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On a set of sine and cosine Fourier transforms of nested functions

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Abstract

It has been recently shown that Bell's polynomials can be used to compute the Laplace Transform of nested analytic functions. In this article we extend said methodology to the case of sine and cosine Fourier transforms of composed functions. Graphical verifications of the proposed technique are shown in the last section.

1 Introduction

The Bell's polynomials [1, 5, 6, 9, 13] have been introduced to represent the subsequent derivatives of composed functions, but they turned out to be useful in several fields of Analysis and applied Mathematics.
They are exploited in very different frameworks, which range from number theory [10, 15, 16] to operators theory [14], and from differential equations [9] to integral transforms [3, 11]. It is almost useless to recall how important is the Fourier Transform (FT) in the theory of ordinary and partial differential equations.
The Fourier transform (FT) is a linear operator which maps a function of a real variable t, typically representing the time (with real or complex values), into a function of a complex variable ω (representing the complex frequency). It has many applications in Physics and Engineering [2], especially in signal theory and imagine processing, as it allows decomposing a signal in terms of its frequencies. We use the classical definition

$$\mathcal{F}(f) := \int_{-\infty}^{+\infty} f(t)\, e^{-i\,\omega\, t}\, dt = F(\omega).$$

The FT can be applied to $L^1(-\infty, +\infty)$ functions. Extensions to more general function spaces (distributions) are also considered.
A large number of FTs, together with the relevant anti-transforms, are reported in the literature (see e.g. [8]).
Similarly to what was recently shown in the case of the Laplace Transform, Bell's polynomials can be used for the evaluation of sine and cosine FTs of nested functions. To this end one can make use of the Maclaurin expansion of the given nested function. Since the coefficients of this expansion are represented in terms of Bell's polynomials, the computation of the relevant sine or cosine FT it is straightforwardly reduced to a series of classical FTs. Obviously, since the considered integral is convergent, the obtained series is convergent too.
This methodology is applied in this paper to the case of sine or cosine FT of nested functions, starting from the simpler case of the nested exponential functions.
All the numerical tests were obtained by using the computer algebra program Mathematica©.

2 Recalling the Bell polynomials

The Bell's polynomials express the nth derivative of a composed function $\Phi(t) := f(g(t))$ in terms of the successive derivatives of the (sufficiently smooth) component functions $x = g(t)$ and $y = f(x)$. More precisely, if

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t),$$

then the nth derivative of $\Phi(t)$ is represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n),$$

where Y_n denotes the nth Bell polynomial.
The first few Bell polynomials are:

$$\begin{aligned} &Y_1(f_1, g_1) = f_1 g_1 \\ &Y_2(f_1, g_1; f_2, g_2) = f_1 g_2 + f_2 g_1^2 \\ &Y_3(f_1, g_1; f_2, g_2; f_3, g_3) = f_1 g_3 + f_2(3 g_2 g_1) + f_3 g_1^3 \\ &\ldots \end{aligned} \tag{1}$$

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More general results can be found in [13], p. 49.
The Bell polynomials [5] are given by:

$$\boxed{Y_n(f_1,g_1;f_2,g_2;\ldots;f_n,g_n)=\sum_{k=1}^{n}B_{n,k}(g_1,g_2,\ldots,g_{n-k+1})\,f_k\,.} \tag{2}$$

where $B_{n,k}$ satisfy the recursion [5]:

$$B_{n,k}(g_1,g_2,\ldots,g_{n-k+1})=\sum_{h=0}^{n-k}\binom{n-1}{h}B_{n-h-1,k-1}(g_1,g_2,\ldots,g_{n-h-k+1})\,g_{h+1}\,. \tag{3}$$

The $B_{n,k}$ functions for any $k=1,2,\ldots,n$ are homogeneous polynomials of degree k in the $g_1,g_2,\ldots,g_n$ variables. They are *isobaric* of weight n (i.e. they are linear combinations of monomials $g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n}$ whose weight is constantly given by $k_1+2k_2+\ldots+nk_n=n$). Therefore we have the equations

$$B_{n,k}(\alpha\beta\, g_1,\alpha\beta^2 g_2,\ldots,\alpha\beta^{n-k+1}g_{n-k+1})=\alpha^k\beta^n B_{n,k}(g_1,g_2,\ldots,g_{n-k+1})\,, \tag{4}$$

and

$$Y_n(f_1,\beta\, g_1;f_2,\beta^2 g_2;\ldots;f_n,\beta^n g_n)=\beta^n\,Y_n(f_1,g_1;f_2,g_2;\ldots;f_n,g_n)\,. \tag{5}$$

3 A set of sine and cosine Fourier transforms

Let $f(g(t))$ be a composed function that is analytic in a neighborhood of the origin, so that it can be expressed by the Taylor's expansion

$$f(g(t))=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}\,,\qquad a_n=D_t^n[f(g(t))]_{t=0}\,. \tag{6}$$

According to the preceding section, it results

$$\begin{aligned} a_0&=f(\mathring{g}_0)\,,\\ a_n&=D_t^n[f(g(t))]_{t=0}=\sum_{k=1}^{n}B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\,\mathring{f}_k\,,\quad (n\geq 1)\,, \end{aligned} \tag{7}$$

where

$$\mathring{f}_k:=D_x^k f(x)|_{x=g(0)},\qquad \mathring{g}_h:=D_t^h g(t)|_{t=0}. \tag{8}$$

We consider Fourier transforms of the type

$$\int_{-\infty}^{+\infty}H(t)\,f(g(t))\,e^{-at}\,e^{-i\omega t}\,dt=\int_{0}^{+\infty}f(g(t))\,e^{-at}\,e^{-i\omega t}\,dt\,,$$

where $H(t)$ is the Heaviside distribution.
Therefore we have the sine and cosine FT in the form

$$\int_0^{+\infty}f(g(t))\,e^{-at}\,\sin(\omega t)\,dt \tag{9}$$

and

$$\int_0^{+\infty}f(g(t))\,e^{-at}\,\cos(\omega t)\,dt\,. \tag{10}$$

We have the following result.
Theorem 1. The sine Fourier transform of the function

$$H(t)f(g(t))\,e^{-at}\,,$$

where a is a given constant, is given by

$$\begin{aligned} &\int_0^{+\infty}f(g(t))\,e^{-at}\,\sin(\omega t)\,dt=\\ &=\frac{f(\mathring{g}_0)\,\omega}{\omega^2+a^2}+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\,\mathring{f}_k\right)\frac{\sin[(n+1)\arctan(\frac{\omega}{a})]}{(\omega^2+a^2)^{\frac{n+1}{2}}}\,, \end{aligned} \tag{11}$$

and the cosine Fourier transform writes

$$\int_0^{+\infty} f(g(t))\,e^{-at}\,\cos(\omega t)\,dt = \tag{12}$$
$$= \frac{f(\mathring{g}_0)\,a}{\omega^2+a^2} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\,\mathring{f}_k\right)\frac{\cos[(n+1)\arctan(\frac{\omega}{a})]}{(\omega^2+a^2)^{\frac{n+1}{2}}}\,.$$

Proof. Representing the coefficients of the Taylor expansion in (6) in terms of Bell polynomials, and using the uniform convergence of series, we find for the sin FT,

$$\int_0^{+\infty} f(g(t))\,e^{-at}\,\sin(\omega t)\,dt = \int_0^{+\infty} f(\mathring{g}_0)\,e^{-at}\,\sin(\omega t)\,dt +$$
$$+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\,\mathring{f}_k\right)\int_0^{+\infty}\frac{t^n}{n!}\,e^{-at}\,\sin(\omega t)\,,$$

and for the cosine FT,

$$\int_0^{+\infty} f(g(t))\,e^{-at}\,\cos(\omega t)\,dt = \int_0^{+\infty} f(\mathring{g}_0)\,e^{-at}\,\cos(\omega t)\,dt +$$
$$+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\,\mathring{f}_k\right)\int_0^{+\infty}\frac{t^n}{n!}\,e^{-at}\,\cos(\omega t)\,,$$

so that the result follow from the basic rules of the sine or cosine FT.

3.1 The particular case of the exponential function

In the particular case where $f(x)=e^x$, and therefore we are considering the composed function $e^{g(t)}$, and upon assuming that $g(0)=0$, we have the simplified form

$$\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\,\mathring{f}_k = \sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1}) = B_n(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_n)\,, \tag{13}$$

where the B_n are the *complete Bell's polynomials*. It results $B_0(g_0):=f(g_0)$, and the first few values of B_n, for $n=1,2,\ldots,5$, are given by

$$\begin{aligned}
B_1(g_1) &= g_1,\\
B_2(g_1,g_2) &= g_1^2+g_2,\\
B_3(g_1,g_2,g_3) &= g_1^3+3g_1g_2+g_3,\\
B_4(g_1,g_2,g_3,g_4) &= g_1^4+6g_1^2g_2+4g_1g_3+3g_2^2+g_4,\\
B_5(g_1,g_2,g_3,g_4,g_5) &= g_1^5+10g_1^3g_2+15g_1g_2^2+10g_1^2g_3+10g_2g_3+5g_1g_4+g_5\,.
\end{aligned} \tag{14}$$

Further values are reported in [12], Appendix I.
The values of the complete Bell polynomials for particular parameter choices can be found in [10].
The complete Bell polynomials satisfy the identity (see e.g. [9])

$$B_{n+1}(g_1,\ldots,g_{n+1}) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}(g_1,\ldots,g_{n-k})\,g_{k+1}. \tag{15}$$

In this case, using (13), we arrive at the following result

Corollary 1. The sine FT of the composed exponential function $\exp[g(t)]$, with $g(0)=0$, is given by

$$\int_0^{+\infty} e^{g(t)}\,e^{-at}\,\sin(\omega t)\,dt = \tag{16}$$
$$= \frac{\omega}{\omega^2+a^2} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\right)\frac{\sin[(n+1)\arctan(\frac{\omega}{a})]}{(\omega^2+a^2)^{\frac{n+1}{2}}}\,,$$

and the cosine FT by

$$\int_0^{+\infty} e^{g(t)}\,e^{-at}\,\cos(\omega t)\,dt = \tag{17}$$
$$= \frac{a}{\omega^2+a^2} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1,\mathring{g}_2,\ldots,\mathring{g}_{n-k+1})\right)\frac{\cos[(n+1)\arctan(\frac{\omega}{a})]}{(\omega^2+a^2)^{\frac{n+1}{2}}}\,.$$

In what follows we illustrate the approximation of sine and cosine FT of nested functions. These were obtained by the first author using the computer algebra program Mathematica©.

3.1.1 The sine Fourier case

Let $f(x) = e^x$ and $g(t) = J_0(t)$, the Bessel function of the first kind and order 0. Upon assuming $a = 4$, we find

$$\begin{aligned}\int_0^{+\infty} e^{J_0(t)}\, e^{-4t} \sin(\omega t)\, dt = {} & \frac{e}{128\,\omega\,(16+\omega^2)^{9/2}} \Big(512(16+\omega^2)^{9/2} - \\ & 64(16+\omega^2)^4 \sin[3\arctan(\omega/4)] + 144(16+\omega^2)^3 \sin[5\arctan(\omega/4)] - \\ & 640(16+\omega^2)^2 \sin[7\arctan(\omega/4)] + 4585(16+\omega^2)\sin[9\arctan(\omega/4)] - \\ & 47124 \sin[11\arctan(\omega/4)] \Big) + O\left(\frac{1}{\omega^{11}}\right).\end{aligned} \tag{18}$$

Graphical results are depicted in Figures 1 and 2.

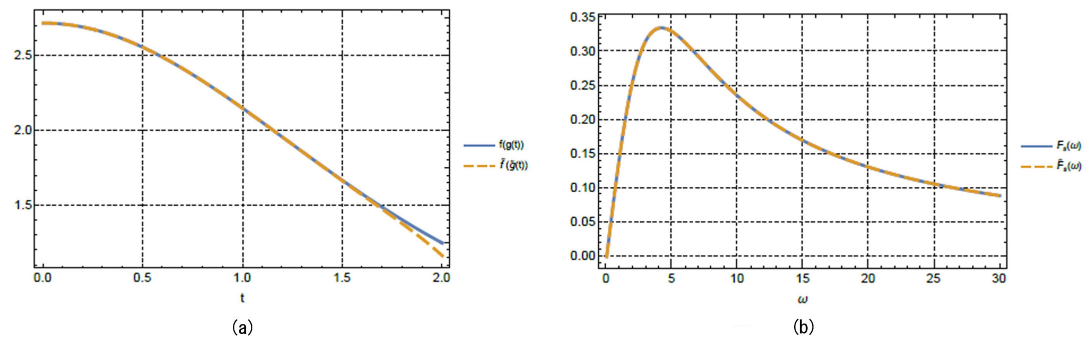


Figure 1: Function $\exp(J_0(t))$) (a) and its sine Fourier transform (b) as evaluated through the approximant $\tilde{F}(s)$ and the rigorous analytical expression $F(s)$.

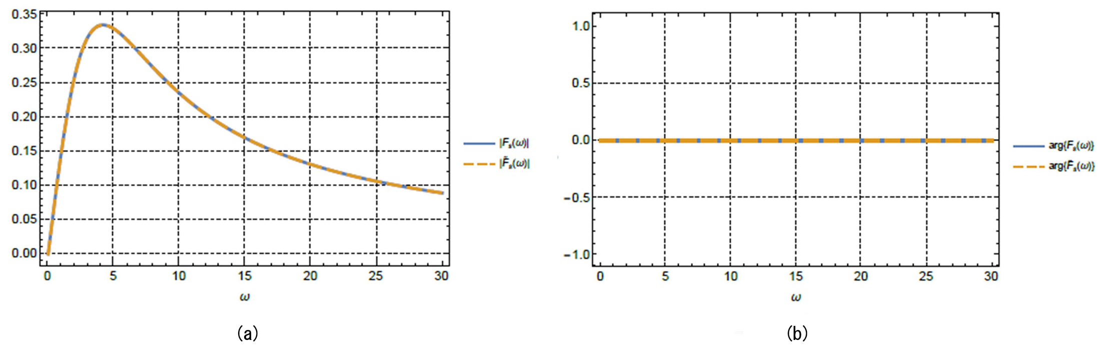


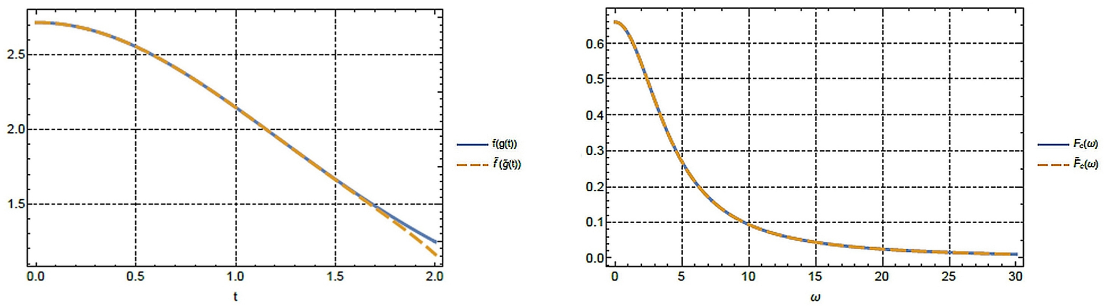
Figure 2: Magnitude (a) and argument (b) of the sine Fourier transform of the function $\exp(J_0(t))$) as evaluated through our approximant and its rigorous analytical expression.

3.1.2 The cosine Fourier case

Let $f(x) = e^x$ and $g(t) = J_0(t)$. Upon assuming $a = 4$, we find

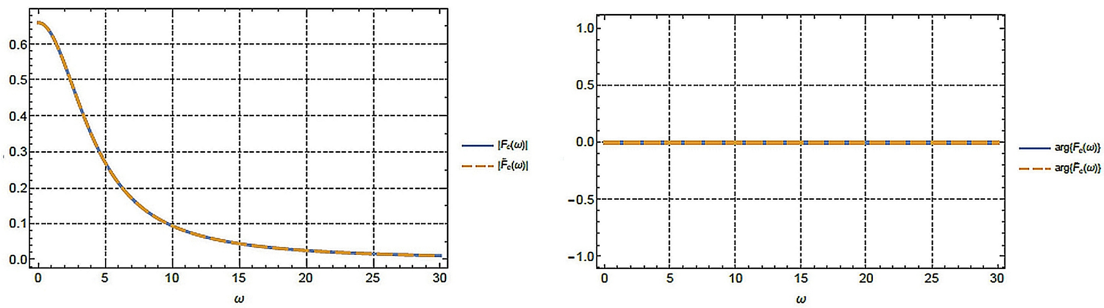
$$\int_0^{+\infty} e^{J_0(t)}\, e^{-4t} \cos(\omega t)\, dt = \frac{e}{128(16+\omega^2)^{9/2}} \Big(512(16+\omega^2)^{9/2} -$$
$$64(16+\omega^2)^4 \cos[3\arctan(\omega/4)] + 144(16+\omega^2)^3 \cos[5\arctan(\omega/4)] - \tag{19}$$
$$640(16+\omega^2)^2 \cos[7\arctan(\omega/4)] + 4585(16+\omega^2) \cos[9\arctan(\omega/4)] -$$
$$47124 \cos[11\arctan(\omega/4)] \Big) + O\left(\frac{1}{\omega^{10}}\right).$$

Graphical results are depicted in Figures 3 and 4.



(a) (b)

Figure 3: Function $\exp(J_0(t))$ (a) and its cosine Fourier transform (b) as evaluated through the approximant $\tilde{F}(s)$ and the rigorous analytical expression $F(s)$.



(a) (b)

Figure 4: Magnitude (a) and argument (b) of the cosine Fourier transform of the function $\exp(J_0(t))$ as evaluated through our approximant and its rigorous analytical expression.

3.2 The general sine Fourier case

3.2.1 Example 1

Let $f(x) = \log x$ and $g(t) = \cosh(t)$. Upon assuming $a = 2\pi$, we find

$$\begin{aligned} \int_0^{+\infty} \log\cosh(t)\, e^{-2\pi t} \sin(\omega t)\, dt = \frac{1}{(4\pi^2+\omega^2)^{11/2}} \Bigg(& (4\pi^2+\omega^2)^4 \sin[3\,\mathrm{arccot}(2\pi/\omega)] - \\ & 2(4\pi^2+\omega^2)^3 \sin[5\,\mathrm{arccot}(2\pi/\omega)] + 16\Big((4\pi^2+\omega^2)^2 \sin[7\,\mathrm{arccot}(2\pi/\omega)] - \\ & 17(4\pi^2+\omega^2)\sin[9\,\mathrm{arccot}(2\pi/\omega)] + 496\sin[11\,\mathrm{arccot}(\omega/4)]\Big)\Bigg) + O\left(\frac{1}{\omega^{12}}\right). \end{aligned} \tag{20}$$

Graphical results are depicted in Figures 5 and 6.

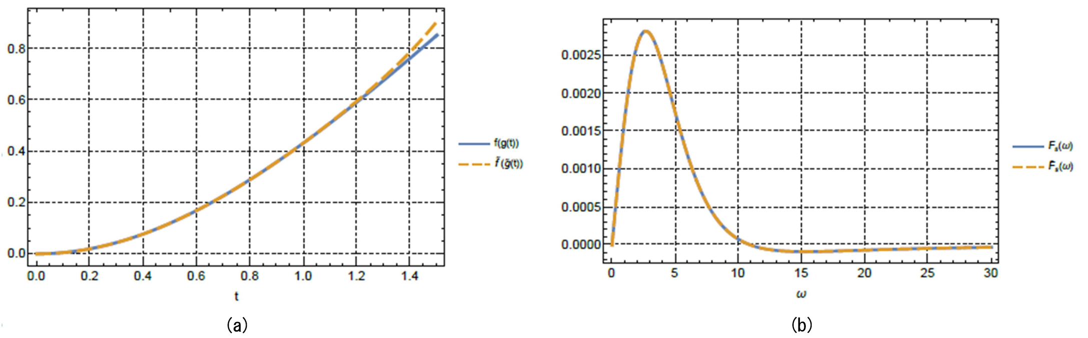


Figure 5: Function log(cosh(t) (a) and its cosine Fourier transform (b) as evaluated through the approximant $\tilde{F}(s)$ and the rigorous analytical expression $F(s)$.

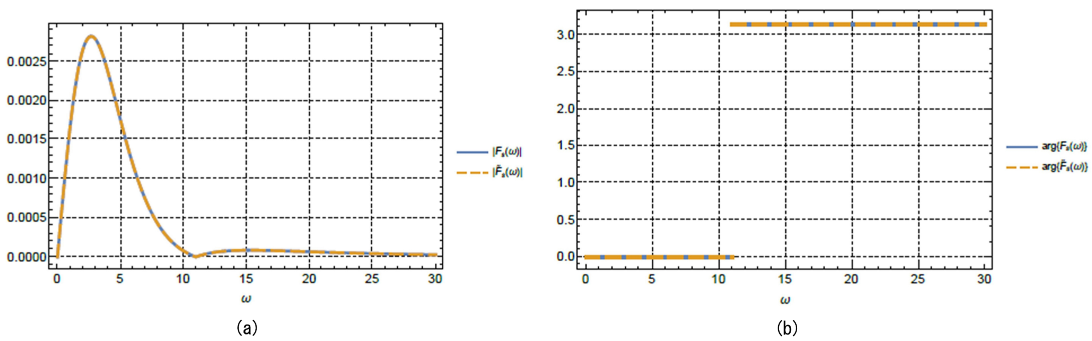


Figure 6: Magnitude (a) and argument (b) of the cosine Fourier transform of the function log(cosh(t) as evaluated through our approximant and its rigorous analytical expression.

3.2.2 Example 2

Let $f(x) = \operatorname{arcsinh} x$ and $g(t) = t^2$. Upon assuming $a = 7$, we find

$$\begin{aligned} \int_0^{+\infty} \operatorname{arcsinh}(t^2)\, e^{-7t} \sin(\omega t)\, dt = \frac{2}{(49+\omega^2)^{11/2}} \Big(& (49+\omega^2)^4 \sin[3\,\arctan(\omega/7)] - \\ & 60(49+\omega^2)^2 \sin[7\,\arctan(\omega/7)] + 136080\sin[11\,\arctan(\omega/7)]\Big) + O\left(\frac{1}{\omega^{12}}\right). \end{aligned} \tag{21}$$

Graphical results are depicted in Figures 7 and 8.

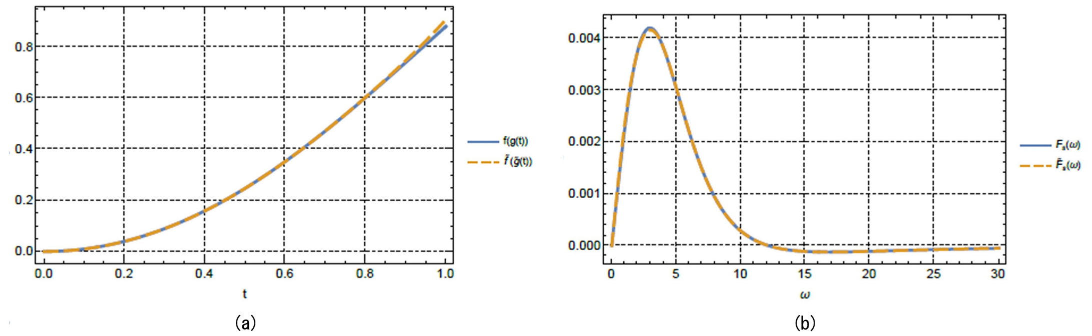


Figure 7: Function $\operatorname{arcsinh}(t^2)$ (a) and its sine Fourier transform (b) as evaluated through the approximant $\tilde{F}(s)$ and the rigorous analytical expression $F(s)$.

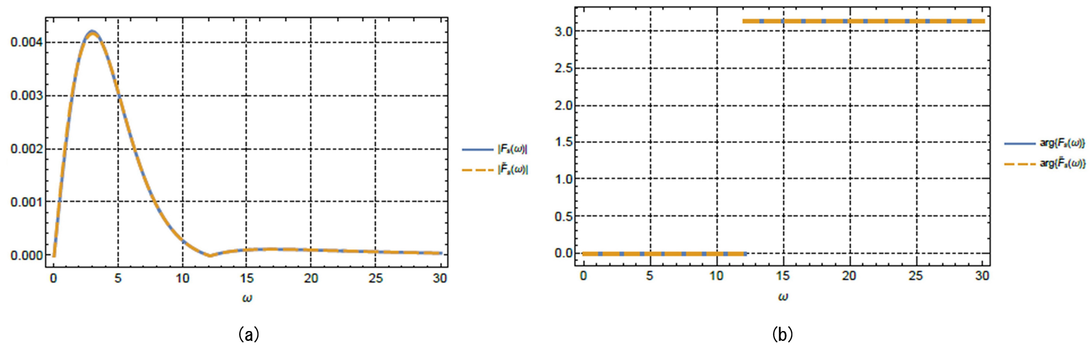


Figure 8: Magnitude (a) and argment (b) of the sine Fourier transform of the function $\operatorname{arcsinh}(t^2)$ as evaluated through our approximant and its rigorous analytical expression.

3.3 The general cosine Fourier case

3.3.1 Example 3

Let $f(x) = \log x$ and $g(t) = \cosh(t)$. Upon assuming $a = 2\pi$, we find

$$\int_0^{+\infty} \log\cosh(t)\, e^{-2\pi t} \cos(\omega t)\, dt = \frac{1}{(4\pi^2+\omega^2)^{11/2}} \Bigg((4\pi^2+\omega^2)^4 \cos[3\operatorname{arccot}(2\pi/\omega)] - 2(4\pi^2+\omega^2)^3 \cos[5\operatorname{arccot}(2\pi/\omega)] + 16\Big((4\pi^2+\omega^2)^2 \cos[7\operatorname{arccot}(2\pi/\omega)] - 17(4\pi^2+\omega^2)\cos[9\operatorname{arccot}(2\pi/\omega)] + 496\cos[11\operatorname{arccot}(\omega/4)] \Big) \Bigg) + O\left(\frac{1}{\omega^{12}}\right). \tag{22}$$

Graphical results are depicted in Figures 9 and 10.

3.3.2 Example 4

Let $f(x) = \operatorname{arcsinh} x$ and $g(t) = t^2$. Upon assuming $a = 7$, we find

$$\int_0^{+\infty} \operatorname{arcsinh}(t^2)\, e^{-7t} \cos(\omega t)\, dt = \frac{2}{(49+\omega^2)^{11/2}} \Big((49+\omega^2)^4 \cos[3\arctan(\omega/7)] - 60(49+\omega^2)^2 \cos[7\arctan(\omega/7)] + 136080\cos[11\arctan(\omega/7)] \Big) + O\left(\frac{1}{\omega^{12}}\right).. \tag{23}$$

Graphical results are depicted in Figures 11 and 12.

4 Conclusion

We have shown how to compute the sine and cosine FT of nested analytic functions by using Bell's polynomials. We started considering the Maclaurin expansion of the given function in a neighborhood of the origin, while representing the relevant

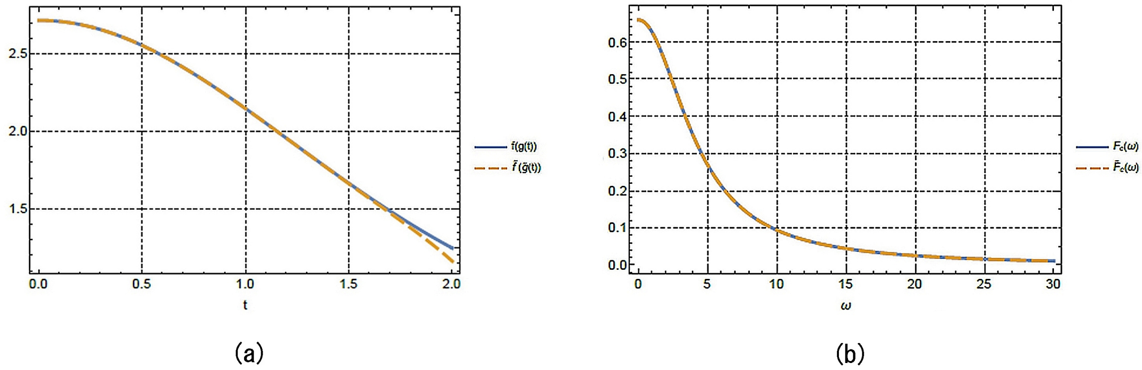


Figure 9: Function log(cosh(t) (a) and its cosine Fourier transform (b) as evaluated through the approximant $\tilde{F}(s)$ and the rigorous analytical expression $F(s)$.

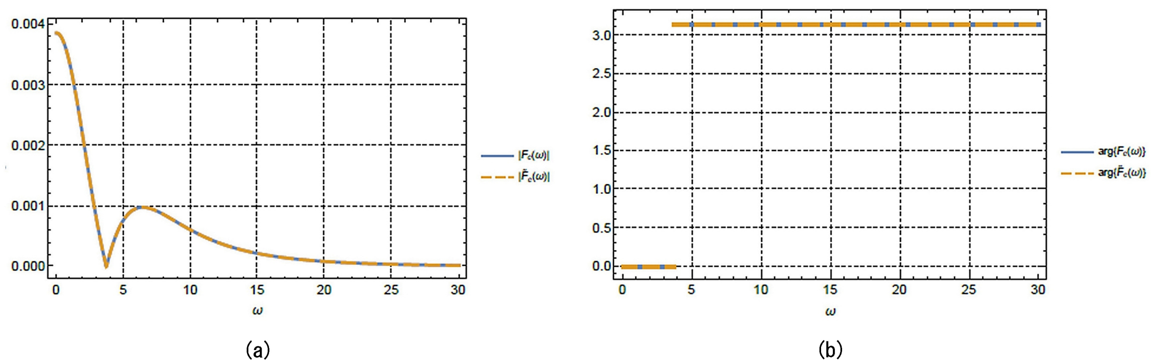


Figure 10: Magnitude (a) and argment (b) of the cosine Fourier transform of the function log(cosh(t) as evaluated through our approximant and its rigorous analytical expression.

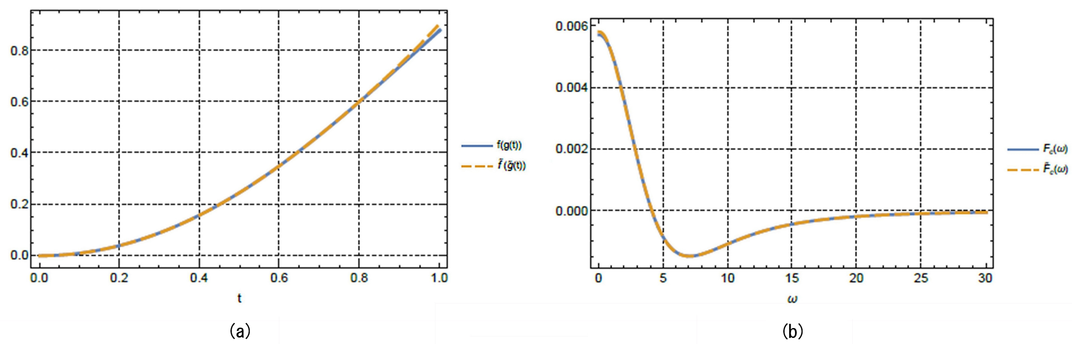


Figure 11: Function arcsinh(t^2) (a) and its cosine Fourier transform (b) as evaluated through the approximant $\tilde{F}(s)$ and the rigorous analytical expression $F(s)$.

coefficients in terms of Bell's polynomials. As a consequence, the sine and cosine FT can be reduced to the computation of an approximating series, which obviously converges if the integral converges.

This methodology can be applied to a great variety of functions starting from the simpler case of analytic nested exponential functions.

The same technique can be applied to higher-order nested analytic functions by exploiting higher-order Bell polynomials already introduced in [7]. We want to stress that the basic subject examined in this article has never been considered in the literature, a gap that in our opinion is worthy of being filled.

In the last section, a graphical verification of the proposed technique has been performed by making use of the computer algebra program Mathematica©.

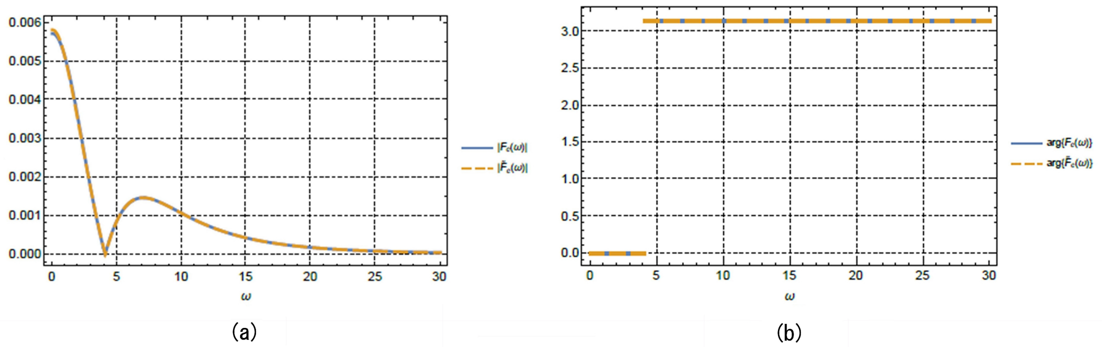


Figure 12: Magnitude (a) and argment (b) of the cosine Fourier transform of the function arcsinh(t^2) as evaluated through our approximant and its rigorous analytical expression.

Compliance with ethical standards

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