Edge preserving and noise reducing reconstruction for magnetic particle imaging

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Magnetic particle imaging (MPI)

MPI is an emerging imaging modality based on non-linear response of magnetic nanoparticles to an applied magnetic field

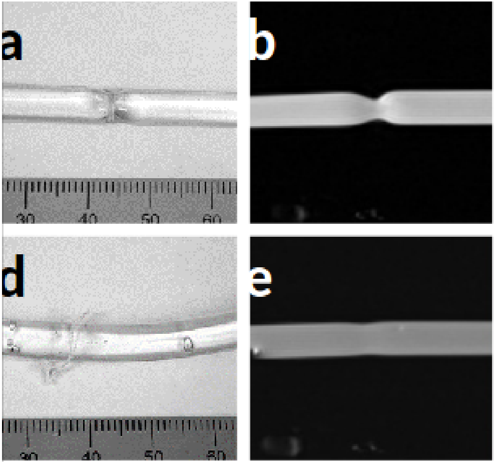
- Invention in 2005 by Gleich and Weizenecker
- First in-vivo experiment in 2009

Features of MPI

- *MPI*. Spatial resolution: 1mm ; measurement time: 0.1 sec.
- *MRT*. Spatial resolution: 1mm ; measurement time: 10 sec -30 min.
- *PET*. Spatial resolution: 4mm ; measurement time: 1 min.

Potential applications: cancer detection, blood flow monitoring, tracking of instruments in cardiovascular intervention

Stenosis phantom before (top) and after (bottom) the intervention by a balloon catheter



Forward model of MPI as linear system

$$Au \approx f$$

where

- $f \in \mathbb{R}^M$ acquired (noisy) data
- u unknown particle concentration
- A linear system function $A : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^M$, as matrix $\widetilde{A} \in \mathbb{R}^{N \times M}$

System function A

- here: measurementent-based using a delta probe
- alternative: model-based reconstruction (Knopp '10; Goodwill/Conolly '10; März/Weinmann '15)

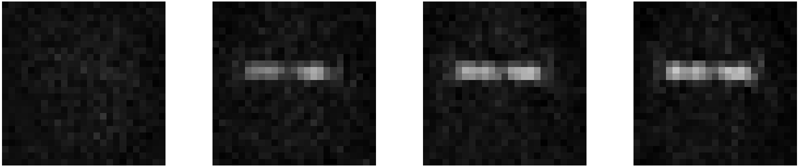
Image reconstruction ill-posed inverse problem (Knopp '08; März/Weinmann '15)
$\rightsquigarrow$ regularization necessary

State-of-the-art reconstruction in MPI

Currently used reconstruction method: Tikhonov regularization with non-negativity constraints

$$u_\rho = \arg \min_{u \geq 0} \rho \|u\|^2 + \frac{1}{2} \|Au - f\|^2$$

+ u_ρ solution of constrained linear system $\rightsquigarrow$ standard solvers
– Prior $\rho \|u\|^2$ not well matched to image properties $\rightsquigarrow$ limited noise suppression, loss of contrast, smoothes out edges



Inflating balloon catheter over time, maximum intensity projection of reconstruction result

Fused lasso regularization

Image characteristics

- Catheter and background have approximately homogeneous concentration
- Catheter volume is small compared to background
- Physical assumption: particle concentration is non-negative

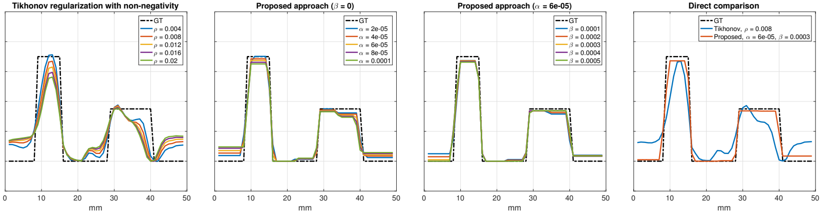
Fused lasso model (Tibshirani et al. '05) **with non-negativity constraint**

$$u^{\#} = \arg\min_{u \geq 0} \ \alpha \, \mathrm{TV}(u) + \beta \, \|u\|_1 + \frac{1}{2} \|Au - f\|_2^2 \, .$$

Discrete version in 1D

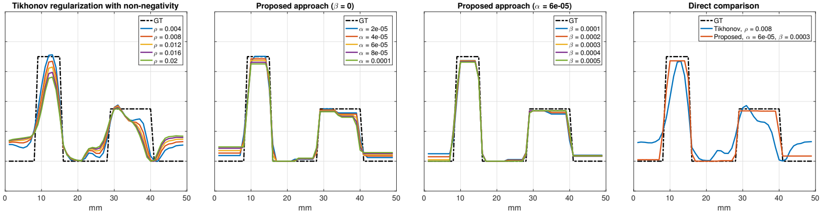
$$u^{\#} = \arg \min_{u \in (\mathbb{R}_0^+)^n} \ \alpha \sum_i |u_{i+1} - u_i| + \beta \sum_i |u_i| + \frac{1}{2} \sum_i |(Au)_i - f_i|^2 .$$

Simulated data

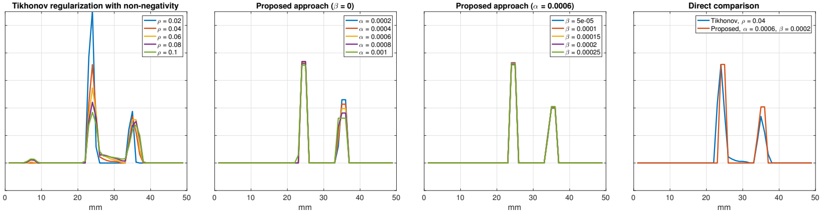


Reconstruction example for 1D MPI

Simulated data



Real data of a physical phantom
(two homogeneous spots of length 4 mm in a 5 mm distance, left spot has double particle concentration right spot)



Finite difference discretization (Blake/Zisserman '87; Chambolle '99)

$$\mathrm{TV}_{\mathrm{2D}}(u) = \sum_{s=1}^{S} \omega_s \|\nabla_{a_s} u\|_1 = \sum_{s=1}^{S} \sum_{ij} \omega_s |u_{ij} - u_{(i,j)+a_s}|,$$

with a finite difference system $\mathcal{N} = \{a_1, ..., a_S\} \subset \mathbb{Z}^2 \setminus \{0\}$ and weights $\omega_1, ..., \omega_S > 0$.

Common finite difference systems

$$\mathcal{N}_0 = \{(1,0),(0,1)\},$$

$$\mathcal{N}_1 = \{(1,0),(0,1),(1,1),(1,-1)\},$$

$$\mathcal{N}_2 = \{(1,0),(0,1),(1,1),(1,-1),(-2,-1),(-2,1),(2,1),(2,-1)\}.$$

Derivation of weights

Discretization (system $\mathcal{N}$, weights ω) gives rise to a metric on $\mathbb{R}^2$ induced by

$$\|a\|_{\mathcal{N}} = \sum_{s=1}^{S} \omega_s |\langle a, a_s \rangle|, \quad a \in \mathbb{R}^2.$$

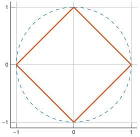
Proposed design criterion for ω (Storath/Weinmann/Frikel/Unser ’15)

$$\|a\|_{\mathcal{N}} \overset{!}{=} \|a\|_2 \quad \text{for all } a \in \mathcal{N}$$

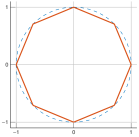
$\rightsquigarrow$ linear system

$$T\omega = q$$

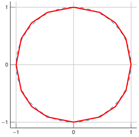
with $T_{rs} = |\langle a_r, a_s \rangle|$ and $q_r = \|a_r\|_2$



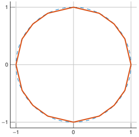
Anisotropic $\mathcal{N}_0$ ($E \approx 0.71$).



With diagonals $\mathcal{N}_1$ ($E \approx 0.92$)



“Knight moves” $\mathcal{N}_2$, weights of Chambolle ’99 ($E \approx 0.95$)



“Knight moves” $\mathcal{N}_2$, proposed weights ($E \approx 0.97$)

Measure of isotropy E: Ratio of shortest and longest vector on the red line ($E = 1$ is optimal) (Chambolle ’99)

Discretization of the form

$$\mathrm{TV}_{3\mathrm{D}}(u) = \sum_{s=1}^{S} \omega_s \|\nabla_{a_s} u\|_1 = \sum_{s=1}^{S} \sum_{ijk} \omega_s |u_{ijk} - u_{(i,j,k)+a_s}|.$$

with 26-connected neighborhood

$$\mathcal{N}_{3\mathrm{D}} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,-1,0),(1,0,1),\\ (1,0,-1),(0,1,1),(0,1,-1),(1,1,1),(1,1,-1),(1,-1,-1),(-1,1,-1)\}$$

Adaption to MPI acquisition geometry in 3D (I)

Discretization of the form

$$\mathrm{TV}_{3\mathrm{D}}(u) = \sum_{s=1}^{S} \omega_s \|\nabla_{a_s} u\|_1 = \sum_{s=1}^{S} \sum_{ijk} \omega_s |u_{ijk} - u_{(i,j,k)+a_s}|.$$

with 26-connected neighborhood

$$\begin{aligned}\mathcal{N}_{3\mathrm{D}} = \{&(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,-1,0),(1,0,1),\\ &(1,0,-1),(0,1,1),(0,1,-1),(1,1,1),(1,1,-1),(1,-1,-1),(-1,1,-1)\}\end{aligned}$$

MPI scanner configuration: ***z*-direction has double resolution of *x*-*y* resolution**

$\rightsquigarrow$ incorporate into weight design $T\omega = q$ by setting right hand side to

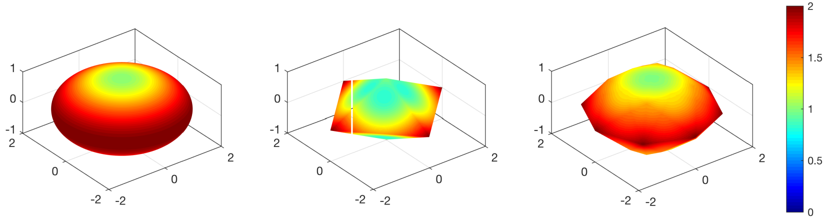
$$q_s = (\sum_{i=1}^{3} (\delta_i \cdot (a_s)_i)^2)^{1/2}.$$

where δ vector of voxel face areas, $\delta = (\Delta y \Delta z, \Delta x \Delta z, \Delta x \Delta y)$

Problem: Solving $T\omega = q$ can lead to negative weights

$\rightsquigarrow$ solve in least squares sense with non-negativity constraint

$$\min_{\omega} \|T\omega - q\|_2^2, \quad \text{s.t.} \quad \omega \geq 0.$$



Left: Desired distances, *Center:* anisotropic discretization (8-NH), *Right:* proposed discretization (based on 26-NH)

Splitting of the fused lasso functional

Discretization of fused lasso problem

$$u^{\#} = \arg\min_{u} \; \alpha \sum_{s=1}^{S} \omega_s \|\nabla_{a_s} u\|_1 + \beta \|u\|_1 + \mathcal{I}_+(u) + \frac{1}{2}\|Au - f\|_2^2,$$

where $\mathcal{I}_+(u)$ is equal to 0 if $u_{ij} \geq 0$ for all i, j, and equal to ∞ otherwise.

Proposed splitting

$$u^{\#} = \arg\min_{u} \; \sum_{s=1}^{S} \underbrace{\left(\alpha\omega_s \|\nabla_{a_s} u\|_1 + \frac{\beta}{S}\|u\|_1\right)}_{G_s(u)} + \mathcal{I}_+(u) + \underbrace{\frac{1}{2}\|Au - f\|_2^2}_{F(u)},$$

Minimization using generalized forward backward algorithm (Raguet/Fadili/Peyré '13)

Proposed splitting for the fused lasso problem

$$u^{\#} = \arg\min_u \sum_{s=1}^{S} \underbrace{\left(\alpha\omega_s \|\nabla_{a_s} u\|_1 + \frac{\beta}{S}\|u\|_1\right)}_{G_s(u)} + \mathcal{I}_+(u) + \underbrace{\frac{1}{2}\|Au - f\|_2^2}_{F(u)},$$

Minimization using a **generalized forward backward algorithm** (Raguet/Fadili/Peyré '13)

Iteration

$$z_1^{(k+1)} = z_1^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma G_1}{r_1}} \left(2u^{(k)} - z_1^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$\vdots$$

$$z_S^{(k+1)} = z_S^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma G_S}{r_S}} \left(2u^{(k)} - z_S^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$z_{S+1}^{(k+1)} = z_{S+1}^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma \mathcal{I}_+}{r_{S+1}}} \left(2u^{(k)} - z_{S+1}^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$u^{(k+1)} = \sum_{s=1}^{S+1} r_s z_s^{(k+1)}.$$

Idea of a (basic, non-relaxed) forward backward algorithm

$$u^{\#} = \arg\min_u \; f_1(u) + f_2(u)$$

with

f_1 convex, potentially not differentiable, $\quad$ f_2 convex, differentiable.

Iteration

$$u^{(k+1)} = \underbrace{\text{prox}_{\gamma f_1}}_{\text{backward step}} \Big(\underbrace{u^{(k)} - \gamma \nabla f_2(u^{(k)})}_{\text{forward step}} \Big),$$

where

$$\text{prox}_{\alpha f}(u) = \arg\min_x \; \alpha f(x) + \frac{1}{2}\|x - u\|_2^2.$$

Relaxation

$$u^{(k+1)} = u^{(k)} + \lambda_k \left(\text{prox}_{f_1/\beta}\left(u^{(k)} - \tfrac{1}{\beta}\nabla f_2(u^{(k)}) \right) - u^{(k)} \right).$$

Convergence for $\beta \geq \frac{1}{L}$, L Lipschitz constant of ∇f_2, $\lambda_k \in [\varepsilon, 3/2 - \varepsilon]$, $\varepsilon > 0$ (cf. Combettes '06).

Generalized forward backward algorithm

Iteration (Generalized forward backward algorithm)

$$z_1^{(k+1)} = z_1^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma G_1}{r_1}} \left(2u^{(k)} - z_1^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$\vdots$$

$$z_S^{(k+1)} = z_S^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma G_S}{r_S}} \left(2u^{(k)} - z_S^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$z_{S+1}^{(k+1)} = z_{S+1}^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma \mathcal{I}_+}{r_{S+1}}} \left(2u^{(k)} - z_{S+1}^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$u^{(k+1)} = \sum_{s=1}^{S+1} r_s z_s^{(k+1)}.$$

Algorithmic parameters and convergence

$\gamma = 1/\|A^*A\|_{\mathrm{op}}$, $\lambda_k = 1$, and $r_1 = \ldots = r_{S+1} = 1/(S+1)$, guarantee convergence (Raguet/Fadili/Peyré '13)

Advantage. We can explicitly compute the involved proximal mappings.

Subproblems of the algorithm

Iteration (Generalized forward backward algorithm)

$$\vdots$$

$$z_{S+1}^{(k+1)} = z_{S+1}^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma \mathcal{I}_+}{r_{S+1}}} \left(2u^{(k)} - z_{S+1}^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$\vdots$$

Gradient step in F

$$\nabla F(u) = A^*(Au - f).$$

Only matrix vector multiplication $O(MN)$
$\rightsquigarrow$ NO linear system for (large and dense) system matrix A needs to be solved!

Iteration (Generalized forward backward algorithm)

$$\vdots$$

$$z_{S+1}^{(k+1)} = z_{S+1}^{(k)} + \lambda_k \left(\operatorname{prox}_{\frac{\gamma \mathcal{I}_+}{r_{S+1}}} \left(2u^{(k)} - z_{S+1}^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$\vdots$$

Gradient step in F

$$\nabla F(u) = A^*(Au - f).$$

Only matrix vector multiplication $O(MN)$
$\rightsquigarrow$ NO linear system for (large and dense) system matrix A needs to be solved!

Proximal mappings of $\mathcal{I}_+$ $\rightsquigarrow$ cutting negative values

$$(\operatorname{prox}_{\gamma \mathcal{I}_+ / r_{S+1}}(u))_{ij} = \arg\min_{r \in \mathbb{R}} \gamma \frac{\mathcal{I}_+(r)}{r_{S+1}} + \frac{1}{2} \|r - u_{ij}\|_2^2 = \max(0, u_{ij}).$$

where $\mathcal{I}_+(u)$ is equal to 0 if $u_{ij} \geq 0$ for all i, j, and equal to ∞ otherwise.

Subproblems of the algorithm

Iteration (Generalized forward backward algorithm)

$$z_1^{(k+1)} = z_1^{(k)} + \lambda_k \left(\mathrm{prox}_{\frac{\gamma G_1}{r_1}} \left(2u^{(k)} - z_1^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$\vdots$$

$$z_S^{(k+1)} = z_S^{(k)} + \lambda_k \left(\mathrm{prox}_{\frac{\gamma G_S}{r_S}} \left(2u^{(k)} - z_S^{(k)} - \gamma \nabla F(u^{(k)}) \right) - u^{(k)} \right),$$

$$\vdots$$

where

$$G_s(u) = \alpha \omega_s \|\nabla_{a_s} u\|_1 + \frac{\beta}{S} \|u\|_1.$$

Proximal mappings of $G_s \rightsquigarrow$ decomposes into pathwise univariate fused lasso problems of the form

$$v^{\#} = \arg\min_{v \in \mathbb{R}^n} \; \alpha' \sum_{i=1}^{n-1} |v_{i+1} - v_i| + \beta' \sum_{i=1}^{n} |v_i| + \frac{1}{2} \sum_{i=1}^{n} (v_i - f_i')^2.$$

Solver for univariate fused lasso problems

Exact solution $v^{\#}$ of 1D fused lasso problem in two stages

$$v^{\#} = \arg\min_{v \in \mathbb{R}^n} \; \alpha' \sum_{i=1}^{n-1} |v_{i+1} - v_i| + \beta' \sum_{i=1}^{n} |v_i| + \frac{1}{2} \sum_{i=1}^{n} (v_i - f_i')^2.$$

(i) solve TV problem (i.e. fused lasso with $\beta' = 0$)

$$u^0 = \arg\min_{v \in \mathbb{R}^n} \; \alpha' \sum_{i=1}^{n-1} |v_{i+1} - v_i| + \frac{1}{2} \sum_{i=1}^{n} (v_i - f_i')^2$$

using taut string algorithm (Davies/Kovac '01; Condat '13)

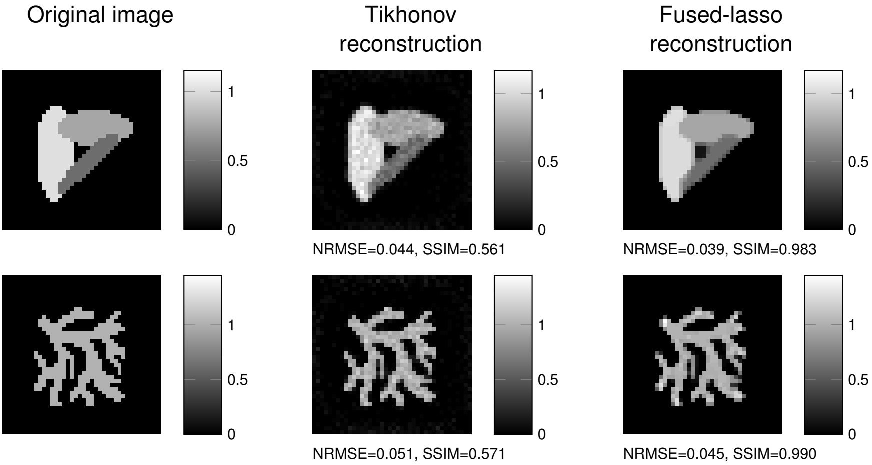
(ii) fused lasso solution by soft-thresholding of u^0 (Friedman et al. '07)

$$v_i^{\#} = \mathrm{ST}_{\beta'}(u_i^0) = \mathrm{sign}(u_i^0) \max(|u_i^0| - \beta', 0), \quad i = 1, \dots, n.$$

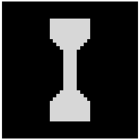
Key for fast algorithm: both steps have linear complexity

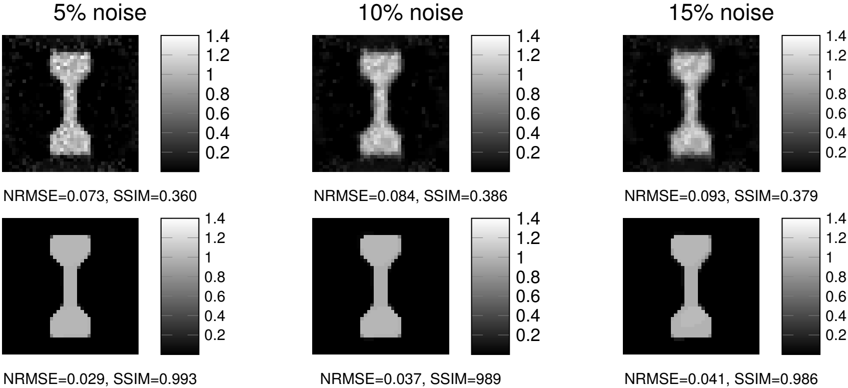
Fast implementation of taut string algorithm by L. Condat
http://www.gipsa-lab.grenoble-inp.fr/~laurent.condat/software.html

Reconstruction from simulated data



Reconstruction from simulated data with different noise levels

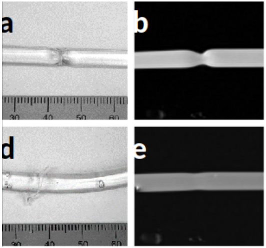




Scanner setup

- Preclinical MPI scanner from Philips/Bruker
    - selection field gradient: $1.5\text{Tm}^{-1}\mu_0^{-1}$ in z and $0.75\text{Tm}^{-1}\mu_0^{-1}$ in x,y-direction
- Vessel phantom: polyvinyl chloride tube
- Balloon catheter inflated with ferudextran (pressure between 4.5 to 20 bar)
- 20000 frames without averaging
- Matrix measured with a $2 \times 2 \times 1$ mm^3 delta probe filled with Resovist
    - Field of view $25 \times 25 \times 25$

$\Rightarrow$ System matrix of size 13104×15625



Stenosis phantom before and after experiment

3D real data: reconstruction

Results visualized as maximum intensity projection along z-axis

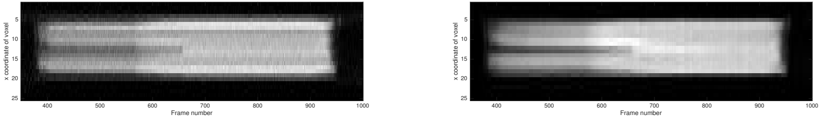
baseline method

proposed method

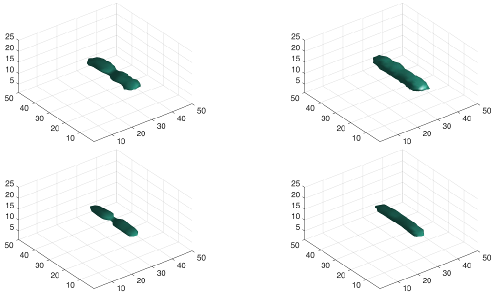
SNR = 14.1 dB
0.7 seconds per frame

SNR = 30.0 dB
5.0 seconds per frame

3D real data: reconstruction



Intensity profile at central axis over time (left: baseline, right: proposed)



Isosurface rendering of catheter intensity (top: baseline; bottom: proposed)

Summary

- Fused lasso model for noise suppressing and edge preserving reconstruction in 3D+time MPI
- Quasi-isotropic discretization adapted to 3D acquisition geometry
- Efficient minimization algorithm (parallelizable, no linear system to be solved explicitly)

Main reference:
M. Storath, C. Brandt, M. Hofmann, T. Knopp, J. Salamon, A. Weber, A. Weinmann
“Edge preserving and noise reducing reconstruction for magnetic particle imaging”
IEEE T Medical Imaging, 2017