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The multivariate Durrmeyer-sampling type operators in functional spaces

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Abstract

This paper deals with the study of the convergence of the family of multivariate Durrmeyer-sampling type operators in the general setting of Orlicz spaces. The above result implies also the convergence in remarkable subcases, such as in Lebesgue, Zygmund and exponential spaces. Convergence results have been established also in case of continuous functions, where pointwise and uniform convergence theorems, including some quantitative estimates, have been achieved. Finally, several examples with graphical representations are given.

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Key Words: Orlicz spaces, modular convergence, order of approximation, Durrmeyer-sampling operators

1 Introduction

Sampling-type operators have been introduced in order to establish an approximate version of the celebrated classical sampling theorem, known in literature as Wittaker-Kotel'nikov-Shannon sampling theorem. An important family of sampling operators is represented by the well-known generalized-sampling type operators, introduced by P. L. Butzer and his school in Aachen in the eighties (see, e.g., [12, 13, 14, 11, 9, 35, 3]), and defined by

$$\left(S_w^{\varphi} f\right)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k) f\left(\frac{k}{w}\right), \;\; w>0,\, x\in\mathbb{R}, \tag{1}$$

where φ denotes a suitable kernel function, that is, in fact, a discrete approximate identity. Due to the pointwise nature of the above operators S_w^{φ}, they revealed to be very suitable in order to approximate continuous signals. By the study of the above operators, several applications, especially in signal theory, have been developed. However, from the applications point of view, most of the real world signals (such as digital images) are not represented mathematically by continuous functions and moreover the values $f\left(\frac{k}{w}\right)$ are not always representative of the reconstruction process, as we will see later. Hence, other suitable versions of operators, as those in (1), have been introduced with the aim of including even the possibility to approximate also not necessarily continuous functions, i.e., functions belonging to more general functional spaces. To this aim, in the year 2007, a Kantorovich version of (1) has been provided (see [4]). The idea is to replace the sample values $f\left(\frac{k}{w}\right)$ by an integral mean of f in a neighbourhood of the node $\frac{k}{w}$, so as to have to deal with the Kantorovich-sampling type operators of the form

$$\left(K_w^{\varphi} f\right)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k)\, w \int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)\,du, \;\; w>0,\, x\in\mathbb{R}. \tag{2}$$

Thanks to their definition, the above operators are now well-defined and continuous not only in L^p-spaces, but also in the more general setting of Orlicz spaces. As it is well-known, the latter spaces have been introduced for the first time in the 30s by the Polish mathematician W. Orlicz, as a natural extension of Lebesgue spaces. The literature treating the problem of convergence for (2) is very wide, both in the one-dimensional and in the multidimensional case (see, e.g., [20, 39, 21, 22, 23, 19]).
An interesting extension of (2) was given by C. Bardaro et al. (inspirited by the work of J.L. Durrmeyer [30, 29]), with the introduction of the following modification of the sampling series, by using two different kernel functions φ and ψ, namely

$$\left(S_w^{\varphi,\psi} f\right)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k)\, w \int_{\mathbb{R}} \psi(wu-k) f(u)\,du, \;\; w>0,\, x\in\mathbb{R}, \tag{3}$$

(see, e.g., [7, 6]). The operators $S_w^{\varphi,\psi}$ have been named *Durrmeyer-sampling type operators*. In literature, due to the approximation properties of the Durrmeyer-type operators, they have attracted the attention of many mathematicians (see, e.g., [33, 16, 32, 17, 1, 43, 31, 34, 2]). Recently, in [18] we have investigated the approximation properties and the problem of convergence of (3) in the univariate setting.
The main purpose of the present paper is to provide a unifying approach for the convergence of the multivariate version of (3)

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(see, e.g., [8]) in the general setting of Orlicz spaces. By using this approach, it is possible to deduce the convergence in several functional spaces, such as Lebesgue, Zygmund and exponential spaces, covering in this way a large class of functions, even discontinuous.
Moreover, it is possible to show that the operators (3), by suitable choices of the functions φ and ψ, contain, as particular cases, both the operators (1) and (2) in their multidimensional version.
The study of the convergence in the multivariate frame is crucial mainly from the applications point of view: indeed, in signal theory, and especially in image processing, we have to work with multivariate signals (see, e.g., [20]). Moreover, convergence results that include also the case of not necessarily continuous functions, turns out to be particularly useful in the multivariate setting, since images, for instance, are represented mathematically by bivariate functions with discontinuities in correspondence of the edges of the image itself, where jumps of grey levels occur.

Here we give the plan of the paper. In Section 2 we introduce the main notations and definitions and in Section 3 we recall the definition of the multivariate version of (3). Section 4 contains the main approximation results, starting from the study of a modular convergence theorem in the multivariate Orlicz spaces. Then, we show its applications to the particular cases of Lebesgue spaces L^p with $1 \leq p < +\infty$, Zygmund spaces $L^\alpha \log^\beta L$ and exponential spaces. Furthermore, for the sake of completeness of the theory, we also provide a uniform convergence theorem, for bounded continuous and uniformly continuous functions, with respect to the usual uniform norm. Here, we also furnish quantitative estimates for the order of approximation, in terms of the modulus of continuity of the involved function. In order to better understand the approximation properties of the multivariate Durrmeyer-sampling type operators and the theory here developed, the last section is devoted to examples with graphical representations, by using special multivariate kernels, such as Jackson, Bochner-Riesz kernels, and others. Among these, suitable linear combinations of shifted kernels will also be presented, both to improve the order of approximation and, in some cases, to predict the signal only by a finite number of samples taken from the past.

2 Preliminaries and notations

From now on, let $\mathbb{N}^n$ denote the set of all n-tuples $\underline{k} = (k_1, \dots, k_n)$ of elements of $\mathbb{N}$; $\mathbb{Z}^n$ and $\mathbb{R}^n$ are defined analogously. In particular, $\mathbb{R}^n$ is the Euclidean space endowed, e.g., with the norm $\|\underline{u}\|_2 = (u_1^2 + \dots + u_n^2)^{1/2}$, where $\underline{u} = (u_1, \dots, u_n)$, $u_i \in \mathbb{R}$, $i = 1, \dots, n$. Further, $B(\underline{x}, r)$ denotes the closed ball of $\mathbb{R}^n$ of center $\underline{x}$ and radius $r > 0$ containing all the vectors $\underline{u} \in \mathbb{R}^n$ such that $\|\underline{x} - \underline{u}\|_2 \leq r$. Furthermore, we denote by $C(\mathbb{R}^n)$ the space of all uniformly continuous and bounded functions $f : \mathbb{R}^n \to \mathbb{R}$ endowed with the usual norm $\| \cdot \|_{C(\mathbb{R}^n)} = \| \cdot \|_\infty$.
Moreover, by $M(\mathbb{R}^n)$ we denote the space of all (Lebesgue) measurable real functions over $\mathbb{R}^n$ and let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex φ-function, i.e., φ satisfies the following assumptions:

1. φ is convex in $\mathbb{R}_0^+$;
2. $\varphi(0) = 0$ and $\varphi(u) > 0$, for every $u > 0$.

For every fixed φ, we can consider the functional $I^\varphi : M(\mathbb{R}^n) \to \widetilde{\mathbb{R}}$ defined by

$$I^\varphi[f] := \int_{\mathbb{R}^n} \varphi(|f(\underline{x})|) d\underline{x}, \quad f \in M(\mathbb{R}^n).$$

It is well-known that I^φ is a modular functional on $M(\mathbb{R}^n)$ (see [37]), for every given φ-function φ, which generates the Orlicz space

$$L^\varphi(\mathbb{R}^n) = \{f \in M(\mathbb{R}^n) : I^\varphi[\lambda f] < +\infty, \text{for some } \lambda > 0\}.$$

In $L^\varphi(\mathbb{R}^n)$ we introduce the notion of *modular convergence*: a net $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ converges modularly to a function $f \in L^\varphi(\mathbb{R}^n)$ if

$$\lim_{w \to +\infty} I^\varphi[\lambda(f_w - f)] = 0,$$

for *some* $\lambda > 0$.
For further details concerning Orlicz spaces, see, e.g., [38, 40, 10].

3 The multivariate generalized Durrmeyer-sampling type operators

Here we recall the definition of the family (net) of Durrmeyer-sampling type operators in the multidimensional setting, introduced in [8]. First of all, let us consider two functions $\varphi, \psi \in L^1(\mathbb{R}^n)$, that we will call as *discrete* and *continuous kernel*, respectively, such that φ is bounded in a neighborhood of the origin, and satisfying

$$\sum_{\underline{k} \in \mathbb{Z}^n} \varphi(\underline{u} - \underline{k}) = 1, \text{ for every } \underline{u} \in \mathbb{R}^n, \text{ and } \int_{\mathbb{R}^n} \psi(\underline{u}) d\underline{u} = 1. \tag{4}$$

As it is well-known, ψ defines an approximate identity (see, e.g., [15, 41, 36]) by the formula $\psi_w(\underline{u}) := w\psi(w\underline{u})$, $\underline{u} \in \mathbb{R}^n$ and $w > 0$, i.e., it is a Fejér-type approximate identity.
Similarly to the one-dimensional case, for any real $\nu \geq 0$, the discrete and continuous absolute moments of order ν are defined by

$$M_\nu(\varphi) := \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} \left| \varphi(\underline{u} - \underline{k}) \right| \|\underline{u} - \underline{k}\|_2^\nu \tag{5}$$

and

$$\widetilde{M}_\nu(\psi) := \int_{\mathbb{R}^n} \left|\psi(\underline{u})\right| \|\underline{u}\|_2^\nu d\underline{u},$$

respectively. From now on, we always assume that the discrete moment of order 0 of φ is finite, i.e., $M_0(\varphi) < +\infty$. Later on, in Section 6 we give examples of kernels satisfying the previous assumptions.
For $w > 0$ and for discrete and continuous kernels φ and ψ, we define a net of operators $\left(S_w^{\varphi,\psi}\right)_{w>0}$ by

$$\left(S_w^{\varphi,\psi} f\right)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x}-\underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u}-\underline{k}) f(\underline{u}) d\underline{u}, \qquad \underline{x} \in \mathbb{R}^n, \tag{6}$$

for any given function $f : \mathbb{R}^n \to \mathbb{R}$, such that the above series is convergent, for every $\underline{x} \in \mathbb{R}^n$. $S_w^{\varphi,\psi}$ are called the *multivariate Durrmeyer-sampling type operators* based on φ and ψ.

Remark 1. We note that $S_w^{\varphi,\psi} f$ are well-defined for every $f \in L^\infty(\mathbb{R}^n)$. Indeed,

$$\left|\left(S_w^{\varphi,\psi} f\right)(\underline{x})\right| \leq M_0(\varphi)\widetilde{M}_0(\psi)\|f\|_\infty, \ \underline{x} \in \mathbb{R}^n.$$

Thus, the Durrmeyer-sampling type operator is a bounded linear operator mapping $L^\infty(\mathbb{R}^n)$ into itself.

4 Modular convergence in Orlicz spaces

This section contains the main results of the present study, since it is focused on the problem of convergence for the family of multivariate Durrmeyer-sampling type operators $(S_w^{\varphi,\psi})_{w>0}$ in the general setting of Orlicz spaces.
Here, we always denote by η a convex φ-function. Now, we can prove the following.

Theorem 4.1. *Let ψ be a kernel such that $M_0(\psi) < +\infty$, and $f \in L^\eta(\mathbb{R}^n)$ be fixed. Then there exists $\lambda > 0$ such that*

$$I^\eta[\lambda S_w^{\varphi,\psi} f] \leq \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\widetilde{M}_0(\psi)} I^\eta[\lambda M_0(\varphi)\widetilde{M}_0(\psi) f], \ w > 0.$$

In particular, $S_w^{\varphi,\psi} f$ is well-defined and belongs to $L^\eta(\mathbb{R}^n)$, for every $w > 0$.

Proof. Since $f \in L^\eta(\mathbb{R}^n)$, there exists a positive parameter $\overline{\lambda}$ such that $I^\eta[\overline{\lambda} f] < +\infty$. We choose now $\lambda > 0$ such that

$$\lambda \leq \frac{\overline{\lambda}}{M_0(\varphi)\widetilde{M}_0(\psi)}.$$

Applying Jensen inequality twice and Fubini-Tonelli theorem, we can write what follows:

$$\begin{aligned}
I^\eta\left[\lambda\left(S_w^{\varphi,\psi} f\right)\right] &= \int_{\mathbb{R}^n} \eta\left(\lambda\left|\left(S_w^{\varphi,\psi} f\right)(\underline{x})\right|\right) d\underline{x} \\
&= \int_{\mathbb{R}^n} \eta\left(\lambda\left|\sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x}-\underline{k})\left[w^n \int_{\mathbb{R}^n} \psi(w\underline{u}-\underline{k}) f(\underline{u}) d\underline{u}\right]\right|\right) d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \eta\left(\lambda \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right|\left[w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \left|f(\underline{u})\right| d\underline{u}\right]\right) d\underline{x} \\
&\leq \frac{1}{M_0(\varphi)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \eta\left(\lambda M_0(\varphi)\|\psi\|_1 w^n \int_{\mathbb{R}^n} \frac{\left|\psi(w\underline{u}-\underline{k})\right|}{\|\psi\|_1} \left|f(\underline{u})\right| d\underline{u}\right) \left|\varphi(w\underline{x}-\underline{k})\right| d\underline{x} \\
&\leq \frac{1}{M_0(\varphi)\|\psi\|_1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right| d\underline{x}\, w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \eta\left(\lambda M_0(\varphi)\|\psi\|_1 \left|f(\underline{u})\right|\right) d\underline{u} \\
&= \frac{1}{M_0(\varphi)\|\psi\|_1} \int_{\mathbb{R}^n} \left|\varphi(\underline{y})\right| d\underline{y} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\psi(w\underline{u}-\underline{k})\right| \eta\left(\lambda M_0(\varphi)\|\psi\|_1 \left|f(\underline{u})\right|\right) d\underline{u} \\
&\leq \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \int_{\mathbb{R}^n} \left|\varphi(\underline{y})\right| d\underline{y} \int_{\mathbb{R}^n} \eta\left(\lambda M_0(\varphi)\|\psi\|_1 \left|f(\underline{u})\right|\right) d\underline{u} \\
&= \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \|\varphi\|_1 I^\eta[\lambda M_0(\varphi)\|\psi\|_1 f] \\
&= \frac{M_0(\psi)}{M_0(\varphi)\widetilde{M}_0(\psi)} \|\varphi\|_1 I^\eta[\lambda M_0(\varphi)\widetilde{M}_0(\psi) f] < +\infty,
\end{aligned}$$

with the change of variable $w\underline{x} - \underline{k} = \underline{y}$. □

Note that a direct consequence of the previous inequality, taking into account the linearity of $S_w^{\varphi,\psi}$, is the modular continuity for the net $\left(S_w^{\varphi,\psi}\right)_{w>0}$. This means that for every modularly convergent sequence $\left(f_j\right)_j \subset L^\eta(\mathbb{R}^n)$, with $f_j \to f \in L^\eta(\mathbb{R}^n)$ as $j \to +\infty$, it turns out that

$$I^\eta\left[\lambda\left(S_w^{\varphi,\psi} f - S_w^{\varphi,\psi} f_j\right)\right] \to 0, \text{ as } j \to +\infty.$$

Now, we establish the main convergence result.

Theorem 4.2. *Let ψ be a kernel such that $M_0(\psi) < +\infty$, and let $f \in L^\eta(\mathbb{R}^n)$ be fixed. Then*

$$\lim_{w\to\infty} I^\eta\left[\lambda\left(S_w^{\varphi,\psi} f - f\right)\right] = 0,$$

for some $\lambda > 0$.

Proof. Since f belongs to $L^\eta(\mathbb{R}^n)$, it is possible to choose $\lambda_1, \lambda_2 > 0$ such that $I^\eta[\lambda_1 f] < +\infty$, and

$$I^\eta[\lambda_2\left(f(\cdot) - f(\cdot + \underline{h})\right)] \to 0, \text{ as } \underline{h} \to \underline{0},$$

(see, e.g., [37, 10]). Let now $\varepsilon > 0$ be fixed. Hence, there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^n} \eta\left(\lambda_2 \left|f(\underline{u} + \underline{h}) - f(\underline{u})\right|\right) d\underline{u} < \varepsilon, \tag{7}$$

for every $\underline{h} \in \mathbb{R}^n$ with $\|\underline{h}\|_2 \le \delta$.
Let us now choose $\lambda > 0$, sufficiently small, in such a way that

$$\lambda \le \min\left\{\frac{\lambda_2}{2M_0(\varphi)\|\psi\|_1}, \frac{\lambda_1}{4M_0(\varphi)\|\psi\|_1}\right\}.$$

Now, we can write what follows:

$$\begin{aligned}
I^\eta\left[\lambda\left(S_w^{\varphi,\psi} f - f\right)\right] &= \int_{\mathbb{R}^n} \eta\left(\lambda \left|\left(S_w^{\varphi,\psi} f\right)(\underline{x}) - f(\underline{x})\right|\right) d\underline{x} \\
&= \int_{\mathbb{R}^n} \eta\left(\lambda \left|\left(S_w^{\varphi,\psi} f\right)(\underline{x}) - \sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x} - \underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u} - \underline{k}) f\left(\underline{u} + \underline{x} - \frac{\underline{k}}{w}\right) du \right.\right. \\
&\left.\left. + \sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x} - \underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u} - \underline{k}) f\left(\underline{u} + \underline{x} - \frac{\underline{k}}{w}\right) d\underline{u} - f(\underline{x})\right|\right) d\underline{x} \\
&\le \frac{1}{2}\left\{\int_{\mathbb{R}^n} \eta\left(2\lambda \left|\left(S_w^{\varphi,\psi} f\right)(\underline{x}) - \sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x} - \underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u} - \underline{k}) f\left(\underline{u} + \underline{x} - \frac{\underline{k}}{w}\right) d\underline{u}\right|\right) d\underline{x} \right. \\
&\left. + \int_{\mathbb{R}^n} \eta\left(2\lambda \left|\sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x} - \underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u} - \underline{k}) f\left(\underline{u} + \underline{x} - \frac{\underline{k}}{w}\right) d\underline{u} - f(\underline{x})\right|\right) d\underline{x}\right\} \\
&=: \frac{1}{2}\left\{T_1 + T_2\right\}.
\end{aligned}$$

We begin estimating T_1. Applying Jensen inequality twice (as made in the proof of Theorem 4.1), the change of variable $w\underline{x} - \underline{k} = \underline{t}$ and Fubini-Tonelli theorem, we get

$$\begin{aligned}
|T_1| &\le \int_{\mathbb{R}^n} \eta\left(2\lambda \sum_{\underline{k}\in\mathbb{Z}^n} |\varphi(w\underline{x} - \underline{k})| w^n \int_{\mathbb{R}^n} |\psi(w\underline{u} - \underline{k})| \left|f\left(\underline{u} + \underline{x} - \frac{\underline{k}}{w}\right) - f(\underline{u})\right| d\underline{u}\right) d\underline{x} \\
&\le \frac{1}{M_0(\varphi) w^n} \int_{\mathbb{R}^n} |\varphi(\underline{t})| \left[\sum_{\underline{k}\in\mathbb{Z}^n} \eta\left(2\lambda M_0(\varphi) w^n \int_{\mathbb{R}^n} |\psi(w\underline{u} - \underline{k})| \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right) - f(\underline{u})\right| d\underline{u}\right)\right] d\underline{t} \\
&\le \frac{1}{M_0(\varphi)\|\psi\|_1} \int_{\mathbb{R}^n} |\varphi(\underline{t})| \left[\sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\psi(w\underline{u} - \underline{k})| \, \eta\left(2\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right) - f(\underline{u})\right|\right) d\underline{u}\right] d\underline{t} \\
&\le \frac{1}{M_0(\varphi)\|\psi\|_1} \int_{\mathbb{R}^n} |\varphi(\underline{t})| \left[\int_{\mathbb{R}^n} M_0(\psi) \eta\left(2\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right) - f(\underline{u})\right|\right) d\underline{u}\right] d\underline{t} \\
&= \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \left\{\int_{\|\underline{t}\|_2 \le \delta w} + \int_{\|\underline{t}\|_2 > \delta w}\right\} |\varphi(\underline{t})| \left[\int_{\mathbb{R}^n} \eta\left(2\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right) - f(\underline{u})\right|\right) d\underline{u}\right] d\underline{t} \\
&=: T_{1,1} + T_{1,2},
\end{aligned}$$

where $\delta > 0$ is that obtained in (7) and corresponding to $\varepsilon > 0$.
Now, for $T_{1,1}$, we obtain

$$\left|T_{1,1}\right| \leq \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \int_{\|\underline{t}\|_2 \leq \delta w} \left|\varphi(\underline{t})\right| \left[\int_{\mathbb{R}^n} \eta\left(\lambda_2 \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right) - f(\underline{u})\right|\right) d\underline{u}\right] d\underline{t} \leq \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1}\varepsilon,$$

for every $w > 0$, in view of (7).
Further, by $T_{1,2}$, we have

$$\begin{aligned}
\left|T_{1,2}\right| \leq & \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \int_{\|\underline{t}\|_2 > \delta w} \left|\varphi(\underline{t})\right| \frac{1}{2}\left[\int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right)\right|\right) d\underline{u}\right. \\
& \left. + \int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u}\right)\right|\right) d\underline{u}\right] d\underline{t} \\
\leq & \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \int_{\|\underline{t}\|_2 > \delta w} \left|\varphi(\underline{t})\right| d\underline{t} \int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u}\right)\right|\right) d\underline{u} \\
\leq & \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} I^\eta\left[\lambda_1 f\right] \int_{\|\underline{t}\|_2 > \delta w} \left|\varphi(\underline{t})\right| d\underline{t} \\
< & \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} I^\eta\left[\lambda_1 f\right] \varepsilon,
\end{aligned}$$

for every $w > 0$ sufficiently large, since $\varphi \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u} + \frac{\underline{t}}{w}\right)\right|\right) d\underline{u} = \int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{u}\right)\right|\right) d\underline{u},$$

for every $\underline{t} \in \mathbb{R}^n$ and $w > 0$.
Now, we proceed estimating T_2. By the change of variable $\underline{t} = \underline{u} - \frac{\underline{k}}{w}$, applying Jensen inequality twice and Fubini-Tonelli theorem, it turns out that

$$\begin{aligned}
|T_2| = & \int_{\mathbb{R}^n} \eta\left(2\lambda \left|\sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x} - \underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u} - \underline{k}) \left[f\left(\underline{u} + \underline{x} - \frac{\underline{k}}{w}\right) - f(\underline{x})\right] d\underline{u}\right|\right) d\underline{x} \\
\leq & \int_{\mathbb{R}^n} \eta\left(2\lambda \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x} - \underline{k})\right| w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{t})\right| \left|f\left(\underline{t} + \underline{x}\right) - f(\underline{x})\right| d\underline{t}\right) d\underline{x} \\
\leq & \frac{1}{M_0(\varphi)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x} - \underline{k})\right| \eta\left(2\lambda M_0(\varphi) w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{t})\right| \left|f\left(\underline{t} + \underline{x}\right) - f(\underline{x})\right| d\underline{t}\right) d\underline{x} \\
\leq & \frac{1}{M_0(\varphi)} \int_{\mathbb{R}^n} M_0(\varphi) \eta\left(2\lambda M_0(\varphi) \int_{\mathbb{R}^n} \left|\psi(\underline{y})\right| \left|f\left(\frac{\underline{y}}{w} + \underline{x}\right) - f(\underline{x})\right| d\underline{y}\right) d\underline{x} \\
= & \frac{1}{\|\psi\|_1} \int_{\mathbb{R}^n} \left|\psi(\underline{y})\right| \left[\int_{\mathbb{R}^n} \eta\left(2\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\frac{\underline{y}}{w} + \underline{x}\right) - f(\underline{x})\right|\right) d\underline{x}\right] d\underline{y} \\
= & \frac{1}{\|\psi\|_1} \left\{\int_{\|\underline{y}\|_2 \leq \delta w} + \int_{\|\underline{y}\|_2 > \delta w}\right\} \left|\psi(\underline{y})\right| \left[\int_{\mathbb{R}^n} \eta\left(2\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\frac{\underline{y}}{w} + \underline{x}\right) - f(\underline{x})\right|\right) d\underline{x}\right] d\underline{y} \\
=: & T_{2,1} + T_{2,2},
\end{aligned}$$

where $\delta > 0$ is again that given in (7). Hence, arguing as in the first part of the proof, we obtain

$$\left|T_{2,1}\right| \leq \frac{1}{\|\psi\|_1} \int_{\|\underline{y}\| \leq \delta w} \left|\psi(\underline{y})\right| \left[\int_{\mathbb{R}^n} \eta\left(\lambda_2 \left|f\left(\frac{\underline{y}}{w} + \underline{x}\right) - f(\underline{x})\right|\right) d\underline{x}\right] d\underline{y} < \varepsilon,$$

for every $w > 0$.
Further,

$$\begin{aligned}
\left|T_{2,2}\right| \leq & \frac{1}{\|\psi\|_1} \int_{\|\underline{y}\|_2 > \delta w} \left|\psi(\underline{y})\right| \frac{1}{2}\left[\int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{x} + \frac{\underline{y}}{w}\right)\right|\right) d\underline{x}\right. \\
& \left. + \int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f\left(\underline{x}\right)\right|\right) d\underline{x}\right] d\underline{y} \\
\leq & \frac{1}{\|\psi\|_1} \int_{\mathbb{R}^n} \eta\left(4\lambda M_0(\varphi)\|\psi\|_1 \left|f(\underline{x})\right|\right) d\underline{x} \int_{\|\underline{y}\|_2 > \delta w} \left|\psi(\underline{y})\right| d\underline{y} \\
\leq & \frac{1}{\|\psi\|_1} \int_{\|\underline{y}\|_2 > \delta w} \left|\psi(\underline{y})\right| d\underline{y}\, I^\eta[\lambda_1 f] < \frac{\varepsilon}{\|\psi\|_1} I^\eta[\lambda_1 f],
\end{aligned}$$

for every $w > 0$ sufficiently large, since $\psi \in L^1(\mathbb{R}^n)$.
Now, rearranging all the above estimates, we finally get

$$I^\eta\left[\lambda\left(S_w^{\varphi,\psi} f - f\right)\right] \leq \frac{M_0(\psi)}{2\|\psi\|_1}\left[\frac{\|\varphi\|_1}{M_0(\varphi)} + \frac{\|\psi\|_1}{M_0(\psi)} + I^\eta[\lambda_1 f]\left(\frac{1}{M_0(\varphi)} + \frac{1}{M_0(\psi)}\right)\right]\varepsilon,$$

for every sufficiently large $w > 0$. Thus, the proof follows by the arbitrariness of ε. □

Now, we want to apply the modular convergence theorem to remarkable cases of Orlicz spaces. We recall that the Orlicz spaces have been introduced as a natural extension of the Lebesgue spaces. Indeed, if we consider $\eta(u) = u^p$ for $u \geq 0$ and $1 \leq p < +\infty$, the resulting Orlicz space is $L^\eta(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, with $1 \leq p < +\infty$.
Thus, by Theorem 4.1 and Theorem 4.2, the following corollary immediately follows.

Corollary 4.3. *Let ψ be such that $M_0(\psi) < +\infty$. For every $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$, we have:*

(a) $\|S_w^{\varphi,\psi} f\|_p \leq M_0(\psi)^{\frac{1}{p}} M_0(\varphi)^{\frac{p-1}{p}} \|\varphi\|_1^{1/p} \widetilde{M}_0(\psi)^{\frac{p-1}{p}} \|f\|_p$, $w > 0$. *In particular, $S_w^{\varphi,\psi} f$ is well-defined in $L^p(\mathbb{R}^n)$ and $S_w^{\varphi,\psi} f \in L^p(\mathbb{R}^n)$ whenever $f \in L^p(\mathbb{R}^n)$;*

(b) *the family $S_w^{\varphi,\psi} f$ converges to f in $L^p(\mathbb{R}^n)$, i.e.,*

$$\lim_{w\to+\infty} \|S_w^{\varphi,\psi} f - f\|_p = 0.$$

Let now consider the φ-function $\eta_\alpha(u) = e^{u^\alpha} - 1$, $u \geq 0$ for some $\alpha > 0$. The resulting Orlicz space is known as the *exponential type space* and it contains all the functions $f \in \mathcal{M}(\mathbb{R}^n)$ for which

$$I^{\eta_\alpha}[\lambda f] = \int_{\mathbb{R}^n} \left(\exp\left(\lambda\left|f(\underline{x})\right|\right)^\alpha - 1\right) d\underline{x} < +\infty,$$

for some $\lambda > 0$. Note that, for this space the classical Δ_2-condition is not satisfied and therefore the modular convergence is weaker than the Luxemburg convergence (see, e.g., [37]).
By Theorem 4.1 and Theorem 4.2, we can obtain the following.

Corollary 4.4. *Let ψ be such that $M_0(\psi) < +\infty$. For every $f \in L^{\eta_\alpha}(\mathbb{R}^n)$, there holds:*

(a) *the modular continuity inequality*

$$\int_{\mathbb{R}^n} \left(\exp\left(\lambda\left|S_w^{\varphi,\psi} f(\underline{x})\right|\right)^\alpha - 1\right) d\underline{x} \leq \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\widetilde{M}_0(\psi)} \int_{\mathbb{R}^n} \left(\exp\left(\lambda M_0(\varphi)\widetilde{M}_0(\psi)\left|f(\underline{x})\right|\right)^\alpha - 1\right) d\underline{x},$$

for some $\lambda > 0$. In particular, $S_w^{\varphi,\psi} f$ is well-defined in $L^{\eta_\alpha}(\mathbb{R}^n)$ and $S_w^{\varphi,\psi} f \in L^{\eta_\alpha}(\mathbb{R}^n)$ whenever $f \in L^{\eta_\alpha}(\mathbb{R}^n)$;

(b) *there exists $\lambda > 0$ such that*

$$\lim_{w\to\infty} \int_{\mathbb{R}^n} \left(\exp\left(\lambda\left|S_w^{\varphi,\psi} f(\underline{x}) - f(\underline{x})\right|\right)^\alpha - 1\right) d\underline{x} = 0.$$

Finally, another remarkable case of Orlicz space, is that generated by the φ-function $\eta_{\alpha,\beta}(u) = u^\alpha \log^\beta(e+u)$, $u \geq 0$ for $\alpha \geq 1$ and $\beta > 0$. The corresponding Orlicz spaces are the so-called *interpolation spaces* defined by the set of functions $f \in \mathcal{M}(\mathbb{R}^n)$ for which

$$I^{\eta_{\alpha,\beta}}[\lambda f] = \int_{\mathbb{R}^n} \left(\lambda\left|f(\underline{x})\right|\right)^\alpha \log^\beta(e + \lambda\left|f(\underline{x})\right|) d\underline{x} < +\infty,$$

for some $\lambda > 0$; they are denoted by $L^\alpha \log^\beta L(\mathbb{R}^n)$. As made before, a corollary similary to Corollary 4.3 and Corollary 4.4 can also be formulated in case of interpolation spaces.

5 Convergence results and quantitative estimates in $C(\mathbb{R}^n)$

For the sake of completeness, in this section we consider approximation results for the multivariate Durrmeyer-sampling type operators when continuous functions are considered. From now on, we always assume that the discrete kernels φ satisfies the following additional condition, that is

$$M_r(\varphi) < +\infty, \quad \text{for some } r > 0.$$

We start with the following lemma.

Lemma 5.1 ([20]). *Under the above assumptions on the kernel φ, there hold*

(a) $M_0(\varphi) < +\infty$;

(b) *for every* $\gamma > 0$,

$$\lim_{w\to+\infty} \sum_{\|w\underline{x}-\underline{k}\|_2>\gamma w} \left|\varphi(w\underline{x}-\underline{k})\right| = 0,$$

uniformly with respect to $\underline{x} \in \mathbb{R}^n$.

Now, we establish the following pointwise and uniform convergence theorem.

Theorem 5.2. *Let $f \in L^\infty(\mathbb{R}^n)$. Then*

$$\lim_{w\to+\infty} (S_w^{\varphi,\psi} f)(\underline{x}) = f(\underline{x})$$

at any point $\underline{x}$ of continuity of f. Moreover, if $f \in C(\mathbb{R}^n)$, then

$$\lim_{w\to+\infty} \|S_w^{\varphi,\psi} f - f\|_\infty = 0.$$

Proof. For the sake of simplicity, we prove only the second part of the theorem, since the first part can be established by similar arguments. Let $\varepsilon > 0$ be fixed. Then, by the uniformity continuity of f, there exists $\delta > 0$ such that $\left|f(\underline{x})-f(\underline{y})\right| < \varepsilon$ as $\|\underline{x}-\underline{y}\|_2 < \delta$. Let now $\underline{x} \in \mathbb{R}^n$ be fixed. Using (4), we can immediately write

$$\begin{aligned}
\left|(S_w^{\varphi,\psi} f)(\underline{x}) - f(\underline{x})\right| &= \left|\sum_{\underline{k}\in\mathbb{Z}^n} \varphi(w\underline{x}-\underline{k}) w^n \int_{\mathbb{R}^n} \psi(w\underline{u}-\underline{k}) f(\underline{u})\, d\underline{u} - f(\underline{x})\right| \\
&\le \left\{ \sum_{\|w\underline{x}-\underline{k}\|_2 \le \frac{\delta}{2}w} + \sum_{\|w\underline{x}-\underline{k}\|_2 > \frac{\delta}{2}w} \right\} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \left|f(\underline{u})-f(\underline{x})\right| d\underline{u} \\
&=: T_1 + T_2,
\end{aligned}$$

where $\delta > 0$ is the parameter of the uniform continuity of f corresponding to $\varepsilon > 0$. Concerning T_1, we can also write

$$\begin{aligned}
T_1 &= \sum_{\|w\underline{x}-\underline{k}\|_2 \le \frac{\delta}{2}w} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \left\{ \int_{\|w\underline{u}-\underline{k}\|_2 < \frac{\delta}{2}w} + \int_{\|w\underline{u}-\underline{k}\|_2 \ge \frac{\delta}{2}w} \right\} \left|\psi(w\underline{u}-\underline{k})\right| \left|f(\underline{u})-f(\underline{x})\right| d\underline{u} \\
&=: T_{1,1} + T_{1,2}.
\end{aligned}$$

Now, for each $\underline{u} \in \mathbb{R}^n$ such that $\|w\underline{u}-\underline{k}\|_2 < \frac{\delta}{2}w$, if $\|w\underline{x}-\underline{k}\|_2 \le \frac{\delta}{2}w$, we have

$$\|\underline{u}-\underline{x}\|_2 \le \left\|\underline{u} - \frac{\underline{k}}{w}\right\|_2 + \left\|\frac{\underline{k}}{w} - \underline{x}\right\|_2 < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

then, it turns out that

$$\begin{aligned}
T_{1,1} &< \varepsilon \sum_{\|wx-k\|_2 \le \frac{\delta}{2}w} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \int_{\|w\underline{u}-\underline{k}\|_2 < \frac{\delta}{2}w} \left|\psi(w\underline{u}-\underline{k})\right| d\underline{u} \\
&< M_0(\varphi) \|\psi\|_1 \varepsilon,
\end{aligned}$$

for every $w \ge 0$.

Moreover, if we consider the change of variable $w\underline{u} - \underline{k} = \underline{y}$, we can observe that

$$\int_{\|w\underline{u}-\underline{k}\|_2 > \frac{\delta}{2}w} w^n \left|\psi(w\underline{u}-\underline{k})\right| d\underline{u} = \int_{\|\underline{y}\|_2 > \frac{\delta}{2}w} \left|\psi(\underline{y})\right| d\underline{y} \to 0, \text{ as } w \to +\infty,$$

since $\psi \in L^1(\mathbb{R}^n)$. Hence

$$T_{1,2} \le 2\|f\|_\infty M_0(\varphi)\varepsilon, \text{ for } w > 0 \text{ sufficiently large.}$$

By similar reasoning, we immediately obtain the following inequality

$$T_2 \le 2\|f\|_\infty \|\psi\|_1 \sum_{\|w\underline{x}-\underline{k}\|_2 > \frac{\delta}{2}w} \left|\varphi(w\underline{x}-\underline{k})\right| < \varepsilon,$$

as $w \to +\infty$, as a consequence of Lemma 5.1, and thus the proof follows by the arbitrariness of $\varepsilon > 0$. □

Now, we investigate the problem of the order of uniform convergence for $f \in C(\mathbb{R}^n)$. For this purpose, we first recall the notion of modulus of continuity, defined by

$$\omega(f,\delta) = \sup\left\{ \left| f(\underline{x}) - f(\underline{y}) \right| : \|\underline{x}-\underline{y}\|_2 < \delta,\, \underline{x},\underline{y} \in \mathbb{R}^n \right\}.$$

Thus, we can prove what follows.

Theorem 5.3. *Suppose that φ and ψ are such that $M_1(\varphi) + \widetilde{M}_1(\psi) < +\infty$ and let $f \in C(\mathbb{R}^n)$. Then we have*

$$\|S_w^{\varphi,\psi} f - f\|_\infty \le C^{\varphi,\psi}\,\omega\left(f, \frac{1}{w}\right),$$

for every $w > 0$, where $C^{\varphi,\psi} = M_0(\varphi)\left(\widetilde{M}_0(\psi) + \widetilde{M}_1(\psi)\right) + M_1(\varphi)\widetilde{M}_0(\psi)$.

Proof. Let $\underline{x} \in \mathbb{R}^n$ be fixed. Proceeding as in the proof of Theorem 5.2, we immediately have

$$\begin{aligned}
\left|(S_w^{\varphi,\psi} f)(\underline{x}) - f(\underline{x})\right| &\le \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \left|f(\underline{u}) - f(\underline{x})\right| d\underline{u} \\
&\le \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \omega(f, \|u-x\|_2)\, d\underline{u} \\
&\le \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \omega\left(f, \frac{1}{w}\right)(1 + w\|u-x\|_2) d\underline{u} \\
&= \omega\left(f, \frac{1}{w}\right) \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| (1 + w\|\underline{u}-\underline{x}\|_2) d\underline{u} \\
&= \omega\left(f, \frac{1}{w}\right) \sum_{\underline{k}\in\mathbb{Z}^n} \left|\varphi(w\underline{x}-\underline{k})\right| w^n \left\{ \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| d\underline{u} \right. \\
&\quad \left. + \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| w\|\underline{u}-\underline{x}\|_2 d\underline{u} \right\},
\end{aligned}$$

for every $w > 0$, where in the previous estimate we used the well-known inequality $\omega(f, \lambda\delta) \le (\lambda+1)\omega(f,\delta)$ with $\lambda, \delta > 0$. Now, we can observe that

$$w^n \int_{\mathbb{R}^n} \left|\psi(w\underline{u}-\underline{k})\right| \|w\underline{u} - w\underline{x}\|_2 d\underline{u} \le \widetilde{M}_1(\psi) + \|\underline{k} - w\underline{x}\|_2 \widetilde{M}_0(\psi)$$

for every $\underline{k} \in \mathbb{Z}^n$. Thus, we finally have the following estimate

$$\left|(S_w^{\varphi,\psi} f)(\underline{x}) - f(\underline{x})\right| \le \omega\left(f, \frac{1}{w}\right) \left\{ M_0(\varphi)\left(\widetilde{M}_0(\psi) + \widetilde{M}_1(\psi)\right) + M_1(\varphi)\widetilde{M}_0(\psi) \right\}.$$

Now, setting $C^{\varphi,\psi} := M_0(\varphi)\left(\widetilde{M}_0(\psi) + \widetilde{M}_1(\psi)\right) + M_1(\varphi)\widetilde{M}_0(\psi)$, we get the thesis. □

In conclusion, recalling the definition of Lipschitz classes $Lip\,\alpha$, i.e.,

$$Lip\,\alpha := \{ f \in C(\mathbb{R}^n) : \omega(f,\delta) = \mathcal{O}(\delta^\alpha),\ \text{as } \delta \to 0^+ \},$$

with $0 < \alpha \le 1$, and using Theorem 5.3, we can deduce what follows.

Corollary 5.4. *Under the assumptions of Theorem 5.3, and assuming in addition that $f \in Lip\,\alpha$, $0 < \alpha \le 1$, then*

$$\|S_w^{\varphi,\psi} f - f\|_\infty = \mathcal{O}(w^{-\alpha}), \ \textit{as } w \to +\infty.$$

Remark 2. We note that in the results of this section, the assumption $\varphi \in L^1(\mathbb{R}^n)$ could be avoided. Moreover, we highlight that assumption $M_1(\varphi) + \widetilde{M}_1(\psi) < +\infty$ is easily satisfied by several examples of kernels.

6 Examples with special multivariate kernels φ and ψ

The choice of the kernels assumes a central role in the results given in the previous section. However, in general it is not very easy to verify if a multivariate function satisfies the assumptions on the moments. Here we show two different approaches to define suitable instances of multivariate kernels.

6.1 Product type kernels and others

We consider $\xi_1, \ldots, \xi_n \in L^1(\mathbb{R})$, such that

$$M_0(\xi_i) = \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\xi_i(u-k)| < +\infty,$$

and we assume that $\sum_{k\in\mathbb{Z}} \xi_i(u-k) = 1$, for every $u \in \mathbb{R}$, and $\displaystyle\int_{\mathbb{R}} \xi_i(u)du = 1$, for $i = 1, \ldots, n$. Setting

$$\xi(\underline{u}) := \prod_{i=1}^{n} \xi_i(u_i),$$

it is well-known that $\xi \in L^1(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} \xi(\underline{u})d\underline{u} = \int_{\mathbb{R}^n} \prod_{i=1}^{n} \xi_i(u_i)du_1 \ldots du_n = \prod_{i=1}^{n} \int_{\mathbb{R}} \xi_i(u_i)du_i = 1.$$

Moreover,

$$M_0(\xi) = \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} |\xi(\underline{u} - \underline{k})| = \prod_{i=1}^{n} M_0(\xi_i) < +\infty.$$

Further, we also have that

$$\sum_{\underline{k} \in \mathbb{Z}^n} \xi(\underline{u} - \underline{k}) = \prod_{i=1}^{n} \sum_{k_i \in \mathbb{Z}} \xi_i(u_i - k_i) = 1,$$

for every $\underline{u} \in \mathbb{R}^n$, i.e., ξ is a multivariate discrete and continuous kernel (in the sense defined in Section 3).
For example, we can consider the Fejér kernel, defined by

$$F(x) := \frac{1}{2}\operatorname{sinc}^2\left(\frac{x}{2}\right), \; x \in \mathbb{R},$$

where the *sinc*-function is given by

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \in \mathbb{R} \setminus 0, \\ 1, & x = 0. \end{cases}$$

We easily observe that F is bounded and non-negative on $\mathbb{R}$, belongs to $L^1(\mathbb{R})$ and satisfies $\displaystyle\int_{\mathbb{R}} F(u)du = 1$. Now, in order to check the assumptions on the discrete absolute moments, we recall the following remark.

Remark 3. If a function φ is bounded in a neighborhood of the origin and $\varphi(\underline{u}) = \mathcal{O}\left(\|\underline{u}\|_2^{-\alpha}\right)$, as $\|\underline{u}\|_2 \to +\infty$, with $0 < \nu < \alpha - 1$, then

$$M_\mu(\varphi) < +\infty, \text{ for every } 0 \le \mu \le \nu.$$

The previous remark shows that it is easy to find examples of kernels satisfying the required assumption on the finiteness of the discrete absolute moments introduced in sections 3, 4 and 5. In the following, we concentrate on examples of operators approximating discontinuous functions, according to Section 4.
Returning to the Fejér kernel, the moment condition $M_0(F) < +\infty$ is trivially fulfilled in view of the previous remark with $0 \le \mu < 1$. Moreover, since its Fourier transform is given by

$$\widehat{F}(v) = \begin{cases} 1 - \left|\frac{v}{\pi}\right|, & v \le \pi, \\ 0, & v > \pi, \end{cases}$$

and considering that

$$\widehat{F}(2k\pi) = \begin{cases} 1, & k = 0, \\ 0, & k \in \mathbb{Z} \setminus 0, \end{cases}$$

it follows that the discrete singularity assumption (4) on φ is satisfied, as a consequence of the Poisson summation formula. For more details, see, e.g., [15]. Then, the corresponding multivariate version of Fejér kernel is given by

$$\mathcal{F}_n(\underline{x}) := \prod_{i=1}^{n} F(x_i), \; \underline{x} \in \mathbb{R}^n.$$

Now, we show other well-known classes of kernels, for which the above results hold. First of all, we recall the definition of the one-dimensional central B-spline of order N (see, e.g., [42]):

$$\sigma_N(x) := \frac{1}{(N-1)!} \sum_{i=0}^{N} (-1)^i \binom{N}{i} \left(\frac{N}{2} + x - i\right)_+^{N-1}, \; x \in \mathbb{R},$$

which are examples of continuous kernels with compact support contained in the interval $\left[-\frac{N}{2}, \frac{N}{2}\right]$ (here, $(x)_+ = \max\{x, 0\}$ denotes the positive part of x).
Moreover, the singularity assumption (4) follows again as a consequence of the Poisson summation formula, taking into account that $\widehat{\sigma_N}$, i.e., the Fourier transform of σ_N, is such that $\widehat{\sigma_N}(2\pi k) = 0$, if $k \in \mathbb{Z} \setminus 0$, and $\widehat{\sigma_N}(0) = 1$, since

$$\widehat{\sigma_N}(v) = \mathrm{sinc}^N\left(\frac{v}{2\pi}\right), \; v \in \mathbb{R}.$$

The corresponding multivariate version of the central B-spline of order N (see Figure 1) is given by

$$\mathcal{S}_N^n(\underline{x}) := \prod_{i=1}^{n} \sigma_N(x_i), \; \underline{x} \in \mathbb{R}^n.$$

Another useful class of kernels is given by the so-called Jackson type kernels of order N, defined in the univariate case by

$$J_N(x) := c_N \mathrm{sinc}^{2N}\left(\frac{x}{2N\pi\alpha}\right), \; x \in \mathbb{R},$$

with $N \in \mathbb{N}$, $\alpha \geq 1$, and c_N is a non-zero normalization coefficient, given by

$$c_N := \left[\int_{\mathbb{R}} \mathrm{sinc}^{2N}\left(\frac{u}{2N\pi\alpha}\right) du\right]^{-1}.$$

Similarly to the Fejér kernel, also the Jackson kernel has an unbounded support and it can be generated by powers of the sinc-function; thus, it is easy to prove that all the required assumptions are again satisfied. In order to extend this definition in the multivariate setting, we proceed as before, hence the multivariate Jackson type kernels of order N (see Figure 1) is given by

$$\mathcal{J}_N^n(\underline{x}) := \prod_{i=1}^{n} J_N(x_i), \; \underline{x} \in \mathbb{R}^n.$$

Lastly, we mention a further class of kernels, defined by some characteristic functions. For instance, we can take into account the

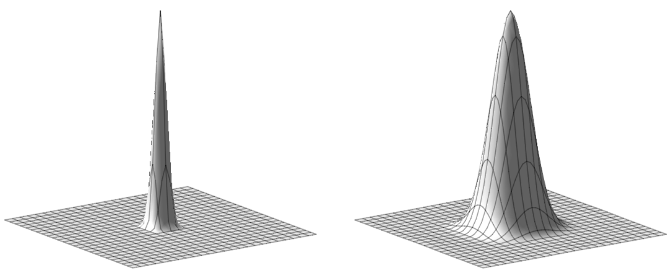

Figure 1: The bivariate B-spline kernel $\mathcal{S}_3^2$, with $N = 3$ (on left) and the bivariate Jackson kernel $\mathcal{J}_2^2$, with $N = 2$ and $\alpha = 1$ (on right).

characteristic function of the unitary hypercube of $\mathbb{R}^n$, i.e.,

$$\chi_n(\underline{x}) := \prod_{i=1}^{n} \chi_{[0,1]}(x_i), \; \underline{x} \in \mathbb{R}^n.$$

We can also consider a similar version of the above kernel, defined using symmetric characteristic functions of the form

$$\chi_n^s(\underline{x}) := \frac{1}{2^n} \prod_{i=1}^{n} \chi_{[-1,1]}(x_i), \; \underline{x} \in \mathbb{R}^n,$$

where $\frac{1}{2^n}$ is a normalization constant.

Now, in order to support the theory through graphical examples, let now consider several operators based on specific kernels that we will apply to a particular discontinuous function $f \in L^p(\mathbb{R}^2)$, with $1 \leq p < +\infty$. First of all, we analyse in details the case of Durrmeyer-sampling type operators based on $\varphi := J_N^n$ and on $\psi := \chi_n$, respectively the Jackson kernel of order $N > 0$ and the kernel defined by the product of characteristic functions in $[0, 1]$ given above, in the bivariate case $n = 2$. Note that, with this choice of the kernel ψ, we obtain in particular a family of Kantorovich-sampling operators of the form (2).
The corresponding multivariate Durrmeyer-Sampling type operators of a general $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$, is given by

$$\begin{aligned}\left(S_w^{\mathcal{J}_N^n, \chi_n} f\right)(\underline{x}) &= \sum_{\underline{k} \in \mathbb{Z}^n} \mathcal{J}_N^n(w\underline{x} - \underline{k}) w^n \int_{\mathbb{R}^n} \chi_n(w\underline{u} - \underline{k}) f(\underline{u}) d\underline{u} \\ &= \sum_{\underline{k} \in \mathbb{Z}^n} \mathcal{J}_N^n(w\underline{x} - \underline{k}) w^n \int_{B_n} f(\underline{u}) d\underline{u},\end{aligned}$$

where $B_n = \prod_{i=1}^{n} [\frac{k_i}{w}, \frac{k_i+1}{w}]$. Note that, in this case ($n = 2$), the convolution product in $S_w^{\mathcal{J}_N^n,\chi_n} f$ reduces to the mean value of f given by $w^2 \int_{B_2} f(\underline{u}) d\underline{u}$, where $B_2 = \left[\frac{k_1}{w}, \frac{k_1+1}{w}\right] \times \left[\frac{k_2}{w}, \frac{k_2+1}{w}\right]$. Now, we apply the above operator $S_w^{\mathcal{J}_2^2,\chi_2}$ (where we consider the order $N = 2$ and $\alpha = 1$) to a specific function belonging to $L^p(\mathbb{R}^2)$, $1 \leq p < +\infty$, (see Figure 2), defined by

$$f(x, y) := \begin{cases} 2, & -1 \leq x \leq 1 \; and \; -1 \leq y \leq 1, \\ 6 \ln\left(\frac{1}{x^2+y^2} + 1\right), & otherwise. \end{cases} \tag{8}$$

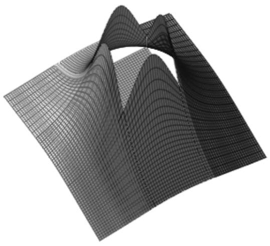

Figure 2: Graph of the function f defined in (8).

The two-dimensional Durrmeyer-sampling type operators $S_w^{\mathcal{J}_2^2,\chi_2}$ for the function f defined in (8) in case of $w = 5$ and $w = 10$ are given in Figure 3, in a same octant of the plane. The graphical representations show that via the above operators we are able to reconstruct an example of discontinuous surface in $\mathbb{R}^3$, which is represented in our case by the graph of the function f, according to the theory developed in Section 4.

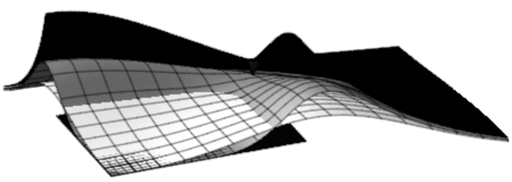

Figure 3: f (black), $S_5^{\mathcal{J}_2^2,\chi_2} f$ (grey), $S_{10}^{\mathcal{J}_2^2,\chi_2} f$ (dark grey).

Now, leaving the kernel ψ fixed, we consider the two-dimensional Durrmeyer-sampling type operators generated by the B-spline of order $N = 3$, $\mathcal{S}_3^2$, for the function f defined in (8) in case of $w = 5$ and $w = 10$. As before, in order to give a graphical representation also in this case, we plot the function f, the operators $S_5^{\mathcal{S}_3^2,\chi_2} f$ and $S_{10}^{\mathcal{S}_3^2,\chi_2} f$ all together in a same octant of the plane (Figure 4). In this case, since both spline and the kernel generated by characteristic functions are compactly supported, it is easy to see that the quantitative results stated in Section 5 also hold.

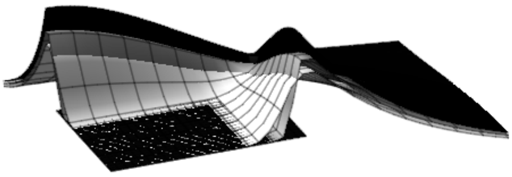

Figure 4: f (black), $S_5^{\mathcal{S}_3^2,\chi_2} f$ (grey), $S_{10}^{\mathcal{S}_3^2,\chi_2} f$ (dark grey).

Furthermore, in order to improve the rate of approximation, we may consider as φ a kernel generated by suitable linear combination of shifted Jackson kernel J_2, of the form

$$\tau_{J_2}(x) := \frac{1}{2}\{J_2(x) + J_2(x-1)\}, \quad x \in \mathbb{R}.$$

Thus, the corresponding bivariate version will be given by

$$\mathcal{T}_{J_2}^2(\underline{x}) := \prod_{i=1}^{2} \tau_{J_2}(x_i), \quad \underline{x} = (x_1, x_2) \in \mathbb{R}^2.$$

The operators $S_w^{\mathcal{T}_{J_2}^2,\chi_2}$, generated by the above kernels, are useful since in the one-dimensional case (see, e.g., [24]), it has been proved that they reach an order of approximation up to 2. Moreover, it is possible to achieve a higher order up to 3 in the one-dimensional case, by using the following linear combination of Jackson kernel (see [24])

$$\widetilde{\tau}_{J_2}(x) := \left[\left(\frac{1}{3} - \frac{1}{2}k\right) J_2(x-1) + \left(k + \frac{5}{6}\right) J_2(x) - \left(\frac{1}{6} + \frac{1}{2}k\right) J_2(x+1)\right], \quad x \in \mathbb{R},$$

where k is the algebraic continuous moment of order 2 of J_2, given by $k := \int_{\mathbb{R}} u^2 J_2(u)du$. As before, in the following graphical examples, we use the corresponding bivariate extension obtained by the product, that we denote by $\widetilde{\mathcal{T}}_{J_2}^2$.

Now, in order to better understand the improvement in terms of approximation, we may compare the series $S_w^{\mathcal{J}_2^2,\chi_2}$, $S_w^{\mathcal{T}_{J_2}^2,\chi_2}$ and $S_w^{\widetilde{\mathcal{T}}_{J_2}^2,\chi_2}$ in a same octant of the plane (see Figure 5), considering $w = 10$ fixed.

At this point, it could be interesting to compare the graphs plotted by operators based on compactly supported kernels, with those ones obtained by using kernels with unbounded support. Indeed, from a graphical viewpoint, the operator based on a spline kernel, such as $S_w^{\mathcal{S}_3^2,\chi_2}$, allows us to reach a faster approximation by taking into account fewer terms of the sampling series, not only compared to the case of $S_w^{\mathcal{J}_2^2,\chi_2}$, but even to $S_w^{\widetilde{\mathcal{T}}_{J_2}^2,\chi_2}$ (see Figure 6). This can be traduced in an advantage from the applications point of view: the implementation of the approximation process turns out to be faster, when discrete kernels φ with compact support are considered.

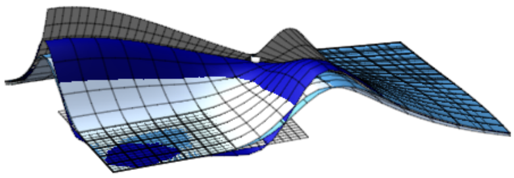

Figure 5: f (grey), $S_{10}^{\mathcal{J}_2^2,\chi_2} f$ (light blue), $S_{10}^{\mathcal{T}_{J_2}^2,\chi_2} f$ (medium blue) and $S_{10}^{\widetilde{\mathcal{T}}_{J_2}^2,\chi_2} f$ (blue).

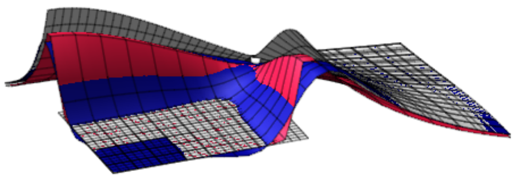

Figure 6: f (grey), $S_{10}^{\widetilde{\mathcal{T}}_{J_2}^2,\chi_2} f$ (blue) and $S_{10}^{\mathcal{S}_3^2,\chi_2} f$ (red).

Finally, in an analogous way, we may construct the class of Durrmeyer-sampling type operators by different choices of the kernel ψ. For instance, taking $\psi := \delta$, where δ is the Dirac delta distribution, the operators reduce to those of the form (1) (see, e.g., [18]). For the sake of brevity, let us give directly an example of kernel defined by a linear combination of shifted B-splines σ_3 which realizes an order of approximation up to 3 and which in addition has been constructed in such a way as to predict the function at a certain instant. To this aim, we can take as φ the bivariate version of the following kernel

$$\tau_{\sigma_3}(x) := \frac{1}{8}\{47\sigma_3(x-2) - 62\sigma_3(x-3) + 23\sigma_3(x-4)\}, \quad x \in \mathbb{R}.$$

Indeed, as it is proved in [5], the support of τ_{σ_3} is the interval $[\frac{1}{2}, \frac{11}{2}]$, which means that through this kernel, we are also able to predict the function $f(t)$ at a certain instant t, by taking into account only five samples $\frac{k}{w}$ for which $t - \frac{11}{2w} < \frac{k}{w} < t - \frac{1}{2w}$. The kernel τ_{σ_3} has been constructed according to Theorem 6.2 of [5]; in particular, its coefficients are the unique solutions of a suitable linear system involving the Fourier transform $\widehat{\sigma_3}$. As made before, the corresponding bivariate version $\mathcal{T}_{\sigma_3}^2$ will be given by the product of two one-dimensional kernels as above (see, e.g., [9]). In conclusion, we give a graphical representation of $S_w^{\mathcal{T}_{\sigma_3}^2,\delta} f$ in the case $w = 10$, with f defined in (8) (see Figure 7).

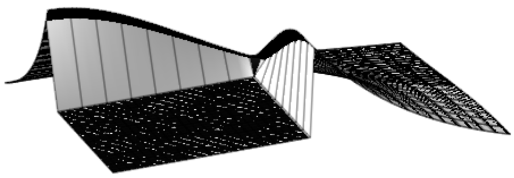

Figure 7: f (black) and $S_{10}^{\mathcal{T}^2_{\sigma_3},\delta} f$ (grey).

For the sake of completeness, we conclude the present subsection by treating a further example of multivariate kernel which is not of product type in general. Therefore, we are going to present and investigate the well-known box spline kernels, first introduced in [26]. Let A be an $n \times m$-matrix ($n \leq m$) with column vectors $A_j \in \mathbb{Z}^n \setminus 0$, $j = 1, 2, \ldots, m$ and $rank(A) = n$. The box spline B_A is defined by

$$\int_{\mathbb{R}^n} B_A(\underline{u}) g(\underline{u}) d\underline{u} = \int_{\mathcal{Q}^m} g\left(A\underline{x}\right) d\underline{x}, \tag{9}$$

where $\mathcal{Q}^m = \left[-\frac{1}{2}, \frac{1}{2}\right]$ is the m-dimensional unit cube and $g \in C(\mathbb{R}^n)$ (see, e.g., [9]). We highlight that, by a suitable choice of the matrix A, the box spline kernel B_A can be regarded as a particular case of B-spline kernel of product type (see Example 1 of [13], pag. 182). It follows that

$$B_A(\underline{u}) \geq 0,\ \underline{u} \in \mathbb{R}^n, \text{ and } supp(B_A) = A\mathcal{Q}^m.$$

Thus, since B_A has compact support, we deduce that $B_A \in L^1(\mathbb{R}^n)$ and the absolute moments $M_\nu(B_A)$ turn out to be finite for every $\nu \geq 0$. If $\rho = \rho(A)$ is the largest integer for which all submatrices generated from A by deleting ρ columns have rank n, then $B_A \in C^{\rho-1}(\mathbb{R}^n)$. Further, it is possible to see that B_A are piecewise polynomials, i.e., polynomial splines of total degree $m - n$. For other details and properties about box spline, one can see [27, 28] and [26].
In order to investigate if also assumption (4) holds, as made in case of the central B-spline kernels considered before, we now recall the Fourier transform of B_A, defined by

$$\begin{aligned}\widehat{B_A}(\underline{v}) &= \frac{1}{(\sqrt{2\pi})^n} \prod_{j=1}^m \operatorname{sinc}\left(\frac{1}{2\pi}\sum_{i=1}^n v_i a_{ij}\right)\\ &= \frac{1}{(\sqrt{2\pi})^n} \prod_{j=1}^m \operatorname{sinc}\left(\frac{\underline{v}\cdot A_j}{2\pi}\right), \qquad \underline{v} \in \mathbb{R}^n,\end{aligned}$$

where a_{ij} are the entries of A and " $\cdot$ " denotes the usual scalar product. Now, if we take as discrete kernel $\varphi(\underline{x}) := (\sqrt{2\pi})^n B_A(\underline{x})$, then $\widehat{\varphi}(\underline{0}) = 1$ and $\widehat{\varphi}(2\pi\underline{k}) = \prod_{j=1}^m \operatorname{sinc}\left(\underline{k}\cdot A_j\right) = 0$, $\underline{k} \in \mathbb{Z}^n \setminus 0$, because the entries of A are integers and $rank(A) = n$. Hence, singularity assumption (4) on φ is satisfied, in view of the equivalence arising from Poisson summation formula. Moreover, since $\|B_A\|_1 = 1$, we may choose the box spline kernel also as continuous kernel ψ. In summary, in view both of the moment condition $M_\nu(B_A) < +\infty$, $\nu \geq 0$, and kernels assumption (4), the box spline kernel B_A turns out to be suitable to obtain all the convergence and quantitative results presented herein for the multidimensional Durrmeyer operators. Moreover, we point out that it is possible to define linear combinations of shifted box spline kernels in order to enhance the order of approximation, similarly as in the case of the above product kernels. The coefficients in the linear combinations are again given by the solutions of a system of linear equations involving the Fourier transform (see, e.g., Theorem 5.3 of [13]). In the case of Durrmeyer operators, we remark again that the latter method holds if ψ is the Dirac delta distribution, that corresponds to considering operators of the form (1).

6.2 Radial type kernels

An important example of radial type kernel is represented by the so called Bochner-Riesz kernel of order $N > 0$, defined as follows:

$$r_N^n(\underline{x}) := \frac{2^N}{\sqrt{(2\pi)^n}}\Gamma(N+1)\|\underline{x}\|_2^{-N-n/2} J_{N+n/2}\left(\|\underline{x}\|_2\right), \ \underline{x} \in \mathbb{R}^n,$$

for $N > (n-1)/2$, where J_λ is the Bessel function of order λ and Γ is the usual Euler gamma function.
Observing that the corresponding Fourier transform is given by

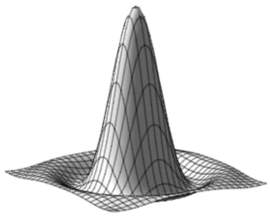

Figure 8: The graph of the bivariate Bochner-Riesz kernel of order $N = 1$.

$$\widehat{r_N^n}(\underline{v}) = \begin{cases} (1-\|\underline{v}\|_2^2)^N, & \|\underline{v}\|_2 \le 1, \\ 0, & \|\underline{v}\|_2 > 1, \end{cases} \qquad \underline{v} \in \mathbb{R}^n,$$

we can deduce that the Bochner-Riesz kernel r_N^n belongs to the class B_π, i.e., the class of all function $g \in L^1(\mathbb{R}^n)$ which are entire functions of the exponential type π (see, e.g., [13]). Moreover, it is easy to see that the discrete absolute moment $M_0(r_N^n)$ is finite, in view of Remark 3. Thus, all the assumptions made in sections 3 and 4 are satisfied, while those in Section 5 turn out to be fulfilled if $N > \frac{3}{2}$. The latter follows by arguing similarly as in Section 5 of [8].
Now, we put $\varphi := r_N^n$ and again $\psi := \chi_n$, the characteristic function of the hypercube $[0,1]^n$ previously defined.
Now, considering again the two-dimensional framework ($n = 2$), we take as φ the bivariate Bochner-Riesz kernel of order $N = 1$, briefly denoted by r_1^2, and χ_2 as ψ (see Figure 8). Thus, we apply the corresponding bivariate operator $S_w^{r_1^2,\chi_2}$ to the function f defined in (8), in both the cases $w = 5$ and $w = 10$ in the same octant of the plane (see Figure 9).

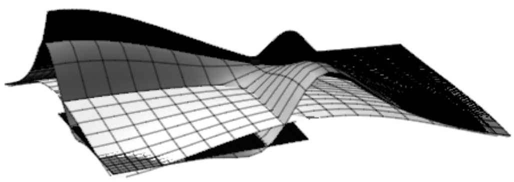

Figure 9: f (black), $S_5^{r_1^2,\chi_2} f$ (grey) and $S_{10}^{r_1^2,\chi_2} f$ (dark grey).

Other examples of radial kernels are the so-called Wendland kernels [44]. These compactly supported kernels emulate in one-dimensional case the central B-splines both in terms of the smoothness properties and the shape of the kernels themselves. Now, we briefly recall the definition. We denote by

$$\mathcal{W}_{l,0}(\rho) := (1-\rho)_+^l, \qquad \rho = \|\underline{x}\|_2,$$

$\underline{x} \in \mathbb{R}^n$, being $l \in \mathbb{N}$ the power of the positive part of the radial function $1-\rho$. Then, the Wendland kernels based upon parameters k and l, are defined recursively by the following radial functions:

$$\mathcal{W}_{l,k+1}(\rho) := \int_\rho^{+\infty} u\mathcal{W}_{l,k}(u)du, \qquad k = 0, 1, \dots, \tag{10}$$

with $\rho = \|\underline{x}\|_2$, $\underline{x} \in \mathbb{R}^n$. It is easy to see that $\mathcal{W}_{l,k+1}$ are compactly supported kernels, since $supp\left(\mathcal{W}_{l,k+1}\right) \subset [-1,1]^n$ if we consider $\mathcal{W}_{l,k+1}$ as a function of $\underline{x}$. This leads to state that $\mathcal{W}_{l,k+1}(\underline{x}) \in L^1(\mathbb{R}^n)$ and the absolute moments $M_\nu(\mathcal{W}_{l,k+1})$ turn out to be finite for every $\nu \geq 0$. Among the best known classes of Wendland kernels, we can find the following instances

$$\mathcal{W}_{1,0}(\rho) = (1-\rho)_+,$$

or

$$\mathcal{W}_{3,1}(\rho) = (1-\rho)_+^4(4\rho+1),$$

especially used in the image processing (see, e.g., [25]). In conclusion, we may use this type of radial kernels as continuous kernel ψ in order to obtain the previous results for multivariate Durrmeyer operators, by setting $\psi := C\mathcal{W}_{l,k+1}$, where $C > 0$ is a suitable normalization constant.

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Conflict of interest/Competing interests

The authors declare that they have no conflict of interest and competing interest.

Availability of data and material and Code availability

Not applicable.

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