}{2}\right|$ relevant to $f(t) = e^t\,\Gamma(t)$ as evaluated, through the approximant $\tilde{F}_a(s)$ and by means of the rigorous theoretical expression $F_a(s)$, for $s = \sigma + \mathrm{i}\,\omega$ with $\omega = -1$.

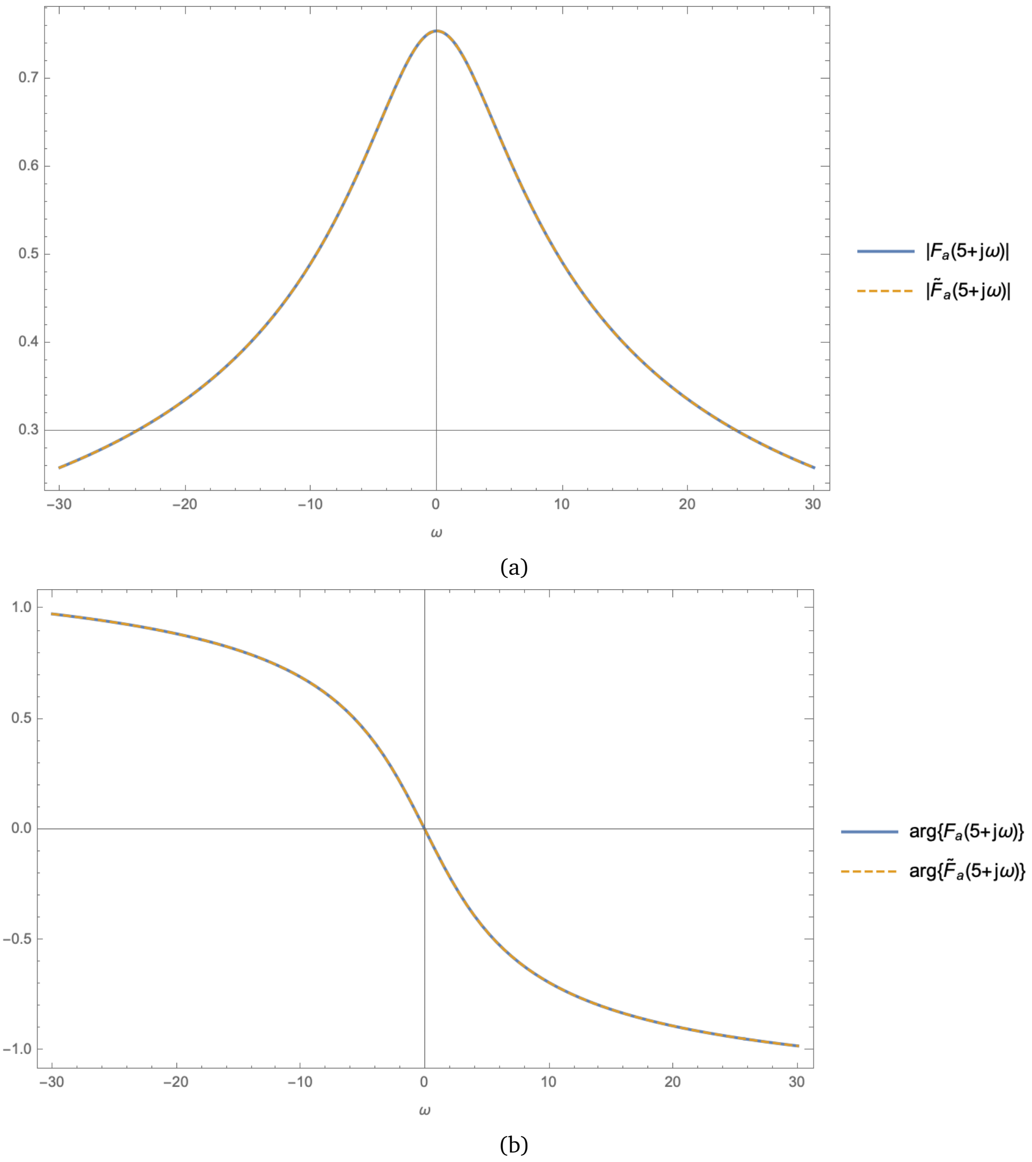


Figure 8: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the umbral symbol $a = \{a_k\}$ with $a_k = \frac{1}{k!}\left|\cos\frac{k\pi}{2}\right|$ relevant to $f(t) = e^t\,\Gamma(t)$ as evaluated, through the approximant $\tilde{F}_a(s)$ and by means of the rigorous theoretical expression $F_a(s)$, for $s = \sigma + \mathrm{i}\,\omega$ with $\sigma = 5$.

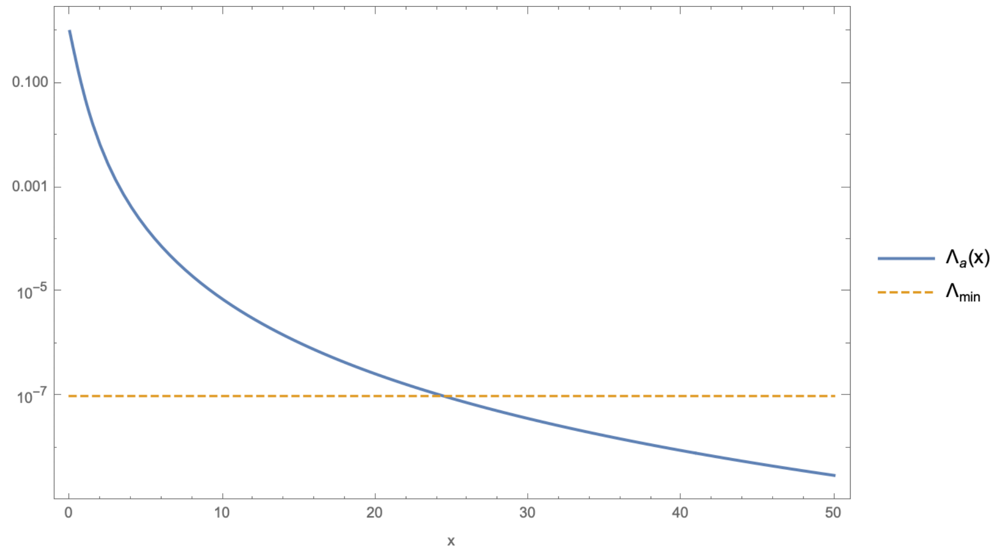
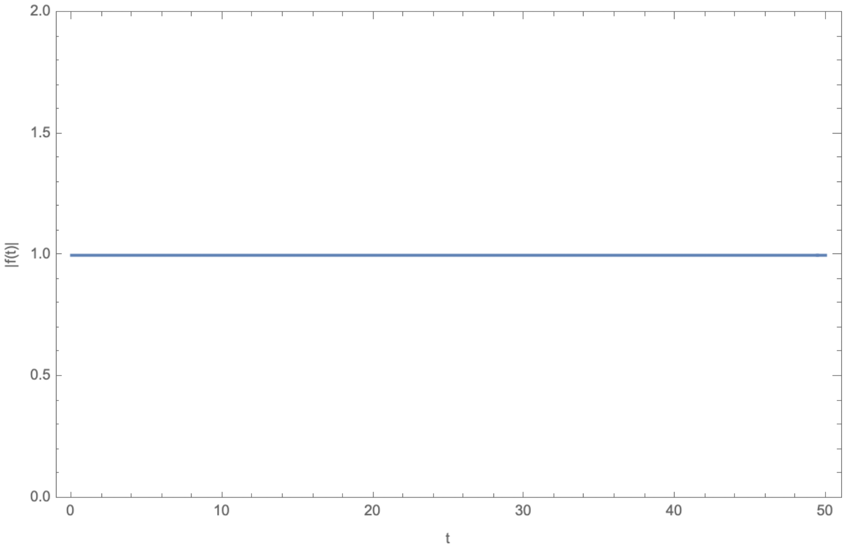
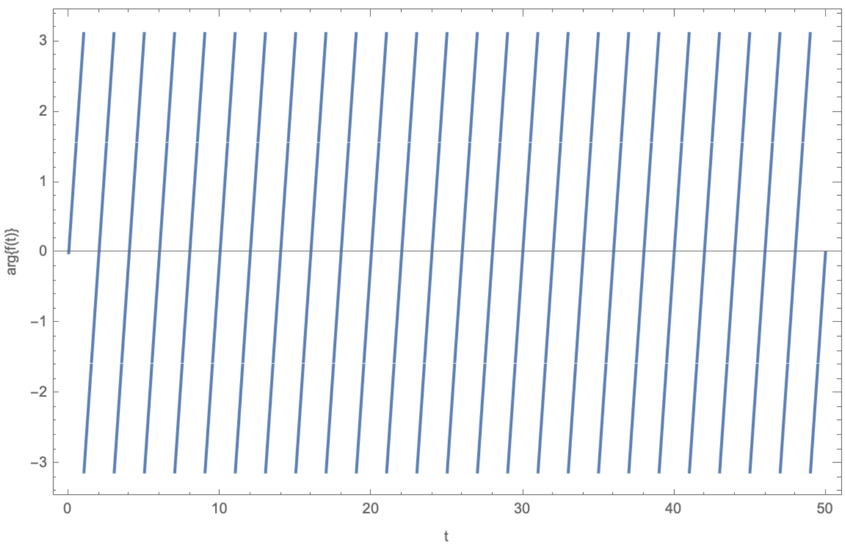


Figure 9: Distribution of the kernel $\Lambda_a(x)$ relevant to the Bell-Laplace transform defined by the sequence $a = (1, 1 + \sqrt{5}, 2(3+\sqrt{5}), 6(3+\sqrt{5}), 24(1+\sqrt{5}), 120, 0, 0, 0, \ldots)$.



(a)



(b)

Figure 10: Magnitude (a) and argument (b) of the function $f(t) = e^{i\pi t}$.

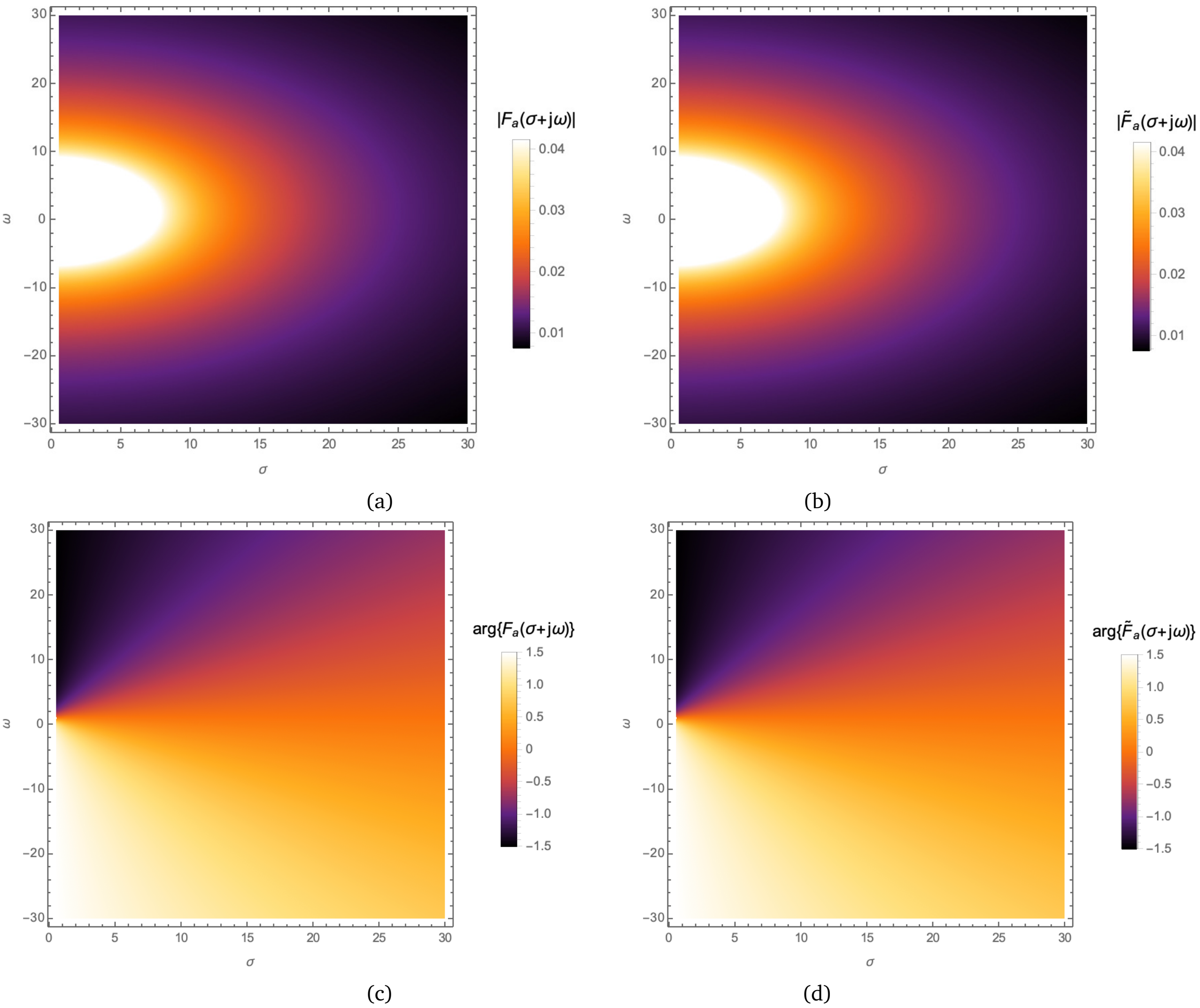


Figure 11: Magnitude and argument of the Bell-Laplace transform defined by the sequence $a = (1, 1+\sqrt{5}, 2(3+\sqrt{5}), 6(3+\sqrt{5}), 24(1+\sqrt{5}), 120, 0, 0, 0, \dots)$ relevant to $f(t) = e^{i\pi t}$ as evaluated, through the approximant $\tilde{F}_a(s)$ and by means of the rigorous theoretical expression $F_a(s)$, in the domain of the complex variable $s = \sigma + i\omega$.

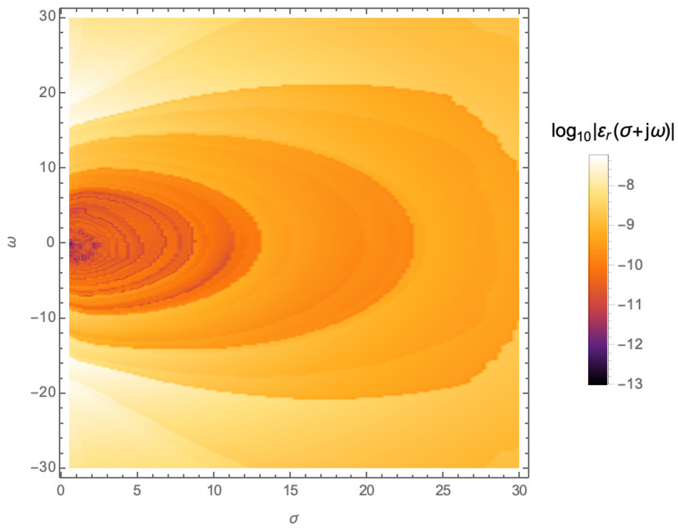
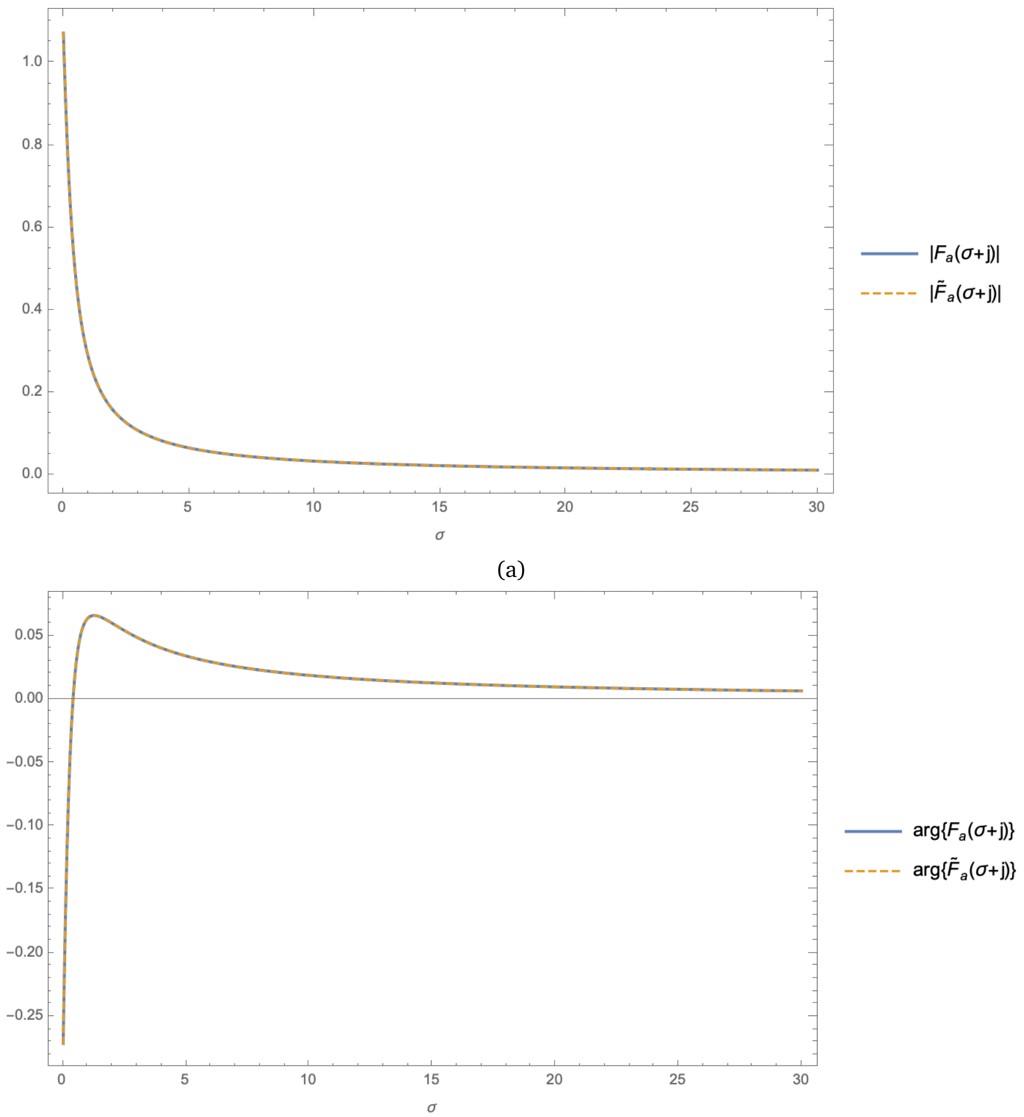


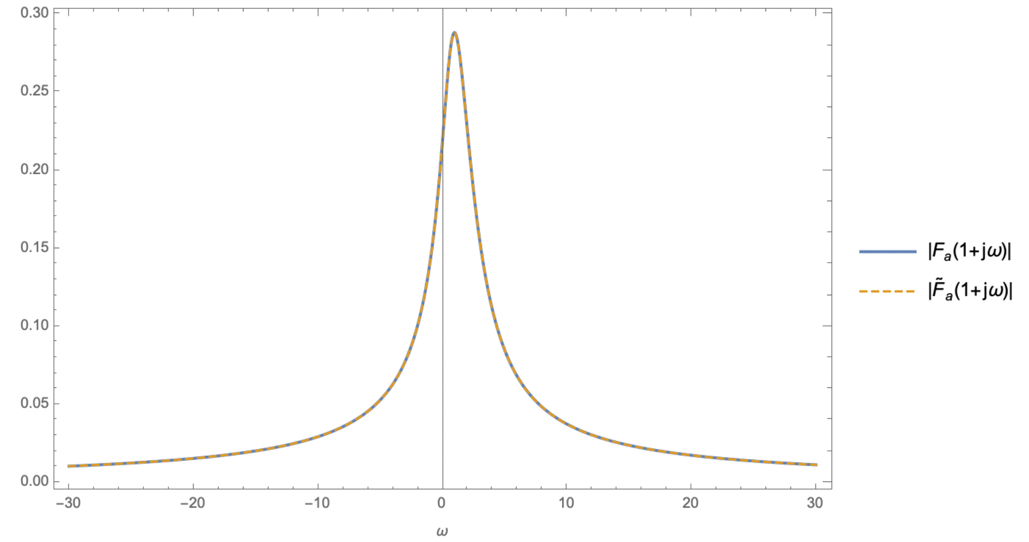
Figure 12: Magnitude in logarithmic scale of the deviation $\varepsilon_r(s)$ between the approximants $\tilde{F}_a(s)$ of the Bell-Laplace transform of $f(t) = e^{i\pi t}$ as calculated by the Gauss-Kronrod and trapezoidal quadrature formulas, respectively, relative to the rigorous theoretical expression $F_a(s)$.



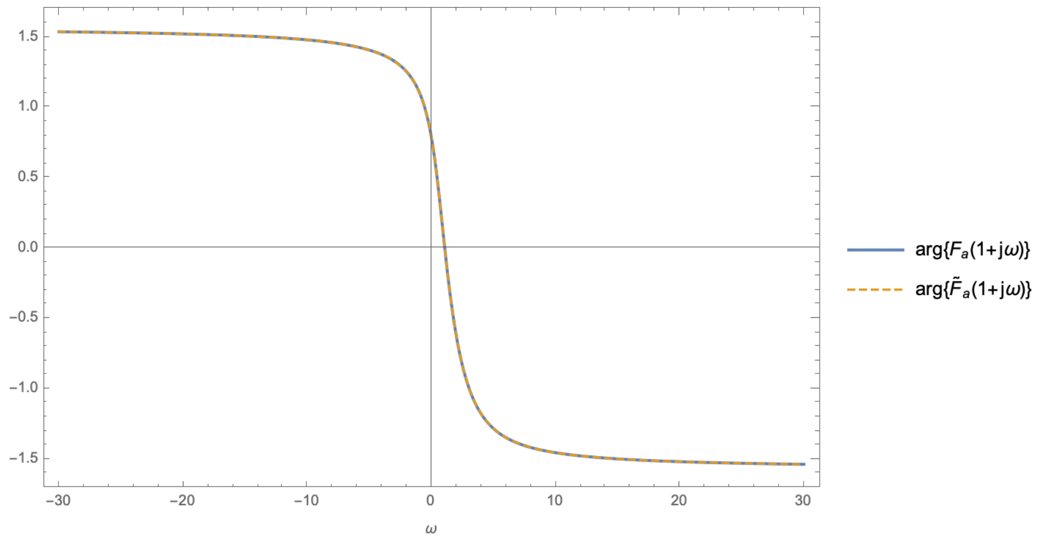
(a)

(b)

Figure 13: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the sequence $a = \left(1, 1+\sqrt{5}, 2\left(3+\sqrt{5}\right), 6\left(3+\sqrt{5}\right), 24\left(1+\sqrt{5}\right), 120, 0, 0, 0, \ldots\right)$ relevant to $f(t) = e^{\mathrm{i}\pi t}$ as evaluated, through the approximant $\tilde{F}_a(s)$ and by means of the rigorous theoretical expression $F_a(s)$, for $s = \sigma + \mathrm{i}\,\omega$ with $\omega = 1$.



(a)



(b)

Figure 14: Magnitude (a) and argument (b) of the Bell-Laplace transform defined by the sequence $a = (1, 1 + \sqrt{5}, 2\left(3+\sqrt{5}\right), 6\left(3+\sqrt{5}\right), 24\left(1+\sqrt{5}\right), 120, 0, 0, 0, \ldots)$ relevant to $f(t) = e^{\mathrm{i}\pi t}$ as evaluated, through the approximant $\tilde{F}_a(s)$ and by means of the rigorous theoretical expression $F_a(s)$, for $s = \sigma + \mathrm{i}\,\omega$ with $\sigma = 1$.