Introduction to Lissajous curves and d-dimensional polynomial interpolation

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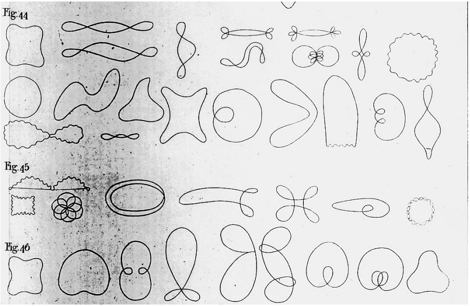
Outline of the talk

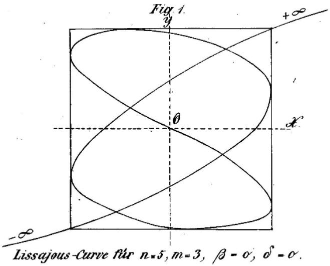
- Motivation
  - A short history of Lissajous curves
  - Lissajous trajectories and MPI
- Interpolation on Lissajous-Chebyshev nodes $\underline{\mathbf{LC}}^{(\boldsymbol{n})}$
  - Some description of Lissajous curves
  - Characterization of the node points of Lissajous curves
  - Interpolation and Quadrature on $\underline{\mathbf{LC}}^{(\boldsymbol{n})}$
  - Numerical condition of the polynomial interpolation
  - Convergence and fast algorithms

A short history of Lissajous curves

- 1800 Thomas Young
- 1815 Nathaniel Bowditch
- 1827 A.C. Wheatstone
- 1857 Jules A. Lissajous



Thomas Young, Outlines of Experiments and Inquiries Respecting Sound and Light. Philosophical Transaction of the Royal Society, London 1800, 106-150, Plate VI, Fig. 44-46



Wilhelm Braun, Die Singularitäten der Lissajous'schen Stimmgabelkurven, Dissertation, Erlangen 1875

- 1875 Wilhelm Braun
- 1902 Edward A. Hook (Multiple points of Lissajous curves in two and three dimensions)

We will consider d-dimensional Lissajous curves

$$\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}} : \mathbb{R} \to \mathbb{R}^{\mathrm{d}}$$

in the parametrized form

$$\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}(t) = \left(u_1 \cos\left(\frac{\mathrm{lcm}[\underline{\boldsymbol{m}}] \cdot t - \kappa_1 \pi}{m_1} \right), \cdots, u_{\mathrm{d}} \cos\left(\frac{\mathrm{lcm}[\underline{\boldsymbol{m}}] \cdot t - \kappa_{\mathrm{d}} \pi}{m_{\mathrm{d}}} \right) \right),$$

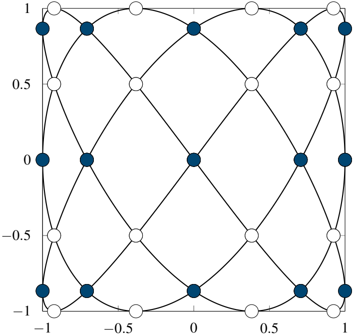
where

- $\underline{\boldsymbol{m}} = (m_1, \ldots, m_{\mathrm{d}}) \in \mathbb{N}^{\mathrm{d}}$ are ‘frequency dividers’,
- $\underline{\boldsymbol{u}} \in \{-1, 1\}^{\mathrm{d}}$ are ‘reflection parameters’,
- $\mathrm{lcm}[\underline{\boldsymbol{m}}]$ is the least common multiple of $m_1, \ldots, m_{\mathrm{d}}$,
- $\underline{\boldsymbol{\kappa}} = (\kappa_1, \ldots, \kappa_{\mathrm{d}}) \in \mathbb{R}^{\mathrm{d}}$ specifies additional phase shifts.

The definition guarantees that in any case the minimal period of $\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}$ is 2π.

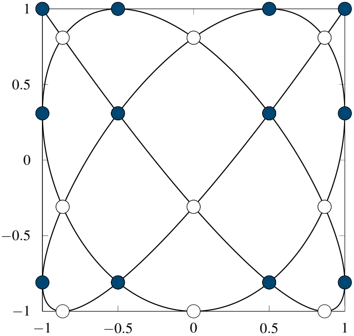
Two examples

$\underline{\ell}^{(8,6)}_{(4,3),(1,1)}(t) = (\sin 3t, \sin 4t)$



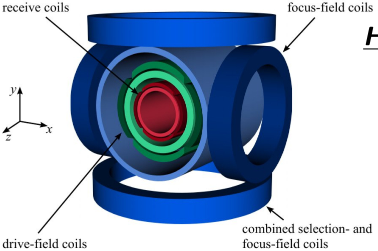
Non-degenerate Lissajous curve used in Magnetic Particle Imaging [5].

$\underline{\ell}^{(6,5)}_{(0,0),(1,1)}(t) = (\cos 5t, \cos 6t)$



Degenerate Lissajous curve generating the Padua points [1, 2].

MPI and Lissajous trajectories



From: T. Knopp and T.M. Buzug, *Magnetic Particle Imaging*, Springer, 2012 [7]

A typical FFP-3D MPI scanner applies magnetic fields of the form

$$\underline{\boldsymbol{H}}((x,y,z),t) = G \begin{pmatrix} -x \\ -y \\ 2z \end{pmatrix} + \begin{pmatrix} A_1 \sin 2\pi t/n_1 \\ A_2 \sin 2\pi t/n_2 \\ A_3 \sin 2\pi t/n_3 \end{pmatrix}$$

In this way a field free point (FFP) is generated moving along a Lissajous trajectory inside a rectangular field of view (FOV) [7].

In two dimensions the corresponding trajectory looks as on the left hand side of the previous slide.

Questions considered in this tutorial:

- Which functions (polynomials) in subsets of $\mathbb{R}^d$ can be reconstructed if data values are available on a Lissajous trajectory?
- What are 'good' points on the trajectory from which a suitable reconstruction of the original function is possible?
- What is a suitable interpolation procedure for data on the curve?
- How many data points on the trajectory are necessary to obtain a good resolution? How large are the approximation errors?

In this tutorial, we are focusing on polynomial interpolation. Why?

- Algebraic polynomials restricted to the Lissajous-curves correspond to trigonometric polynomials on the curve. In FFP-MPI the Fourier coefficients of the magnetization signal are measured. Therefore, spaces of algebraic polynomials are interesting for modeling in MPI.
- Polynomial interpolation on the node points of Lissajous-curves can be implemented easily and efficiently.

Previous work in the literature

Most important influences for our work:

- Interpolation on Padua points: these are the node points generated by a particular family of Lissajous curves [1].
- Interpolation and quadrature on Morrow-Patterson-Xu points [8].

L. Bos, M. Caliari, S. De Marchi, M. Vianello, Y. Xu
Bivariate Lagrange interpolation at the Padua points: the generating curve approach.
Journal of Approximation Theory 143, (2006), 15 – 25.

Y. Xu
Lagrange interpolation on Chebyshev points of two variables.
Journal of Approximation Theory 87, (1996), 220 – 238.

General assumption throughout this work:

The vector $\underline{\boldsymbol{n}} = (n_1, \ldots, n_{\mathsf{d}})$ consists of pairwise relatively prime natural numbers $n_1, \ldots, n_{\mathsf{d}}$.

Then, the Lissajous curves with $\underline{\boldsymbol{m}} = \underline{\boldsymbol{n}}$ have the simpler form

$$\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{n}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}(t) = \left(u_1 \cos\left(\frac{\mathsf{p}[\underline{\boldsymbol{n}}] \cdot t - \kappa_1 \pi}{n_1} \right), \cdots, u_{\mathsf{d}} \cos\left(\frac{\mathsf{p}[\underline{\boldsymbol{n}}] \cdot t - \kappa_{\mathsf{d}} \pi}{n_{\mathsf{d}}} \right) \right),$$

where

$$\mathsf{p}[\underline{\boldsymbol{n}}] = \prod_{\mathsf{i}=1}^{\mathsf{d}} n_{\mathsf{i}}.$$

Definition: The Lissajous curve $\underline{\boldsymbol{\ell}}^{(\boldsymbol{n})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}$ is called *degenerate* if there exist $t' \in \mathbb{R}$ and $\underline{\boldsymbol{u}}' \in \{-1,1\}^{\mathsf{d}}$ such that

$$\underline{\boldsymbol{\ell}}^{(\boldsymbol{n})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}(t-t') = \underline{\boldsymbol{\ell}}^{(\boldsymbol{n})}_{\underline{\mathbf{0}},\underline{\boldsymbol{u}}'}(t).$$

Proposition - Characterization of degenerate Lissajous curves

The Lissajous curve $\underline{\boldsymbol{\ell}}^{(\boldsymbol{n})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}$ is degenerate if and only if

$$\kappa_{\mathsf{i}} - \kappa_{\mathsf{j}} \in \mathbb{Z} \quad \text{for all } \mathsf{i},\mathsf{j} \in \{1,\ldots,\mathsf{d}\}.$$

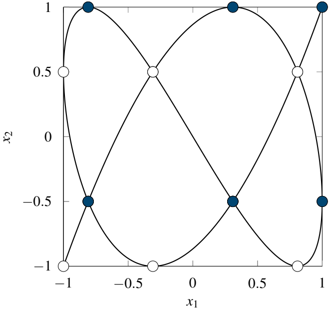
For degenerate curves, we can therefore restrict our attention to the Lissajous curves $\underline{\boldsymbol{\ell}}^{(\boldsymbol{n})}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}(t)$. This is done in the following.

We sample $\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{n}})}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}(t)$ along the $2\mathrm{p}[\underline{\boldsymbol{n}}]$ equidistant points

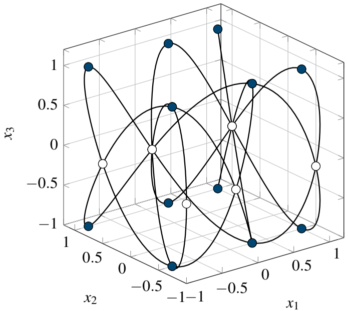
$$t_j^{(\underline{\boldsymbol{n}})} = \frac{\pi j}{\mathrm{p}[\underline{\boldsymbol{n}}]}, \quad j = 0, \ldots, 2\mathrm{p}[\underline{\boldsymbol{n}}] - 1,$$

and obtain the node point set

$$\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})} = \left\{ \underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{n}})}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}(t_j^{(\underline{\boldsymbol{n}})}) \;\middle|\; j = 0, \ldots, 2\mathrm{p}[\underline{\boldsymbol{n}}] - 1 \right\}.$$



(a) $\mathbf{LC}^{(5,3)}$ and $\ell^{(5,3)}_{(0,0),(1,1)}$



(b) $\mathbf{LC}^{(5,3,2)}$ and $\ell^{(5,3,2)}_{(0,0,0),(1,1,1)}$

Multiple points of degenerate Lissajous curves

For $t \in [0, 2\pi)$, let $\mathcal{A}^{(\underline{\boldsymbol{n}})}(t)$ be the set of all $s \in [0, 2\pi)$ with

$$\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{n}})}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}(s) = \underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{n}})}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}(t).$$

Theorem

a) We have

$$\begin{aligned}
\#\mathcal{A}^{(\underline{\boldsymbol{n}})}(t) &= 1 \quad \text{if} \quad t \in \{t_0^{(\underline{\boldsymbol{n}})}, t_{\mathrm{p}[\underline{\boldsymbol{n}}]}^{(\underline{\boldsymbol{n}})}\}, \\
\#\mathcal{A}^{(\underline{\boldsymbol{n}})}(t) &= 2 \quad \text{if} \quad t \in [0, 2\pi) \setminus \{\, t_j^{(\underline{\boldsymbol{n}})} \mid j \in \{0, \ldots, 2\mathrm{p}[\underline{\boldsymbol{n}}] - 1\} \,\}, \\
\#\mathcal{A}^{(\underline{\boldsymbol{n}})}(t) &\geq 2 \quad \text{if} \quad t \in \{\, t_j^{(\underline{\boldsymbol{n}})} \mid j \in \{1, \ldots, 2\mathrm{p}[\underline{\boldsymbol{n}}] - 1\} \setminus \{\mathrm{p}[\underline{\boldsymbol{n}}]\} \,\}.
\end{aligned}$$

i.e. all self-intersection points of the curve $\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{n}})}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}$ are contained in $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$.

For $\mathrm{M} \subseteq \{1, \ldots, \mathrm{d}\}$, consider the $\#\mathrm{M}$-faces

$$\underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}} = \left\{ \underline{\boldsymbol{x}} \in [-1,1]^{\mathrm{d}} \mid \mathrm{i} \in \mathrm{M} \Longleftrightarrow x_{\mathrm{i}} \in (-1,1) \right\}.$$

Theorem

b) Let $\mathrm{M} \subseteq \{1, \ldots, \mathrm{d}\}$ and $j \in \{0, \ldots, 2\mathrm{p}[\underline{\boldsymbol{n}}] - 1\}$. We have

$$\underline{\boldsymbol{\ell}}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}^{(\underline{\boldsymbol{n}})}(t_j^{(\underline{\boldsymbol{n}})}) \in \underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}} \quad \Longrightarrow \quad \#\mathcal{A}^{(\underline{\boldsymbol{n}})}(t_j^{(\underline{\boldsymbol{n}})}) = 2^{\#\mathrm{M}}.$$

Further,

$$\#\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})} = \frac{1}{2^{\mathrm{d}-1}} \mathrm{p}[\underline{\boldsymbol{n}} + \underline{\mathbf{1}}], \quad \#(\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})} \cap \underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}}) = \frac{1}{2^{\#\mathrm{M}-1}} \prod_{\mathrm{i} \in \mathrm{M}} (n_{\mathrm{i}} - 1).$$

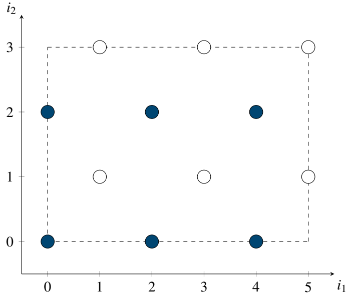
Example: If $\mathrm{M} = \{1, \ldots, \mathrm{d}\}$, then $\underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}} = (-1,1)^{\mathrm{d}}$ and $\#\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})} \cap \underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}} = \frac{1}{2^{\mathrm{d}-1}} \prod_{\mathrm{i}=1}^{\mathrm{d}} (n_{\mathrm{i}} - 1)$ is the number of self-intersection points of $\underline{\boldsymbol{\ell}}_{\underline{\mathbf{0}},\underline{\mathbf{1}}}^{(\underline{\boldsymbol{n}})}$ in the interior of the hypercube.

A second characterization of the points $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

To parametrize the point sets $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$, we can use the index sets

$$\underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})} = \underline{\mathbf{I}}_0^{(\underline{\boldsymbol{n}})} \cup \underline{\mathbf{I}}_1^{(\underline{\boldsymbol{n}})} \text{ with the sets } \underline{\mathbf{I}}_{\mathfrak{r}}^{(\underline{\boldsymbol{n}})},\ \mathfrak{r} \in \{0,1\}, \text{ given by}$$

$$\underline{\mathbf{I}}_{\mathfrak{r}}^{(\underline{\boldsymbol{n}})} = \left\{\, \underline{\boldsymbol{i}} \in \mathbb{N}_0^{\mathsf{d}} \;|\, \forall\, \mathsf{j}:\ 0 \leq i_{\mathsf{j}} \leq n_{\mathsf{j}} \text{ and } i_{\mathsf{j}} \equiv \mathfrak{r} \mod 2 \,\right\}.$$



The index set $\underline{\mathbf{I}}^{(5,3)}$.

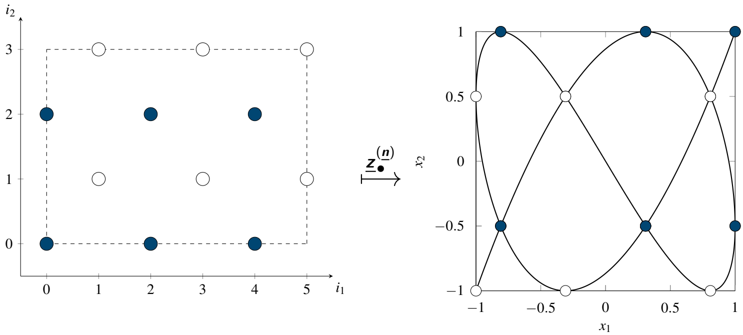
A second characterization of the points $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

Theorem

$$\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})} = \left\{ \underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} \;\middle|\; \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})} \right\}.$$

Chebyshev-Gauss-Lobatto points:

$$\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} = \left(z_{i_1}^{(n_1)}, \ldots, z_{i_d}^{(n_d)} \right), \;\; z_i^{(n)} = \cos\left(i\pi/n \right).$$



Lagrange interpolation in 1D

Given $n+1$ node points $x_0 < \cdots < x_n$ and values $f_0, \cdots, f_n \in \mathbb{R}$.

Find a univariate polynomial P_f of degree at most n such that

$$P_f(x_i) = f_i, \quad i = 0, \ldots, n.$$

This interpolation problem has the unique solution

$$P_f(x) = \sum_{i=0}^{n} f_i L_i(x),$$

where

$$L_i(x) = \prod_{\substack{0 \le m \le n \\ m \neq i}} \frac{x - x_m}{x_i - x_m} = \frac{(x - x_0)}{(x_i - x_0)} \cdots \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \frac{(x - x_{i+1})}{(x_i - x_{i+1})} \cdots \frac{(x - x_n)}{(x_i - x_n)}.$$

The polynomials L_i are called *fundamental solutions* of Lagrange interpolation.

In multivariate polynomial interpolation, we have additional difficulties:

- Spaces of multivariate polynomials can be defined in several ways.
- For given node sets, interpolation is not necessarily unique.

Orthogonal basis polynomials simplify the considerations.

$\Rightarrow$ Use *multivariate Chebyshev polynomials*

$$T_{\underline{\gamma}}(\underline{\boldsymbol{x}}) = T_{\gamma_1}(x_1) \cdot \ldots \cdot T_{\gamma_d}(x_d), \quad \underline{\boldsymbol{x}} \in [-1,1]^d, \underline{\gamma} \in \mathbb{N}_0^d,$$

as basis elements. They form an orthogonal basis of the space of the d-variate polynomials with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\pi^d} \int_{[-1,1]^d} f(\underline{\boldsymbol{x}}) \overline{g(\underline{\boldsymbol{x}})} w_d(\underline{\boldsymbol{x}}) \, \mathrm{d}\underline{\boldsymbol{x}}, \quad w_d(\underline{\boldsymbol{x}}) = \prod_{i=1}^{d} \frac{1}{\sqrt{1 - x_i^2}}.$$

The spectral index set $\underline{\mathbf{\Gamma}}^{(\underline{n})}$

Consider the polynomial spaces

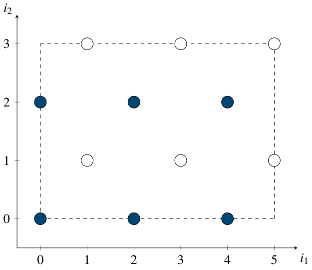
$$\Pi^{(\underline{n})} = \mathrm{span}\left\{\, T_{\underline{\gamma}} \,\middle|\, \underline{\gamma} \in \underline{\mathbf{\Gamma}}^{(\underline{n})} \,\right\}$$

based on the spectral index sets

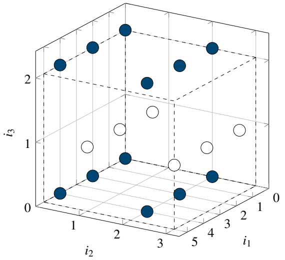
$$\underline{\mathbf{\Gamma}}^{(\underline{n})} = \left\{ \underline{\gamma} \in \mathbb{N}_0^{\mathsf{d}} \;\middle|\; \begin{array}{lll} \forall\, \mathsf{i} : & & \gamma_{\mathsf{i}} < n_{\mathsf{i}}, \\ \forall\, \mathsf{i} \neq \mathsf{j} & : & \gamma_{\mathsf{i}}/n_{\mathsf{i}} + \gamma_{\mathsf{j}}/n_{\mathsf{j}} < 1 \end{array} \right\} \cup \{(0, \ldots, 0, n_{\mathsf{d}})\}\,.$$

Proposition

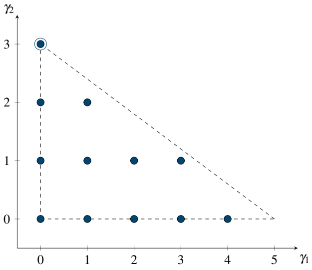
$\#\underline{\mathbf{\Gamma}}^{(\underline{n})} = \#\underline{\mathbf{LC}}^{(\underline{n})}$.



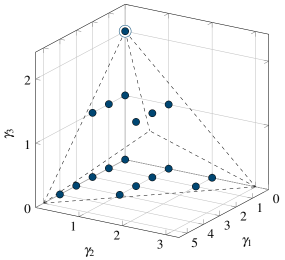
(a) Index set $\mathbf{I}^{(5,3)}$



(b) Index set $\mathbf{I}^{(5,3,2)}$



(c) Spectral index set $\mathbf{\Gamma}^{(5,3)}$



(d) Spectral index set $\mathbf{\Gamma}^{(5,3,2)}$

For $\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}$, we define the weights

$$\mathfrak{w}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} = 2^{\#\mathrm{M}}/(2\mathrm{p}[\underline{\boldsymbol{n}}]) \quad \text{if} \;\; \underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} \in \underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})} \cap \underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}},$$

and the measure $\omega^{(\underline{\boldsymbol{n}})}$ on the power set of $\underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}$ by $\omega^{(\underline{\boldsymbol{n}})}(\{\underline{\boldsymbol{i}}\}) = \mathfrak{w}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}$.

Further, we define the Lagrange polynomials

$$L_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{x}}) = \mathfrak{w}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} \left(K^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}) - T_{n_{\mathrm{d}}}(x_{\mathrm{d}})\, T_{n_{\mathrm{d}}}(z_{i_{\mathrm{d}}}^{(n_d)}) \right), \quad \underline{\boldsymbol{x}} \in [-1,1]^{\mathrm{d}},$$

with the reproducing kernel

$$K^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}') = \sum_{\underline{\boldsymbol{\gamma}} \in \underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})}} \frac{1}{\| T_{\underline{\boldsymbol{\gamma}}} \|^2} T_{\underline{\boldsymbol{\gamma}}}(\underline{\boldsymbol{x}})\, T_{\underline{\boldsymbol{\gamma}}}(\underline{\boldsymbol{x}}'), \quad \underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}' \in [-1,1]^{\mathrm{d}}.$$

Interpolation on $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

Denote by $\mathcal{L}(\underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})})$ the set of the functions $h : \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})} \to \mathbb{C}$.

Theorem (D., E. [3])

For $h \in \mathcal{L}(\underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})})$, the polynomial

$$P_h^{(\underline{\boldsymbol{n}})} = \sum_{\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}} h(\underline{\boldsymbol{i}}) L_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}$$

is the unique element in the polynomial space $\Pi^{(\underline{\boldsymbol{n}})}$ that satisfies

$$P_h^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}) = h(\underline{\boldsymbol{i}}) \quad \text{for all} \quad \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}.$$

Further,

$$\operatorname{span}\{\, P_h^{(\underline{\boldsymbol{n}})} \mid h \in \mathcal{L}(\underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}) \,\} = \Pi^{(\underline{\boldsymbol{n}})}$$

and the polynomials $L_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}$, $\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}$, form a basis of $\Pi^{(\underline{\boldsymbol{n}})}$.

Idea of the proof

Show that the functions

$$\chi_{\underline{\gamma}}^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{i}}) = T_{\underline{\gamma}}(\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}) = \prod_{\mathrm{i}=1}^{\mathrm{d}} \cos(\gamma_{\mathrm{i}} i_{\mathrm{i}} \pi / n_{\mathrm{i}}), \quad \underline{\gamma} \in \underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})},$$

form an orthogonal basis of $\mathcal{L}(\underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})})$ with respect to the discrete inner product

$$\langle f, g \rangle_{\omega^{(\underline{\boldsymbol{n}})}} = \sum_{\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}} f(\underline{\boldsymbol{i}})\, \overline{g(\underline{\boldsymbol{i}})}\, \mathfrak{w}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}.$$

Examples:

(i) $\mathrm{d} = 1$: Gauß-Chebyshev-Lobatto interpolation

(ii) $\mathrm{d} = 2$ and $\underline{\boldsymbol{n}} = (n, n+1)$ or $\underline{\boldsymbol{n}} = (n+1, n)$:
Bivariate polynomial interpolation on Padua points, see [1, 2]

A quadrature rule on $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

As a side result of our interpolation theorem we get a quadrature formula.

Theorem

Let P be a d-variate polynomial.

Assume that for all

$$\underline{\gamma} \in \mathbb{N}_0^{\mathrm{d}} \setminus \{\underline{\mathbf{0}}\} \text{ satisfying } \frac{\gamma_{\mathrm{i}}}{n_{\mathrm{i}}} \in \mathbb{N}_0,\ \mathrm{i} = 1, \ldots, \mathrm{d}, \text{ and } \sum_{\mathrm{i}=1}^{\mathrm{d}} \frac{\gamma_{\mathrm{i}}}{n_{\mathrm{i}}} \in 2\mathbb{N}$$

we have

$$\langle P, T_{\underline{\gamma}} \rangle = 0.$$

Then

$$\frac{1}{\pi^{\mathrm{d}}} \int_{[-1,1]^{\mathrm{d}}} P(\underline{\boldsymbol{x}}) w(\underline{\boldsymbol{x}}) \, \mathrm{d}\underline{\boldsymbol{x}} = \sum_{\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}} \mathfrak{w}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} P(\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}).$$

Efficient computation of the interpolating polynomial

We consider the expansion

$$P_h^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{x}}) = \sum_{\underline{\boldsymbol{\gamma}} \in \underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})}} c_{\underline{\boldsymbol{\gamma}}}(h)\, T_{\underline{\boldsymbol{\gamma}}}(\underline{\boldsymbol{x}}).$$

We introduce

$$g^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{i}}) = \begin{cases} \mathfrak{w}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})} h(\underline{\boldsymbol{i}}), & \text{if } \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}, \\ 0, & \text{if } \underline{\boldsymbol{i}} \in \underline{\mathbf{J}}^{(\underline{\boldsymbol{n}})} \setminus \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}, \end{cases} \qquad \underline{\mathbf{J}}^{(\underline{\boldsymbol{n}})} = \bigtimes_{\mathrm{i}=1}^{\mathrm{d}} \{0, \ldots, n_{\mathrm{i}}\},$$

and, recursively, for $\mathrm{i} = 1, \ldots, \mathrm{d}$,

$$g^{(\underline{\boldsymbol{n}})}_{(\gamma_1,\ldots,\gamma_{\mathrm{i}})}(i_{\mathrm{i}+1}, \ldots, i_{\mathrm{d}}) = \sum_{i_{\mathrm{i}}=0}^{n_{\mathrm{i}}} g^{(\underline{\boldsymbol{n}})}_{(\gamma_1,\ldots,\gamma_{\mathrm{i}-1})}(i_{\mathrm{i}}, \ldots, i_{\mathrm{d}}) \cos(\gamma_{\mathrm{i}} i_{\mathrm{i}} \pi / n_{\mathrm{i}}).$$

Then, we have

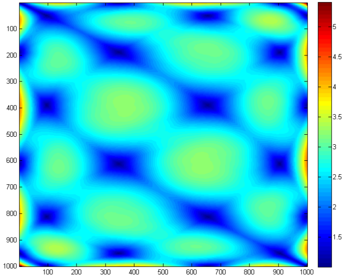
$$c_{\underline{\boldsymbol{\gamma}}}(h) = \frac{\langle h, \chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{n}})} \rangle_{\omega^{(\underline{\boldsymbol{n}})}}}{\|\chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{n}})}\|^2_{\omega^{(\underline{\boldsymbol{n}})}}} = \frac{g_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{n}})}}{\|\chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{n}})}\|^2_{\omega^{(\underline{\boldsymbol{n}})}}}.$$

Using FFT this can be done in $\mathcal{O}(\mathrm{p}[\underline{\boldsymbol{n}}] \log \mathrm{p}[\underline{\boldsymbol{n}}])$ steps.

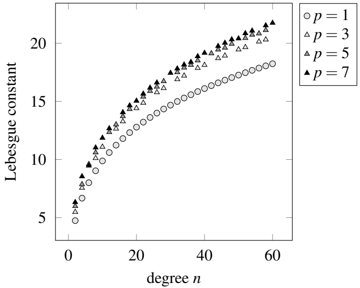
Numerical condition of the interpolation

The absolute condition of the interpolation problem with respect to the uniform norm is given by the Lebesgue constant

$$\Lambda^{(\underline{\boldsymbol{n}})} = \max_{\underline{\boldsymbol{x}} \in [-1,1]^{\mathrm{d}}} \sum_{\underline{\boldsymbol{i}} \in \mathbf{I}^{(\underline{\boldsymbol{n}})}} |L_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{n}})}(\underline{\boldsymbol{x}})|.$$



Lebesgue function for $\underline{\boldsymbol{n}} = (5, 7)$



$\Lambda^{(2\underline{\boldsymbol{n}})}$ for $\underline{\boldsymbol{n}} = (n + p, n)$, see [5]

Theorem (D., E., Kolomoitsev, Lomako [4])

The Lebesgue constant $\Lambda^{(\underline{\boldsymbol{n}})}$ is bounded by

$$C_{\Lambda,1} \prod_{\mathrm{i}=1}^{\mathrm{d}} \ln(n_{\mathrm{i}}+1) \leq \Lambda^{(\underline{\boldsymbol{n}})} \leq C_{\Lambda,2} \prod_{\mathrm{i}=1}^{\mathrm{d}} \ln(n_{\mathrm{i}}+1)$$

with constants $C_{\Lambda,1}, C_{\Lambda,2} > 0$ independent of $\underline{\boldsymbol{n}}$.

Let $P^{(\underline{\boldsymbol{n}})} f$ be the unique polynomial in $\Pi^{(\underline{\boldsymbol{n}})}$ satifying

$$(P^{(\underline{\boldsymbol{n}})} f)(z_{\underline{\boldsymbol{i}}}) = f(z_{\underline{\boldsymbol{i}}}), \quad \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{n}})}.$$

For $f \in C([-1,1]^{\mathrm{d}})$ we have

$$\|f - P^{(\underline{\boldsymbol{n}})} f\|_{\infty} \leq \left(C_{\Lambda,2} \prod_{\mathrm{i}=1}^{\mathrm{d}} \ln(n_{\mathrm{i}}+1) + 1 \right) E^{(\underline{\boldsymbol{n}})}(f),$$

where $E^{(\underline{\boldsymbol{n}})}(f)$ denotes the best approximation of f in $\Pi^{(\underline{\boldsymbol{n}})}$.

Idea of the proof

For the spectral index set $\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})}$ define its symmetrization $\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})\,*}$ by

$$\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})\,*} = \left\{ \underline{\gamma} \in \mathbb{Z}^{\mathsf{d}} \;\middle|\; (|\gamma_1|, \ldots, |\gamma_{\mathsf{d}}|) \in \underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})} \right\}.$$

and define the *Fourier-Lebesgue constant* $\mathrm{L}\left(\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})\,*}\right)$ as

$$\mathrm{L}\left(\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})\,*}\right) = \frac{1}{(2\pi)^{\mathsf{d}}} \int_{[-\pi,\pi)^{\mathsf{d}}} \left| \sum_{\underline{\gamma} \in \underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})\,*}} \mathrm{e}^{\mathrm{i}(\underline{\gamma}, \underline{\boldsymbol{t}})} \right| \mathrm{d}\underline{\boldsymbol{t}},$$

where

$$(\underline{\gamma}, \underline{\boldsymbol{t}}) = \sum_{\mathsf{i}=1}^{\mathsf{d}} \gamma_{\mathsf{i}} t_{\mathsf{i}}.$$

Using a Marcinkiewicz-Zygmund inequality, it is possible to prove

Theorem

For all $\underline{\boldsymbol{n}} \in \mathbb{N}^{\mathsf{d}}$*, we have*

$$\Lambda^{(\underline{\boldsymbol{n}})} \lesssim \mathrm{L}\left(\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}}),*}\right) + \prod_{\mathsf{i}=1}^{\mathsf{d}} \ln(n_{\mathsf{i}}+1), \qquad \mathrm{L}\left(\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}}),*}\right) \lesssim \Lambda^{(\underline{\boldsymbol{n}})}.$$

Theorem

For all $\underline{\boldsymbol{n}} \in \mathbb{N}^{\mathsf{d}}$*, we have*

$$\mathrm{L}\left(\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})}\right) \asymp \mathrm{L}\left(\underline{\boldsymbol{\Gamma}}^{(\underline{\boldsymbol{n}})\,*}\right) \asymp \prod_{\mathsf{i}=1}^{\mathsf{d}} \ln(n_{\mathsf{i}}+1).$$

The proof of this Theorem is the technically more sophisticated part. Here, a suitable decomposition of the polyhedral spectra is necessary.

Dini-Lipschitz criterion for convergence

Theorem (D., E., Kolomoitsev, Lomako [4])

Let $\underline{\boldsymbol{s}} \in \mathbb{N}_0^{\mathrm{d}}$ and

$$\frac{\partial^{s_{\mathrm{j}}} f}{\partial x_{\mathrm{j}}^{s_{\mathrm{j}}}} \in C([-1,1]^{\mathrm{d}}), \quad \mathrm{j} \in \{1, \ldots, \mathrm{d}\}.$$

Then, we have

$$\|f - P^{(\underline{\boldsymbol{n}})} f\|_\infty \lesssim \left(\prod_{\mathrm{i}=1}^{\mathrm{d}} \ln(n_{\mathrm{i}} + 1) \right) \sum_{\mathrm{j}=1}^{\mathrm{d}} \frac{\omega\left(\frac{\partial^{s_{\mathrm{j}}} f}{\partial x_{\mathrm{j}}^{s_{\mathrm{j}}}}; 0, \ldots, 0, \frac{1}{n_{\mathrm{j}}+1}, 0, \ldots, 0 \right)}{(n_{\mathrm{j}} + 1)^{s_{\mathrm{j}}}},$$

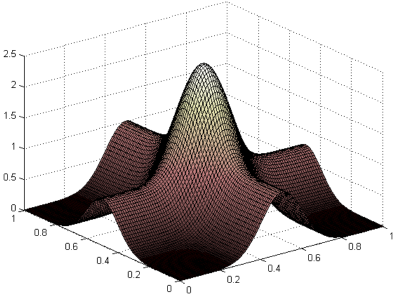
where

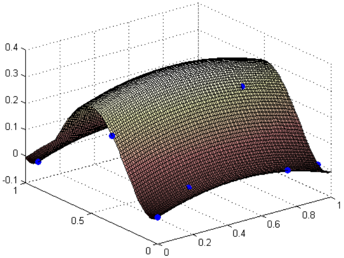
$$\omega(f; \underline{\boldsymbol{\delta}}) = \sup_{\substack{\underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}' \in [-1,1]^{\mathrm{d}} \\ \forall\, \mathrm{i} \in \{1,\ldots,\mathrm{d}\}:\, |x_{\mathrm{i}}' - x_{\mathrm{i}}| \leq \delta_{\mathrm{i}}}} |f(\underline{\boldsymbol{x}}') - f(\underline{\boldsymbol{x}})|$$

denotes the modulus of continuity of f on $[-1,1]^{\mathrm{d}}$.

Convergence of interpolation on $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

Approximation of $f \in C([0,1]^2)$ with polynomial $P^{(\underline{\boldsymbol{n}})} f$, $\underline{\boldsymbol{n}} = (3,5)$.

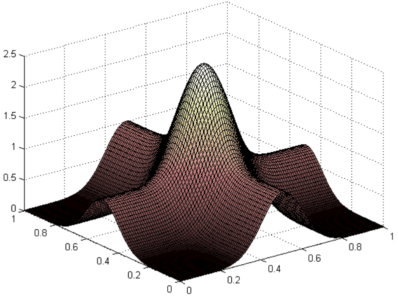


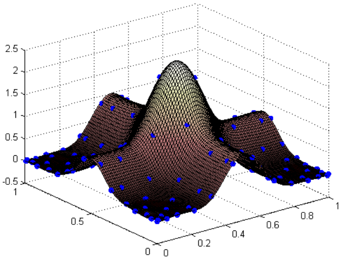


Error: $\|f - P^{(\underline{\boldsymbol{n}})} f\|_\infty = 2.1319$.

Convergence of interpolation on $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

Approximation of $f \in C([0,1]^2)$ with polynomial $P^{(\underline{\boldsymbol{n}})}f$, $\underline{\boldsymbol{n}} = (13, 15)$.

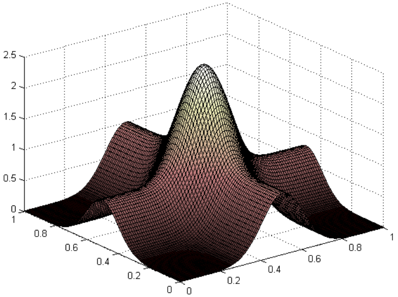


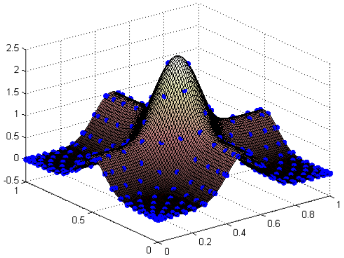


Error: $\|f - P^{(\underline{\boldsymbol{n}})}f\|_\infty = 0.1005$.

Convergence of interpolation on $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

Approximation of $f \in C([0,1]^2)$ with polynomial $P^{(\underline{\boldsymbol{n}})}f$, $\underline{\boldsymbol{n}} = (23, 25)$.

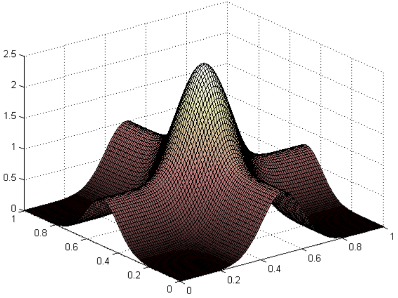


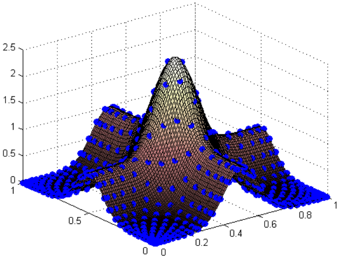


Error: $\|f - P^{(\underline{\boldsymbol{n}})}f\|_\infty = 0.0019$.

Convergence of interpolation on $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{n}})}$

Approximation of $f \in C([0,1]^2)$ with polynomial $P^{(\underline{\boldsymbol{n}})} f$, $\underline{\boldsymbol{n}} = (33, 35)$.





Error: $\|f - P^{(\underline{\boldsymbol{n}})} f\|_\infty = 1.0035 \cdot 10^{-5}$.

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