Dolomites Research Notes on Approximation

Integration and Approximation with Fibonacci lattice points

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Abstract

We study the properties of a special rank-1 point set in 2 dimensions — Fibonacci lattice points. We present the analysis of these point sets for cubature and approximation of bivariate periodic functions with decaying spectral coefficients. We are interested in truncating the frequency space into index sets based on different degrees of exactness. The numerical results support that the Lebesgue constant of these point sets grows like the conjectured optimal rate $\ln^2(N)$, where N is the number of sample points.

1 Introduction

Rank-1 lattice points are very well known for their usefulness in multivariate cubature, see [7, 17, 18]. They have also been used for function approximation in [12, 13, 15]. Here we focus on a special case of these lattice points, namely Fibonacci lattice points. These are point sets in two dimensions, with the special property that the number of points in the set is a Fibonacci number. These were first introduced by Bhakhvalov [2] in 1959. He proved that these point sets are the best for integrating smooth bivariate periodic functions (with respect to a certain quality criterion). Fibonacci lattice points have since been widely used for the integration of bivariate functions in various settings. For instance, these point sets appear in [20] in the context of having a good so called *Zaremba or hyperbolic cross index*. More recent studies of Fibonacci lattices include [3], where the *trigonometric degree* of these point sets is calculated. In [16], these points were used for the integration of nonperiodic bivariate functions. Also in [8], extensions to Fibonacci lattices to improve the trigonometric degree have been introduced. In [11], it is shown that the first few Fibonacci lattices are the unique global minimizers of the worst case integration error in a space of bivariate periodic functions with bounded mixed derivatives; see [10] for more related results. Fibonacci lattice points have also been used for integration over the unit sphere, e.g., in [1] and [14].

Motivated by the above results, we study Fibonacci lattices for various degrees of exactness (defined in the next section) for cubature as well as function approximation. We are also interested in the *Lebesgue constant* of trigonometric interpolation operators that use these point sets. In [19], upper and lower bounds for the Lebesgue constant for generic lattice point sets are given. We numerically verify that Fibonacci lattices have the conjectured optimal growth in the Lebesgue constant of order $\ln^2(N)$ for N up to 2178309, similar to the well known *Padua points* which are used in bivariate polynomial interpolation, see [4].

We will first start with basic concepts needed in this paper. Throughout the paper we work with bivariate smooth periodic functions that have absolutely convergent Fourier series expansions of the form

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^2} \widehat{f}(\boldsymbol{h}) \exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x}), \tag{1}$$

where $\widehat{f}(\boldsymbol{h})$ are the Fourier coefficients given by

$$\widehat{f}(\boldsymbol{h}) = \int_{[0,1]^2} f(\boldsymbol{x}) \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{x}) \, d\boldsymbol{x}. \tag{2}$$

For *cubature points* or *sampling points* (for function approximation), we are going to use points from a rank-1 lattice point set, which is defined as follows:

Definition 1.1. For a given $N \in \mathbb{N}$ and $\boldsymbol{z} \in \mathbb{Z}_N^d$, a *rank-1 lattice point set* $\Lambda(\boldsymbol{z}, N)$ in d dimensions is given by

$$\Lambda(\boldsymbol{z}, N) := \left\{ N^{-1} n \boldsymbol{z} \bmod 1 : n = 0, 1, \ldots, N-1 \right\},$$

where $\boldsymbol{z}$ is called the *generating vector*.

Let F_k be the k-th Fibonacci number defined by $F_1 = 1, F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $2 < k \in \mathbb{N}$. A Fibonacci lattice is then a rank-1 lattice point set with $N = F_k$ points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ or $\boldsymbol{z} = [1, F_{k-2}]$. Figure 1 shows two Fibonacci lattice point sets with $N = F_8 = 21$ and $N = F_9 = 34$ points with generating vectors $\boldsymbol{z} = [1, 13]$ and $\boldsymbol{z} = [1, 21]$ respectively.

2 Cubature

We will consider N-point cubature rules Q_N that are linear combinations of function values to approximate integrals (and inner products):

$$Q_N(f) = Q_N(f; \{w_0, \ldots, w_{N-1}\}, \{\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(N-1)}\}) := \sum_{n=0}^{N-1} w_n f(\boldsymbol{x}^{(n)}), \tag{3}$$

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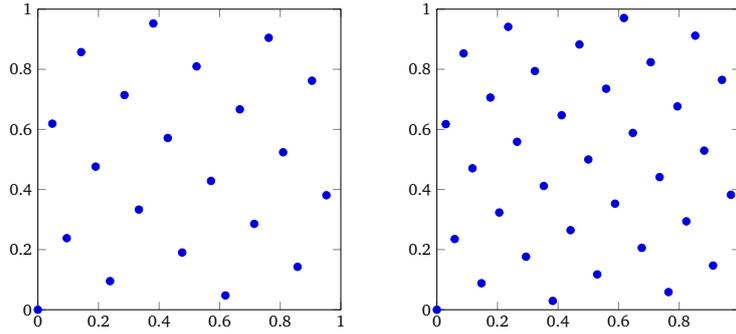
Figure 1: Fibonacci lattice points with $N = F_8 = 21$, $\boldsymbol{z} = [1, 13]$ and $N = F_9 = 34$, $\boldsymbol{z} = [1, 21]$.

where $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(N-1)}$ are the sampling points and $w_0, \ldots, w_{N-1}$ are the weights. A *lattice rule* is then a cubature rule that uses the points from a lattice $\Lambda(\boldsymbol{z}, N)$ with equal weights $1/N$. We will denote its application to a function f by $Q_N(f; \boldsymbol{z})$. The following *character property* with respect to the Fourier basis is useful:

$$Q_N(\exp(2\pi i(\boldsymbol{h} \cdot \boldsymbol{x})); \boldsymbol{z}) = \begin{cases} 1 & \text{if } \boldsymbol{h} \cdot \boldsymbol{z} \equiv 0 \pmod{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The set of *frequency indices* $\boldsymbol{h} \in \mathbb{Z}^d$ for which the cubature sum is equal to 1 is called the *dual lattice* and we denote it by $\Lambda^\perp(\boldsymbol{z}, N)$. A Fibonacci lattice rule uses points from a Fibonacci lattice; the following rules are equivalent Fibonacci lattice rules (in the sense that $x_2^{(i)}$ is replaced by $1 - x_2^{(i)}$):

$$Q_{F_k}(f; [1, F_{k-1}]) = \frac{1}{F_k} \sum_{n=0}^{F_k-1} f\left(\frac{n}{F_k}, \frac{nF_{k-1}}{F_k}\right),$$

$$Q_{F_k}(f; [1, F_{k-2}]) = \frac{1}{F_k} \sum_{n=0}^{F_k-1} f\left(\frac{n}{F_k}, \frac{nF_{k-2}}{F_k}\right).$$

All results in this paper hold for both formulations of the Fibonacci rule. To describe the quality of a cubature rule, we use the largest *degree of exactness* with respect to a certain index set $\mathcal{A}_d(T)$, with $T \in \mathbb{N}$ the dilation. In Figure 2 we illustrate the different shapes of $\mathcal{A}_d$ that we consider.

Definition 2.1. The *cubature degree of exactness* $\mathcal{T}(Q_N) = \mathcal{T}(Q_N, \mathcal{A}_d(\cdot), \{\exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x})\}_{\boldsymbol{h}})$ of a cubature rule Q_N with respect to an index set $\mathcal{A}_d(T) \subset \mathbb{Z}^d$ and the set of basis functions $\{\exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x})\}_{\boldsymbol{h}}$ is the largest $T \in \mathbb{N}$ for which

$$Q_N(\exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x})) = \int_{[0,1]^d} \exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x}) \, d\boldsymbol{x} \qquad \text{for all } \boldsymbol{h} \in \mathcal{A}_d(T).$$

We will consider three standard shapes of index sets for the quality measures of cubature rules, see, e.g., [5, 6].

Definition 2.2. For *trigonometric degree of exactness* T, we define a cross polytope

$$\mathcal{T}_d(T) := \left\{ \boldsymbol{h} \in \mathbb{Z}^d : \sum_{j=1}^d |h_j| \leq T \right\}.$$

Definition 2.3. For *hyperbolic cross degree of exactness* T, or *Zaremba index* $T+1$, we define a hyperbolic cross

$$\mathcal{H}_d(T) := \left\{ \boldsymbol{h} \in \mathbb{Z}^d : \prod_{j=1}^d \max(1, |h_j|) \leq T \right\}.$$

Definition 2.4. For *product trigonometric degree of exactness* T, we define the index set

$$\mathcal{P}_d(T) := \left\{ \boldsymbol{h} \in \mathbb{Z}^d : \max_{1 \leq j \leq d} |h_j| \leq T \right\}.$$

We now present the results for these degrees of exactness for cubature using Fibonacci lattice rules. The degrees of rules with F_k points follow a different pattern based on whether k is even or odd. Therefore we will differentiate between these cases. We first state known results from [3] and [20], and then add a similar result for the product trigonometric degree which will be of use later on.

Theorem 2.1. *The trigonometric degree of a Fibonacci lattice rule with F_k points, where $k \geq 2$, and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ is $2F_{k/2} - 1$ for even k and $F_{\lfloor k/2 \rfloor + 2} - 1$ for odd k.*
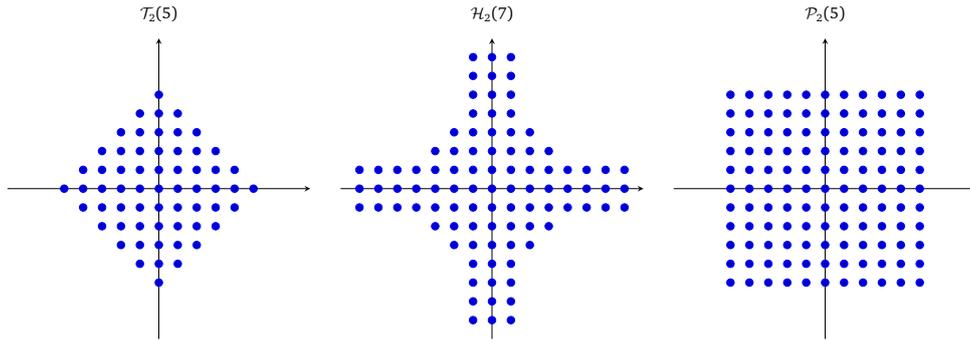
Figure 2: Index sets associated with the trigonometric, hyperbolic cross and product trigonometric degrees of exactness.

Proof. See [3] for a proof. ∎

Theorem 2.2. *For a Fibonacci lattice rule with F_k points, where $k \geq 3$, and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ the hyperbolic cross degree is $F_{k-2} - 1$, i.e., the Zaremba index is F_{k-2}.*

Proof. See [20] for a proof. ∎

We now add results on the product trigonometric degree for Fibonacci lattice rules with odd and even indexed number of points F_k.

Lemma 2.3. *The product trigonometric degree of a Fibonacci rule with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$, where $3 \leq k = 2m + 1$, is $F_{m+1} - 1$.*

Proof. A dual generating matrix is (see [3])

$$B = \begin{pmatrix} F_m & (-1)^{m+1} F_{m+1} \\ F_{m+1} & (-1)^m F_m \end{pmatrix}.$$

The points of the dual lattice are thus given by $((j_1 F_m + j_2 F_{m+1}), (-1)^m (-j_1 F_{m+1} + j_2 F_m))$ with $j_1, j_2 \in \mathbb{Z}$. Now we have 4 cases:

- When $j_1 = 0, j_2 \neq 0$, we have $|j_2 F_{m+1}| > F_{m+1} - 1$.
- Similarly, when $j_1 \neq 0, j_2 = 0$, we have $|(-1)^m j_1 F_{m+1}| > F_{m+1} - 1$.
- When $j_1 j_2 > 0$, we have $|(j_1 F_m + j_2 F_{m+1})| > F_{m+1} - 1$.
- When $j_1 j_2 < 0$, we have $|(-1)^m (-j_1 F_{m+1} + j_2 F_m)| > F_{m+1} - 1$.

On the other hand, for $j_1 = 0$ and $j_2 = 1$, we have a point of the dual lattice at $(F_{m+1}, (-1)^m F_m)$. This completes the proof. ∎

Lemma 2.4. *The product trigonometric degree of a Fibonacci lattice rule with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$, where $2 \leq k = 2m$, is $F_m - 1$.*

Proof. A dual generating matrix is (see [3])

$$B = \begin{pmatrix} F_m & (-1)^m F_m \\ F_{m-1} & (-1)^{m+1} F_{m+1} \end{pmatrix}.$$

The points of the dual lattice are thus given by $((j_1 F_m + j_2 F_{m-1}), (-1)^m (j_1 F_m - j_2 F_{m+1}))$ with $j_1, j_2 \in \mathbb{Z}$. We again have 4 cases:

- When $j_1 = 0, j_2 \neq 0$, we have $|(-1)^m j_2 F_{m+1}| > F_m - 1$.
- Similarly, when $j_1 \neq 0, j_2 = 0$, we have $|j_1 F_m| > F_m - 1$.
- When $j_1 j_2 > 0$, we have $|(j_1 F_m + j_2 F_{m-1})| > F_m - 1$.
- When $j_1 j_2 < 0$, we have $|(-1)^m (-j_1 F_{m+1} + j_2 F_m)| > F_m - 1$.

On the other hand, for $j_1 = 1$ and $j_2 = 0$, we have a point of the dual lattice at $(F_m, (-1)^m F_m)$. This completes the proof. ∎

We now combine the two previous results.

Theorem 2.5. *The product trigonometric degree of a Fibonacci lattice rule with F_k points, where $k \geq 2$, and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ is $F_{\lceil k/2 \rceil} - 1$.*

Proof. See Lemma 2.3 and Lemma 2.4. ∎

We now present an interesting result on the number of points of a type of rule derived from a specific Fibonacci lattice rule. It is known (see, e.g., [9]) that the number of points N of a cubature rule with trigonometric degree T is bounded from below by

$$N \geq t(d, \lfloor T/2 \rfloor),$$

where $t(d, \lfloor T/2 \rfloor)$ denotes the number of monomials in d variables of degree at most $\lfloor T/2 \rfloor$. An improved (i.e., higher) lower bound is known for odd degree T. For $d = 2$, these bounds reduce to $2(m+1)^2$ for $T = 2m + 1$ and $2m^2 + 2m + 1$ for $T = 2m$ and are attainable. Examples are the Fibonacci lattice rule with $N = F_5 = 5$ points having trigonometric degree 2 and the Fibonacci lattice rule with $N = F_6 = 8$ points having trigonometric degree 3. In the following lemma we give an infinite sequence of cubature rules with optimal number of points. These are ℓ^2 copy rules based on the Fibonacci rule with 8 points. Such an ℓ^2 copy rule (see, e.g., [18]) basically scales the rule in each direction by ℓ^{-1}, $\ell \in \mathbb{N}$, and fills the unit cube by copies of the scaled rule. Written for general rank-1 rules in d dimensions, this is

$$Q_{\ell^d \times N}^{\text{copy}}(f; \boldsymbol{z}) = \frac{1}{N\ell^d} \sum_{\boldsymbol{m} \in \mathbb{Z}_\ell^d} \sum_{n=0}^{N-1} f\left(\frac{n\boldsymbol{z}/N \bmod 1}{\ell} + \frac{\boldsymbol{m}}{\ell}\right).$$

Somehow surprisingly, the lower bound on the number of points for trigonometric degree is attained by copy rules based on a Fibonacci rule with 8 points.

Theorem 2.6. *For $\ell \in \mathbb{N}$ with $\gcd(8, \ell) = 1$, consider the sequence of ℓ^2 copy rules of the Fibonacci lattice rule with $N = F_6 = 8$ points and generating vector $\boldsymbol{z} = [1, 5]$ given by*

$$Q_{\ell^2 \times 8}^{\text{copy}}(f; [1, 5]) = \frac{1}{8\ell^2} \sum_{\boldsymbol{m} \in \mathbb{Z}_\ell^2} \sum_{n=0}^{7} f\left(\left(\frac{n}{8\ell}, \frac{5n \bmod 8}{8\ell}\right) + \frac{\boldsymbol{m}}{\ell}\right).$$

These rules have a trigonometric degree of exactness of $4\ell - 1$ and have the optimal number of points for the trigonometric degree that they achieve.

Proof. The trigonometric degree of the Fibonacci rule with 8 points is 3. Hence, we would find a point of the dual lattice which is at a Manhattan distance of 4 from the origin. The dual lattice of an ℓ^2 copy of this rule is simply scaled by ℓ and hence, for the copy rule, we would find the closest point of the dual lattice at a distance of 4ℓ. The degree of the ℓ^2 copy rule is thus $4\ell - 1$.

The optimal number of points to attain a degree of $4\ell - 1$ is $2(m+1)^2$ where $4\ell - 1 = 2m + 1$ (so $m = 2\ell - 1$). This is equal to $8\ell^2$, which is exactly the number of points in a ℓ^2 copy of the rule with 8 points. ∎

3   Approximation

A function f is approximated by truncating its Fourier series expansion with respect to a suitable index set $\mathcal{A}_d(T)$ and approximating the Fourier coefficients using samples from a specific point set. Denoting by $\widehat{f_a}$ the approximation of the Fourier coefficients of a function f, we can define an approximation f_a by

$$f_a(\boldsymbol{x}) := \sum_{\boldsymbol{h} \in \mathcal{A}_d(T)} \widehat{f_a}(\boldsymbol{h}) \exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x}), \tag{5}$$

where the Fourier coefficients are approximated by a cubature rule

$$\widehat{f_a}(\boldsymbol{h}) := Q_N(f(\boldsymbol{x}) \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{x})).$$

Substitution of the Fourier expansion (1) into (2) yields

$$\widehat{f}(\boldsymbol{h}) = \int_{[0,1]^d} \left(\sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} \widehat{f}(\boldsymbol{\ell}) \exp(2\pi i \boldsymbol{\ell} \cdot \boldsymbol{x})\right) \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{x}) \, d\boldsymbol{x}$$

$$= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} \widehat{f}(\boldsymbol{\ell}) \int_{[0,1]^d} \exp(2\pi i (\boldsymbol{\ell} - \boldsymbol{h}) \cdot \boldsymbol{x}) \, d\boldsymbol{x}, \tag{6}$$

where the last integral is unity for $\boldsymbol{h} = \boldsymbol{\ell}$ and vanishes otherwise due to the orthonormality of the basis. On the other hand, for the approximation of the coefficients, we have

$$\widehat{f_a}(\boldsymbol{h}) = Q_N\left(\left(\sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} \widehat{f}(\boldsymbol{\ell}) \exp(2\pi i \boldsymbol{\ell} \cdot \boldsymbol{x})\right) \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{x})\right)$$

$$= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} \widehat{f}(\boldsymbol{\ell}) Q_N(\exp(2\pi i (\boldsymbol{\ell} - \boldsymbol{h}) \cdot \boldsymbol{x})). \tag{7}$$

From (6) and (7), we note that calculating $\widehat{f_a}(\boldsymbol{h})$ involves approximating the integrals in the inner products of the basis function. The contamination of $\widehat{f_a}(\boldsymbol{h})$ by other coefficients is called *aliasing*. In what follows, the concept of a *non-aliasing index set* will be useful.

Definition 3.1. We call a set $\mathcal{A}_d \subset \mathbb{Z}^d$ a *non-aliasing index set* of a lattice $\Lambda(\boldsymbol{z}, N)$ if no two distinct vectors in $\mathcal{A}_d$ differ by a vector that is in the dual lattice; i.e.,

$$\boldsymbol{h} - \boldsymbol{h}' \notin \Lambda^{\perp}(\boldsymbol{z}, N) \qquad \text{for all } \boldsymbol{h}, \boldsymbol{h}' \in \mathcal{A}_d, \tag{8}$$

or equivalently,

$$Q_N(\exp(2\pi i (\boldsymbol{h} - \boldsymbol{h}') \cdot \boldsymbol{x})) = \int_{[0,1]^d} \exp(2\pi i (\boldsymbol{h} - \boldsymbol{h}') \cdot \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{h}, \boldsymbol{h}' \in \mathcal{A}_d. \tag{9}$$

Note that if $\mathcal{A}_d$ is a non-aliasing index set, then $|\mathcal{A}_d| \leq N$. If we define a difference set $\mathcal{D} = \{\boldsymbol{h} - \boldsymbol{h}' : \boldsymbol{h}, \boldsymbol{h}' \in \mathcal{A}_d\}$, it will contain no points that are in the dual lattice ($\mathcal{D} \cap \Lambda^{\perp}(\boldsymbol{z}, N) = \emptyset$).

3.1 Degrees of exactness for approximation

As in the case of cubature, we define an *approximation degree of exactness* based on the exact approximation of the inner products of the basis functions.

Definition 3.2. The *approximation degree* $\mathcal{T}(Q_N) = \mathcal{T}(Q_N, \mathcal{A}_d(\cdot), \{\exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x})\}_{\boldsymbol{h}})$ of a cubature rule Q_N with respect to an index set $\mathcal{A}_d(T)$ and the set of basis functions $\{\exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{x})\}_{\boldsymbol{h}}$ is the largest $T \in \mathbb{N}$ for which $\mathcal{A}_d(T)$ is a non-aliasing index set, i.e., for which condition (8), or equivalently (9), is met.

It turns out that the approximation degree of exactness strongly depends on that of cubature, and hence we observe a similar pattern with respect to the even and odd indexed Fibonacci numbers.

Theorem 3.1. *The trigonometric approximation degree of a Fibonacci lattice rule with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ is $\lfloor (F_{\lfloor k/2 \rfloor + 2} - 1)/2 \rfloor$ for odd $k \geq 3$ and $\lfloor (2F_{k/2} - 1)/2 \rfloor$ for even $k \geq 2$.*

Proof. Note that the difference set of a cross polytope is itself a cross polytope of twice the degree, i.e., the degree of the difference set of a cross polytope of degree of $\lfloor (F_{\lfloor k/2 \rfloor + 2} - 1)/2 \rfloor$ is at most $(F_{\lfloor k/2 \rfloor + 2} - 1)$. To have an approximation degree of $\lfloor (F_{\lfloor k/2 \rfloor + 2} - 1)/2 \rfloor$, the rule needs to have a trigonometric degree for cubature of at least $(F_{\lfloor k/2 \rfloor + 2} - 1)$, which is exactly the degree of the rule with F_k points with odd k. On the other hand, for a cross polytope of degree greater than $\lfloor (F_{\lfloor k/2 \rfloor + 2} - 1)/2 \rfloor$, the difference set will not fit into the trigonometric degree for the rule with F_k points. For even k, the same argument holds. ∎

Theorem 3.2. *The product trigonometric approximation degree of a Fibonacci lattice rule with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ is $\lfloor (F_{\lceil k/2 \rceil} - 1)/2 \rfloor$, where $k \geq 2$.*

Proof. As before, the degree of the difference set of the product trigonometric index set of degree $\lfloor (F_{\lceil k/2 \rceil} - 1)/2 \rfloor$ is at most $(F_{\lceil k/2 \rceil} - 1)$. To have an approximation degree of $\lfloor (F_{\lceil k/2 \rceil} - 1)/2 \rfloor$, the rule needs to have a product trigonometric degree for cubature of at least $(F_{\lceil k/2 \rceil} - 1)$, which is exactly the degree of the rule with F_k points. On the other hand, for an index set of degree greater than $\lfloor (F_{\lceil k/2 \rceil} - 1)/2 \rfloor$, the difference set will not fit into the product trigonometric degree for the rule with F_k points. ∎

Finding the hyperbolic cross degree for approximation is a more tedious task. We first need the following lemma.

Lemma 3.3. *Consider the hyperbolic cross $\mathcal{H}_2(T) = \{\boldsymbol{k} \in \mathbb{Z}^2 : \max(1, |k_1|) \max(1, |k_2|) < T\}$ where $T = F_{m+1} - 2$. Then the difference set $\mathcal{D} = \{\boldsymbol{\ell} - \boldsymbol{k} : \boldsymbol{\ell}, \boldsymbol{k} \in \mathcal{H}_2(T)\}$ has the following properties: for every $\boldsymbol{h} \in \mathcal{D}$, we have*

1.

$$-2T \leq h_1, h_2 \leq 2T,$$

2. and

$$|h_1| \leq T + \frac{T}{\max(|h_2| - 1, 1)} \quad \text{and} \quad |h_2| \leq T + \frac{T}{\max(|h_1| - 1, 1)}, \tag{10}$$

3. and at least one of the following is true:

    (a) either

$$\boldsymbol{h} \in [-F_{m+1} + 1, F_{m+1} - 1]^2,$$

    (b) or

$$\max(|h_1|, 1) \max(|h_2|, 1) \leq F_{2m-1} - 1 \qquad \text{for } m \geq 2,$$

    (c) or

$$|h_1| + |h_2| \leq F_{m+2} - 1 \qquad \text{for } m \geq 3.$$

Proof. The first point is trivial to prove as the bounding box of the hyperbolic cross is $[-T, T]^2$, the bounding box of the difference set will be $[-2T, 2T]^2$.

We have to tackle the second point in several steps. First, it suffices to handle the case for $h_1, h_2 \geq 0$ because the difference set inherits the symmetry of the hyperbolic cross; it is symmetric under the exchange of variables and sign change. This can be seen from the illustration in Figure 3. Second, since we are dealing with the upper bound, it is enough to consider cases where $\boldsymbol{h}$ is as large as possible. Let $\boldsymbol{h} = \boldsymbol{p} - \boldsymbol{q}$, then we only need to consider the case when $\boldsymbol{p}$ is in the first quadrant $(p_1, p_2 > 0)$ and $\boldsymbol{q}$ is in the third quadrant, we will take $\boldsymbol{q} = (-q_1, -q_2)$, $q_1, q_2 > 0$. All other points will follow because they are interior points or by symmetry. So we have $\boldsymbol{h} = (p_1 + q_1, p_2 + q_2)$. More specifically,

$$
\begin{aligned}
h_2 &= p_2 + q_2 \\
&\leq T\left(\frac{1}{p_1} + \frac{1}{q_1}\right) \\
&\leq T + \frac{T}{\max(p_1 + q_1 - 1, 1)} \\
&\leq T + \frac{T}{\max(h_1 - 1, 1)}.
\end{aligned}
$$

The second inequality above holds as, for the cases $(p = 1, q \geq 1)$ and $(p \geq 1, q = 1)$ we have equality, and if $p, q \geq 2$, the right hand side is always larger than the left hand side. Because of symmetry under the exchange of variables, the following also holds:

$$
h_1 \leq T + \frac{T}{\max(h_2 - 1, 1)}.
$$

To extend this to all the quadrants, we use the absolute value and that completes the proof of this part.

For the third point, again due to the symmetry conditions, we only need to consider the case for $0 \leq h_1 \leq T+1$ and $0 \leq h_2 \leq 2T$; please refer to the shaded regions in Figure 3. This is constrained by the second part of (10), $(h_2 - T)\max(h_1 - 1, 1) \leq T$. We need to consider the following cases:

- If $0 \leq h_2 \leq T + 1$, then $\boldsymbol{h} \in [-F_{m+1} + 1, F_{m+1} - 1]^2$, hence Item *3.(a)* holds.
- $h_2 = 2T$ implies $h_1 = 0, 1$ or 2 and

$$
\max(|h_1|, 1)\max(|h_2|, 1) \leq 4T = 4(F_{m+1} - 2).
$$

Subtracting this from $F_{2m-1} - 1$, we get

$$
\begin{aligned}
F_{2m-1} - 1 - 4F_{m+1} + 8 &= F_{m-1}^2 + F_m^2 - 4(F_m + F_{m-1}) + 7 \\
&= F_{m-1}(F_{m-1} - 4) + F_m(F_m - 4) + 7 \\
&\geq 0 \text{ for } m \geq 2.
\end{aligned}
$$

Hence Item *3.(b)* holds.

- The substitution $t_1 = h_1 - 1$ and $t_2 = h_2 - T$ reduces the constraint $(h_2 - T)\max(h_1 - 1, 1) \leq T$ to $t_1 t_2 \leq T$ and we are left with the case where $t_1 \geq 2$ and $t_2 \geq 2$. We want to prove that in this situation, Item *3.(c)* holds. We have

$$
1 + t_1 + T + t_2 = t_1 + t_2 + F_{m+1} - 1
$$

and to show that $t_1 + t_2 + F_{m+1} - 1 \leq F_{m+2} - 1$, we need to show that $t_1 + t_2 \leq F_m$. Considering the indices on the boundary, note that $t_1 + t_2 = t_1 + \frac{T}{t_1}$ has its minima at $t_1 = \sqrt{T}$. So it is enough to consider one of the corner indices for this case which is $t_1 = 2, t_2 = \lfloor T/2 \rfloor$.

$$
\begin{aligned}
2 + \lfloor T/2 \rfloor &\leq 2 + \frac{F_{m+1} - 2}{2} \\
&= \frac{F_m + F_{m-1} + 2}{2} \\
&\leq F_m \text{ for } m \geq 3.
\end{aligned}
$$

This completes this part and the proof. ∎

Lemma 3.4. *The hyperbolic cross approximation degree of a Fibonacci lattice point set with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$, where $7 \leq k = 2m + 1$, is $F_{m+1} - 2$.*

Proof. The proof directly follows by Lemma 3.3, which shows that the difference set of a hyperbolic cross with index $F_{m+1} - 2$ has no points from the dual in it. The indices in the boxes described in Lemma 3.3 *3.(a)*, *3.(b)*, and *3.(c)* have product trigonometric degrees, hyperbolic cross degrees and trigonometric degrees less than the respective degrees of the Fibonacci lattice rule with F_k points. On the other hand, taking $T = F_{m+1} - 1$ will result in at least one dual point in the box described in Lemma 3.3 *3.(a)*. Hence this is the highest degree achievable. ∎

Lemma 3.5. *The hyperbolic cross approximation degree of a Fibonacci lattice point set with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$, where $8 \leq k = 2m$, is $F_m - 2$.*
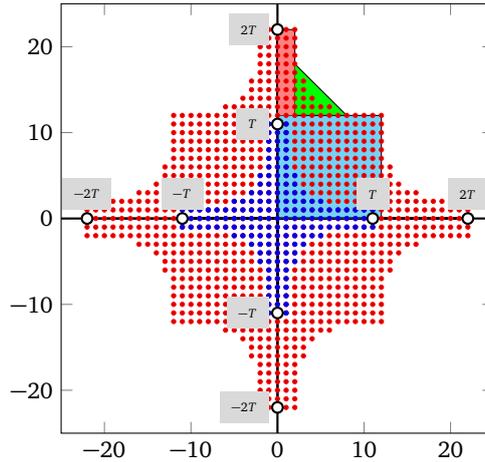
Figure 3: An example of $\mathcal{H}_2(T)$ (the blue dots) and its difference set (the red dots). The blue square, the red rectangle and the green triangle illustrate the claims stated in *3.(a)*, *3.(b)*, and *3.(c)* of Lemma 3.3.

Proof. A similar proof as before follows. The difference set of the corresponding hyperbolic cross is covered by the index sets with appropriate trigonometric, product trigonometric and hyperbolic cross degrees for cubature. ∎

Theorem 3.6. *The hyperbolic cross approximation degree of a Fibonacci lattice rule with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$, where $k \geq 7$, is $F_{\lceil k/2 \rceil} - 2$.*

Proof. This follows from Lemma 3.4 and Lemma 3.5. ∎

3.2 Lebesgue constant

In this section, we will numerically calculate the Lebesgue constant of a trigonometric interpolation operator using points from a Fibonacci lattice. Let f^* be the best approximation of f onto a space $\mathcal{H}_d$ and let L be the interpolation operator. We have

$$\|Lf - f\| \leq \|f - f^*\| + \|L(f - f^*)\| \leq (\|L\| + 1)\|f - f^*\|.$$

We are interested in the behavior of the Lebesgue constant $\mathcal{L} := \|L\|$. The following theorems are from [19].

Theorem 3.7. *Let $\mathcal{A}_d$ be a non-aliasing index set of the lattice $\Lambda(\boldsymbol{z}, N)$ such that its cardinality is N. Then the trigonometric interpolating operator using $\Lambda(\boldsymbol{z}, N)$ as the interpolating point set is given by*

$$(Lf)(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{A}_d} \tilde{c}_{\boldsymbol{k}} \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x}) \quad \text{where} \quad \tilde{c}_{\boldsymbol{k}} = \sum_{\boldsymbol{\ell} \in \Lambda^\perp(\boldsymbol{z}, N)} c_{\boldsymbol{k}+\boldsymbol{\ell}},$$

and the lattice interpolation space $\mathcal{H}_d$ is spanned by $\{\exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x})\}_{\boldsymbol{k} \in \mathcal{A}_d}$.

Once we know the interpolation space and the basis functions it is spanned by, we arrive at the following theorem.

Theorem 3.8. *Consider a rank-1 lattice point set $\Lambda(\boldsymbol{z}, N)$ and its d-dimensional Dirichlet kernel*

$$D_{\mathcal{A}_d}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathcal{A}_d} \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{x}).$$

Then for any two points $\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)} \in \Lambda(\boldsymbol{z}, N)$,

$$\frac{1}{N} D_{\mathcal{A}_d}(\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(j)}) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

We can then define Lagrange functions for $\Lambda(\boldsymbol{z}, N)$ as

$$L_i(\boldsymbol{x}) = \frac{1}{N} D_{\mathcal{A}_d}(\boldsymbol{x} - \boldsymbol{x}^{(i)}), \quad \forall \boldsymbol{x}^{(i)} \in \Lambda(\boldsymbol{z}, N),$$

and the Lebesgue constant is

$$\mathcal{L}_N = \frac{1}{N} \max_{\boldsymbol{x} \in [0,1]^d} \sum_{\boldsymbol{x}^{(i)} \in \Lambda(\boldsymbol{z}, N)} |L_i(\boldsymbol{x})|. \tag{12}$$
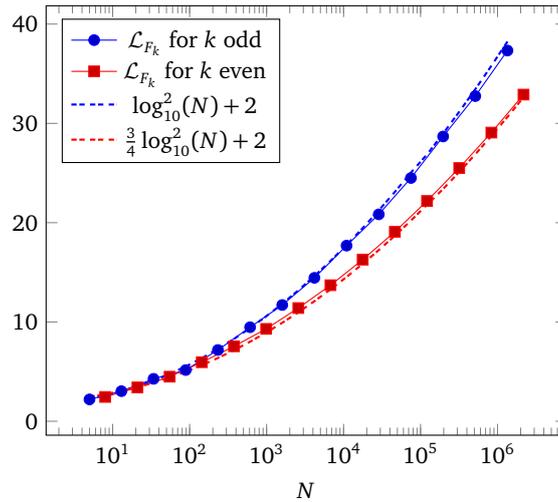
Figure 4: Growth of the Lebesgue constant of Fibonacci lattice points.

Although the discrete orthogonality condition in (11) is proven only for prime N in [19], a slight modification to [5, Theorem 7.3] will give the result for all N.

It is conjectured in [19] that

$$C_1 \ln^d(N) \le \mathcal{L}_N \le C_2 \ln^d(N).$$

We numerically evaluate $\mathcal{L}_N$ from (12) for Fibonacci lattice points on a very fine grid (with 1000 equidistant grid points in each direction). Some observations follow.

In Figure 4, the numerical Lebesgue constant for consecutive Fibonacci lattices, denoted by $\mathcal{L}_{F_k}$, is plotted against $\log_{10}(N)$ for convenience. Note that the Dirichlet kernel for Fibonacci numbers is complex and we take the complex norm (modulus) while calculating the Lebesgue constant. We verify that this quantity grows at a rate of $\log_{10}^2(N)$ for k up to 32. Further, the curve connecting the results for point sets corresponding to Fibonacci numbers with an even index lies below the one connecting those with an odd index; however, they both have a growth of $c \ln^2(N)$ but with different constants c.

In Figure 5, we present the contour plots of the function

$$\sum_{\boldsymbol{x}^{(i)} \in \Lambda(\boldsymbol{z},N)} \left| D_{\mathcal{A}_2}(\boldsymbol{x} - \boldsymbol{x}^{(i)}) \right| \tag{13}$$

appearing in (12) for $N = F_{10} = 55$ and $N = F_{11} = 89$. We observe from these and other plots that the minima of this function are at the lattice points themselves (i.e., $N = F_k$ and the generating vector $\boldsymbol{z} = [1, F_{k-1}]$). These minima are the dark blue centers in both plots. A possible explanation could be Property (11). Unsurprisingly, the value of these minima is 1 for all Fibonacci point sets.

Note that the expression in (13) already suggests that the maxima are obtained by shifting the Fibonacci lattice by a fixed vector $\boldsymbol{x} \in [0,1]^2$. The contour plots also substantiate this; the black dots superimposed on the peaks are from the original lattice with an appropriate shift. These observations are summarized in the following conjecture.

Conjecture 3.9. *Consider a Fibonacci point set with F_k points and generating vector $\boldsymbol{z} = [1, F_{k-1}]$ and the corresponding non-aliasing index set $\mathcal{A}_2$. Then, the Lebesgue constant of this point set is given by*

$$\mathcal{L}_N = \frac{1}{N} \sum_{\boldsymbol{x}^{(i)} \in \Lambda([1,F_{k-1}],F_k)} \left| D_{\mathcal{A}_2}(\boldsymbol{y} - \boldsymbol{x}^{(i)}) \right|,$$

where $\boldsymbol{y} = \mathrm{mod}\,(\boldsymbol{x} + \boldsymbol{\Delta}, 1)$, for any $\boldsymbol{x} \in \Lambda([1, F_{k-1}], F_k)$ and some $\boldsymbol{\Delta} \in [0,1]^2$. Further, we conjecture that

$$\boldsymbol{\Delta} = \begin{cases} \boldsymbol{x}^{(F_{k-1})}/2 & \text{if } k \equiv 0 \pmod{6}, \\ (1/2, 1/2) & \text{if } k \equiv 1, 5 \pmod{6}, \\ (1/2, 0) & \text{if } k \equiv 2, 3, 4 \pmod{6}. \end{cases}$$

The above expression for the Lebesgue constant can be further simplified by taking into account the periodicity of the Dirichlet kernel, but this is beyond the scope of this manuscript.

4 Conclusions

We presented different cubature degrees of exactness for Fibonacci lattice rules; in particular we derived a new result for the product trigonometric degree. We also proved that the sequence of ℓ^2 copies of the Fibonacci lattice rule with 8 points achieves the lower bound on the number of points: to achieve a trigonometric degree of $4\ell - 1$ it uses $8\ell^2$ points.
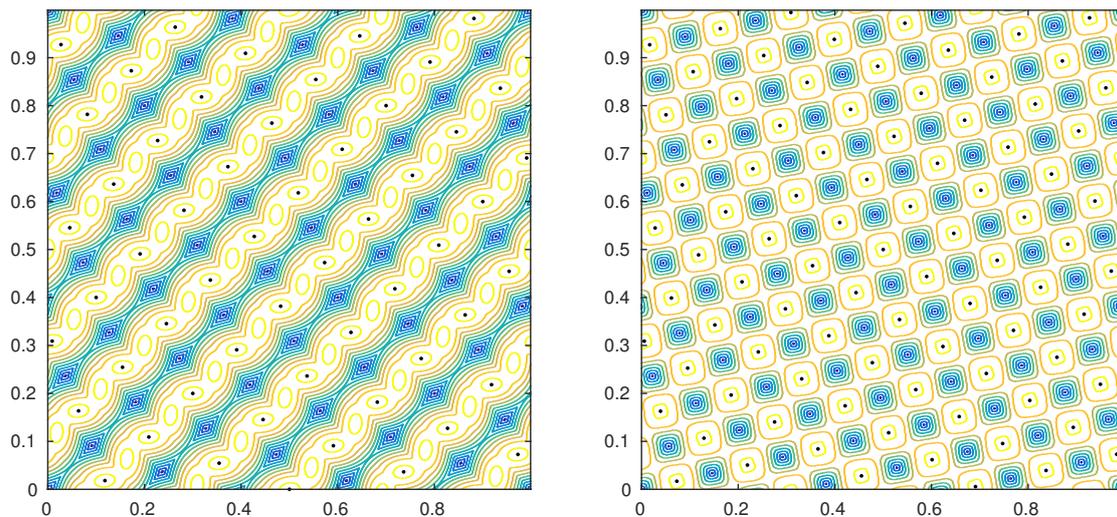
Figure 5: Contour plots of the function (13) for F_k points with $k = 10$ (left) and $k = 11$ (right).

Next, we used Fibonacci lattice points to approximate a function by first truncating its Fourier series expansion to an index set $\mathcal{A}_2(T)$ and then calculating the Fourier coefficients. In this context, we derived the trigonometric, product trigonometric, and hyperbolic cross approximation degrees.

Finally, we presented numerical results on the Lebesgue constant in using Fibonacci lattice points for approximation of periodic bivariate functions. It is conjectured in [19] that this Lebesgue constant is bounded above and below by curves with growth rate $\ln^d(N)$. Our numerical results show that Fibonacci lattice points follow this bound for the considered range of numbers. In addition we made a conjecture about the location of the maxima in the expression for the Lebesgue constant.

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