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Non standard properties of m-subharmonic functions

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To Norm Levenberg on the occasion of his 60th birthday.

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Abstract

We survey elements of the nonlinear potential theory associated to m-subharmonic functions and the complex Hessian equation. We focus on properties which distinguish m-subharmonic functions from plurisubharmonic ones.

Introduction

Plurisubharmonic functions arose as multidimensional generalizations of subharmonic functions in the complex plane (see [LG]). Thus it is not surprising that these two classes of functions share many similarities. There are however many subtler properties which make a plurisubharmonic function in $\mathbb{C}^n, n > 1$ differ from a general subharmonic function. Below we list some of the basic ones:

Liouville type properties. it is known ([LG]) that an entire plurisubharmonic function cannot be bounded from above unless it is constant. The function $u(z) = \frac{-1}{||z||^{2n-2}}$ in $\mathbb{C}^n$, $n > 1$ is an example that this is not true for subharmonic ones;

Integrability. Any plurisubarmonic function belongs to L^p_{loc} for any $1 \leq p < \infty$. For subharmonic functions this is true only for $p < \frac{n}{n-1}$ as the function u above shows.

Symmetries. Any holomorphic mapping preserves plurisubharmonic functions in the sense that a composition of a plurisubharmonic function with a holomporhic mapping is still plurisubharmonic. This does not hold for subharmonic functions in $\mathbb{C}^n, n > 1$.

The notion of m-subharmonic function (see [Bl1], [DK2, DK1]) interpolates between subharmonicity and plurisubharmonicity. It is thus expected that the corresponding nonlinear potential theory will share the joint properties of potential and pluripotential theories.

Indeed in the works of Li, Blocki, Chinh, Abdullaev and Sadullaev, Dhouib and Elkhadhra, Nguyen and many others the m-subharmonic potential theory was thoroughly developed. In particular S. Y. Li [Li] solved the associated smooth Dirichlet problem under suitable assumptions, proving thus an analogue of the Caffarelli-Nirenberg-Spruck theorem [CNS] who dealt with the real setting. Z. Blocki [Bl1, Bl3] noted that the Bedford-Taylor apparatus from [BT1] and [BT2] can be adapted to m-subharmonic setting. He also described the domain of definition of the complex Hessian operator. L. H. Chinh developed the variational apporach to the complex Hessian equation [Chi1] and studied the associated viscosity theory of weak solutions in [Chi3]. He also developed the theory of m-subharmonic Cegrell classes [Chi1, Chi2]. Abdullaev and Sadullaev in [AS] defined the corresponding m-capacities (this was done also independently by Chinh in [Chi2] and the authors in [DK2]). A. Dhouib and F. Elkhadhra investigated m-subharmonicity with respect to a current [DE] and noticed several interesting phenomena. N. C. Nguyen in [N] investigated existence of solutions to the Hessian equations if a subsolution exists.

Arguably the most interesting part of the theory is the one that **differs** from its pluripotential counterpart. This involves not only new phenomena but also requires new tools. Obviously there are good reasons for such a discrepancy. The very notion of plurisubharmonicity is independent of the Kähler metric in sharp contrast to m-subharmonicity. The fundamental solution for the m-Hessian equation is $-\frac{1}{|z|^{2\frac{n}{m}-2}}$, hence there is stronger than logarithmic singularity at the origin and the function is bounded at infinity. Also it is only L^p integrable for $p < \frac{nm}{n-m}$.

The goal of this survey note is to gather such distinctive results for m-subharmonic functions. Our choice is of course subjective and we do not cover many important issues such as m-polar sets or m-subharmonic functions on compact manifolds. First we deal with the symmetries of m-sh functions. We show in particular that the set these symmetries coincides with the set of holomorphic and antiholomorphic orthogonal affine maps for any $1 < m < n$ in sharp contrast to the borderline cases. We also investigate the analogues of upper level sets of Lelong numbers. Following the arguments of Harvey and Lawson ([HL1]) and Chu ([Ch]) we present the proof of the stunning fact that the upper level sets are discrete for $m < n$. This again is drastically different from the plurisubharmonic case where Siu's theorem implies the analyticity of such sets when $m = n$.

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The note is organized as follows: the basic notions and tools are listed in Section 1. In particular we have covered the linear algebraic and potential theoretic properties of m-sh functions. We have also included a fairly brief subsection devoted to weak solutions of general elliptic PDEs. The first part of Section 2 is devoted to the symmetries of m-sh functions. In the second one we construct a particular nonlinear operator $\mathcal{P}_m$. We show that all m-sh functions are subsolutions for $\mathcal{P}_m$ and, more importantly, $\mathcal{P}_m$ has the same fundamental solution as the m-Hessian operator. We wish to point out that $\mathcal{P}_m$ is an example of a much more general construction of an uniformly elliptic operator with the same *Riesz characteristic* as defined by Harvey and Lawson (see [HL1]). Finally in Section 3 we investigate the upper level sets of analogues of Lelong numbers of m-sh functions. This section depends on the general agruments of Harvey and Lawson ([HL1, HL2]) and Chu ([Ch]). As we deal with the concrete case of m-sh functions our argument is slightly simpler but the main ideas are the same.

Dedication. It is our pleasure to dedicate this article to Norm, a great friend and mathematician.

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1 Preliminaries

In this section we recall the notions and tools appearing in the potential theory of m-subharmonic functions.

1.1 Linear algebra.

Denote by $\mathcal{M}_n$ the set of all Hermitian symmetric $n \times n$ matrices. Fix a matrix $M \in \mathcal{M}_n$. By $\lambda(M) = (\lambda_1, \lambda_2, ..., \lambda_n)$ denote its eigenvalues arranged in the decreasing order.

Definition 1.1. The m-th symmetric polynomial associated to M is defined by

$$S_m(M) = S_m(\lambda(M)) = \sum_{0<j_1<...<j_m\le n} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_m}.$$

We recall that $S_1(M)$ is the trace of M, whereas $S_n(M)$ is the determinant of M.

Next one can define the positive cones Γ_m as follows

$$\Gamma_m = \{\lambda \in \mathbb{R}^n |\ S_1(\lambda) > 0,\ \cdots,\ S_m(\lambda) > 0\}. \tag{1.1}$$

The following two properties of these cones are classical:

1. (Maclaurin's inequality) If $\lambda \in \Gamma_m$ then $(\frac{S_j}{\binom{n}{j}})^{\frac{1}{j}} \ge (\frac{S_i}{\binom{n}{i}})^{\frac{1}{i}}$ for $1 \le j \le i \le m$;
2. (Gårding's inequality, [Ga]) Γ_m is a convex cone for any m and the function $S_m^{\frac{1}{m}}$ is concave when restriced to Γ_m;

We refer the Reader to [Bl1] or [W] for further properties of these cones.

1.2 Potential theoretic aspects of m-subharmonic functions.

We restirct our considerations to a relatively compact domain $\Omega \subset \mathbb{C}^n$. We assume $n \ge 2$ in what follows.

Denote by $d = \partial + \bar\partial$ and
$d^c := i(\bar\partial - \partial)$ the standard exterior differentiation operators. By $\beta := dd^c|z|^2$ we denote the canonical Kähler form in $\mathbb{C}^n$.

We now define the smooth m-subharmonic functions.

Definition 1.2. Given a $\mathcal{C}^2(\Omega)$ function u we call it m-subharmonic in Ω if for any $z \in \Omega$ the Hessian matrix $\frac{\partial^2 u}{\partial z_i \partial \bar z_j}(z)$ has eigenvalues forming a vector in the closure of the cone Γ_m.

The geometric properties of the eigenvalue vector can be stated more analytically in the language of differential forms: u is m-subharmonic if and only if the following inequalities hold:

$$(dd^c u)^k \wedge \beta^{n-k} \ge 0,\ k = 1,\ \cdots,\ m.$$

Note that these inequalities depend on the background Kähler form β if $n - k \ge 1$. Thus it is meaningful to define m-subharomicity with respect to a general Kähler form ω (see [DK1] for details). In this survey however we shall deal only with the standard Kähler form β.

In ([Bl1]) Z. Błocki proved, one can relax the smoothness requirement on u and develop a non linear version of potential theory for Hessian operators just as Bedford and Taylor did in the case of plurisubharmonic functions ([BT1], [BT2]).

In general m-sh functions are defined as follows:

Definition 1.3. Let u be a subharmonic function on a domain $\Omega \in \mathbb{C}^n$. Then u is called m-subharmonic (m-sh for short) if for any collection of $\mathcal{C}^2$-smooth m-sh functions $v_1,\ \cdots,\ v_{m-1}$ the inequality

$$dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \ge 0$$

holds in the weak sense of currents.

The set of all $m-\omega$-sh functions is denoted by $\mathcal{SH}_m(\Omega)$.

Remark 1. In the case $m = n$ the m-sh functions are simply plurisubharmonic ones. Also it is enough to test m-subharmonicity of u against a collection of m-sh quadratic polynomials (see [Bl1]).

Using the approximating sequence u_j from the definition one can follow the Bedford and Taylor construction from [BT2] of the wedge products of currents given by locally bounded m-sh functions. They are defined inductively by

$$dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge \beta^{n-m} := dd^c(u_1 \wedge \cdots \wedge dd^c u_p \wedge \beta^{n-m}).$$

It can be shown (see [Bl1]) that analogously to the pluripotential setting these currents are continuous under monotone or uniform convergence of their potentials.

Given an m-sh function u one can always construct locally a dereasing sequence of smooth m-sh approximants through the standard regularizations $u * \rho_\varepsilon$ with ρ_ε being a family of smooth mollifiers.

Unlike classical elliptic PDEs one cannot apply the maximum principle for m-sh functions directly as we deal with non-smooth functions in general. Instead one can use the so-called *comparison principles* which are standard tools in pluripotential theory. Their proofs follow essentially from the same arguments as in the plurisubharmonic case $m = n$ (see [K]):

Theorem 1.1. *Let u, v be continuous m-sh functions in a domain $\Omega \subset \mathbb{C}^n$. Suppose that $\liminf_{z\to\partial\Omega}(u-v)(z) \geq 0$ then*

$$\int_{\{u<v\}} (dd^c v)^m \wedge \beta^{n-m} \leq \int_{\{u<v\}} (dd^c u)^m \wedge \beta^{n-m}.$$

Theorem 1.2. *Let u, v be continuous m-sh functions in a domain $\Omega \subset \mathbb{C}^n$. Suppose that $\liminf_{z\to\partial\Omega}(u-v)(z) \geq 0$ and $(dd^c v)^m \wedge \beta^{n-m} \geq (dd^c u)^m \wedge \beta^{n-m}$. Then $v \leq u$ in Ω.*

In particular the Dirichlet problem associated to the m-Hessian operator can have at most one solution. As for the existence we have the following fundamental existence theorem due to S. Y. Li ([Li]):

Theorem 1.3. *Let Ω be a smoothly bounded relatively compact domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is $(m-1)$-pseudoconvex (that means that the Levi form at any point $p \in \partial\Omega$ has its $n-1$ eigenvalues in the cone Γ_{m-1}). Let φ be a smooth function on $\partial\Omega$ and f a strictly positive and smooth function in the closure of Ω. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\bar{\Omega}); \\ (dd^c u)^m \wedge \beta^{n-m} = f; \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a smooth solution u.

m-sh functions, just as plurisubharmonic ones may not be bounded from below. Indeed it can be checked by direct computation that the function

$$G(z) := -\frac{1}{|z|^{2n/m-2}} \tag{1.2}$$

is m-sh on $\mathbb{C}^n$ if $m < n$ and is obviously unbounded at 0. It can also be checked that the Hessian measure

$$(dd^c G)^m \wedge \beta^{n-m}$$

is up to a normalizing constant equal to the Dirac delta at zero.

In pluripotential theory there are different tools for measuring the pointwise singularities of plurisubharmonic functions. Among the basic ones (see [De]) is the *Lelong number*:

Definition 1.4 (Lelong number)**.** Let u be a plurisubharmonic function defined in a neighbourhood of a point $z_0 \in \mathbb{C}^n$. Then the limit $lim_{r\to 0^+}$ of the quantity

$$\int_{|z-z_0|\leq r} dd^c u \wedge (dd^c log|z-z_0|)^{n-1} = \frac{1}{r^{2n-2}} \int_{|z-z_0|\leq r} dd^c u \wedge \beta^{n-1}$$

is called a Lelong number of the function u at z_0.

Remark 2. There are many other equivalent definitions of the Lelong numbers. We have chosen this one since it is easily adjustable to m-sh functions.

Note that unless u is unbounded near z_0 the Lelong number vanishes. This is however not a sufficient condition as the plurisubharmonic function

$-log(-log|z|)$ near zero shows. Intuitively speaking the Lelong number measures whether u has *logarithmic* singularity at z_0- these are the strongest singularities that plurisubharmonic functions can have.

The equality (whose proof can be found in [De]) in particular implies that the quantity $\frac{1}{r^{2n-2}} \int_{|z-z_0|\leq r} dd^c u \wedge \beta^{n-1}$ (which is up to a universal multiplicative constant equal to $\frac{1}{r^{2n-2}} \int_{|z-z_0|\leq r} \Delta u$) is increasing with r. This implies that the set

$$E_c(u) := \{z | u \text{ has a Lelong number at least } c \text{ at } z\}$$

is small for any $c > 0$. More precisely for any $\varepsilon > 0$ it has zero $2n - 2 + \varepsilon$ Hausdorff measure.

It turns out however that more is true: a deep theorem of Siu [S] states that the sets $E_c(u)$ are always analytic for $c > 0$:

Theorem 1.4 (Siu)**.** *Let u be a plurisubharmonic function in a domain $\Omega \subset \mathbb{C}^n$. Then for any $c > 0$ the set $E_c(u)$ is an analytic subset of Ω.*

Returning to m-sh functions the following definition of an m-sh Lelong number is natural:

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Definition 1.5 (m-sh Lelong number)**.** let u be an m-sh function defined in a neighbourhood of a point $z_0 \in \mathbb{C}^n$. Then the limit $lim_{r\to 0^+}$ of the quantity

$$\int_{|z-z_0|\le r} dd^c u \wedge (dd^c \frac{-1}{|z-z_0|^{2n/m-2}})^{m-1} \wedge \beta^{n-m}$$

is called the m-sh Lelong number of the function u at z_0.

Just as in the plurisubharmonic case integration by parts implies the equality

$$\int_{|z-z_0|\le r} dd^c u \wedge (dd^c \frac{-1}{|z-z_0|^{2n/m-2}})^{m-1} \wedge \beta^{n-m} = \frac{1}{r^{2n-2n/m}} \int_{|z-z_0|\le r} dd^c u \wedge \beta^{n-1}. \tag{1.3}$$

In particular the latter quantity is increasing in r.

We also define the level sets of m-sh Lelong numbers:

Definition 1.6. let u be an m-sh function defined in a domain $\Omega \subset \mathbb{C}^n$. We define the set

$$E_c^m(u,\Omega) = E_c^m(u) := \{z \in \Omega | u \text{ has a } m-\text{sh Lelong number at least } c \text{ at } z\}.$$

The properties of the set $E_c^m(u)$ will be studied in Section 3.

Definition 1.7. An m-sh function u is called maximal in a domian $\Omega \subset \mathbb{C}^n$ if for every m-sh function v the implication $v \le u$ off a compact subset K of Ω implies $v \le u$ in the whole Ω.

Remark 3. For $m = 1$ maximal subharmonic functions are of course the harmonic ones. For $m > 1$ any *bounded* maximal m-sh function u satisfies

$$(dd^c u)^m \wedge \beta^{n-m} = 0.$$

On the other hand there exist maximal m-sh functions for which the m-Hessian measure is not well defined.

1.3 Uniformly elliptic PDEs

Hessian equations just like the Monge-Ampère one are examples of *degenerate elliptic equations*. Recall that an equation

$$S(x, D^2u(x)) = 0$$

is said to be degenerate elliptic at $(x, D^2u(x))$ if for any positive semidefinite Hermitian matrix $M \ge 0$ one has

$$S(x, D^2u(x) + M) \ge 0. \tag{1.4}$$

The operator is degenerate elliptic for a class $\mathcal{F}$ of functions if it is degenerate elliptic for any $u \in \mathcal{F}$. Note that complex m-Hessian equation is degenerate elliptic for the class of m-subharmonic functions.

A great deal of problems in the regularity theory of the solutions to the complex Hessian equations are caused by the lack of more quantitive control in the inequality (1.4). We shall need the notion of a uniformly elliptic operator. We define it in the Hermitian setting which is the one we shall need later on. Recall that $\mathcal{M}_n$ is the set of Hermitian $n \times n$ matrices.

Definition 1.8. An operator $F : \mathcal{M}_n \to \mathbb{R}$ is said to be uniformly elliptic at A if there are constants $c, C > 0$, such that for any matrix $P \ge 0$ one has

$$c \times trace(P) \le F(A+P) - F(A) \le C \times trace(P).$$

The equation $S(x, D^2u(x)) = 0$ is said to be uniformly elliptic if there are constants $c, C > 0$ such that the operator $S(x, \cdot)$ is uniformly elliptic with constants c, C for any x.

Remark 4. It is straightforward to verify that the Laplacian operator is uniformly elliptic whereas the n-th root of the Monge-Ampère operator is uniformly elliptic at a *fixed* strictly positive matrix A but is not globally uniformly elliptic.

The m-Hessian operator and the associated m-subharmonic functions admit a very effective theory of weak solutions based on pluripotential theory. This however relies on many properties of the Hessian operator, the most basic one being the ability of reformulating the operator in the language of exterior powers of closed differential forms. As this is impossible for a general operator there is no natural way to develop a pluripotential theory for non-smooth solutions of the equation $S(D^2u) = 0$. Instead a very general machinery of *viscosity* solutions can be used. We shall briefly list the most relevant notions and refer to [CC] for the details:

Let Ω be a bounded domain in $\mathbb{C}^n$. Consider the following equation:

$$S[u] := S(x, D^2u(x)) = 0, \tag{1.5}$$

where we assume that S is a uniformly elliptic operator.

Definition 1.9. An upper semicontinuous function $u \in L^{\infty}(\Omega)$ is a viscosity subsolution of (1.5) if for any $z_0 \in \Omega, q \in C^2(\Omega)$ satisfying $u \leq q$, $u(z_0) = q(z_0)$, we have $S[q](z_0) \geq 0$. We also say that $S[u] \geq 0$ in the viscosity sense and q is an upper (differential) test for u at z_0.

A lower semicontinuous function $v \in L^{\infty}(\Omega)$ is a viscosity supersolution of equation (1.5) if for any $z_0 \in \Omega, q \in C^2(\Omega)$ satisfying $u \geq q$, $u(z_0) = q(z_0)$, we have $S[q](z_0) \leq 0$. We also say that $S[u] \leq 0$ in the viscosity sense and q is a lower (differential) test for u at z_0.

A viscosity solution is a function which is both a sub- and supersolution.

Remark 5. In [CC] it is already assumed that sub- and supersolutions are continuous. In our applications it is however important to relax these conditions. Note however that viscosity solutions are by definition continuous.

Remark 6. If S is merely degenerate elliptic the analogous definition still works. However if S is elliptic with respect to a class of functions $\mathcal{F}$ (such as the m-Hessian operator) the subsolution is defined as above but the notion of a supersolution is much more subtle- we refer to [DDT] for details.

We state the following important result about solutions of uniformly elliptic equations without proof. We refer to [CC] for the details:

Theorem 1.5. *Let the locally bounded function u solve in viscosity sense the uniformly elliptic equation*

$$S(D^2u) = 0.$$

Then $u \in \mathcal{C}^1$ and the local Lipschitz constant depends on S and the domain where u is defined. In particular the set of locally uniformly bounded solutions is (locally) equicontinuous.

Just like for m-sh and plurisubharmonic functions one can naturally coin a notion of maximality for any elliptic operator (see [HL1]):

Definition 1.10. An upper semicontinuous function u defined on a domain $\Omega \subset \mathbb{C}^n$ is said to be maximal with respect to an operator $S(x,\cdot)$ if it is a viscosity subsolution, and for any compact set $K \subset \Omega$ and any subsolution v if $v \leq u$ on $\Omega \setminus K$ then $v \leq u$ in Ω.

Note that maximal functions need not be (viscosity) solutions to the equation $S(x, D^2u(x)) = 0$, since a priori they are merely upper semicontinuous. It follows from standard arguments however that if u is a viscosity solution to $S(x, D^2u(x)) = 0$, then it is maximal with respect to S.

All this is in line with standard pluripotential theory where locally bounded maximal plurisubharmonic functions have vanishing Monge-Ampère measures.

The following theorem shows that if S is uniformly elliptic, then locally bounded maximal functions are continuous (in sharp contrast to pluripotential theory). This suffices to show that they are indeed *viscosity solutions*.

Theorem 1.6. *Let $S(x,\cdot)$ be an uniformly elliptic operator. Then any locally bounded maximal function u is continuous. Additionally it is a solution to $S(x, D^2u(x)) = 0$ in the viscosity sense.*

Proof. We follow the argument from [HL1]. Assume that u is a function defined on a domain Ω which is locally bounded and maximal there. Fix a closed ball $\overline{B} = \overline{B}_r(z)$ contained in Ω. There is a sequence $\varphi_j \in \mathcal{C}(\partial B)$ decreasing on ∂B to $u|_{\partial B}$. Let $u_j \in \mathcal{C}(\overline{B})$ be the Perron envelope defined by

$$u_j(x) = sup\{v(x) | v \text{ is } S-\text{subsolution, } limsup_{w\to w_0 \in \partial B} v(w) \leq \varphi_j(w_0)\}.$$

Using the Perron method (see [HL1]) in a standard way it is easy to check that $S(x, D^2u_j(x)) = 0$ in the viscosity sense in the ball B. Furthermore $u_j \in \mathcal{C}(\overline{B})$ and $u_j|_{\partial B} = \varphi_j$. But then u_j is maximal, hence $u \leq u_j$. Also u_j form a decreasing sequence of continuous subsolutions, so the limit $v(x) = lim_{j\to\infty} u_j(x)$ is a subsolution with respect to S. Therefore $u \leq v$ on $\overline{B}$ with equality in ∂B. Thus the function

$$\tilde{v} = \begin{cases} u \text{ on } \Omega \setminus \overline{B}; \\ v \text{ on } \overline{B} \end{cases}$$

is a global S-subsolution, such that $\tilde{v} \leq u$ off a compact set. By the maximality of u it follows that $u = v$ in $\overline{B}$, hence u is locally a decreasing limit of continuous S-solutions.

All this works for plurisubharmonic functions and the Monge-Ampère operator as well. Now we invoke Theorem 1.5 which says in particular that the sequence u_j is equicontinuous on $B_{\frac{r}{2}}(z)$. Hence the limit u has to be continuous in $B_{\frac{r}{2}}(z)$. The ball B was chosen arbitrarily, hence $u \in \mathcal{C}(\Omega)$.

It remains to show that u is a supersolution in Ω. Suppose the contrary. Then there exists a point $z_0 \in \Omega$ and a $\mathcal{C}^2$ smooth function q such that $u \geq q$ with equality at z_0 and $S(z_0, D^2q(z_0)) > 0$. Note that $q_\varepsilon(z) := q(z) - \varepsilon|z - z_0|^2$ will also be a lower differental test for u at z_0 and taking ε sufficiently small we still have $S(z_0, D^2q_\varepsilon(z_0)) > 0$. By the smoothness of q_ε this inequality remains true at least in a small ball $B_\delta(z_0)$ for some $\delta > 0$. On $\partial B_\delta(z_0)$ a strict inequality $u > q_\varepsilon$ holds, hence there is an $\eta > 0$, such that $u \geq q_\varepsilon + \eta$ there but then the function

$$\tilde{q} = \begin{cases} u \text{ on } \Omega \setminus \overline{B}_\delta(z_0); \\ max\{q_\varepsilon + \eta, u\} \text{ on } \overline{B}_\delta(z_0) \end{cases}$$

will be a subsolution which is majorized by u off a compact set. Thus, by the maximality of u once again $\bar{q} \leq u$ everywhere. Then at z_0 we end up with

$$u(z_0) < \eta + u(z_0) = (q - \varepsilon|z - z_0|^2 + \eta)|_{z=z_0} = (\bar{q})(z_0) \leq u(z_0),$$

a contradiction.

Thus u is a continous solution. □

2 Interpolating between subharmonicity and plurisubharmonicity

Up to now we have seen numerous properties shared for all m-sh functions $1 \leq m \leq n$. In this section we will focus on properties which differ from the extreme (i.e. $m = 1$ and $m = n$) cases.

2.1 Symmetries of m-subharmonic functions

The role of plurisubharmonic functions in complex analysis is emphasized by the fact that they are *preserved* by holomorphic mappings- a composition of a plurisubharmonic function and a holomorphic mapping is still plurisubharmonic. In a sense the holomorphic mappings play the role of *symmetries* in pluripotential theory.

Definition 2.1 (Symmetry)**.** A bijective mapping $F : \Omega_1 \to \Omega_2$, $\Omega_1,\ \Omega_2 \subset \mathbb{C}^n$ is called a symmetry for m-sh functions ($1 \leq m \leq n$) if for every m-sh function u defined on Ω_2 the composition $u \circ F$ is m-sh on Ω_1.

A natural question arises how to describe *all* the symmetries of m-sh functions for fixed m. The answer is known for the extreme cases $m = 1$ and $m = n$: namely in the second case all the functions satisfying this property are precisely the holomorphic and antiholomorphic diffeomorphisms, while in the first case all the maps F are described by the following two properties:

1. If $F = (F^1, F^2, \cdots, F^{2n})$ (in real notation), then the (real) Jacobian is an orthogonal matrix at every point,
2. $\triangle F^j = 0,\ j = 1, 2, \cdots, 2n$.

For more details we refer to [BI] or [IM]. We only mention that there exist many functions satisfying the above conditions which are neither holomorphic or antiholomorphic and vice versa.

In general there is no reason why there should be inclusions for the sets of maps preserving m-sh functions. Nevertheless it is interesting to have such a description also for the intermediate cases. One reason for that is the study of global diffeomorphisms between domains or automorphisms of a given domain. In particular if the group of smooth diffeomorphisms preserving m-sh functions in the unit ball was transitive, this would come in handy in proving interior $\mathcal{C}^{1,1}$- estimates for solutions of Monge-Ampère type equations (as in [BT1] for plurisubharmonic functions)- this was pointed out in [Bl1], where however a conjecture was made that this should hold only in the $m = n$ case. The results in this subsection confirm this conjecture. On the other hand it is rather surprising that the group is the same for all intermediate cases $1 < m < n$.

Our first result in this vein is quite unexpected and in drastic contrast with the subharmonic case:

Theorem 2.1. *Let $F = (F^1, F^2, \cdots, F^n) \in \mathcal{C}^2(\mathbb{C}^n, \mathbb{C}^n)$ is a germ of a smooth mapping that preserves r-subharmonic functions for $r > 1$. Then F is holomorphic or antiholomorphic.*

Proof. Let $u \in r$-sh be a smooth function (so $S_1(u) \geq 0,\ S_2(u) \geq 0,\ \cdots,\ S_r(u) \geq 0$). F preserves r-subharmonicty, hence $S_1(u \circ F) \geq 0,\ S_2(u \circ F) \geq 0,\ \cdots, S_r(u \circ F) \geq 0$.

Let us compute S_1. We shall write $u_k := \frac{\partial u}{\partial z_k},\ u_{\bar{k}} := \frac{\partial u}{\partial \bar{z}_k}$.

$$\begin{aligned} S_1(u \circ F)(z) &= \sum_j (u \circ F)_{j\bar{j}}(z) = \sum_j \sum_k (u_k(F(z))F^k_{\bar{j}} + u_{\bar{k}}(F(z))\overline{F}^k_{\bar{j}})_j \qquad (2.1)\\ &= \sum_j (\sum_k u_k F^k_{j\bar{j}} + u_{\bar{k}} \overline{F}^k_{j\bar{j}}) + \sum_j (\sum_{k,s} u_{ks} F^k_{\bar{j}} F^s_j + u_{k\bar{s}} F^k_{\bar{j}} \overline{F}^s_j + u_{\bar{k}s} \overline{F}^k_{\bar{j}} F^s_j \\ &\quad + u_{\bar{k}\bar{s}} \overline{F}^k_{\bar{j}} \overline{F}^s_j) \geq 0. \end{aligned}$$

This should hold for any $u \in r$-sh. Take in particular $u_1 = \Re(a_m z_m)$, where $a_m \in \mathbb{C}$ is an arbitrary constant and $\Re(w)$ stands for the real part of w. Then (2.1) simplifies to

$$\sum_j \frac{1}{2} a_m F^m_{j\bar{j}} + \frac{1}{2} \overline{a_m} \overline{F}^m_{j\bar{j}} = \Re \sum_j a_m F^m_{j\bar{j}} \geq 0.$$

Now, since a_m is arbitrary we get

$$\sum_j F^m_{j\bar{j}} = 0,\ m \in \{1, \cdots, n\}. \qquad (2.2)$$

Take then $u_2 = \Re(a_{ks} z_k z_s)$ (again a_{ks} are complex constants). Then (2.1) (using (2.2)) simplifies to

$$\begin{aligned} &\sum_j \frac{1}{2} a_{ks} F^k_{\bar{j}} F^s_j + \frac{1}{2} \bar{a}_{ks} \overline{F}^k_{\bar{j}} \overline{F}^s_j + \frac{1}{2} a_{ks} F^s_{\bar{j}} F^k_j + \frac{1}{2} \bar{a}_{ks} \overline{F}^k_j \overline{F}^s_{\bar{j}} = \\ &= \Re a_{ks} (\sum_j F^k_{\bar{j}} F^s_j + F^s_{\bar{j}} F^k_j) \end{aligned}$$

and as in (2.2) we get

$$\sum_j F^k_{\bar{j}} F^s_j + F^s_{\bar{j}} F^k_j = 0, \text{ for any } k, s \in \{1, 2, \cdots, n\}, \tag{2.3}$$

so that (2.1) simplifies to

$$\sum_j \sum_{k,s} u_{k\bar{s}} F^k_{\bar{j}} \overline{F}^s_j + u_{\bar{k}s} \overline{F}^k_{\bar{j}} F^s_j \geq 0. \tag{2.4}$$

Now we turn our attention to $S_2(u \circ F)$. By definition

$$S_2(u \circ F) = \sum_{i<j} \{(u \circ F)_{i\bar{i}}(u \circ F)_{j\bar{j}} - (u \circ F)_{i\bar{j}}(u \circ F)_{j\bar{i}}\} = \tag{2.5}$$

$$= \frac{1}{2} \sum_{i,j} \{(u \circ F)_{i\bar{i}}(u \circ F)_{j\bar{j}} - (u \circ F)_{i\bar{j}}(u \circ F)_{j\bar{i}}\} \geq 0.$$

>From (2.4) the first term is equal to

$$\sum_{i,j} \sum_{k,s} \sum_{m,n} (u_{k\bar{s}} u_{m\bar{n}} F^k_{\bar{j}} \overline{F}^s_j F^m_{\bar{i}} \overline{F}^n_i + u_{k\bar{s}} u_{\overline{m}n} F^k_{\bar{j}} \overline{F}^s_j \overline{F}^m_{\bar{i}} F^n_i + \tag{2.6}$$

$$+ u_{\bar{k}s} u_{m\bar{n}} \overline{F}^k_{\bar{j}} F^s_j F^m_{\bar{i}} \overline{F}^n_i + u_{\bar{k}s} u_{\overline{m}n} \overline{F}^k_{\bar{j}} F^s_j \overline{F}^m_{\bar{i}} F^n_i).$$

Before we compute the second term of (2.5) let us make some observations. First of all the only term in (2.5) that does not involve second order derivatives of u is

$$-\sum_{i,j} \sum_k \sum_m (u_k F^k_{i\bar{j}} + u_{\bar{k}} \overline{F}^k_{i\bar{j}})(u_m F^m_{j\bar{i}} + u_{\overline{m}} \overline{F}^m_{j\bar{i}}) \tag{2.7}$$

Note that using again $u_3 := \Re(a_p z_p)$ the quantity (2.7) (and hence (2.5)) simplifies to

$$-\sum_{i,j} \frac{1}{4}(a_p F^p_{i\bar{j}} + \bar{a}_p \overline{F}^p_{i\bar{j}})(a_p F^p_{j\bar{i}} + \bar{a}_p \overline{F}^p_{j\bar{i}}) = -\sum_{i,j} \frac{1}{4} |a_p F^p_{i\bar{j}} + \bar{a}_p \overline{F}^p_{i\bar{j}}|^2 \geq 0,$$

but this clearly implies

$$F^p_{i\bar{j}} = 0 \text{ for all } i, j, p \in \{1, 2, \cdots, n\}. \tag{2.8}$$

Now, due to (2.8) and (2.3) the second term of (2.6) is equal to

$$-\sum_{i,j} \sum_{k,s} \sum_{m,n} (u_{ks} u_{\overline{m}\bar{n}} F^k_{\bar{j}} \overline{F}^s_i \overline{F}^m_{\bar{i}} \overline{F}^n_j + u_{k\bar{s}} u_{m\bar{n}} F^k_{\bar{j}} \overline{F}^s_i F^m_{\bar{i}} \overline{F}^n_j +$$

$$+ u_{\bar{k}s} u_{\overline{m}n} \overline{F}^k_{\bar{j}} F^s_i \overline{F}^m_i F^n_j + u_{\bar{k}\bar{s}} u_{mn} \overline{F}^k_{\bar{j}} \overline{F}^s_i F^m_{\bar{i}} F^n_j).$$

Let us investigate the terms that do not involve mixed derivatives. Take $u_4 := \Re(a_{pq} z_p z_q)$. Note that all terms with mixed derivatives vanish and (2.5) turns into

$$-\sum_{ij} \frac{1}{4} |a_{pq} F^p_{\bar{j}} F^q_i + a_{pq} F^q_{\bar{j}} F^p_i + \bar{a}_{pq} \overline{F}^p_{\bar{j}} \overline{F}^q_i + \bar{a}_{pq} \overline{F}^q_{\bar{j}} \overline{F}^p_i|^2 \geq 0,$$

so

$$F^p_{\bar{j}} F^q_i + F^q_{\bar{j}} F^p_i = 0 \text{ for all } i, j, p, q \in \{1, 2, \cdots, n\}. \tag{2.9}$$

Now put $p = q$ in the above equation. We obtain $F^p_{\bar{j}} F^p_i = 0$. Also if we put $i = j$ we get $F^p_{\bar{j}} F^q_j + F^q_{\bar{j}} F^p_j = 0$. Suppose now that for some p, j we have $F^p_{\bar{j}} \neq 0$. Then we have $F^p_i = 0$ and hence $F^q_i = 0$ for all $i, q \in \{1, 2, \cdots, n\}$, so F is antiholomorphic. Otherwise F is holomorphic and that finishes the proof. □

Remark 7. Note that in the proof we have used in fact only 2-subharmonicity, since all the testing functions were plurisubharmonic (pluriharmonic in fact). Until now we have not used the whole information we have.

Having the above result in mind, we now can describe the diffeomorphisms completely:

Theorem 2.2. *Let F be a C^2 smooth diffeomorphism that preserves m-sh functions, and $1 < m < n$. Then F is holomorphic or antiholomorphic and its complex Hessian is a pointwise orthogonal matrix. Conversely all mappings satisfying these two conditions are symmetries for m-sh functions.*

Proof. We shall work with the assumption that F is holomorphic (the antiholomorphic case is analogous). Before we proceed further recall again that until now we have used only plurisubharmonic testing functions, hence information we gained is not enough to prove the claimed result. Indeed, the picture for plurisubharmonic functions is different (one does not need the orthogonality).

Since $F \in \mathcal{O}$ we have a simplified formula for the complex Hessian of $u \circ F$ for $u \in\ m-sh$ which in matrix notation takes the form

$$[(u \circ F)''(z)]_{i,j} = [\overline{F'(z)}^T]_{i,r}[u''(z)](F(z))_{\bar{r},l}[F'(z)]_{l,j}. \tag{2.10}$$

By the algebraic properties of the k-th symmetric functions of the eigenvalues we know that

$$S_k(A) = S_k(BAB^{-1})$$

for any invertible matrix B. Now since F is a holomorphic diffeomorphism by a classical result in complex analysis we know that F' is pointwise invertible. Using this we get

$$S_k([(u \circ F)''(z)]) = S_k([u''](F(z))[F'(z)][\overline{F'(z)}^T]). \tag{2.11}$$

Take $u(z) := (\sum_{t=1}^k |z_{i_t}|^2) - \frac{1}{k}|z_{i_{k+1}}|^2$. To simplify the notation we assume $\{i_1, \cdots, i_{k+1}\} = \{1, 2, \cdots, k+1\}$ and the general result will follow with obvious modifications.

Note that $u \in k$-sh$\setminus(k+1)$-sh (here $k < n$).

By elementary calculations

$$[u''](F(z))[F'(z)][\overline{F'(z)}^T] = \begin{pmatrix} (F^1, F^1) & \cdots & (F^1, F^n) \\ \vdots & & \vdots \\ (F^k, F^1) & \cdots & (F^k, F^n) \\ -\frac{1}{k}(F^{k+1}, F^1) & \cdots & -\frac{1}{k}(F^{k+1}, F^n) \\ 0 & \cdots & 0 \\ & \vdots & \end{pmatrix},$$

where

$$(F^i, F^j) := \langle \nabla F^i, \nabla F^j \rangle = \sum_{t=1}^n F^i_t \overline{F^j_t}.$$

Denote by $V(F; i_1, \cdots, i_k)$ the Gramm determinant

$$V(F; i_1, \cdots, i_k) = det \begin{pmatrix} (F^{i_1}, F^{i_1}) & \cdots & (F^{i_1}, F^{i_k}) \\ \vdots & & \vdots \\ (F^{i_k}, F^{i_1}) & \cdots & (F^{i_k}, F^{i_k}) \end{pmatrix}.$$

Then we have

$$S_k((u \circ F)''(z)) = V(F; i_1, \cdots, i_k) - \sum_{j=1}^k \frac{1}{k} V(F; i_1, \cdots, \widehat{i_j}, \cdots, i_{k+1}). \tag{2.12}$$

As $S_k((u \circ F)''(z)) \geq 0$ the formula (2.12) implies that $V(F; i_1, \cdots, i_k) = const$ is independent of the choice of $i_1 < i_2 < \cdots < i_k$.

The second testing function we choose is

$$u(z) = |z_{i_1}|^2 + \cdots + |z_{i_k}|^2 + 2\Re(\lambda \frac{1}{\sqrt{k-1}} z_{i_s} \overline{z}_{i_{k+1}}),\ s \in \{1, \cdots, k\},$$

$$\lambda \in \mathbb{C},\ |\lambda| = 1.$$

Again $u \in k$-sh$\setminus(k+1)$-sh for $k < n$. Assume, just as before, that $\{i_1, \cdots, i_{k+1}\} = \{1, 2, \cdots, k+1\}$ and $s = 1$. The Hessian of u is

$$\begin{pmatrix} 1 & \cdots & 0 & \frac{\lambda}{\sqrt{k-1}} & 0 \\ 0 & \ddots & \vdots & 0 & \vdots \\ \vdots & \cdots & 1 & \vdots & \\ \frac{\bar{\lambda}}{\sqrt{k-1}} & 0 & \cdots & 0 & \\ 0 & \cdots & & & 0 \end{pmatrix}$$

and

$$S_k((u\circ F)''(z)) = S_k(\begin{pmatrix} 1 & \cdots & 0 & \frac{\lambda}{\sqrt{k-1}} & 0 \\ 0 & \ddots & \vdots & 0 & \vdots \\ \vdots & \cdots & 1 & \vdots & \\ \frac{\bar{\lambda}}{\sqrt{k-1}} & 0 & \cdots & 0 & \\ 0 & \cdots & & & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} (F^1,F^1) & \cdots & (F^1,F^n) \\ \vdots & & \vdots \\ & & \\ (F^n,F^1) & \cdots & (F^n,F^n) \end{pmatrix}) = S_k(b_{i\bar{j}}),$$

where

$$b_{i\bar{j}} = \begin{cases} (F^1,F^j) + \frac{\lambda}{\sqrt{k-1}}(F^{k+1},F^j), & \text{for } i=1; \\ (F^i,F^j), & \text{for } 1<i\leq k; \\ \frac{\bar{\lambda}}{\sqrt{k-1}}(F^1,F^j), & \text{for } i=k+1; \\ 0, & \text{for } k+1<i\leq n. \end{cases}$$

So one can compute

$$\begin{aligned} S_k((u\circ F)''(z)) &= V(F;1,\cdots,k) \\ &+ \frac{\lambda}{\sqrt{k-1}} det \begin{pmatrix} (F^{k+1},F^1) & \cdots & (F^{k+1},F^k) \\ (F^2,F^1) & & (F^2,F^k) \\ \vdots & & \vdots \\ (F^k,F^1) & \cdots & (F^k,F^k) \end{pmatrix} \\ &- \frac{1}{k-1}\sum_{i=2}^{k} V(F;1,\cdots,\hat{i}\cdots,k+1) \\ &+ \frac{\bar{\lambda}}{\sqrt{k-1}} det \begin{pmatrix} (F^2,F^2) & \cdots & (F^2,F^{k+1}) \\ \vdots & & \vdots \\ (F^k,F^2) & & (F^k,F^{k+1}) \\ (F^1,F^2) & \cdots & (F^1,F^{k+1}) \end{pmatrix}. \end{aligned}$$

By (2.12) and since λ is arbitrary point from the unit circle we get that both determinants (which are complex conjugate to each other) must vanish.

In general we obtain

$$det \begin{pmatrix} (F^{i_{k+1}},F^{i_1}) & \cdots & (F^{i_{k+1}},F^{i_k}) \\ (F^{i_2},F^{i_1}) & & (F^{i_2},F^{i_k}) \\ \vdots & & \vdots \\ (F^{i_k},F^{i_1}) & \cdots & (F^{i_k},F^{i_k}) \end{pmatrix} = 0. \tag{2.13}$$

Note however that $V(F;i_1,\cdots,i_k)\neq 0$ (otherwise $det\ F'(z)=0$ contrary to the fact that F is a holomorphic diffeomorphism).

Summing up

$$A = [(F^i,F^j)]_{i,j=1\cdots,n}$$

is a matrix whose all main k-minors are equal to a nonzero positive constant b and all minors which are formed by rows $i_1,i_2,\cdots,i_k$ and columns $i_2,i_3,\cdots,i_{k+1}$ are 0. This means that the vector

$$v_{k+1} := ((F^{i_{k+1}},F^{i_1}),\cdots,(F^{i_{k+1}},F^{i_k}))$$

belongs to all spaces spanned by $(v_1,\cdots,\hat{v_i},\cdots,v_k)$, where

$$v_s := ((F^{i_s},F^{i_1}),\cdots,(F^{i_s},F^{i_k})).$$

But since $v_1,\cdots,v_k$ are linearly independent we get that v_{k+1} is the zero vector. So all entries of A except those on the diagonal are 0. Those on the diagonal are equal to $\sqrt[k]{b}$ so A is proportional to Id and hence F' is an orthogonal matrix.

On the other hand if F' is orthogonal using (2.11) we get

$$S_k((u\circ F)'') = bS_k([u''](F(z))) \geq 0.$$

□

Using the result above we can easily confirm a conjecture from [Bl1]:

Theorem 2.3. *The group of smooth diffeomorphisms of the unit ball $f \in \mathcal{C}^\infty(\mathbb{B}^n, \mathbb{B}^n)$ preserving m-sh functions is not transitive for $1 \le m < n$.*

Proof. For $m = 1$ the result can be easily deduced from results in [BI] or [IM], as it was mentioned in [Bl1]. For $m > 1$, we know that F must be holomorphic or antiholomorphic. It suffices to check the holomorphic authomorphisms. These automorphisms of the unit ball are known (see e.g. [Ru]) and their form is

$$T_{a,\phi}(z) := U \circ e^{i\phi}\frac{(\frac{\langle z,a\rangle a}{||a||^2}) + \sqrt{1-||a||^2}(z - \frac{\langle z,a\rangle a}{||a||^2})}{1-\langle z,a\rangle} \tag{2.14}$$

where $a \in \mathbb{B}^n$, $\phi \in (0, 2\pi)$ and U is an orthogonal complex linear isomorphism. It is straightforward to check which of these diffeomorphisms have the property of pointwise orthogonal Jacobians: in fact it is enough to check the Jacobian at 0. Since U does not change orthogonality it is sufficient to check this when $U = Id$ is the identity. By elementary calculations

$$T^k_{a,\phi;j}(0) = e^{i\phi}(\frac{\overline{a}_j a_k(1-||a||^2)}{||a||^2} - \frac{\overline{a}_j a_k\sqrt{1-||a||^2}}{||a||^2} + \delta_{j,k}\sqrt{1-||a||^2})$$

with $\delta_{j,k}$ denoting the Cronecker delta. Hence

$$< \nabla T^k(0), \nabla T^s(0) >= \sum_j T^k_j(0)\overline{T^s_j(0)} = \cdots = (1-||a||^2)a_k\overline{a}_s + \delta_{k,s}(1-||a||^2).$$

This should be equal to $C\delta_{k,s}$, and so $a_k\overline{a}_s = 0$ for $k \neq s$, and $|a_k| = \text{const}$. Hence $a = 0$. One can check that $T_{0,\phi}(z) = e^{i\phi}z$ has pointwise orthogonal Jacobians, but these functions are not enough to give transitivity (in paricular they all send 0 to 0). □

Remark 8. In a very recent preprint Ahag, Czyz and Hed ([ACH]) investigated more generally maps from $\mathbb{C}^{n_1}$ to $\mathbb{C}^{n_2}$ which preserve m-sh functions.

2.2 The Laplacian and the Levi form

The Laplacian operator is the canonical operator associated to subharmonic functions. In the plurisubharmonic case there is no such single *linear* operator but instead plurisubharmonicity is characterized in the distributional sense (see [Ho]) through the nonnegativity of the *Levi form* $\mathcal{L}(u;X) \ge 0$, where

$$\mathcal{L}(u;X) := \sum_{i,j}\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}(z)X_i\overline{X}_j,$$

($X \in \mathbb{C}^n$ is an arbitrary vector).

Is there a similar characterization of m-subharmonicity? Below we give a partial answer. We shall construct a linear operator $\mathcal{L}_m$ (mixing the Laplacian and the Levi form) such that every m-sh function u satisfies $\mathcal{L}_m(u,X) \ge 0$. Note however that contrary to the plurisubharmonic case this does *not*
characterize m-subharmonicity.

Theorem 2.4. *Let u be an m-sh $\mathcal{C}^2$ smooth function. Then*

$$(\frac{n}{m} - 1)\Delta u(z)|X|^2 + (n - \frac{n}{m})\mathcal{L}(u(z);X) \ge 0$$

for any $X \in \mathbb{C}^n$.

Remark 9. in the $m = 1$ and $m = n$ cases we discover the natural linear conditions of the Laplacian and Levi form, respectively. Thus such an inequality shows that indeed m-sh functions serve as a bridge between subharmonic and plurisubharmonic functions not only in set-theoretic aspects. The constants in the claim are sharp in the sense that we can get equality if we let m of the eigenvalues of the complex Hessian to be equal to 1, one to be equal to $-\frac{1}{m-1}$ and the rest to be equal to 0.

Proof. Let us fix a point in the domain of u and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the complex Hessian at this point labelled in decreasing order. Without loss of generality we assume $\lambda_n < 0$, otherwise the claim is trivial. Also assume $1 < m < n$. Note that

$$\begin{aligned}&(\frac{n}{m} - 1)\Delta u(z)|X|^2 + (n - \frac{n}{m})\mathcal{L}(u(z);X)\\ &\ge ((\frac{n}{m} - 1)S_1(\lambda_1, \cdots, \lambda_n) + (n - \frac{n}{m})\lambda_n)|X|^2\\ &= ((\frac{n}{m} - 1)S_1(\lambda_1, \cdots, \lambda_{n-1}) + (n-1)\lambda_n))|X|^2,\end{aligned}$$

hence it is enough to prove that the last quantity is nonnegative. Using the inequality

$$S_k(\lambda_1, \cdots, \lambda_n) = \lambda_n S_{k-1}(\lambda_1, \cdots, \lambda_{n-1}) + S_k(\lambda_1, \cdots, \lambda_{n-1}) \ge 0, \tag{2.15}$$

by easy induction we get

$$S_k(\lambda_1, \cdots, \lambda_{n-1}) \ge 0, \text{ for } k \in \{1, \cdots, m\}.$$

Using the Maclaurin inequality twice we obtain

$$S_m(\lambda_1,\cdots,\lambda_{n-1}) \le \frac{\binom{n-1}{m}S_{m-1}^{\frac{m}{m-1}}}{\binom{n-1}{m-1}^{\frac{m}{m-1}}} \le \frac{\binom{n-1}{m}S_{m-1}S_1}{\binom{n-1}{m-1}(n-1)} = S_{m-1}S_1\frac{n-m}{m(n-1)}.$$

Plugging this inequality in (2.15) we obtain

$$\lambda_n + \frac{n-m}{m(n-1)}S_1(\lambda_1,\cdots,\lambda_{n-1}) \ge 0,$$

which is equivalent to the inequality we wanted to prove. □

The inequality that we have shown above holds for *every* vector $X \in \mathbb{C}^n$. Notice also that if $u \in \mathcal{C}^2$ it is satisfied pointwise i.e. it would still hold for any *vector field* defined over the domain of definition of u. Taking the infimum of the Levi form over all unit vectors at each point we construct the following *nonlinear* operator:

Definition 2.2. Let $\mathcal{P}_m$ be an operator defined on $n \times n$ Hermitian matrices by

$$\mathcal{P}_m(A) = \frac{n-m}{n(m-1)}tr(A) + \lambda_{min}(A),$$

where $\lambda_{min}(A)$ denotes the smallest eigenvalue of A.

Remark 10. As defined the operator makes sense for $m > 1$ only. For $m = 1$ we can simply take the trace of A.

Our dicussion above yields the following result:

Proposition 1. Let u be a smooth m-sh function. Then $\mathcal{P}_m(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)) \ge 0$.

Later on we shall need to define $\mathcal{P}_m$ also for nonsmooth functions u. To this end we need to check ellipticity of $\mathcal{P}_m$.

Proposition 2. If $1 < m < n$ then $\mathcal{P}_m$ is a uniformly elliptic operator i.e. for any Hermitan matrix A any Hermitian matrix $M \ge 0$

$$\delta tr(M) \le \mathcal{P}_m(A+M) - \mathcal{P}_m(A) \le (1+\delta)tr(M).$$

Here $\delta = \frac{n-m}{n(m-1)}$.

Proof. The left inequality is simply the fact that $\lambda_{min}(A+M) \ge \lambda_{min}(A)$ while the right one follows from evaluating $\bar{X}^T(A+M)X$ for $X \in \mathbb{C}^n$ being a non zero eigenvector of A associated to its minimum eigenvalue. □

This shows that $\mathcal{C}^2$ smooth m-sh functions are *subsolutions* to the uniformly elliptic equation $\mathcal{P}_m(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)) = 0$. As every locally bounded m-sh function is locally a decreasing limit of smooth m-sh functions it will be a $\mathcal{P}_m$ subsolution in the viscosity sense.

m-sh functions are of course subsolutions to many uniformly elliptic equations, the Laplacian being the most trivial example. The reason why $\mathcal{P}_m$ is special is that the *fundamental solution* $G_m(z) = -\frac{1}{|z|^{\frac{2n}{m}-2}}$ is a solution to $\mathcal{P}_m(G) = 0$ (i.e. it is $\mathcal{P}_m$- maximal) on $\mathbb{C}^n \setminus \{0\}$, as a short computation shows.

Lemma 2.5. *Let $m < n$. Then*

$$\mathcal{P}_m(\frac{\partial^2 G_m}{\partial z_j \partial \bar{z}_k}) = 0$$

on $\mathbb{C}^n \setminus \{0\}$.

Remark 11. In the plurisubharmonic case the problem is that $\mathcal{P}_n(A) = \lambda_{min}(A)$ is not a uniformly elliptic operator anymore.

The next proposition shows that equation (1.3) is applicable not only to m-sh functions but also to $\mathcal{P}_m$ subsolutions:

Proposition 3. Let u be a viscosity subsolution to the equation

$$\mathcal{P}_m(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}) \ge 0.$$

Then

$$dd^c u \wedge (dd^c G_m)^{m-1} \wedge \beta^{n-m} \ge 0$$

in $\mathbb{C}^n \setminus \{0\}$.

Proof. We shall prove the result for smooth u, the general case will follow by approximation. Pick a point $z_0 \in \mathbb{C}^n \setminus \{0\}$. Rotating the coordinates if necessary we may assume that $dd^c G_m$ is diagonal at z_0 i.e.

$$dd^c G_m(z_0) = \theta_1 i dz_1 \wedge d\bar{z}_1 + \cdots + \theta_n i dz_n \wedge d\bar{z}_n,$$

with θ_j's being the eigenvalues of the Hessian matrix of G ordered in the decreasing order. By computation $\theta_1 = \cdots = \theta_{n-1} > 0$, while $\theta_n = (1 - \frac{n}{m})\theta_1 < 0$.

Of course $\beta = \sum_{j=1}^n idz_j \wedge d\bar{z}_j$ regardless of the rotations and we denote $\beta' := \sum_{j=1}^{n-1} idz_j \wedge d\bar{z}_j$. Then

$$\begin{aligned} & dd^c u \wedge (dd^c G_m)^{m-1} \wedge \beta^{n-m}(z_0) \\ &= \theta_1^{m-1} dd^c u \wedge [\beta'^{m-1} - (n/m-1)(m-1) idz_n \wedge d\bar{z}_n \wedge \beta'^{m-2}] \wedge [\beta'^{n-m} \\ &\quad + (n-m) idz_n \wedge d\bar{z}_n \wedge \beta'^{n-m-1}] \\ &= \theta_1^{m-1} dd^c u \wedge [(n/m-1) idz_n \wedge d\bar{z}_n \wedge \beta'^{n-2} + \beta'^{n-1}] \\ &= \theta_1^{m-1}(n-2)![(n/m-1)\sum_{j=1}^{n-1} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} + (n-1)\frac{\partial^2 u}{\partial z_n \partial \bar{z}_n}] dV. \end{aligned}$$

Notice that the latter quantity is nonnegative, since the term in the brackets is equal to

$$\begin{aligned} & (n/m-1)\Delta u(z_0) + (n-n/m)\frac{\partial^2 u}{\partial z_n \partial \bar{z}_n} \\ & \geq (n/m-1)\Delta u(z_0) + (n-n/m)\lambda_{min}(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0)). \end{aligned}$$

□

Integrating by parts just as for m-sh functions we obtain the following corollary:

Corollary 2.6. *For viscosity subsolutions u of $\mathcal{P}_m$ (which are also subharmonic) the quantity*

$$\frac{1}{r^{2n-2n/m}} \int_{|z-z_0|\leq r} \Delta u$$

is increasing in r for any point z_0 in the domain of definition of u.

3 Local behavior near a singularity

Recall the notion of a tangential flow which was coined by Harvey and Lawson (see [HL1, HL2]):

Definition 3.1 (Tangential flow)**.** Let u be a m-sh function defined near a point $z_0 \in \mathbb{C}^n$. For any $r > 0$ the tangential flow at z_0 is defined by

$$\begin{cases} u_{z_0,r}(w) := r^{2n/m-2} u(z_0 + rw) \text{ if } m < n; \\ u_{z_0,r}(w) := u(z_0 + rw) - max_{|w-z_0|\leq r} u \text{ if } m = n. \end{cases} \tag{3.1}$$

Any L^1_{loc} limit of the flow along some subsequence $r_j \searrow 0^+$ is called a tangent at z_0.

Intuitively the tangential flow zooms the domain around z_0. The exponents in the definition are chosen so that in the limit we get the zero function unless u has an essential singularity at z_0. Thus the tangents capture the information about the local behavior of u around a singular point z_0.

Of course any tangent φ is defined as an equivalence class in L^1_{loc}. Taking however the standard upper semicontinuous regularizations

$$\varphi^*(z) := lim_{r\to 0^+} ess\ sup_{B_r(z)} \varphi$$

(with *ess sup* denoting the essential supremum) produces a representative that is an entire m-subharmonic function. Indeed, classical theory (see [Ho], Theorem 3.2.13) shows that the limit is subharmonic, and the nonnegativity of

$$dd^c \varphi \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \wedge \beta^{n-m}$$

for any suitable testing $(1,1)$-forms α_j follows from the m-subharmonicity of $u_{z_0,r}$'s and the continuity of the distributional differentiation with respect to L^1_{loc} convergence.

>From now on by tangent we shall mean the upper semicontinuous representative described above and we shall drop the $*$ sign in the notation.

The next example shows that in the case of plurisubharmonic functions the tangents can have different shapes:

Example 3.1. Let z_0 be the zero vector in $\mathbb{C}^n$. Fix $j \in \{1, \cdots, n\}$ and consider the functions $u_j(z) = log(\sum_{k=1}^j |z_k|^2)$. Then for each u_j the unique tangent at 0 reads

$$\tilde{u}_j(w) = log(\sum_{k=1}^j |z_k|^2)\ (= u_j(w)).$$

The example shows that there are many ways how a plurisubharmonic function may look like around a point with the positive Lelong number.

It is thus quite surprising that in the case of m-sh functions, $m < n$, there is only *one* local model of behavior:

Theorem 3.1 (Strong uniqueness of tangent cones [HL1])**.** *Let u be an m-sh function for some $m < n$ which is defined near $0 \in \mathbb{C}^n$. Then the tangential flow of u converges in L^1_{loc} to the function*

$$-\frac{K}{|z|^{2n/m-2}}$$

for some constant $K \geq 0$.

The constant K is equal to the generalized m-sh Lelong number at 0.

In the following subsections we prove this important result and then collect some of its implications.

3.1 Proof of the strong uniqueness theorem

This subsection is devoted to the proof of Theorem 3.1. The proof consists of several steps. The first one is the monotonicity property of spherical averages.

Definition 3.2. Let u be an m-sh function defined in some ball $B_R(z_0)$. Then for any $r \in (0, R)$ the spherical r-average centered at zero is the quantity

$$S(u, z_0, r) := \frac{1}{\sigma_{2n-1}} \int_{|\xi|=1} u(z_0 + r\xi) dS(\xi), \tag{3.2}$$

with dS denoting the Lebesgue measure on the unit sphere and σ_{2n-1} the total area of the unit sphere in $\mathbb{C}^n$.

Just as in classical pluripotential theory spherical averages are not only increasing with r but also have certain convexity properties.

Proposition 4. Let u be as above for $s \in (-\infty, \frac{-1}{R^{\frac{2n}{m}-2}})$ define the function $r(s) := (-\frac{1}{s})^{m/(2n-2m)}$. Then the function

$$f(s) := S(u, z_0, r(s))$$

is convex.

Proof. The proof is standard. We provide the details for the sake of completeness. We assume that u is smooth (the general case follows from approximation since u is in particular subharmonic and for a sequence of smooth subharmonic approximants u_j we have $lim_{j\to\infty} S(u_j, z_0, r) = S(u, z_0, r)$).

By computation, using Stokes' theorem, and denoting by ξ the real coordinates, we get

$$\begin{aligned} f'(s) &= \frac{r'(s)}{\sigma_{2n-1}} \int_{|\xi|=1} \sum_k \frac{\partial u}{\partial \xi_k}(z_0 + r(s)\xi)\xi_k dS(\xi) \\ &= \frac{r'(s) r(s)}{\sigma_{2n-1}} \int_{|v|\le 1} \Delta u(z_0 + r(s)v) dV(v) \\ &= \frac{r'(s)}{\sigma_{2n-1} r(s)^{2n-1}} \int_{|w-z_0|\le r(s)} \Delta u(w) dV(w) \\ &= \frac{m}{2n-2m} \frac{1}{\sigma_{2n-1} r(s)^{2n-2n/m}} \int_{|w-z_0|\le r(s)} \Delta u(w) dV(w), \end{aligned}$$

where we have used the explicit formula of r to get the last equality. Recalling now equality (1.3) and its implication together with the fact that $r(s)$ is increasing we get that $f'(s)$ is increasing in s. □

Remark 12. Using Corollary 2.6 the same result holds for any $\mathcal{P}_m$ subsolution.

This proposition has a standard application which says that the slopes of secant segments for a convex function are increasing (see [HL1]);

Corollary 3.2. *If* $0 < t_1 < t_2 < R$ *then*

$$\frac{S(u, z_0, t_2) - S(u, z_0, t_1)}{(1/t_1)^{2n/m-2} - (1/t_1)^{2n/m-2}}$$

is increasing in both t_1 *and* t_2.

Another immediate corollary is that the ratio

$$\frac{S(u, z_0, t)}{-(1/t)^{2n/m-2}} \tag{3.3}$$

has a (nonnegative) limit as $t \searrow 0^+$ for any non positive m-subharmonic function in a neighborhood of z_0.

Fix now an m-subharmonic function u defined near a point z_0, which we assume to be the coordinate origin. Without loss of generality assume that u is negative close to the origin. Consider the tangential flow of u at the coordinate center. Then for any sufficiently small r we have

$$S(u_{r,0}, 0, t) = \frac{r^{2n/m-2}}{\sigma_{2n-1}} \int_{|\xi|=1} u(rt\xi) dS(\xi) = \frac{S(u, z_0, rt)}{-(1/rt)^{2n/m-2}} (-1/t^{2n/m-2}).$$

But $\frac{S(u,z_0,rt)}{-(1/rt)^{2n/m-2}}$ converges to some constant K as $r \searrow 0^+$ by the argument above. Hence passing to a limit (along a subsequence) we obtain for any tangent φ at zero

$$S(\varphi, 0, t) = -\frac{K}{t^{2n/m-2}} \tag{3.4}$$

for some constant $K \ge 0$. The constant K is independent of the choice of the subsequence. Note that this equality holds for any $t > 0$ since for any fixed t we can choose r so small so that $u_{r,0}$ is defined near the sphere of radius t centered at zero.

Equality (3.4) implies that φ is a maximal m-subharmonic function on $\mathbb{C}^n \setminus \{0\}$.

Indeed, suppose that there is a function v which is m-subharmonic in $\mathbb{C}^n\setminus\{0\}$ which is not larger than φ except on some compact set K which we assume to be contained in an annular region $A:=\{R_1<|z|<R_2\}$. Consider the function $u(z):=\max\{\varphi(z),v(z)\}$ defined on $\mathbb{C}^n$ (even if v is a priori defined only on $\mathbb{C}^n\setminus\{0\}$). Obviously $S(u,0,t)\geq S(\varphi,0,t)$ for any $t\in(R_1,R_2)$ with equality close to the endpoints of this segment. Note however that $S(\varphi,0,r(p))$ (recall that $r(p):=(-\frac{1}{p})^{m/(2n-2m)}$) is *linear* on $(-\infty,0)$ while $S(u,0,r(p))$ is *convex* in p and matches $S(\varphi,0,r(p))$ ar $r^{-1}(R_1)$ and $r^{-1}(R_2)$.) This is only possible if $S(\varphi,0,t)=S(u,0,t)$ for all $t\in[R_1,R_2]$ which in turn implies $v\leq\varphi$ in K.

Remark 13. The analogous reasoning yields the following stronger statement: φ is maximal for the operator $\mathcal{P}_m$.

The argument up to now works, modulo technical details, also for plurisubharmonic functions.

The argument that follows (taken from [HL1]) is broken into two cases:

Case 1. Assume that φ is locally bounded in $\mathbb{C}^n\setminus\{0\}$.

Let g be any complex rotation of $\mathbb{C}^n$ around the origin. Define

$$u_g(z):=max\{u(z),u(g(z))\}.$$

Observe that u_g is also locally bounded in $\mathbb{C}^n\setminus\{0\}$. It is straightforward to verify that the function $\varphi_g:=max\{\varphi(z),\varphi(g(z))\}$ is tangent to u_g and is also locally bounded on $\mathbb{C}^n\setminus\{0\}$.

Repeating the reasoning above we obtain that φ_g is maximal. At this moment we recall the crucial observation from Remark 13: φ_g is also maximal with respect to $\mathcal{P}_m$!

Now using Theorem 1.6 we get that

$$\mathcal{P}_m\Big(\frac{\partial^2\varphi_g}{\partial z_j\partial\bar{z}_k}\Big)=0 \tag{3.5}$$

in the viscosity sense in $\mathbb{C}^n\setminus\{0\}$. But then Theorem 1.5 implies that φ_g is $\mathcal{C}^1$ on $\mathbb{C}^n\setminus\{0\}$. In particular $\varphi_g=\max\{\varphi(z),\varphi(g(z))\}$ is $\mathcal{C}^1$ on any sphere
$\mathbb{S}_r:=\{z:|z|=r\}$. Observe that this holds for *every* complex rotation. As an easy calculus lemma shows (compare Lemma 13.7 in [HL1]) φ has to be constant on every $\mathbb{S}_r$. This together with equation (3.4) implies that

$$\varphi(z)=-\frac{K}{|z|^{\frac{2n}{m}-2}}$$

and the uniqueness of tangents is proven in this case.

Case 2. If φ is not locally bounded on $\mathbb{C}^n\setminus\{0\}$, then we consider the function

$$u^{(N)}(z):=max\{u(z),-\frac{N}{|z|^{\frac{2n}{m}-2}}\}$$

for each $N\in\mathbb{N}$. It is easy to see that $\varphi^{(N)}(z)=max\{\varphi(z),-\frac{N}{|z|^{\frac{2n}{m}-2}}\}$ is the tangent obtained from the subsequence that yields the tangent φ for u. But $\varphi^{(N)}$ is locally bounded on $\mathbb{C}^n\setminus\{0\}$, hence it is a multiple of the fundamental solution. Taking $N\to\infty$, we see that $\varphi^{(N)}$'s converge decreasingly to φ, hence φ is also a multiple of the fundamental solution.

This finishes the proof.

3.2 Applications of the strong uniqueness theorem

The strong uniqueness theorem has deep implications in the study of singular sets of m-sh functions. The following counterpart of Siu's theorem is due to Harvey and Lawson ([HL1]):

Theorem 3.3 (Harvey-Lawson)**.** *Let u be an m-sh function defined on a domain $\Omega\subset\mathbb{C}^n$. Assume $m<n$. Then the set $E_c^m(u)$ is discrete in Ω for any $c>0$.*

A more quantitive version of this result has been obtained by Chu ([Ch]):

Theorem 3.4 (Chu)**.** *Let u be an m-sh function defined on the ball $B_2(0)\subset\mathbb{C}^n$. Suppose that $||u||_{L^1(B_2(0))}\leq\Lambda$. If $m<n$ then there exists a constant $C=C(m,n,c,\lambda)$ such that*

$$\sharp[E_c^m(u)\cap B_1(0)]\leq C.$$

In the sequel we shall follow the argument of Chu from [Ch]. To this end we need a couple of definitions which are simplified versions of the ones from [Ch].

Definition 3.3. A function $h:\mathbb{C}^n\to\mathbb{R}$ is called homogeneous at a point $z_0\in\mathbb{C}^n$ if the following conditions are satisfied:
(a) h is m-subharmonic;
(b) the tangential flow $h_{r,z_0}(w)$ leaves h invariant i.e. $h_{r,z_0}(w)=h(z_0+w)$ for any $r>0$ and $w\in\mathbb{C}^n$;

The conditions roughly say that h is a model for tangents. Of course once the strong uniqueness for tangents has been established we know that $h(z)=-\frac{K}{|z|^{\frac{2n}{m}-2}}$ but our reasoning below does not depend on this fact.

The next definition captures functions that are close to such models in L^1 sense:

Definition 3.4. A function $u:B_{2r}(z_0)\subset\mathbb{C}^n\to\mathbb{R}$ is said to be (ϵ,r,z_0) homogeneous if there is a homogeneous function h at zero such that

$$||u_{r,z_0}-h||_{B_1(0)}<\epsilon.$$

Now we state a crucial lemma (Lemma 6.5 in [Ch]):

Lemma 3.5. *Let u be an m-sh function on the ball $B_2(0) \in \mathbb{C}^n$. Suppose that $||u||_{L^1(B_2(0))} \leq \Lambda < \infty$. For any $c > 0$ there exists an $\epsilon > 0$ dependent on c, Λ, m and n, such that if u is $(\epsilon, 1, 0)$ homogeneous, then*

$$E_c^m(u) \cap \{1/16 \leq |z| \leq 1/2\} = \emptyset.$$

Proof. The proof proceeds by contradiction. Suppose that for any $j \in \mathbb{N}$ we have a function u_j which is m-subharmonic in the ball $B_2(0)$, such that $||u_j||_{L^1_{B_2(0)}} \leq \Lambda$, u_j is $(1/j, 1, 0)$ homogeneous and there is a point $z_j \in E_c^m(u_j) \cap \{1/16 \leq |z| \leq 1/2\}$.

u_j's are in particular subharmonic, hence form a compact subset with respect to the L^1_{loc} topology. Thus after passing to a subsequence u_j's converge in L^1_{loc} to an m-sh function u. We can also assume, passing once again to a subsequence if necessary, that z_j's converge to some $\tilde{z} \in \{1/16 \leq |z| \leq 1/2\}$. It is easy to see that $\tilde{z} \in E_c^m(u)$. In particular $u(\tilde{z}) = -\infty$.

We claim that u is equal to a homogeneous function on $B_1(0)$. If this is true then u is its own tangent at zero (recall that the tangential flow preserves u). Hence by the strong uniqueness theorem

$$u(w) = -K/|w|^{2n/m-2}$$

for some $K \geq 0$, but this is a contriadiction with $u(\tilde{z}) = -\infty$.

We proceed to prove the claim. By definition there are homogeneous functions h_j such that $||u_j - h_j||_{L^1_{B_1(0)}} \leq 1/j$. Obviously h_j converge to u on $B_1(0)$. We claim that in fact h_j's converge in L^1_{loc} topology to a global homogeneous function h.

To this end note that for any fixed $R > 1$

$$\begin{aligned}
\int_{B_R(0)} |h_i(z) - h_j(z)| dV(z) &= \int_{B_R(0)} |(h_i)_{1/R,0}(z) - (h_j)_{1/R,0}(z)| dV(z) \\
&= \int_{B_R(0)} (1/R)^{2n/m-2} |h_i(z/R) - h_j(z/R)| dV(z) \\
&= \int_{B_1(0)} R^{2n-2n/m+2} |h_i(w) - h_j(w)| dV(w).
\end{aligned}$$

Note that h_j's form a Cauchy sequence in $L^1(B_1(0))$, so the computation above shows that they also form a Cauchy sequence in $L^1(B_R(0))$. Thus extracting a diagonal limit the existence of a global h is shown. Now it is straightforward to check that h is homogeneous.

□

The lemma in particular proves the discreteness of the set $E_c^m(u)$ for any positive c.

Now we are ready to prove Theorem 3.4:

Proof. Without loss of generality we assume that $u \leq 0$ near $B_1(0)$. Denote by S_0 the quantity $\sharp[E_c^m(u) \cap B_1(0)]$.

Following [Ch] we consider a Vitali type covering of $E_c^m(u) \cap B_1(0)$ by balls $B_{1/2}(x_j)$, such that $B_{1/4}(x_j)$ are pairwise disjoint and $x_j \in E_c^m(u) \cap B_1(0)$. Suppose, without loss of generality, that the ball $B_{1/2}(x_1)$ contains the largest number of the points in $E_c^m(u) \cap B_1(0)$. Denote this largest number by S_1.

Two cases may occur: either $S_0 = S_1$ or $S_1 < S_0$. In the latter case recall that the balls in the covering $B_{1/2}(x_j)$ are chosen so that $B_{1/4}(x_j)$'s are disjoint and thus their joint volume is less than the volume of $B_{1+1/4}(0)$. This shows that $S_0 \leq 5^{2n} S_1$.

Assume still that the second case occurs and pick a point $z \in B_{1/2}(x_1)$. If $z \in B_{1/4}(x_1)$ then $B_2(z) \setminus B_{1/4}(z)$ contains a point from $E_c^m(u) \cap B_1(0)$ which is outside $B_{1/2}(x_1)$. If in turn $z \notin B_{1/4}(x_1)$ then $B_2(z) \setminus B_{1/4}(z)$ contains x_1.

In any case we obtain that

$$[B_2(z) \setminus B_{1/4}(z)] \cap E_c^m(u) \cap B_1(0) \neq \emptyset$$

for all $z \in B_{1/2}(x_1)$.

Now we repeat the process by covering $E_c^m(u) \cap B_{2^{-j}}(x_j)$ (x_j being chosen at the previous step as the center of the ball with largest number of points from $E_c^m(u) \cap B_1(0)$) with balls of radius 2^{-j-1} so that the concentric balls of radius 2^{-j-2} are pairwise disjoint.

The discreteness of $E_c^m(u) \cap B_1(0)$ yields that at some stage $S_{j_0} = 1$. S_0 can be estimated by S_{j_0} provided one controls how many times along the process we have $S_{j+1} < S_j$. Obviously if this number is I, then

$$S_0 \leq S_{j_0} 5^{2nI} = 5^{2nI}.$$

Thus we are left with bounding I. Note that from the construction for any j for which $S_j < S_{j-1}$ we have

$$(E_c^m(u) \cap B_1(0)) \cap (B_{2^{-j+2}}(x_{j_0}) \setminus B_{2^{-j-1}}(x_{j_0})) \neq \emptyset. \tag{3.6}$$

Consider now the function $u_{2^{-j+3}, x_{j_0}}(z)$. Each point in $E_c^m(u) \cap B_1(0)$ corresponds to a point in $E_c^m(u_{2^{-j+3}, x_{j_0}}) \cap B_{2^{j-3}}(0)$ after the rescaling.

But then (3.6) implies that

$$E_c^m(u_{2^{-j+3},x_{j_0}}) \cap \{1/16 \leq |z| \leq 1/2\} \neq \emptyset.$$

Invoking Lemma 3.5 we obtain that for some universal ϵ dependent only on c, n, m and $||u_{2^{-j+3},x_{j_0}}||_{L^1(B_2(0))}$ (which is bounded independently of j) the function $u_{2^{-j+3},x_{j_0}}$ is not $(\epsilon, 1, 0)$-homogeneous. But this is impossible for large j as $u_{2^{-j+3},x_{j_0}}$ converges in L^1_{loc} to the tangent.

In conclusion any j such that $S_j < S_{j-1}$ has a universal bound dependent merely on c, n, m and $||u||_{B_2(0)}$ and the proof is complete.

□

References

[ACH] P. Ahag, R. Czyz, L. Hed, The classification of holomorphic (m, n) subharmonic morphisms, preprint arXiv:1806.07756.

[AS] B. Abdullaev B, A. Sadullaev, Potential theory in the class of m-sh functions. *Proc. Steklov Inst. Math.* 279 (2012), 155-180.

[BT1] E. Bedford, B.A. Taylor, The Dirichlet problem for a complex Monge-Ampère equation, *Invent. Math.* 37 (1976), 1-44.

[BT2] E. Bedford, B.A. Taylor, A new capacity for plurisubharmonic functions, *Acta Math.,* 149 (1982), 1-40.

[Bl1] Z. Błocki, Weak solutions to the complex Hessian equation, *Ann. Inst. Fourier (Grenoble)* 55, 5 (2005), 1735-1756.

[Bl3] Z. Błocki, Defining nonlinear elliptic operators for non-smooth functions, Complex Analysis and Digital Geometry proceedings, Uppsala (2009).

[BI] B. Bojarski, T. Iwaniec, Another approach to Liouville theorem, *Math. Nachr.* 107 (1982), 253-262.

[CC] X. Cabre, L. Caffarelli, Fully nonlinear elliptic equations. AMS Colloquium Publications, 43. AMS, Providence, RI, 1995. vi+104 pp. ISBN: 0-8218-0437-5.

[CNS] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, *Acta Math.* 155 (1985), 261-301.

[Ch] J. Chu, Quantitive stratification of F-subharmonic functions, preprint arXiv 1610.09946v2.

[Chi1] L. H. Chinh, A variational approach to complex Hessian equations in $\mathbb{C}^n$. *J. Math. Anal. Appl.* 431 (2015), 228-259.

[Chi2] L. H. Chinh, Solutions to degenerate complex Hessian equations, *J. Math. Pures Appl.* 100 (2013) 785-805.

[Chi3] L. H. Chinh, Viscosity solutions to complex Hessian equations, *J. Funct. Anal.* 264 (2013) 1355-1379.

[De] J,-P.Demailly, Complex Analytic and Differential Geometry (1997), can be found at the www site http://www-fourier.ujf-grenoble.fr/ demailly/books.html

[DE] A. Dhouib, Abir, F. Elkhadhra, m-potential theory associated to a positive closed current in the class of m-sh functions. *Complex Var. Elliptic Equ.* 61 (2016), no. 7, 875-901.

[DDT] S. Dinew, H. S. Do, T. D. To, A viscosity approach to the Dirichlet problem for degenerate complex Hessian type equations, preprint arXiv, 1712.08572, to appear in Analysis PDE.

[DK1] S. Dinew, S. Kołodziej, A priori estimates for complex Hessian equations. *Anal. PDE* 7 (2014), no. 1, 227-244.

[DK2] S. Dinew, S. Kołodziej, Liouville and Calabi-Yau type theorems for complex Hessian equations, *Amer. J. Math.* 139 (2017), no. 2, 403-415.

[Ga] L. Gårding, An inequality for hyperbolic polynomials, *J. Math. Mech.* 8 (1959) 957-965.

[HL1] F. R. Harvey, H. B. Lawson, Tangents to subsolutions- existence and uniqueness, Part I, preprint arXiv 1408.5797v2.

[HL2] F. R. Harvey, H. B. Lawson, Tangents to subsolutions- existence and uniqueness, Part II, J. Geom. Anal. 27 (2017), no. 3, 2190-2223.

[Ho] L. Hörmander, Notions of convexity, Birkhauser, Boston 1994.

[IM] T. Iwaniec, G. Martin, Geometric function theory and non-linear analysis, Clarendon press (2001).

[K] S. Kołodziej, The complex Monge-Ampère equation and pluripotential theory, *Memoirs Amer. Math. Soc.* 178 (2005) 64p.

[LG] P. Lelong, L. Gruman, Entire functions of several complex variables. Grundl. Math. Wiss., 282. Springer-Verlag, Berlin, 1986. xii+270 pp.

[Li] S.-Y. Li, On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian, *Asian J. Math.* 8 (2004), 87-106.

[N] N. C. Nguyen, Subsolution theorem for the complex Hessian equation. *Univ. Iagiellonicae Acta. Math.* 50 (2012), 69-88.

[Ru] W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$. Grundl. Math. Wiss., 241. Springer-Verlag, New York-Berlin, (1980). xiii+436 pp.

[S] Y.-T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. *Invent. Math.* 27 (1974), 53-156.

[W] X.-J. Wang, The k-Hessian equation, *Geometric analysis and PDEs,*177-252, Lect. Not. Math. 1977, Springer, (2009).

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