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# Beyond B-splines: exponential pseudo-splines and subdivision schemes reproducing exponential polynomials

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**Abstract**

The main goal of this paper is to present some generalizations of polynomial B-splines, which include exponential B-splines and the larger family of exponential pseudo-splines. We especially focus on their connections to subdivision schemes. In addition, we generalize a well-known result on the approximation order of exponential pseudo-splines, providing conditions to establish the approximation order of non-stationary subdivision schemes reproducing spaces of exponential polynomial functions.

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## 1 Introduction

It is well known that B-splines form a basis for the space of polynomial splines and possess, in addition to other nice properties, minimal support with respect to a given degree and smoothness (see, e.g., the celebrated book [22]). They are employed in many contexts, including approximation theory, numerical differentiation and integration, signal and image processing. Also, they provide an effective way for constructing curves and surfaces from a given set of control points, thus finding applications in computer-aided design and computer graphics.

The special case of cardinal B-splines, i.e., B-splines with uniformly spaced knots, is widely exploited within the framework of multiresolution, multilevel and subdivision techniques. We recall that all cardinal B-splines of order $N$ (degree $N-1$) with simple integer knots $t_i = i$, $i \in \mathbb{Z}$, are shifted copies of the one supported in $[0, N]$, say $B_N$. A classical way to define $B_N$ is through de Boor's recurrence formula [22]

$$B_n(t) = \frac{t}{n-1} B_{n-1}(t) + \frac{n-t}{n-1} B_{n-1}(t-1), \quad t \in \mathbb{R}, \quad n = 2, \ldots, N,$$

with $B_1 = \chi_{[0,1]}$ being the characteristic function of the unit interval. Cardinal B-splines can also be defined through a convolution approach, namely by

$$B_n(t) = (B_{n-1} * B_1)(t), \quad t \in \mathbb{R}, \quad n = 2, \ldots, N,$$

and also with the help of the Green's function of differential operators [50]. Indeed, denoting by $D^N$ the $N$-th order differential operator and by $\rho$ the corresponding Green's function satisfying $D^N(\rho(t)) = \delta_{t,0}$, where $\delta_{t,0}$ is the Dirac delta function, the cardinal B-spline of order $N$ can be defined as

$$B_N(t) = \Delta^N(\rho(t)), \quad t \in \mathbb{R},$$

where $\Delta$ denotes the discrete difference operator $\Delta f = f - f(\cdot - 1)$.

It is well known (see again [22]) that the functions of the family $\{B_N(\cdot - i),\ i \in \mathbb{Z}\}$ have very nice properties such as compact support of width $N+1$, non-negativity, $C^N$ regularity and partition of unity property, just to mention the most important ones. Nevertheless, in spite of these nice properties, B-splines have several drawbacks. Firstly, they provide a low approximation order which means that, whenever we use them to approximate a function from a certain space, a pre-processing of the data is necessary. Secondly, they are not suitable for approximating causal exponentials, which play a fundamental role for example in classical system theory [50]. Thirdly, their use for modeling manifolds with arbitrary topology is conceptually complex and extremely expensive. Last, but not least, they are not able to exactly reproduce geometries like conic sections which appear very often, e.g., in geometric modeling, biomedical imaging or isogeometric analysis. Figure 1 illustrates several conic sections appearing in different application contexts (in this regard see, e.g., [1, 34, 49]).

In the last two decades, several generalizations of B-splines have been proposed with the aim of overcoming their limitations. The most popular one is certainly given by *non-uniform rational B-splines*, also called *NURBS* [43]. A NURBS curve is a linear

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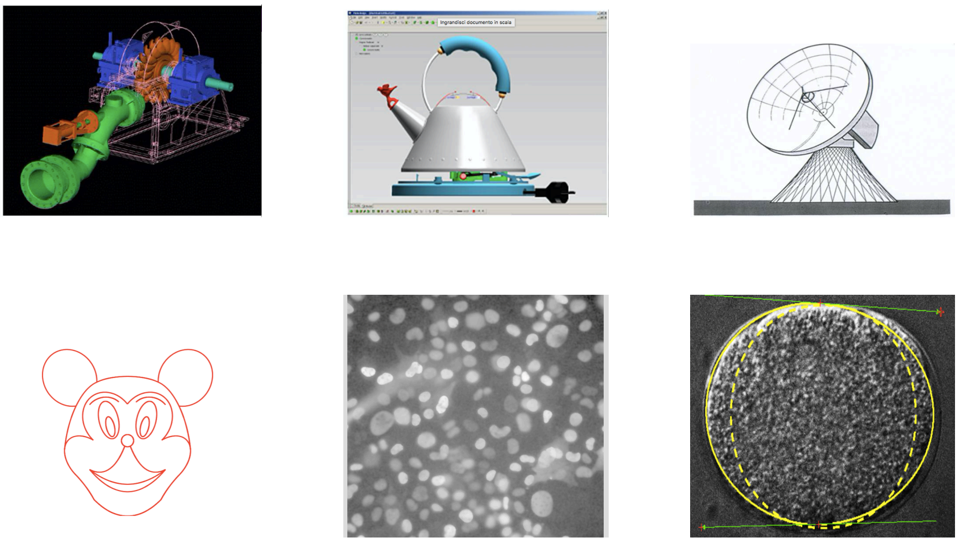

**Figure 1:** Conic section geometries appearing in different contexts.

combination of rational basis functions, expressed as a ratio of B-splines associated to non-uniform knot sequences. Its definition involves additional parameters (or weights), which have to be properly handled. Unfortunately, the rational nature of NURBS makes them unpleasant with respect to differentiation and integration. Moreover, though they allow for exact description of conic sections, not all transcendental curves can be reproduced by NURBS, and the modeling of manifolds with arbitrary topology is still rather complicated.

An attractive alternative to NURBS is given by the so-called *generalized B-splines* [38, 48]. While classical B-splines are piecewise polynomial functions, generalized B-splines are piecewise functions with function segments in more general spaces. With a suitable selection of such spaces (typically including trigonometric or hyperbolic functions), generalized B-splines can allow exact representation of polynomial curves, conic sections or transcendental curves. Generalized B-splines also possess all fundamental properties of polynomial B-splines shared by NURBS, e.g., recurrence relations, minimal support, local linear independence, knot-insertion or degree elevation, but behave in the same way as B-splines with respect to differentiation and integration [4].

A special instance of generalized B-splines with integer knots is given by *exponential B-splines*, which have recently received an increasing attention [10, 24, 25, 37, 44, 45, 46, 47]. Besides their classical applications in geometric modeling and approximation theory, uniform exponential B-splines are indeed very useful in signal processing [49, 50] and in isogeometric analysis [34].

Another interesting generalization of B-splines, which has emerged recently, is given by *pseudo-splines* [16, 26, 27, 28, 29] and, more generally, by *exponential pseudo-splines* [14]. Exponential pseudo-splines form a rich family of basis functions meeting various demands for balancing approximation power, regularity, support size, interpolation and reproduction capabilities. Pseudo-splines have been used, for example, for generating tight wavelet frames to be used as multiresolution analysis techniques in signal and image processing, as discussed in [28].

A common feature shared by B-splines, exponential B-splines and pseudo-splines is the fact that they are basic limit functions of *subdivision schemes* (either of stationary or of non-stationary nature). Subdivision schemes are simple and efficient iterative procedures for generating functions or curves, from a given set of data points. The subdivision framework provides, in addition, a tool for addressing problems connected to the evaluation of such splines, for the characterization of their reproduction properties and the computation of their approximation order [3, 6, 8, 11, 13, 15, 18, 20, 21, 32, 36, 40, 41, 42]. With respect to the latter point, one of our goals is to provide a new result on the approximation order of general classes of subdivision schemes which include exponential pseudo-splines.

This paper is organized as follows. In Section 2 we describe the refinement properties of cardinal B-splines and the corresponding subdivision algorithms. Then the subdivision approach is presented as a general discrete tool for generating refinable basis functions. In Section 3, we present cardinal exponential B-splines and pseudo-splines from the perspective of their refinability properties and associated subdivision schemes. Finally, in the last section, we provide a new result on the approximation order of non-stationary subdivision schemes reproducing certain classes of exponential polynomial functions.

## 2 Cardinal B-splines and subdivision schemes

This section recalls the connection between cardinal B-splines and subdivision schemes (see subsection 2.1). The subdivision approach is then presented as a general discrete tool for generating refinable basis functions and cardinal B-splines in particular (see subsection 2.2).

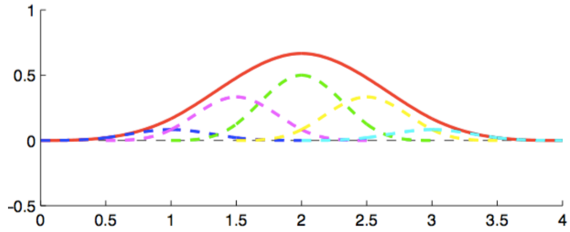


**Figure 2:** Graphical representation of the refinability property of the cardinal cubic B-spline $B_4$.

### 2.1 A subdivision approach to cardinal B-splines

An important property of cardinal (polynomial) B-splines $B_N$ of order $N > 0$ is their refinability, i.e., the fact that they can be written as linear combinations of shifts of dilated versions of themselves

$$B_N = \sum_{j=0}^{N} a_{N;j} B_N(2\cdot - j) \tag{1}$$

with weights

$$a_{N;j} = \frac{1}{2^{N-1}} \begin{pmatrix} N \\ j \end{pmatrix}, \quad j = 0, \dots, N.$$

In the *refinement equation* (1), the finite sequence $\boldsymbol{a}_N = \{a_{N;j} : j = 0, \dots, N\}$ represents the *refinement mask* of $B_N$. For example, for the cubic B-spline the refinement mask is given by $\boldsymbol{a}_4 = \{\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}\}$, and the corresponding refinability property is depicted in Figure 2.

A useful consequence of (1) is the possibility of deriving an iterative algorithm for the computation of cardinal B-splines at dyadic points. In the cubic case, for instance, evaluation of (1) at the variable's value $1, 2, 3$ yields

$$\begin{pmatrix} B_4(1) \\ B_4(2) \\ B_4(3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} & 0 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} B_4(1) \\ B_4(2) \\ B_4(3) \end{pmatrix}.$$

Hence the non-zero values of cardinal B-splines at integer knots can be obtained by solving an eigenvector-eigenvalue problem for a matrix determined only by the refinement coefficients. Once $B_4(k)$, $k = 1, 2, 3$, are known, it is simple to compute the values of the order-4 cardinal B-spline at half-integers as

$$B_4\left(\frac{\cdot}{2}\right) = \frac{1}{8}B_4 + \frac{1}{2}B_4(\cdot - 1) + \frac{3}{4}B_4(\cdot - 2) + \frac{1}{2}B_4(\cdot - 3) + \frac{1}{8}B_4(\cdot - 4),$$

and similarly for the other values at all dyadic points.

Let us now denote with $s$ a polynomial spline of order $N$ and with $\boldsymbol{f}^{[k]} = \{f_j^{[k]},\ j \in \mathbb{Z}\} \in \ell(\mathbb{Z})$ the coefficient sequence connected to its representation in the space spanned by $\{B_N(2^k \cdot - i) : i \in \mathbb{Z}\}$, i.e.

$$s = \sum_{j \in \mathbb{Z}} f_j^{[k]} B_N(2^k \cdot - j).$$

Using (1), for a fixed integer $k \geq 0$, we are able to write $s$ as

$$\begin{aligned} s &= \sum_{j\in\mathbb{Z}} f_j^{[k]} B_N(2^k \cdot - j) = \sum_{j\in\mathbb{Z}} f_j^{[k]} \sum_{i\in\mathbb{Z}} a_{N;i} B_N(2^{k+1} \cdot - 2j - i) \\ &= \sum_{j\in\mathbb{Z}} f_j^{[k]} \sum_{i\in\mathbb{Z}} a_{N;i-2j} B_N(2^{k+1} \cdot - i) = \sum_{i\in\mathbb{Z}} f_i^{[k+1]} B_N(2^{k+1} \cdot - i) \end{aligned}$$

with

$$f_i^{[k+1]} = \sum_{j\in\mathbb{Z}} a_{N;i-2j} f_j^{[k]}, \quad i \in \mathbb{Z}, \quad k \geq 0. \tag{2}$$

Since the support of $B_N(2^k \cdot)$ shrinks as $k$ increases, for $k$ large enough the coefficient sequence $\boldsymbol{f}^{[k]}$ is a good discrete representation of the spline $s$.

More generally, starting with any initial sequence of points $\boldsymbol{f}^{[0]} \in \ell(\mathbb{Z})$, the computation of denser and denser sequences of points can be described in terms of the repeated application of a linear operator $S_{\boldsymbol{a}_N} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ whose action is described by

$$f_i^{[k+1]} = (S_{\boldsymbol{a}_N} \boldsymbol{f}^{[k]})_i = \sum_{j\in\mathbb{Z}} a_{N;i-2j} f_j^{[k]}, \quad i \in \mathbb{Z}. \tag{3}$$

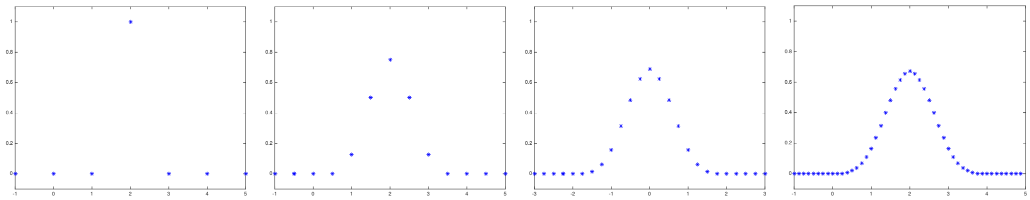

**Figure 3:** A B-spline function as a subdivision limit.

The resulting procedure $\boldsymbol{f}^{[k+1]} = S_{\boldsymbol{a}_N} \boldsymbol{f}^{[k]}$, $k \geq 0$, defines the so-called *subdivision scheme* for polynomial splines of order-$N$ [31]. The corresponding linear operator $S_{\boldsymbol{a}_N}$ is called the *subdivision operator*.

Since (3) splits into two different subdivision rules

$$\begin{cases} f_{2i}^{[k+1]} &= \sum_{j\in\mathbb{Z}} a_{N;2j} f_{i-j}^{[k]}, \\ & \\ f_{2i+1}^{[k+1]} &= \sum_{j\in\mathbb{Z}} a_{N;2j+1} f_{i-j}^{[k]}, \end{cases} \qquad i \in \mathbb{Z}, \quad k \geq 0,$$

the subdivision scheme is said to be *binary*. Note that subdivision schemes of different *arity* can also be considered (see, for example, [16, 33]).

The limit of (3), as $k$ goes to infinity, is the order-$N$ polynomial spline associated to the initial control points $\boldsymbol{f}^{[0]}$, and is often denoted by $S_{\boldsymbol{a}_N}^{\infty} \boldsymbol{f}^{[0]}$ to stress its connection with the subdivision operator $S_{\boldsymbol{a}_N}$.

When the subdivision scheme is applied to the starting sequence $\boldsymbol{\delta} = \{\delta_{i,0} \, : \, i \in \mathbb{Z}\}$, as in Figure 3, the limit is exactly the order-$N$ cardinal B-spline basis function, also called the *basic limit function* of the subdivision scheme.

## 2.2 Subdivision schemes and their basic limit functions

The subdivision idea presented in the case of B-splines can be generalized to define other types of refinable basis functions, not necessarily of piecewise polynomial nature.

A subdivision scheme is identified by a sequence of *subdivision operators* $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$, each based on the mask sequence $\boldsymbol{a}^{[k]} = \{a_i^{[k]} \in \mathbb{R} : i \in \mathbb{Z}\}$, usually assumed to be finite (see [5, 31]). The subdivision scheme consists of the iterative application of such operators as follows:

$$\begin{cases} \text{Input:} \quad \{S_{\boldsymbol{a}^{[k]}},\ k \geq 0\}, \quad \boldsymbol{f}^{[0]} \in \ell(\mathbb{Z}) \\ \text{For} \quad k \geq 0 \\ \qquad \boldsymbol{f}^{[k+1]} = S_{\boldsymbol{a}^{[k]}} \boldsymbol{f}^{[k]} \end{cases}$$

and it is therefore simply denoted by $\{S_{\boldsymbol{a}^{[k]}},\ k \geq 0\}$. In the special situation where the mask does not depend on the level, i.e. $\boldsymbol{a}^{[k]} = \boldsymbol{a}$, for every $k$, and the same operator $S_{\boldsymbol{a}}$ is applied at each step, the scheme is said to be *stationary* and denoted by $S_{\boldsymbol{a}}$. In the other case, it is said a *non-stationary* or *level-dependent* scheme (see [5, 31]).

For a subdivision scheme identified by the subdivision operators $\{S_{\boldsymbol{a}^{[k]}} : k \geq \ell\}$, $\ell \geq 0$, the following definition of convergence can be given.

**Definition 2.1.** For any $\ell \geq 0$, the subdivision scheme $\{S_{\boldsymbol{a}^{[k]}} : k \geq \ell\}$ applied to the initial data $\boldsymbol{f}^{[\ell]} \in \ell^{\infty}(\mathbb{Z})$ is (uniformly) *convergent* if there exists a function $g_{\boldsymbol{f}^{[\ell]}} \in C(\mathbb{R})$, $g_{\boldsymbol{f}^{[\ell]}} \neq 0$, such that

$$\lim_{k\to+\infty} \sup_{i\in\mathbb{Z}} |g_{\boldsymbol{f}^{[\ell]}}(2^{-(k+\ell)}i) - f_i^{[k+\ell]}| = 0.$$

Moreover, the scheme $\{S_{\boldsymbol{a}^{[k]}},\ k \geq \ell\}$ is $C^L$*–convergent* if $g_{\boldsymbol{f}^{[\ell]}} \in C^L(\mathbb{R})$.

In case of convergence, all the schemes based on the operator sequences $\{S_{\boldsymbol{a}^{[k]}} : k \geq \ell\}$, $\ell \geq 0$ (based on masks $\boldsymbol{a}^{[k]}$ of finite length) define compactly supported *basic limit functions* $\phi^{[\ell]}$, $\ell \geq 0$, associated to the initial sequence $\boldsymbol{\delta} = \{\delta_{i,0},\ i \in \mathbb{Z}\}$, hereinafter called the $\boldsymbol{\delta}$-sequence. Even if in most cases these functions are not defined analytically, they are all related through the so called *level-dependent refinement equations*

$$\phi^{[\ell]} = \sum_{i\in\mathbb{Z}} a_i^{[\ell]} \phi^{[\ell+1]}(2 \cdot -i), \quad \ell \geq 0.$$

In the stationary case, the latter ones reduce to a standard *refinement equation*, involving only one basic limit function,

$$\phi = \sum_{i\in\mathbb{Z}} a_i \phi(2 \cdot -i).$$

An important role for the analysis of subdivision schemes is played by the *symbols* of the subdivision masks. The symbol of a finitely supported sequence $\boldsymbol{a}$ is defined as the Laurent polynomial

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4}$$

A stationary scheme is identified by the symbol of its subdivision mask, while a non-stationary scheme by the sequence of mask symbols $\{a^{[k]}(z) : k \geq 0\}$. Many of the properties of a subdivision scheme and of its basic limit functions (e.g., regularity, symmetry, reproduction properties) can be checked using algebraic conditions on the subdivision symbol [5, 31].

## 3 Cardinal exponential B-splines and pseudo-splines

Non-stationary subdivision schemes can be efficiently used to define a useful generalization of cardinal B-splines, namely *cardinal exponential B-splines*. The latter ones turn out to be a "perfect" basis for the space of exponential splines. Let us start by defining the space of exponential polynomials, the space of exponential splines and one of their corresponding bases.

**Definition 3.1.** Let $n \in \mathbb{N}$ and let $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ with $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_i \neq \theta_j$ for $i \neq j$, and $\tau_i \in \mathbb{N}$, $i = 1, \ldots, n$. The corresponding space of exponential polynomials is defined by

$$EP_\Gamma = \mathrm{span}\{\, t^{r_i} e^{\theta_i t},\ r_i = 0, \ldots, \tau_i - 1,\ i = 1, \ldots, n\}\,.$$

Note that for each $i = 1, \ldots, n$, $\tau_i$ denotes the multiplicity of the value $\theta_i$.

The set $EP_\Gamma$ is a linear space of dimension $N = \sum_{\ell=1}^{n} \tau_\ell$ whose elements belong to the null space of the differential operator $L_\Gamma = (D - \theta_1 I)^{\tau_1} \circ \cdots \circ (D - \theta_n I)^{\tau_n}$, with $D$ the first derivative operator. Polynomials of order $N$, belonging to the space $\mathbb{P}_N$, are special instances of exponential polynomials, corresponding to the case $\Gamma = \{(0, N)\}$, so that $\mathbb{P}_N \subseteq EP_\Gamma$.

**Example 3.1.** For $\theta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, consider $\Gamma = \{(0, 1), (\theta, 2), (-\theta, 2)\}$, whose corresponding space of exponential polynomials is of dimension $N = 5$ and given by

$$EP_\Gamma = \mathrm{span}\{1, e^{\theta t}, e^{-\theta t}, te^{\theta t}, te^{-\theta t}\}.$$

For $\theta = 0$, $\theta = \mathrm{i}s$ and $\theta = s$, $EP_\Gamma$ reduces to (see [44])

$$\begin{aligned}
&\mathrm{span}\{1, t, t^2, t^3, t^4\},\\
&\mathrm{span}\{1, \cos(st), \sin(st), t\cos(st), t\sin(st)\},\\
&\mathrm{span}\{1, \cosh(st), \sinh(st), t\cosh(st), t\sinh(st)\}.
\end{aligned}$$

The space of exponential splines, denoted as $S(EP_\Gamma)$, is the space of piecewise functions with pieces belonging to $EP_\Gamma$. It can be represented in terms of the basis of cardinal exponential B-splines. To define them we follow the approach in [50], which makes use of the Green's functions of differential operators. Consider the first order differential operator $D_\theta = D - \theta I$ and its corresponding Green's function $\rho_\theta(t) = e^{\theta t} 1_+(t)$. Take a discretization of $D_\theta$, say $\Delta_\theta f = f - e^\theta f(\cdot - 1)$. Then the order-one (discontinuous) cardinal exponential B-spline, supported in $[0, 1]$, is

$$B_{1,\{(\theta,1)\}}(t) = \Delta_\theta\left(\rho_\theta(t)\right) = \rho_\theta(t) - e^\theta \rho_\theta(t-1), \quad t \in [0, 1].$$

For $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$, higher order cardinal exponential B-splines are defined via convolution of order-one factors

$$B_{1,\{(\theta_i,1)\}}(t) = \rho_{\theta_i}(t) - e^{\theta_i} \rho_{\theta_i}(t-1),$$

that is as

$$B_{N,\Gamma} = \underbrace{B_{1,\{(\theta_1,1)\}} * \cdots * B_{1,\{(\theta_1,1)\}}}_{\tau_1\text{–times}} * \cdots * \underbrace{B_{1,\{(\theta_n,1)\}} * \cdots * B_{1,\{(\theta_n,1)\}}}_{\tau_n\text{–times}}.$$

Cardinal exponential B-splines $B_{N,\Gamma}$ share many important properties with B-splines. In particular, order-$N$ cardinal exponential B-splines are non-negative, have support $[0, N]$, are $C^{N-2}$-convergent and have the shortest possible support for the given smoothness [10]. Moreover, shifted copies of $B_{N,\Gamma}$ are linearly independent and satisfy the partition of unity property [10]. Finally, they can be defined via level-dependent subdivision schemes by direct construction of the corresponding subdivision symbols as follows.

**Proposition 3.1.** *Let $n \in \mathbb{N}$ and $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ with $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_i \neq \theta_j$ for $i \neq j$, and $\tau_i \in \mathbb{N}$, $i = 1, \ldots, n$. Then the Laurent polynomials*

$$\mathcal{B}^{[k]}_{N,\Gamma}(z) = F^{[k]}_N \prod_{\ell=1}^{n} (1 + e^{\frac{\theta_\ell}{2^{k+1}}} z)^{\tau_\ell}, \quad k \geq 0,$$

*with $N = \sum_{\ell=1}^{n} \tau_\ell$ and $F^{[k]}_N \in \mathbb{R}$, are the symbols associated to a convergent subdivision scheme whose basic limit function is the cardinal exponential B-spline $B_{N,\Gamma}$ of order $N$.*

The parameters $F^{[k]}_N \in \mathbb{R}$, $k \geq 0$, play the role of normalization factors (for details see [14, 35]).

When $\Gamma = \{(0, N)\}$ the symbol $\mathcal{B}^{[k]}_{N,\Gamma}(z)$ reduces to the symbol of the order-$N$ (degree $N - 1$) polynomial B-spline given by $\mathcal{B}_N(z) = \frac{1}{2^{N-1}}(1 + z)^N$, and this is exactly the limit of $\mathcal{B}^{[k]}_{N,\Gamma}(z)$ as $k \to +\infty$. In fact, exponential B-spline and B-spline subdivision schemes are *asymptotically similar* in the sense that will be specified in Section 4 (Definition 4.1).

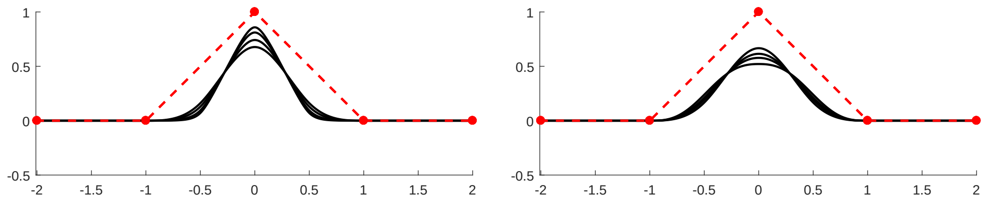


**Figure 4:** From lower to taller functions: basic limit functions for the space $EP_\Gamma = \text{span}\{1, t, e^{\theta t}, e^{-\theta t}\}$ with $\theta \in \{\mathrm{i}, 3\mathrm{i}, 5\mathrm{i}, 7\mathrm{i}\}$ (left), $\theta \in \{3, 2.5, 2, 0\}$ (right), obtained from the initial data set represented by a dashed line.

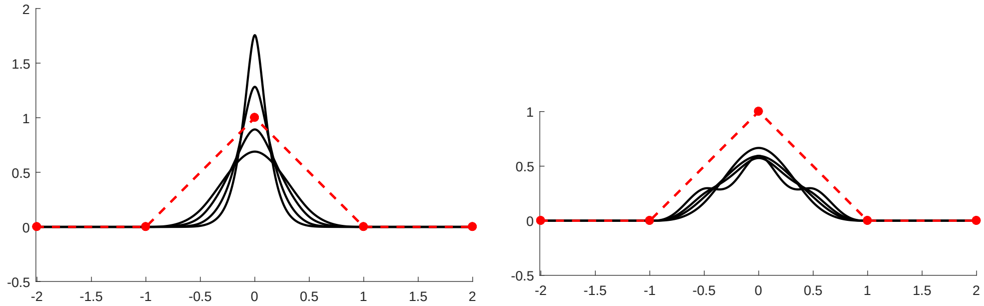


**Figure 5:** From lower to taller functions: basic limit functions for the space $EP_\Gamma = \text{span}\{e^{\theta t}, e^{-\theta t}, te^{\theta t}, te^{-\theta t}\}$ with $\theta \in \{\mathrm{i}, 3\mathrm{i}, 5\mathrm{i}, 7\mathrm{i}\}$ (left), $\theta \in \{3, 2.5, 2, 0\}$ (right), obtained from the initial data set represented by a dashed line.

**Example 3.2.** Consider cardinal exponential B-splines for the space $EP_\Gamma = \text{span}\{1, t, e^{\theta t}, e^{-\theta t}\}$, with $\theta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$. They can be generated from the $\boldsymbol{\delta}$-sequence applying the subdivision scheme specified by the subdivision rules (see [2])

$$\begin{aligned} f_{2i}^{[k+1]} &= \frac{1}{4(v^{[k]}+1)} f_{i-1}^{[k]} + \frac{1+2v^{[k]}}{2(v^{[k]}+1)} f_i^{[k]} + \frac{1}{4(v^{[k]}+1)} f_{i+1}^{[k]} \\ f_{2i+1}^{[k+1]} &= \tfrac{1}{2} f_i^{[k]} + \tfrac{1}{2} f_{i+1}^{[k]}, \end{aligned}$$

where

$$v^{[k]} = \frac{1}{2}\left(e^{\mathrm{i}\frac{\theta}{2^{k+1}}} + e^{-\mathrm{i}\frac{\theta}{2^{k+1}}}\right) = \sqrt{\frac{1+v^{[k-1]}}{2}}, \quad k \geq 0, \quad v^{[-1]} = \cos(\theta) > -1.$$

The free parameter $v^{[-1]}$ slightly influences the final shape of the exponential B-spline as shown in Figure 4.

**Example 3.3.** We now change the exponential-polynomial space and consider $EP_\Gamma = \text{span}\{e^{\theta t}, e^{-\theta t}, te^{\theta t}, te^{-\theta t}\}$, with $\theta \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$. The corresponding exponential B-spline is obtained from the $\boldsymbol{\delta}$-sequence via the subdivision rules (see [14])

$$\begin{aligned} f_{2i}^{[k+1]} &= \frac{1}{2(v^{[k]}+1)^2} f_{i-1}^{[k]} + \frac{4(v^{[k]})^2+2}{2(v^{[k]}+1)^2} f_i^{[k]} + \frac{1}{2(v^{[k]}+1)^2} f_{i+1}^{[k]} \\ f_{2i+1}^{[k+1]} &= \frac{2v^{[k]}}{(v^{[k]}+1)^2} f_i^{[k]} + \frac{2v^{[k]}}{(v^{[k]}+1)^2} f_{i+1}^{[k]}, \end{aligned}$$

where

$$v^{[k]} = \frac{1}{2}\left(e^{\mathrm{i}\frac{\theta}{2^{k+1}}} + e^{-\mathrm{i}\frac{\theta}{2^{k+1}}}\right) = \sqrt{\frac{1+v^{[k-1]}}{2}}, \quad k \geq 0, \quad v^{[-1]} = \cos(\theta) > -1.$$

The influence of the parameter $v^{[-1]}$ on the final shape of the exponential B-spline is illustrated in Figure 5.

As already mentioned, one of the main drawbacks of both polynomial and exponential B-splines is that they have low *approximation order*. It means that, if used to approximate a set of data, a pre-processing of the data is necessary. From the point of view of subdivision, the formal definition of approximation order [23, 39] is given as follows.

**Definition 3.2.** Let $\gamma \in \mathbb{N}$, $f \in C^\gamma(\mathbb{R})$ with $\|f^{(\ell)}\|_\infty < \infty$, $\ell = 0, \ldots, \gamma$. A convergent subdivision scheme $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ is said to have *approximation order* $\gamma$ if the limit function $g_{\boldsymbol{f}^{[0]}}$ obtained from $\boldsymbol{f}^{[0]} = \{f(ih) : i \in \mathbb{Z}\}$, $h \in \mathbb{R}_+$, satisfies

$$\|g_{\boldsymbol{f}^{[0]}} - f\|_{L_\infty(\mathbb{R})} \leq C_f\, h^\gamma,$$

with $C_f$ a positive constant depending only on $f$.

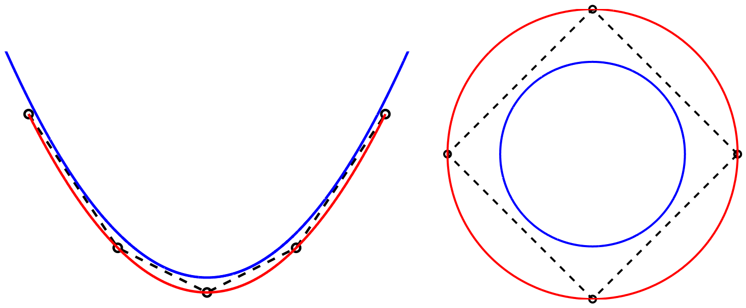

**Figure 6:** Limits of a reproducing subdivision scheme (outer/red lines) and a generating subdivision scheme (inner/blue lines) obtained from the initial data set represented by a dashed line.

To be able to compute $\gamma$ in Definition 3.2, we rely on the fact that the approximation order of any subdivision scheme is closely connected with its generation or reproduction properties [29]. The latter concepts are explained in detail in [18] and recalled in Definition 3.3. Loosely speaking we can say that the higher is the number of exponential polynomials reproduced, the higher is the approximation order of the scheme [19].

**Definition 3.3.** Let $\left\{t_i^{[0]} = i + p : p \in \mathbb{R}, i \in \mathbb{Z}\right\}$ be a nondecreasing *parameter sequence*. A convergent subdivision scheme $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ *generates* $EP_\Gamma$ if for all initial sequences $\boldsymbol{f}^{[0]} = \left\{f\left(t_i^{[0]}\right) : i \in \mathbb{Z}\right\}$, $f \in EP_\Gamma$,

$$\lim_{k\to+\infty} S_{\boldsymbol{a}^{[k]}} S_{\boldsymbol{a}^{[k-1]}} \cdots S_{\boldsymbol{a}^{[0]}} \boldsymbol{f}^{[0]} \in EP_\Gamma.$$

Moreover, $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ *reproduces* $EP_\Gamma$ if

$$\lim_{k\to+\infty} S_{\boldsymbol{a}^{[k]}} S_{\boldsymbol{a}^{[k-1]}} \cdots S_{\boldsymbol{a}^{[0]}} \boldsymbol{f}^{[0]} = f.$$

To better understand the difference between generation and reproduction we can refer to Figure 6, where the limits of a subdivision scheme generating exponential-polynomials and a subdivision scheme reproducing exponential-polynomials are plotted in different colors, for the initial data set represented by the dashed line.

In order to increase the approximation order of a subdivision scheme (hence, of its basic limit function), the above discussion can lead to the idea of constructing a subdivision scheme reproducing as many exponential polynomials as possible. Unfortunately, the higher is the number of exponential polynomials reproduced, the bigger is the size of the support of the corresponding basic limit functions. It has been recently shown that the most effective way to increase the approximation order of B-splines is to construct the family of *pseudo-splines* [26, 27, 28, 29]. Indeed, pseudo-splines give a wide range of choices of refinable functions that meet various demands for balancing approximation order, length of the support and also regularity. They are a family of compactly supported refinable basic limit functions of stationary subdivision schemes not defined analytically. Since the first members of the family of pseudo-splines are cardinal B-splines, they can be also seen as a generalization of B-splines.

By enforcing exponential polynomial reproduction in a purely algebraic way, pseudo-splines as well as their non-stationary generalizations, given by *exponential pseudo-splines* [14], can be constructed in a general subdivision setting. Indeed, for the class of convergent and *non-singular subdivision schemes* (providing a zero limit if and only if the starting sequence is the zero sequence), reproduction of exponential polynomials is equivalent to the following set of algebraic conditions on the corresponding subdivision symbols (see [9] and [18]).

**Theorem 3.2.** *Let $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ be defined as in Definition 3.1 and let $z_\ell^{[k]} = e^{\frac{-\theta_\ell}{2^{k+1}}}$, $\ell = 1, \ldots, n$, $k \geq 0$. A convergent and non-singular subdivision scheme $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ generates $EP_\Gamma$ if and only if*

$$\frac{d^r\, a^{[k]}(-z_\ell^{[k]})}{dz^r} = 0,\ \ r = 0, \ldots, \tau_\ell - 1,\ \ell = 1, \ldots, n, \quad k \geq 0. \tag{5}$$

*The subdivision scheme reproduces $EP_\Gamma$, with respect to the parameters $t_i^{[k]} = \dfrac{i+p}{2^k}$, $p \in \mathbb{R}$, $i \in \mathbb{Z}$, if and only if, in addition to (5),*

$$\frac{d^r\, a^{[k]}(z_\ell^{[k]})}{dz^r} = 2\left(z_\ell^{[k]}\right)^{p-r} \prod_{q=0}^{r-1}(p-q), \quad r = 0, \ldots, \tau_\ell - 1,\ \ell = 1, \ldots, n, \quad k \geq 0.$$

Theorem 3.2 is the algebraic tool to be used for increasing the approximation order of exponential B-splines, by defining the family of exponential pseudo-splines. Referring to [14], their definition can be briefly summarized as follows.

**Definition 3.4.** Let $\Gamma = \{(\theta_1, \tau_1), \ldots (\theta_n, \tau_n)\}$ be defined as in Definition 3.1 and let $\tilde{\Gamma} = \{(\theta_1, \tilde{\tau}_1), \ldots, (\theta_\ell, \tilde{\tau}_\ell)\}$ with $\tilde{\tau}_i \leq \tau_i$ for all $i = 1, \ldots, \ell$ and $\ell \leq n$. The *exponential pseudo-spline* is defined as the basic limit function of a convergent non-stationary subdivision scheme $\{S_{\boldsymbol{a}^{[k]}},\ k \geq 0\}$ with symbols

$$a_{N,\tilde{L}}^{[k]}(z) = \mathcal{B}_{N,\Gamma}^{[k]}(z)\, c_{\tilde{L},\tilde{\Gamma}}^{[k]}(z), \quad k \geq 0 \quad \text{with} \quad N = \sum_{i=1}^{n} \tau_i, \quad \tilde{L} = \sum_{i=1}^{\ell} \tilde{\tau}_i,$$

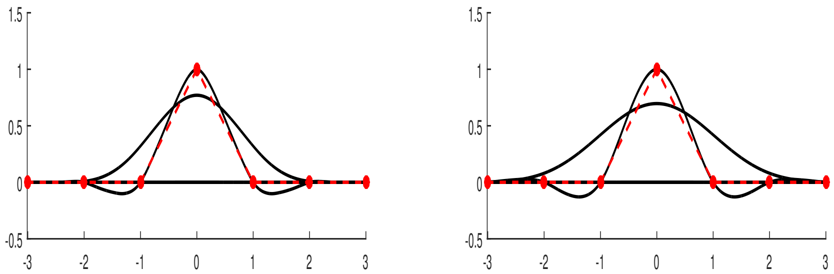


**Figure 7:** Basic limit functions of the extreme members of two different families of exponential pseudo-splines obtained from the initial data set represented by a dashed line.

where the Laurent polynomials $c^{[k]}_{\tilde{L},\tilde{\Gamma}}(z)$ are univocally identified by $\mathcal{B}^{[k]}_{N,\Gamma}(z)$, $\tilde{L}$ and $\tilde{\Gamma}$, and constructed as specified in [14].

As proved in [14], the non-stationary subdivision scheme with symbols in Definition 3.4, generates exponential polynomials in $EP_\Gamma$, reproduces exponential polynomials in $EP_{\tilde{\Gamma}}$ and has minimal support. A complete family of exponential pseudo-splines is identified by taking $\tilde{\Gamma}$ of dimension $\tilde{L}$ ranging from 1 to $N$, so that the approximation order $\tilde{L}+1$ ranges between 2 and $N+1$. When $\tilde{L}=1$, exponential pseudo-splines associated with $\Gamma$ and $\tilde{\Gamma}\subseteq\Gamma$, reduce to cardinal exponential B-splines. But, except for this case, all other exponential pseudo-splines are neither piecewise exponential polynomials nor analytically defined functions. It can be observed that the first and the last members of the family are opposite extreme cases: exponential B-splines have high smoothness and short support, but provide a rather poor approximation order. In contrast, the limit functions corresponding to the case $\tilde{L}=N$ are interpolatory subdivision schemes with optimal approximation order but lower smoothness and larger supports. A plot of the basic limit functions of these two extreme cases for the families of exponential pseudo-splines $a^{[k]}_{N,\tilde{L}}(z)$ obtained from $\mathcal{B}^{[k]}_{N,\Gamma}(z)=\dfrac{(z+z^{-1}+2v^{[k]})^\rho}{2^{2\rho-1}(v^{[k]})^\rho}$ with $\rho=2$ (left), $\rho=3$ (right) and $\Gamma=\{(\mathrm{i},\rho),(-\mathrm{i},\rho)\}$ are given in Figure 7 together with the initial data set used in the subdivision procedure.

## 4 Approximation order of non-stationary subdivision schemes

The aim of this section is to provide some new results for determining the approximation order of non-stationary subdivision schemes [19, 42]. To this purpose we need to recall the notions of asymptotical equivalence [30] and asymptotical similarity [7, 12] and to provide known results related to non-stationary $\{S_{\boldsymbol{a}^{[k]}}:k\geq 0\}$ and to stationary $S_{\boldsymbol{a}}$ schemes. We always assume that the corresponding masks $\{\boldsymbol{a}^{[k]}:k\geq 0\}$ and $\boldsymbol{a}$ have the same finite support.

**Definition 4.1.** The subdivision schemes $\{S_{\boldsymbol{a}^{[k]}}:k\geq 0\}$ and $S_{\boldsymbol{a}}$, with $\mathrm{supp}(\boldsymbol{a}^{[k]})=\mathrm{supp}(\boldsymbol{a})$, for $k\geq 0$, are said to be

i) *asymptotically equivalent* if $\displaystyle\sum_{k=0}^{\infty}\|\boldsymbol{a}^{[k]}-\boldsymbol{a}\|_\infty<\infty$;

ii) *asymptotically similar* if $\displaystyle\lim_{k\to+\infty}\|\boldsymbol{a}^{[k]}-\boldsymbol{a}\|_\infty=0$.

The following result from [30] links the basic limit functions of asymptotically equivalent non-stationary and stationary schemes.

**Proposition 4.1.** *Let $\{S_{\boldsymbol{a}^{[k]}}:k\geq 0\}$ be a convergent, non-stationary subdivision scheme with basic limit functions $\{\phi^{[k]}:k\geq 0\}$ and let $S_{\boldsymbol{a}}$ be a convergent, stationary subdivision scheme with basic limit function $\phi$. If there exists $C>0$ such that $\|\boldsymbol{a}^{[k]}-\boldsymbol{a}\|_\infty\leq C\,2^{-\nu k}$, with $\nu\in\mathbb{N}$, then there exists $\widetilde{C}>0$ such that*

$$\|\phi^{[k]}-\phi\|_\infty\leq\widetilde{C}\,2^{-\nu k}.$$

As a consequence of Proposition 4.1, in view of the fact that $\phi$ is bounded, we obtain that $\{\phi^{[k]}:k\geq 0\}$ is uniformly bounded independently of $k$, i.e. $\|\phi^{[k]}\|_\infty\leq M$, for all $k\geq 0$.

We continue by recalling from [19] a theorem that estimates the approximation order of a non-stationary subdivision scheme $\{S_{\boldsymbol{a}^{[k]}}:k\geq 0\}$ which reproduces exponential polynomials in $EP_\Gamma$ and is asymptotically similar to a stationary one, in case the initial data are sampled from a function $f$ in the Sobolev space $W^N_\infty(\mathbb{R})$. (The latter is the space of all functions in $L_\infty(\mathbb{R})$ with $l$-th derivative in $L_\infty(\mathbb{R})$ for all $l=0,\dots,N$.) For the sake of simplicity, from now on we will write the $N$-dimensional space $EP_\Gamma$ as the space generated by the $N$ functions $\varphi_0,\dots,\varphi_{N-1}$, where each $\varphi_i(t)$, $i=0,\dots,N-1$, corresponds to some exponential polynomial $t^{r_i}e^{\theta_i t}$, as in Definition 3.1, and we will drop the subscript $\Gamma$.

**Theorem 4.2.** *[19, Theorem 21] Let $EP=\mathrm{span}\{\varphi_0,\dots,\varphi_{N-1}\}$ be an $N$-dimensional space of exponential polynomials and $\{S_{\boldsymbol{a}^{[k]}}:k\geq 0\}$ a convergent non-stationary subdivision scheme. Assume that:*

*(i) $\{S_{\boldsymbol{a}^{[k]}}:k\geq 0\}$ reproduces $EP$ and is asymptotically similar to a convergent, stationary subdivision scheme $S_{\boldsymbol{a}}$ with stable basic limit function of Hölder continuity $\alpha\in(0,1)$;*

(*ii*) *the initial data are of the form* $\boldsymbol{f}^{[m]} = \{f_i^{[m]} = f(2^{-m}i),\ i \in \mathbb{Z}\}$ *for some fixed* $m \geq 0$ *and for some function* $f \in W_\infty^N(\mathbb{R})$*, and that* $g_{\boldsymbol{f}^{[m]}}$ *is the limit of the scheme for such initial data;*

(*iii*) *the Wronskian matrix* $\mathcal{W} = \left( \dfrac{\varphi_s^{(r)}(0)}{r!} : r,s = 0,\dots,N-1 \right)$ *is invertible.*

*Then*

$$\|g_{\boldsymbol{f}^{[m]}} - f\|_{L_\infty(\mathbb{R})} \leq C_f\, 2^{-Nm}, \quad m \geq 0,$$

*where* $C_f > 0$ *is a constant depending only on* $f$.

We point out that this result limits the approximation order of a non-stationary subdivision scheme to be equal to the number of exponential polynomials reproduced. We next prove that the approximation order can be higher. Indeed we show that it coincides with the minimum between the approximation order of the asymptotically equivalent stationary scheme and the sum $N+\nu$ of the number $N$ of reproduced exponential polynomials and the rate $\nu$ of convergence of the sequence of level-dependent masks.

**Theorem 4.3.** *Let* $EP = \mathrm{span}\{\varphi_0,\dots,\varphi_{N-1}\}$ *be an* $N$*-dimensional space of exponential polynomials. Let* $S_{\boldsymbol{a}}$ *and* $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ *be convergent, respectively stationary and non-stationary, subdivision schemes. Assume that:*

(*i*) $S_{\boldsymbol{a}}$ *reproduces polynomials up to degree* $M-1$*, with* $M \geq N$*;*

(*ii*) $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ *reproduces* $EP$*;*

(*iii*) *the subdivision masks* $\{\boldsymbol{a}^{[k]} : k \geq 0\}$ *and* $\boldsymbol{a}$ *satisfy* $\|\boldsymbol{a}^{[k]} - \boldsymbol{a}\|_\infty \leq C\, 2^{-\nu k}$ *for some* $\nu \in \mathbb{N}$*;*

(*iv*) *the Wronskian matrix* $\mathcal{W} = \left( \dfrac{\varphi_s^{(r)}(0)}{r!} : r,s = 0,\dots,N-1 \right)$ *is invertible;*

(*v*) *the initial data are of the form* $\boldsymbol{f}^{[m]} = \{f_i^{[m]} = f(2^{-m}i) : i \in \mathbb{Z}\}$*, for some fixed* $m \geq 0$ *and for some function* $f \in W_\infty^M(\mathbb{R})$*.*

*Then*

$$\|g_{\boldsymbol{f}^{[m]}} - f\|_{L_\infty(\mathbb{R})} \leq C_f\, 2^{-\sigma m}, \quad \sigma = \min(N+\nu, M), \quad m \geq 0,$$

*with* $C_f$ *a positive constant depending only on* $f$.

*Proof.* The proof generalizes the proof of [19, Theorem 21]. The idea is to fix $x$ in $\mathbb{R}$ and to employ another auxiliary function $\psi$ defined as

$$\psi = \sum_{n=0}^{N-1} d_n \varphi_n(\cdot - x),$$

where the coefficient vector $\boldsymbol{d} = (d_n : n = 0,\dots,N-1)$ is obtained by solving the linear system

$$\psi^{(r)}(x) = f^{(r)}(x), \quad r = 0,\dots,N-1. \tag{6}$$

In matrix form (6) reads as

$$\mathcal{W}\boldsymbol{d}^T = \boldsymbol{f}^T \quad \text{with} \quad \boldsymbol{f} = (f^{(r)}(x),\ r = 0,\dots,N-1)^T.$$

The non-singularity of this linear system is guaranteed by the assumption (*iv*). Clearly, the function $\psi$ belongs to the space $EP_\Gamma$. Then, since the non-stationary scheme $\{S_{\boldsymbol{a}^{[k]}} : k \geq 0\}$ reproduces such functions, we obtain the identity

$$\psi = \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m \cdot - i)\psi(2^{-m}i), \tag{7}$$

with $\{\phi^{[m]} : m \geq 0\}$ denoting the basic limit functions of the non-stationary scheme. By hypothesis, $f_i^{[m]} = f(2^{-m}i)$ for $i \in \mathbb{Z}$. Thus, using the expression of $g_{\boldsymbol{f}^{[m]}}$ in terms of basic limit functions, we estimate the difference $f - g_{\boldsymbol{f}^{[m]}}$. By the construction of $\psi$ in (6), $f(x) = \psi(x)$. Then

$$\begin{aligned} f(x) - g_{\boldsymbol{f}^{[m]}}(x) &= \psi(x) - \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i) f(2^{-m}i) \\ &= \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i)\big(\psi(2^{-m}i) - f(2^{-m}i)\big). \end{aligned}$$

Now, let $T_h = \sum_{\ell=0}^{M-1} (\cdot - x)^\ell \dfrac{h^{(\ell)}(x)}{\ell!}$ be the Taylor polynomial of degree $(M-1)$ of a function $h \in W_\infty^M(\mathbb{R})$ around $x$, and consider the Taylor expansions $T_\psi$ and $T_f$ of both the functions $\psi$ and $f$ around $x$. Using the Lagrange remainder formulas of such expansions, we get

$$(\psi - f)(2^{-m}i) = \sum_{r=0}^{M-1} \frac{(2^{-m}i - x)^r}{r!} (\psi - f)^{(r)}(x) + \frac{(2^{-m}i - x)^M}{M!} (\psi - f)^{(M)}(\xi_i),$$

for some $\xi_i$ between $x$ and $i2^{-m}$.

Due to the conditions $\psi^{(r)}(x) = f^{(r)}(x)$, for $r = 0, \ldots, N-1$, $N \le M$, we have that

$$
\begin{aligned}
f(x) - g_{f^{[m]}}(x) &= \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i) \sum_{r=N}^{M-1} \frac{(2^{-m}i - x)^r}{r!} (\psi - f)^{(r)}(x) \\
&\quad + \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i) \frac{(2^{-m}i - x)^M}{M!} (\psi - f)^{(M)}(\xi_i),
\end{aligned}
$$

for some $\xi_i$ between $x$ and $i2^{-m}$. Therefore,

$$
\begin{aligned}
|f(x) - g_{f^{[m]}}(x)| &\le \frac{1}{N!} \sum_{r=N}^{M-1} \Big| \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i)(2^{-m}i - x)^r \Big| \big( |\psi^{(r)}(x)| + |f^{(r)}(x)| \big) \\
&+ \frac{1}{M!} \Big| \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i)(2^{-m}i - x)^M \Big| \big( |\psi^{(M)}(\xi_i)| + |f^{(M)}(\xi_i)| \big).
\end{aligned}
$$

Since $f \in W_\infty^M(\mathbb{R})$ and $|\psi^{(M)}(\xi_i)|$, $|\psi^{(r)}(x)|$, $r = N, \ldots, M-1$, are bounded, we conclude that

$$
\begin{aligned}
|f(x) - g_{f^{[m]}}(x)| &\le \frac{C}{N!} \Big( \sum_{r=N}^{M-1} \Big| \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i)(2^{-m}i - x)^r \Big| \\
&\quad + \Big| \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i)(2^{-m}i - x)^M \Big| \Big).
\end{aligned}
$$

The boundedness of the derivatives above follows by a Taylor expansion argument (with respect to $x$) applied to $\psi$ combined with equation (7) which guarantees, due to the boundedness of $\phi^{[m]}$, that $\psi$ is also bounded. Next, we know by assumption that $\phi$ reproduces $\mathbb{P}_{M-1}$, that is

$$
\sum_{i\in\mathbb{Z}} (2^{-m}i - x)^r \phi(2^m y - i) = (y - x)^r, \quad 0 \le r \le M-1,
$$

which implies

$$
\sum_{i\in\mathbb{Z}} \phi(2^m x - i)(2^{-m}i - x)^r = 0, \quad N \le r \le M-1.
$$

Then, writing $\phi^{[m]}$ as $(\phi^{[m]} - \phi) + \phi$,

$$
\Big| \sum_{i\in\mathbb{Z}} \phi^{[m]}(2^m x - i)(2^{-m}i - x)^r \Big| \le 2^{-mr} \sum_{i\in\mathbb{Z}} \Big| \phi^{[m]}(2^m x - i) - \phi(2^m x - i) \Big| \, |i - 2^m x|^r.
$$

Now, using asymptotical equivalence and the result in [30] we have

$$
\Big| \phi^{[m]}(2^m x - i) - \phi(2^m x - i) \Big| \le C 2^{-\nu m},
$$

and due to the fact that, from Proposition 4.1, $\phi^{[m]}$ is compactly supported and uniformly bounded independently of $m$, we arrive at

$$
|f(x) - g_{f^{[m]}}(x)| \le C_1 2^{-(N+\nu)m} + C_2 2^{-Mm},
$$

which concludes the proof. □

## 5 Conclusion

This paper discusses several generalizations of polynomial B-splines, including exponential B-splines and the larger family of exponential pseudo-splines, with emphasis on their connections to level-dependent subdivision schemes. In addition, it provides conditions to establish the approximation order of non-stationary subdivision schemes reproducing spaces of exponential polynomial functions.

All the discussion relates to the univariate case even though similar results could be considered in the multivariate case. However, while a very recent notion of bivariate pseudo-splines is presented in [17], to the best of our knowledge, no notion of multivariate exponential pseudo-splines is yet available making any multivariate extension certainly more complex.

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