How to construct your own directional wavelet frame ?

Gerlind Plonka
Institute for Numerical and Applied Mathematics
University of Göttingen

in collaboration with Jianwei Ma

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Outline

- Introduction: well-known wavelet constructions
- Wanted properties of a directional wavelet system
- What can be learned from the one-dimensional case ?
- How to construct curvelets ?
- What are α-molecules ?
- References

Many wavelet (frame) constructions for image analysis

1) Tensor product wavelets
2) steerable wavelets [Freeman and Adelson '91]
3) curvelets [Candes, Donoho '03]
4) shearlets [Labate, Lim, Kutyniok, Weiss '05]
5) contourlets [Do, Vetterli '05]
6) Gabor wavelets [Lee '08]
7) α-molecules [Grohs, Keiper, Kutyniok, Schäfer '14]

Wanted properties of a new wavelet system

What is the purpose of the wavelet system?

We want a representation system $(\psi_\lambda)_{\lambda\in\Lambda}$ for images $f \in L^2(\mathbb{R}^2)$

$$f = \sum_{\lambda\in\Lambda} c_\lambda \, \psi_\lambda$$

that allows a "sparse representation" of the image f.

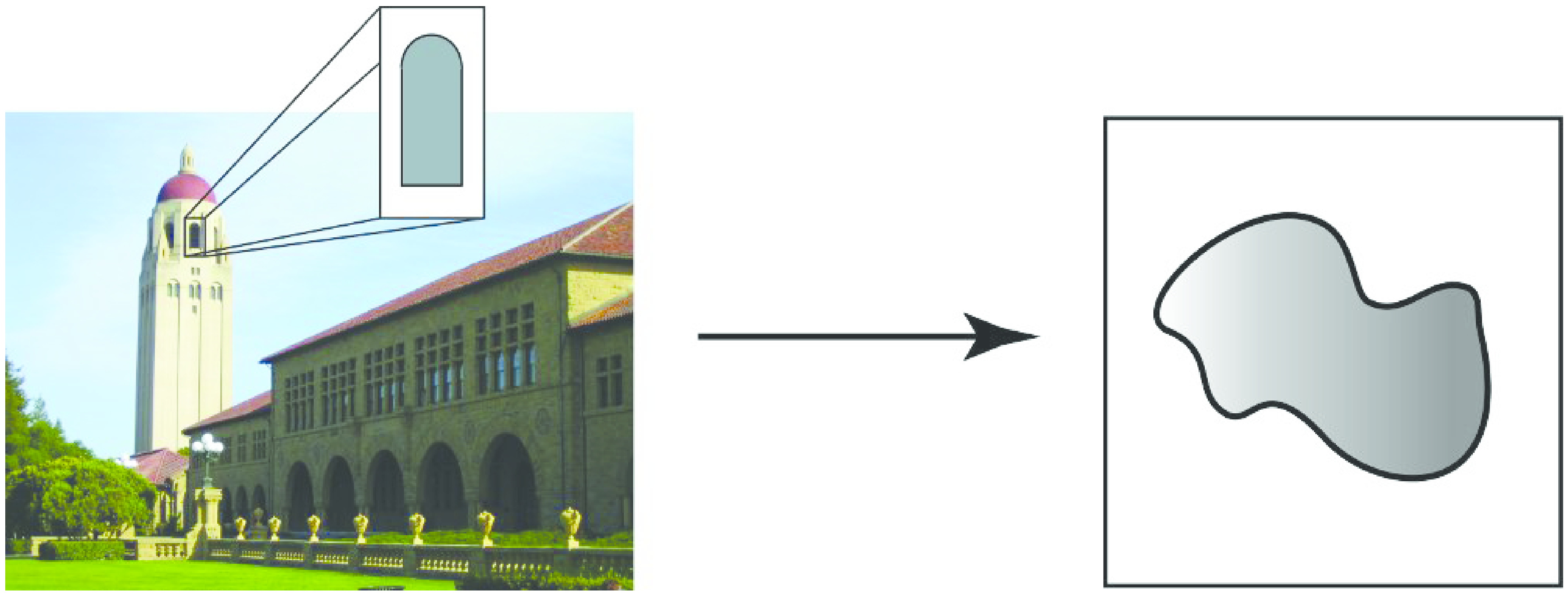
Best N-term approximation $f_N \approx f$

$$f_N = \operatorname{argmin} \left\| f - \sum_{\lambda\in\Lambda_N} c_\lambda \, \psi_\lambda \right\| \quad \text{where} \quad \Lambda_N \subset \Lambda,\ |\Lambda_N| = N.$$

How to model the image data?

How to model the image data ?

Image model: Cartoon-like functions $\mathcal{E}^\beta(\mathbb{R}^2)$ [Donoho '01 ($\beta = 2$)]



Grohs et al '14: The class of cartoon-like functions $\mathcal{E}^\beta(\mathbb{R}^2)$, $\beta \in (1, 2]$, is defined by

$$\mathcal{E}^\beta(\mathbb{R}^2) = \left\{ f \in L^2(\mathbb{R}^2) : \ f = f_0 + f_1 \cdot \chi_B \right\},$$

where $B \subset [0,1]^2$, ∂B a closed C^β-curve, $f_0, \ f_1 \in C_0^\beta([0,1])^2)$.

[Reprinted figure with permission of G. Kutyniok]

Wanted properties of a new wavelet system

- Good space-frequency localization
- "Simple structure" of the wavelet system $\{\psi_\lambda\}_{\lambda\in\Lambda}$ (multiscale approach)
- Orthonormal basis or Parseval frame of $L^2(\mathbb{R}^2)$, i.e.,

$$f = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda\rangle \psi_\lambda$$

and

$$\sum_{\lambda\in\Lambda} |\langle f, \psi_\lambda\rangle|^2 = \|f\|^2_{L^2(\mathbb{R}^2)} \quad \text{for all } f \in L^2(\mathbb{R}^2)$$

(Parseval equation)

- Good approximation properties: If f is in a certain smoothness class, then f can be well approximated by a sparse wavelet frame expansion, such that e.g.

$$\|f - f_N\|_2^2 \le C\, N^{-\beta}$$

for (piecewise) Hölder smooth functions of order β.

Sparse approximation benchmark

Theorem (Donoho '01)

Allowing only polynomial depth search in a dictionary, the approximation rate of the best N-term approximation for $\mathcal{E}^\beta(\mathbb{R}^2)$, $\beta \in (1, 2]$, cannot exceed

$$\|f - f_N\|_2^2 \sim N^{-\beta}.$$

Question: Can this bound be reached?

- Classical wavelet systems achieve $\|f - f_N\|_2^2 \sim N^{-1}$.
- Specifically designed directional representation systems can reach this bound up to log-factors.
- Adaptive wavelet frames can reach this bound.

What can be learned from $\mathbb{R}^1$?

- "Simple structure" of the wavelet system:
  use translations and dilations of only on "mother-wavelet" ψ.

$$\psi_{j,k} = 2^{j/2}\,\psi(2^j \cdot -k), \qquad j,k \in \mathbb{Z}.$$

- Good space-frequency localization:
  ψ should have compact support or fast decay outside in space and frequency domain.
- How to ensure that $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis or a (Parseval) frame in $L^2(\mathbb{R})$?
  Try to achieve that

$$\sum_{j=-\infty}^{\infty} |\hat{\psi}(2^j\omega)|^2 = 1 \qquad \omega \in \mathbb{R}\ a.e.$$

(or $0 < A \le \sum_{j=-\infty}^{\infty} |\hat{\psi}(2^j\omega)|^2 \le B < \infty$)
and has a good frequency localization.

Example: Meyer wavelets

Choose $\hat{\psi}$ with supp $\hat{\psi} \subset [-2, -1/2] \cup [1/2, 2]$ Hence supp $\hat{\psi}(2^{-j}\omega)$ has support $[-2^{j+1}, -2^{j-1}, \cup [2^{j-1}, 2^{j+1}]$.

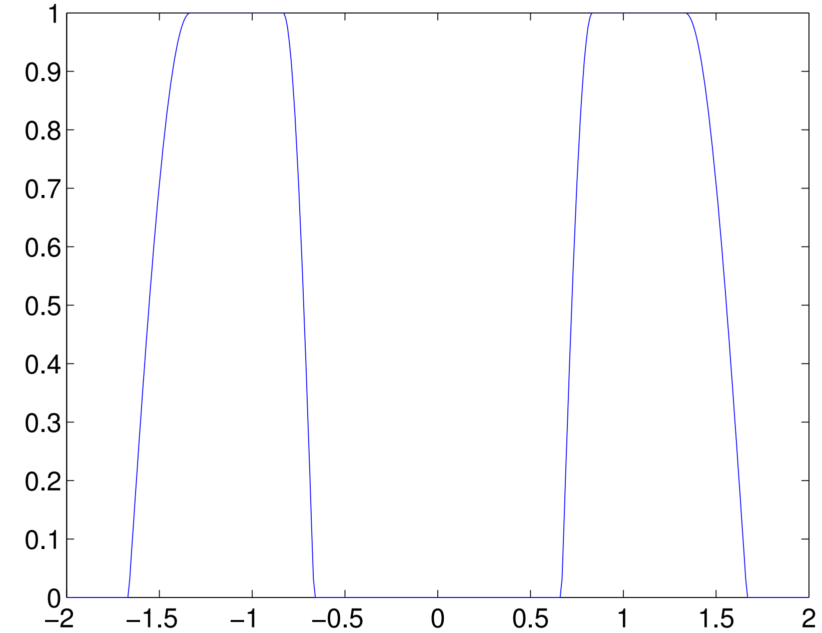
Choose e.g. for $\omega > 0$

$$\hat{\psi}(\omega) = \begin{cases} \cos[\frac{\pi}{2}\nu(5 - 6\omega)] & \frac{2}{3} \leq \omega \leq \frac{5}{6} \\ 1 & \frac{5}{6} \leq \omega \leq \frac{4}{3} \\ \cos[\frac{\pi}{2}\nu(3\omega - 4)] & \frac{4}{3} \leq \omega \leq \frac{5}{3} \\ 0 & \text{else} \end{cases}$$

where ν is smooth and $\nu(x) = 0$ for $x \leq 0$, $\nu(x) = 1$ for $x \geq 1$ and $\nu(x) + \nu(1 - x) = 1$ for $x \in [0, 1]$.

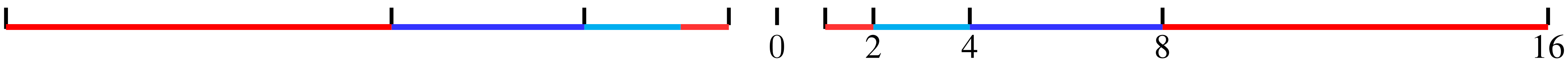
Choose e.g. $\nu(x) = x \cdot \chi_{[0,1]}(x)$

or $\nu(x) = (3x^2 - 2x^3) \cdot \chi_{[0,1]}$ etc.



Corresponding tiling of the frequency domain

one-dimensional case:



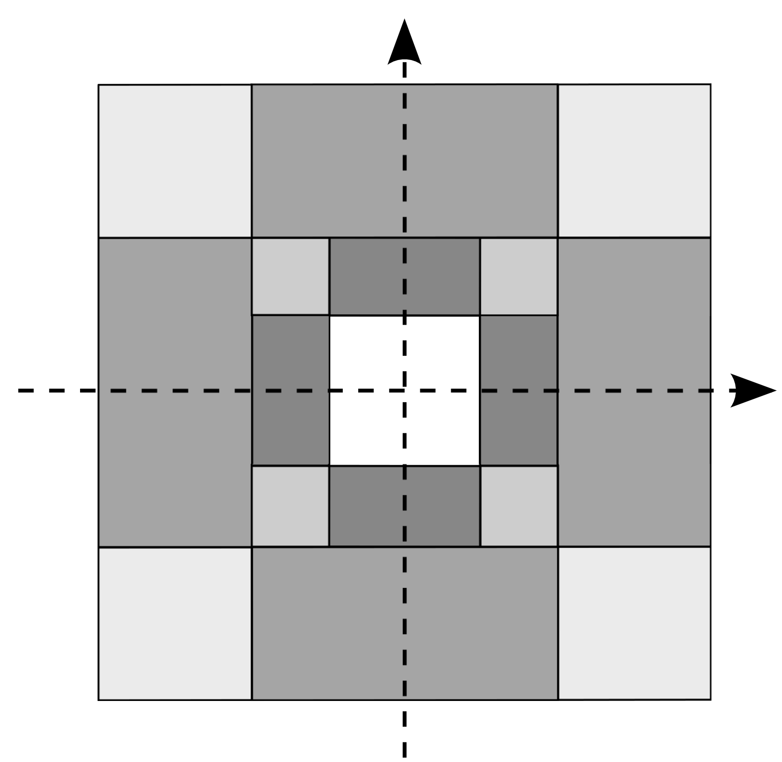
two-dimensional case: tensor-product wavelets

three types of wavelet functions

$\hat{\phi}(\omega_1)\hat{\psi}(\omega_2)$

$\hat{\psi}(\omega_1)\hat{\phi}(\omega_2)$

$\hat{\psi}(\omega_1)\hat{\psi}(\omega_2)$

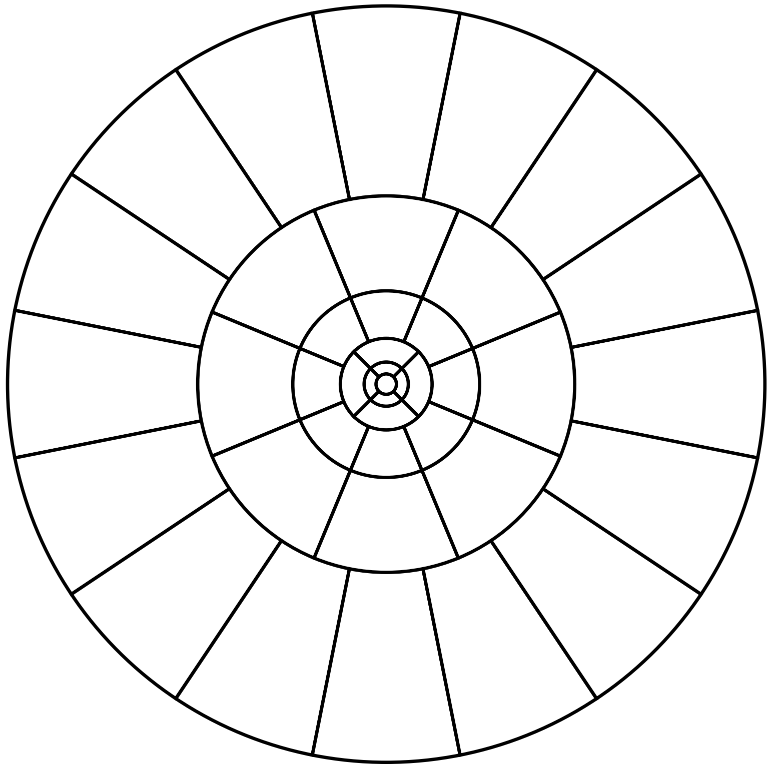


How to construct directional wavelet frames ?

Idea. use translations, dilations and rotations of one "basic function" ψ.

Curvelet construction.

1. Consider polar coordinates in frequency domain
2. Construct curvelet element being locally supported near a wedge.



Curvelet construction

Let $\omega = (\omega_1, \omega_2)^T$, $r := \sqrt{\omega_1^2 + \omega_2^2}$ and $\sigma := \arctan(\omega_1/\omega_2)$.

Ansatz for the dilated basic curvelet:

$$\hat{\psi}_{j,0,0}(r,\sigma) = 2^{-3j/4} W(2^{-j} r)\, V_{N_j}(\sigma), \qquad r \geq 0,\ \sigma \in [0, 2\pi),\ j \in \mathbb{N}_0$$

with suitable window functions W and V_{N_j}, where $N_j = 4 \cdot 2^{\lceil j/2 \rceil}$ indicates the number of wedges in the circular ring at scale 2^{-j}.

We need:

a) $W(r)$ and $V_{N_j}(\sigma) = V_{per}(2^{-\lceil j/2 \rceil}\sigma)$ should have compact support or exponential decay.

b) Partition of frequency domain:

$$\sum_{j=-\infty}^{\infty} |W(2^j r)|^2 = 1$$

$$\sum_{l=0}^{N_j - 1} V_{N_j}^2(\sigma - \frac{2\pi l}{N_j}) = 1 \qquad \text{for all } \sigma \in [0, 2\pi).$$

Indeed we then have

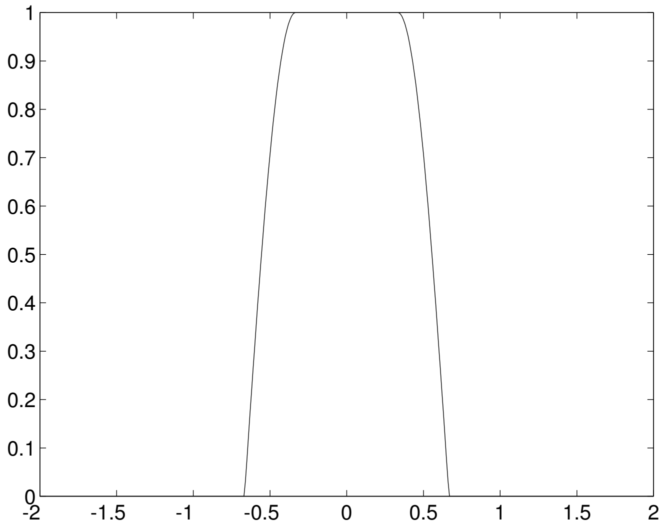
$$\sum_{l=0}^{N_j-1} |2^{3j/4}\hat{\psi}_{j,0,0}(r,\sigma - \frac{2\pi l}{N_j})|^2 = |W(2^{-j}r)|^2 \sum_{l=0}^{N_j-1} V_{N_j}^2(\omega - \frac{2\pi l}{N_j})$$
$$= |W(2^{-j}r)|^2$$

Examples for Window functions.

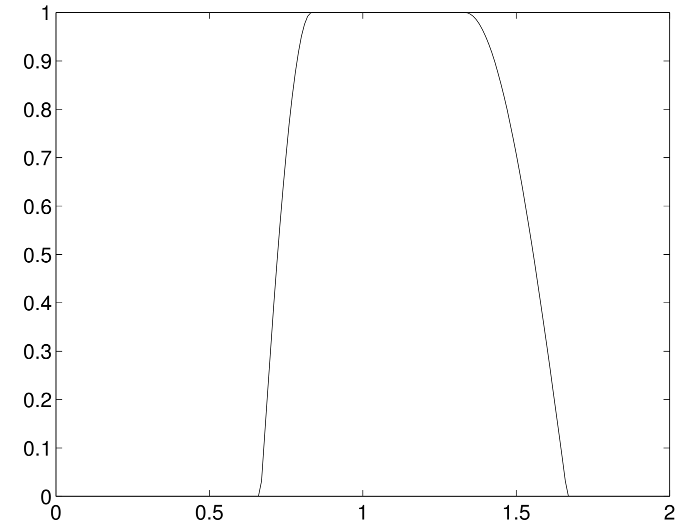
$$V(\sigma) = \begin{cases} 1 & |\sigma| \le \frac{1}{3} \\ \cos(\frac{\pi}{2}\nu(3|\sigma|-1)) & \frac{1}{3} \le |\sigma| \le \frac{2}{3} \\ 0 & \text{else} \end{cases},$$

$$W(r) = \begin{cases} \cos[\frac{\pi}{2}\nu(5-6r)] & \frac{2}{3} \le r \le \frac{5}{6} \\ 1 & \frac{5}{6} \le r \le \frac{4}{3} \\ \cos[\frac{\pi}{2}\nu(3r-4)] & \frac{4}{3} \le r \le \frac{5}{3} \\ 0 & \text{else} \end{cases}$$

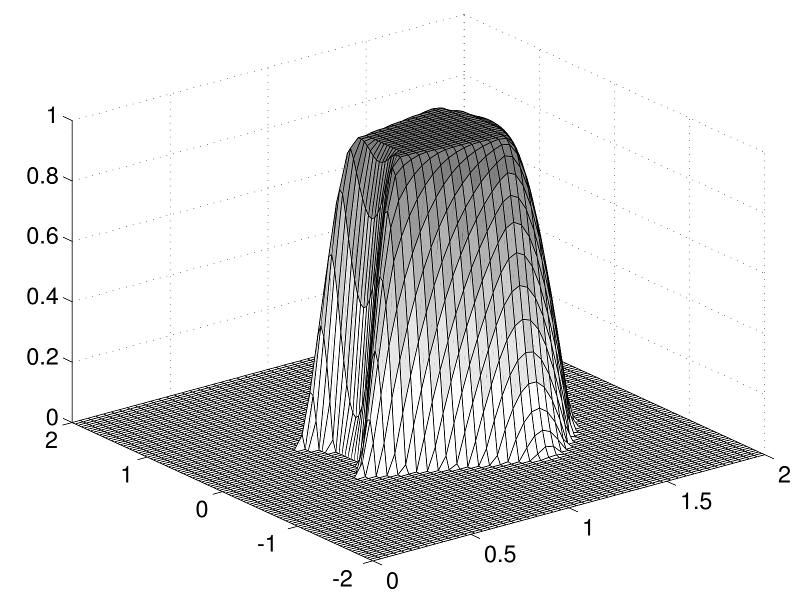
with ν as before.



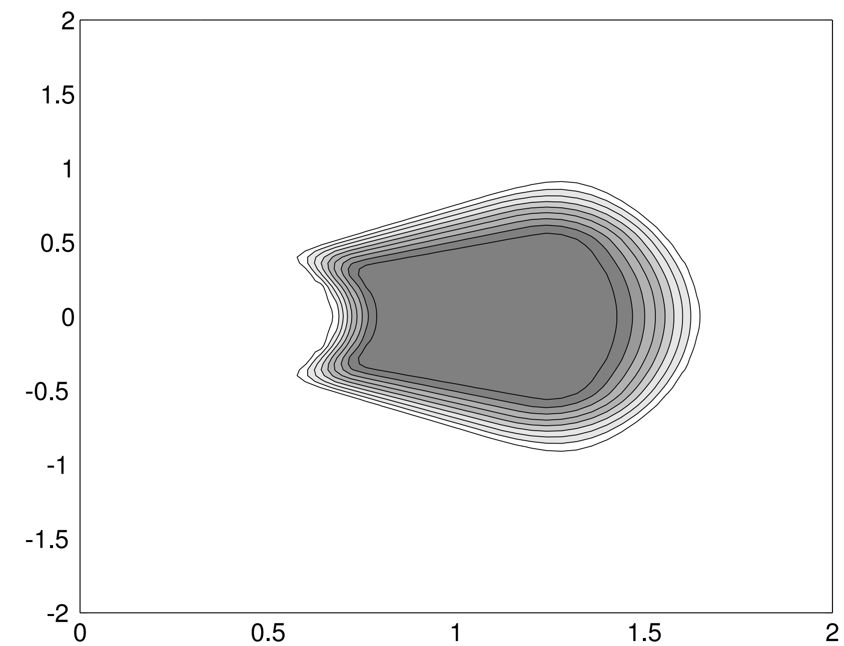
window V



window W



basic curvelet $\hat{\psi}_{0,0,0}$ *in frequency domain*

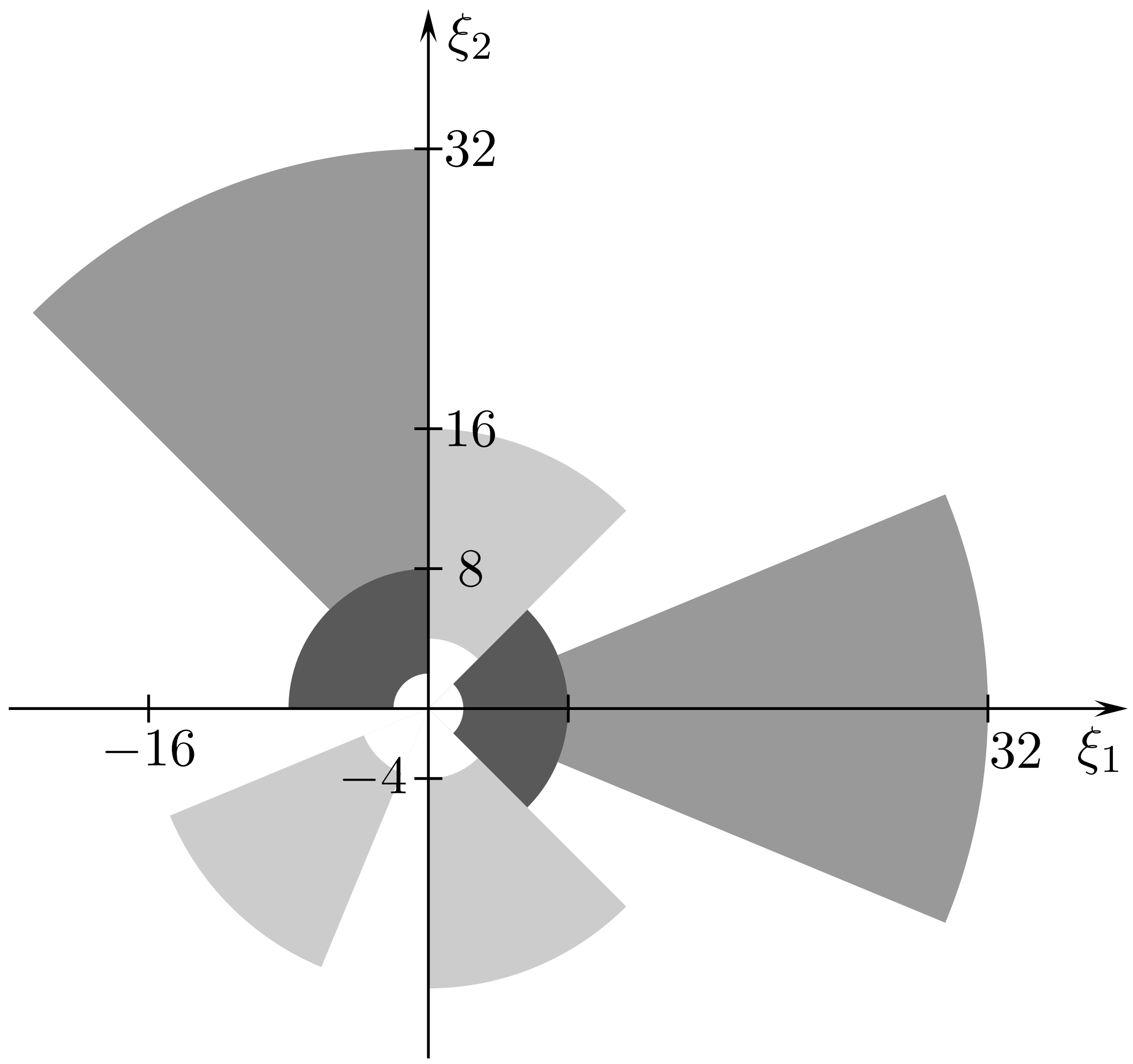


support of $\hat{\psi}_{0,0,0}$

The window V_N is obtained by 2π-periodization of $V(N\sigma/2\pi)$.

With the windows taken above, we have only a small overlap of supports.

Maximal supports of $\hat{\psi}_{2,k,0}$ and $\hat{\psi}_{2,k,5}$ (dark grey); of $\hat{\psi}_{3,k,6}$ and $\hat{\psi}_{3,k,13}$ (light grey); and of $\hat{\psi}_{4,k,0}$ and $\hat{\psi}_{4,k,11}$ (grey). The translation $k \in Z^2$ doe not influence the support of the curve let elements.

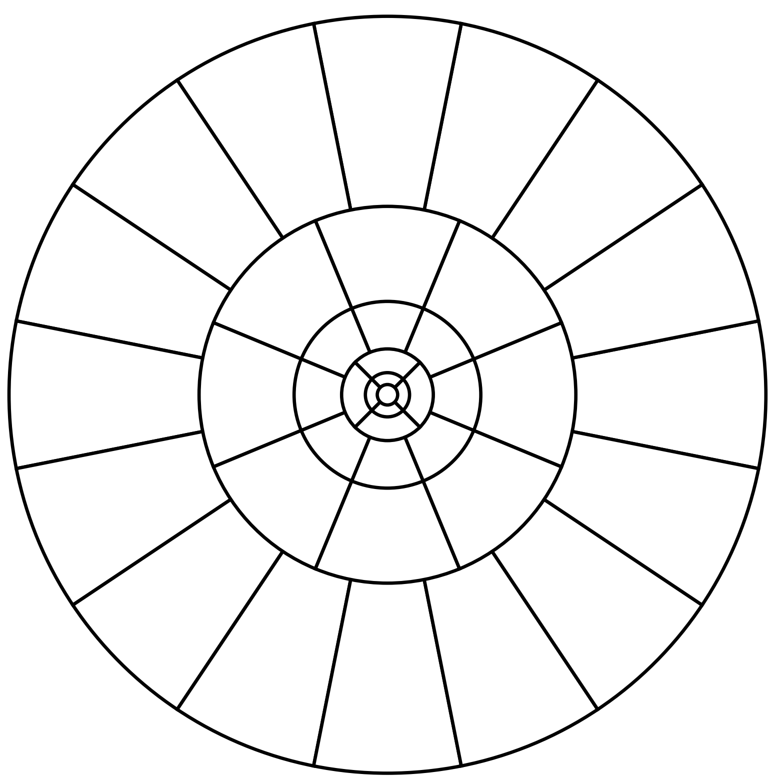


Can we do something else ?

- The window V_N is a low-pass-filter. Any one-dimensional scaling function ϕ (being suitable localized in time and frequency) can serve as the window V and leads to V_{N_j} by 2π-periodization of $\phi(N_j\sigma/2\pi)$.
- The window W is a high-pass filter. Any one-dimensional wavelet function ψ (being suitable localized in time and frequency) can serve as the window W.

How many wedges should be taken in one circular ring ?

- For curvelet construction, choose $N_j = 4 \cdot 2^{\lceil j/2 \rceil}$ wedges in the circular ring with $2^{j-1/2} \leq r \leq 2^{j+1/2}$ (scale 2^{-j}).
- If the number of wedges in a fixed way leads to steerable wavelets.
- If the number of wedges increases like 1/scale (like 2^j), we obtain ridgelets.
- If the number of wedges increases like $\sqrt{1/\text{scale}}$, we obtain curvelets.



The complete set of curvelet elements

We employ rotations and translations of the dilated basic curvelet $\psi_{j,0,0}$. We choose

a) $N_j = 4 \cdot 2^{\lceil j/2 \rceil}$ equidistant rotation angles at level j

$$\theta_{j,l} := \frac{2\pi l}{N_j}, \qquad l = 0, \ldots, N_j - 1.$$

b) the positions

$$\mathbf{b}_{\mathbf{k}}^{j,l} = \mathbf{b}_{k_1,k_2}^{j,l} := \mathbf{R}_{\theta_{j,l}}^{-1}\left(\frac{k_1}{2^j}, \frac{k_2}{2^{j/2}}\right)^T$$

with $k_1, k_2 \in \mathbb{Z}$, $\mathbf{R}_\theta$ rotation matrix with angle θ. Then the family of curvelet functions is given by

$$\psi_{j,k,l}(\mathbf{x}) := \psi_{j,0,0}(\mathbf{R}_{\theta_{j,l}}(\mathbf{x} - \mathbf{b}_{\mathbf{k}}^{j,l})) = \psi_{0,0,0}(\mathbf{A}_{2,2}^{j}\mathbf{R}_{\theta_{j,l}}\mathbf{x} - \mathbf{k})$$

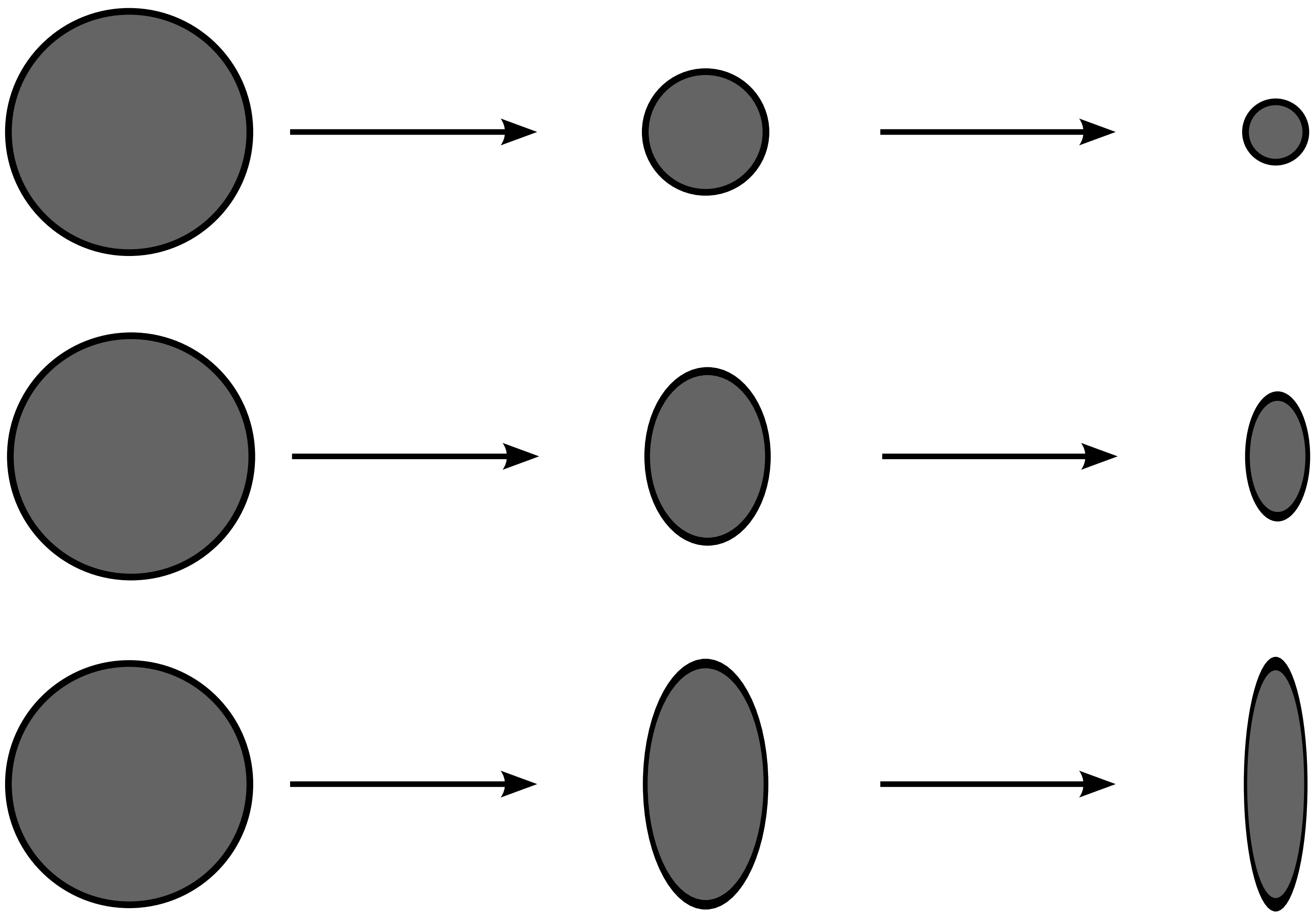
with

$$\mathbf{A}_{2,2}^{j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{\lceil j/2 \rceil} \end{pmatrix}.$$

General directional representation systems (Grohs et al. '14)

- α-scaling matrix: $\mathbf{A}_{\alpha,s} = \begin{pmatrix} s & 0 \\ 0 & s^{\alpha} \end{pmatrix}, \qquad s \in \mathbb{R}_+,\ \alpha \in [0,1]$

- $\alpha = 1$
- $\alpha = \frac{1}{2}$
- $\alpha = 0$



Directional Representation Systems

Basic ingredients. Take a "mother wavelet" $g \in L^2(\mathbb{R}^2)$ and consider

- **Translation**

$$g \to g(\cdot - p), \quad p \in \Lambda \subset \mathbb{R}^2$$

- **Scaling**

$$g \to g(\mathbf{A}_{\alpha,s}\cdot), \quad \mathbf{A}_{\alpha,s} = \begin{pmatrix} s & 0 \\ 0 & s^\alpha \end{pmatrix}, \qquad s \in \mathbb{R}_+$$

- **Orientation**

Rotation: $g \to g(\mathbf{R}_\theta \cdot), \qquad \mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi).$

Shears: $g \to g(\mathbf{S}_a \cdot), \qquad \mathbf{S}_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ or $\mathbf{S}_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ $a \in \mathbb{R}.$

We obtain

$$\psi_{s,\theta,p}(x) = s^{(1+\alpha)/2} g(\mathbf{A}_{\alpha,s}\mathbf{R}_\theta(x-p)).$$

Directional Representation Systems

- Ridgelets (Candes, Donoho '99): Rotations, $s = 2$, $\alpha = 0$
- Curvelets (Candes, Donoho '03): Rotations, $s = 2$ $\alpha = 1/2$
- Shearlets (Kutyniok, Labate '06): Shearings, $s = 2$, $\alpha = 1/2$
- α-Shearlets (Kutyniok et al. '12): Shearings $s > 0$, $\alpha \in [0, 1]$
- α-Curvelets (Grohs et al. '14): Rotations $s > 0$, $\alpha \in [0, 1]$

Common framework $\rightarrow$ α-Molecules (Grohs et al. '14)

Our publications

- Jianwei Ma, Gerlind Plonka.
  The curvelet transform: A review of recent applications.
  IEEE Signal Processing Magazine 27(2) (March 2010), 118-133.
- Jianwei Ma, Gerlind Plonka.
  Computing with Curvelets: From Image Processing to Turbulent Flows.
  Computing in Science and Engineering 11(2) (2009), 72-80.

\thankyou