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A multivariate interpolation problem arising from the scattering of waves in layered media

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Abstract

We present recent results on the scattering of plane waves in piecewise constant layered media, introducing a new geometric perspective. It turns out that the classical inverse scattering problem generically decouples into two separate problems. One problem is to recover a line from the magnitudes of the projections onto the line of a set of lattice points. The second problem is to determine which function best interpolates a given set of data points on the integer lattice, where the interpolating function is chosen from among the translates of a known, highly oscillatory function. In applications such as acoustic imaging it is required that this interpolation problem be solved numerically, necessitating that a solution be both stable and computationally efficient.

1 Introduction

The purpose of this paper is to formulate a multivariate interpolation problem that arises in the context of scattering in piecewise constant layered media. In the process we derive some new results in scattering theory. Most notably, we show that the scattering of normal incidence plane waves in layered media is governed by a single bivariate function, the *total amplitude factor*. More broadly, we present a new picture of the inverse scattering problem whereby it factors naturally into two steps, both of which have a simple geometric description.

- The first step (which is relatively straightforward) is to determine a linear functional given its values on part of the integer lattice. We refer to this step as *inverse projection*, since evaluation of a functional corresponds essentially to orthogonal projection onto a line.
- In the second step one is given data on part of the integer lattice, and the problem is to find a translate of a highly oscillatory function—called the *covering amplitude*—that best interpolates the data. Thus, the second step is a multivariate approximation problem.

There is a substantial literature relating to scattering in layered media, which is important in acoustic, electromagnetic and seismic imaging. The past two decades have seen considerable progress in stochastic methods, involving scaling limits for randomly layered media, and time reversal methods, as detailed in [3]. In the context of seismic imaging, a prevalent approach is to use (Born or Lippman-Schwinger) series approximations, as exemplified in [8] and [5]. By contrast, the methods of the present paper are deterministic and exact.

To begin we make precise the underlying mathematical framework.

1.1 Precise framework

We are interested in the scenario of a normally incident plane wave scattering in a piecewise-constant layered medium. This scenario is important in various applications, including acoustic, seismic and electromagnetic imaging. The following is a brief summary of the requisite technical framework; further details and derivations may be found in [1] and [3], among many other references.

Let (x, y, z) be euclidean coordinates for a three-dimensional solid medium in which the density ρ and bulk modulus K are functions of z alone, referred to as depth. Suppose further that ρ and K are piecewise constant in z, having jumps at the $n+1$ locations

$$z_0 < z_1 < \cdots < z_n$$

and let $z_{-1} < z_0$ be a reference depth in the homogeneous half spaces $z < z_0$. For $0 \le j \le n+1$, let K_j denote the constant value of the bulk modulus in the layer

$$z_{j-1} < z < z_j,$$

and let ρ_j denote the density in the same layer. See Figure 1. Given initial conditions that depend on z only, the particle

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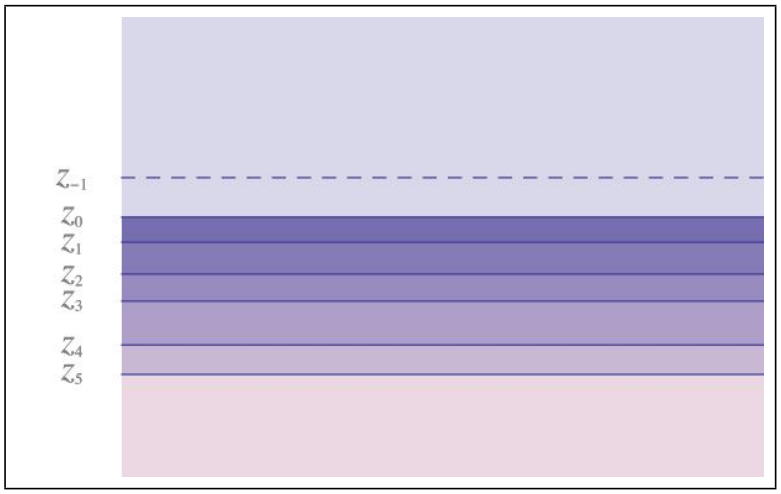

Figure 1: A layered medium, with depth z increasing downward.

velocity $u(t,z)$ and pressure $p(t,z)$ evolve in time t according to the coupled first order equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0 \tag{1.1a}$$

$$\frac{1}{K}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} = 0. \tag{1.1b}$$

For the sake of definiteness we focus on the velocity field $u(t,z)$, although the results can just as easily be formulated in terms of $p(t,z)$. The initial conditions corresponding to a plane wave unit impulse propagating from depth z_{-1} are

$$\begin{aligned} u(0,z) &= \delta(z - z_{-1}) \\ p(0,z) &= \sqrt{K(z_{-1})\rho(z_{-1})}\,\delta(z - z_{-1}). \end{aligned} \tag{1.2}$$

For $t > 0$ this system has a unique solution, $u(t,z)$. Its restriction to depth $z = z_{-1}$ is the reflection Green's function,

$$G(t) = u(t, z_{-1}). \tag{1.3}$$

For $0 \le j \le n+1$, the time it takes a traveling plane wave to go from z_{j-1} to z_j and back is

$$\tau_j = \frac{2(z_j - z_{j-1})}{\sqrt{K_j/\rho_j}}. \tag{1.4}$$

For $0 \le j \le n$, the reflection coefficient for the interface at depth z_j is

$$R_j = \frac{\sqrt{K_j\rho_j} - \sqrt{K_{j+1}\rho_{j+1}}}{\sqrt{K_j\rho_j} + \sqrt{K_{j+1}\rho_{j+1}}}. \tag{1.5}$$

Write

$$R = (R_0, \ldots, R_n) \quad \text{and} \quad \tau = (\tau_0, \ldots, \tau_n).$$

The Green's function G is completely determined by the pair (τ, R). We incorporate this determinacy into the notation, writing

$$G^{(\tau,R)}$$

for the reflection Green's function (1.3). Thus media having a common pair (τ, R) of travel times and reflection coefficients are indistinguishable from one another with respect to reflection of waves at the depth z_{-1}. We shall regard them as the same, and refer to a pair (τ, R) as a medium, letting it be understood that an equivalence class of media is thereby represented.

2 The Green's function

It is well known that in general the Green's function has the form of a delta train,

$$G^{(\tau,R)}(t) = \sum_{j=1}^{\infty} a_j \delta(t - \sigma_j). \tag{2.1}$$

We refer to the numbers σ_j in (2.1) as *arrival times* and the a_j as *amplitudes*. The inverse problem relevant to imaging is to determine (τ, R) given

$$\chi_{(0,T]} G^{(\tau,R)}(t) = \sum_{j=1}^{d} a_j \delta(t - \sigma_j)$$

for some sufficiently large cutoff time $T > \tau_0 + \cdots + \tau_n$, (so that d depends on both τ and T). Thus from the perspective of inverse scattering, the sequences

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d) \quad \text{and} \quad a = (a_1, a_2, \ldots, a_d)$$

comprise the measured data.

The data (σ, a) depend on the medium parameters (τ, R) via scattering sequences, which may described in simple physical terms as follows. When a source pulse traveling from depth z_{-1} reaches z_0, part of the incident pulse is reflected back up toward z_{-1}, and part is transmitted down into the first layer toward z_1. When the transmitted pulse reaches z_1, part of it reflects back up toward z_0 and part is transmitted down into the second layer, and so on. Thus the initial pulse engenders a cascade of successive reflections and transmissions within the various layers, some of which eventually return back to z_{-1}, contributing to terms $a_j \delta(t - \sigma_j)$ in $G^{(\tau,R)}$. Figure 2 depicts an individual scattering sequence that starts and ends at z_{-1}. The arrival time σ_j of such a sequence is determined by the integer vector

$$k = (k_0, \ldots, k_n)$$

in which k_i counts the number of times the scattering sequence crosses back and forth across the ith layer. The arrival time is then simply

$$\sigma_j = \langle \tau, k \rangle. \tag{2.2}$$

The corresponding amplitude a_j is determined by (R, k) and has a much more complicated structure. It is the sum total of amplitudes of scattering sequences having common transit count vector k, of which there may be many. The combinatorial problem of counting the scattering sequences that correspond to a given k and computing their cumulative weights has until recently blocked the way to determining an explicit formula for a_j—but it has now been solved, in [4].

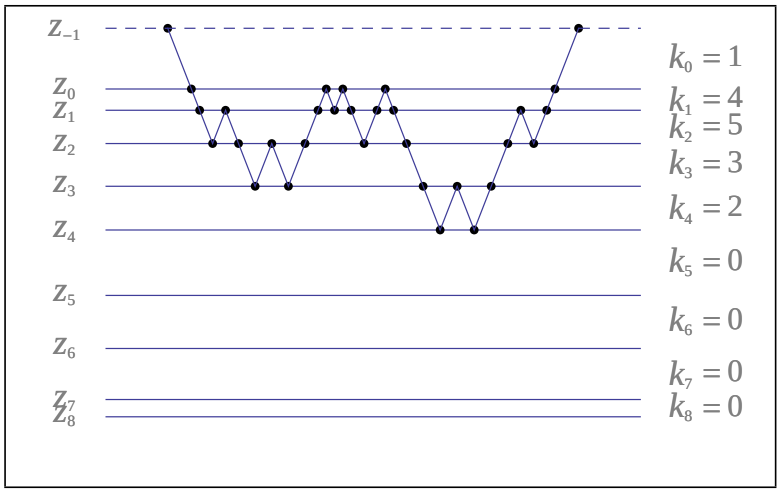

Figure 2: An example of a scattering sequence (time increases to the right). Each index k_j counts the number of times the scattering sequence traverses back and forth across the jth layer.

It follows from the above consideration of scattering sequences that $G^{(\tau,R)}$ may be represented in the form

$$G^{(\tau,R)}(t) = \sum_{k \in \mathbb{Z}^{n+1}} \alpha(R, k) \delta(t - \langle k, \tau \rangle), \tag{2.3}$$

where $\alpha(R, k) = 0$ unless k corresponds to a scattering sequence that starts and ends at z_{-1}. Thus the arrival time data σ up to cutoff time T is the sequence of scalars $\langle k, \tau \rangle < T$ (for appropriate k), arranged in their natural order. The first step of the inverse problem is to determine τ from σ and thereby establish a mapping

$$k \mapsto \langle k, \tau \rangle = \sigma_j, \tag{2.4}$$

between lattice points and arrival times. This is treated in Section 3, below, and in Appendix A.

Since arrival times and amplitudes are naturally paired, (2.4) induces a mapping from lattice points to amplitudes,

$$k \mapsto \sigma_j \mapsto a_j = \alpha(R, k). \tag{2.5}$$

The second step of the inverse problem is to recover R, given the mapping (2.5). We show in Section 4 that this second step may be viewed as a highly structured multivariate interpolation problem.

3 Inverse Projection

In this section we formalize the problem of recovering τ from arrival time data. The full solution is deferred to Appendix A.

By the representations (2.1) and (2.3), recovering τ from arrival times is equivalent to recovering a linear functional from its values on part of the integer lattice. The following notation will be used. Let L_τ denote the linear functional corresponding to a given $\tau \in \mathbb{R}^{n+1}_{>0}$, so that for $x \in \mathbb{R}^{n+1}$,

$$L_\tau(x) = \langle x, \tau \rangle.$$

The symbol $\mathcal{L}_n$ denotes the subset of $\mathbb{Z}^{n+1}_+$ consisting of realizable transit count vectors

$$k = (k_0, \ldots, k_n).$$

(See Figure 2.) That is, for every $k \in \mathbb{Z}^{n+1}_+$, $k \in \mathcal{L}_n$ if and only if $k_0 = 1$ and

$$k_j = 0 \Rightarrow k_{j+1} = 0 \quad (0 \le j \le n-1).$$

Let $\mathbb{1} \in \mathbb{R}^{n+1}$ denote the constant vector, each of whose entries is 1. Given $\tau \in \mathbb{R}^{n+1}_{>0}$, let $\mathcal{L}^\tau$ denote the set

$$\mathcal{L}^\tau = \left\{ k \in \mathcal{L}_n \,\middle|\, 0 < \langle k, \tau \rangle \le \langle \mathbb{1}, \tau \rangle \right\}.$$

Finally, it will be convenient to represent the set $L_\tau(\mathcal{L}^\tau)$ by its elements ordered in a vector, denoted $\Phi(\tau)$. That is, $\sigma = \Phi(\tau)$ means that

$$\sigma_1 < \cdots < \sigma_d \text{ and } \left\{ \sigma_1, \ldots, \sigma_d \right\} = L_\tau(\mathcal{L}^\tau).$$

This defines a mapping

$$\Phi : \bigcup_{n=1}^{\infty} \mathbb{R}^{n+1}_{>0} \to \bigcup_{n=1}^{\infty} \mathbb{R}^{n+1}_{>0}.$$

Note that the mapping Φ commutes with multiplication by a positive scalar: for any $\alpha > 0$ and $\tau \in \bigcup_{n=1}^{\infty} \mathbb{R}^{n+1}_{>0}$,

$$\alpha \Phi(\tau) = \Phi(\alpha \tau). \tag{3.1}$$

The precise problem of interest is to recover τ from $\Phi(\tau)$. Letting ℓ_τ denote the line in $\mathbb{R}^{n+1}$ spanned by τ, the problem of recovering τ from $\Phi(\tau)$ is called an *inverse projection* problem in reference to the fact that the entries of $\Phi(\tau)$ are the norms of the orthogonal projections of the lattice points $\mathcal{L}^\tau$ onto ℓ_τ, rescaled by $||\tau||$.

A full solution to this problem is given in Appendix A. Henceforth we take for granted that the inverse projection problem can be solved, and hence that the mapping (2.5) is known.

4 Interpolation of amplitudes

In this section we show that once τ is known, the inverse scattering problem can be formulated as a highly structured multivariate interpolation problem. The key step, undertaken in Section 4.1, is to express $G^{(\tau,R)}$ in terms of a function on $\mathbb{R}^{n+1}$ that we call a covering amplitude. The interpolation problem is then given in Section 4.2.

4.1 Covering amplitudes

It was shown in [4] that the amplitudes $a_j = \alpha(R,k)$ may be expressed in terms of a collection of polynomials

$$f^{(p,q)} : [-1,1] \mapsto [-1,1],$$

called amplitude factors. Below we encode these collectively into a single bivariate function ψ, called the total amplitude factor, from which the amplitude factors may be recovered by the formula

$$f^{(p,q)}(x) = \psi(p + \tfrac{1}{2\pi} \arcsin x, q). \tag{4.1}$$

We define the function ψ in terms of the classical Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \binom{n+\beta}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j},$$

as follows.

Definition 4.1 (Total amplitude factor). Given $(x,y) \in \mathbb{R}^2$, set $M = \lfloor \max\{x,y\} + \frac{1}{2} \rfloor$ and $m = \lfloor \min\{x,y\} + \frac{1}{2} \rfloor$. Using this notation, define

$$\psi : \mathbb{R}^2 \to [-1,1]$$

according to the table below.

region in $\mathbb{R}^2$	$\psi(x,y)$
$x<-\frac{1}{2}$ or $y<-\frac{1}{2}$	0
$-\frac{1}{2}\le x<\frac{1}{2}\le y$	0
$-\frac{1}{2}\le x,y<\frac{1}{2}$	1
$-\frac{1}{2}\le y<\frac{1}{2}\le x$	$\big(\sin 2\pi x\big)^M$
$\frac{1}{2}\le x\le y$	$(-1)^{M-m}\cos^2 2\pi x\big(\sin 2\pi x\big)^{M-m}P_{m-1}^{(M-m,1)}\big(\cos 4\pi x\big)$
$\frac{1}{2}\le y\le x$	$\frac{M}{m}\cos^2 2\pi x\big(\sin 2\pi x\big)^{M-m}P_{m-1}^{(M-m,1)}\big(\cos 4\pi x\big)$

We call ψ the total amplitude factor.

Note that the dependency of $\psi(x,y)$ on y is through the indices M and m; if $\lfloor y+\frac{1}{2}\rfloor=\lfloor y'+\frac{1}{2}\rfloor$ then $\psi(x,y)=\psi(x,y')$. See Figure 3.

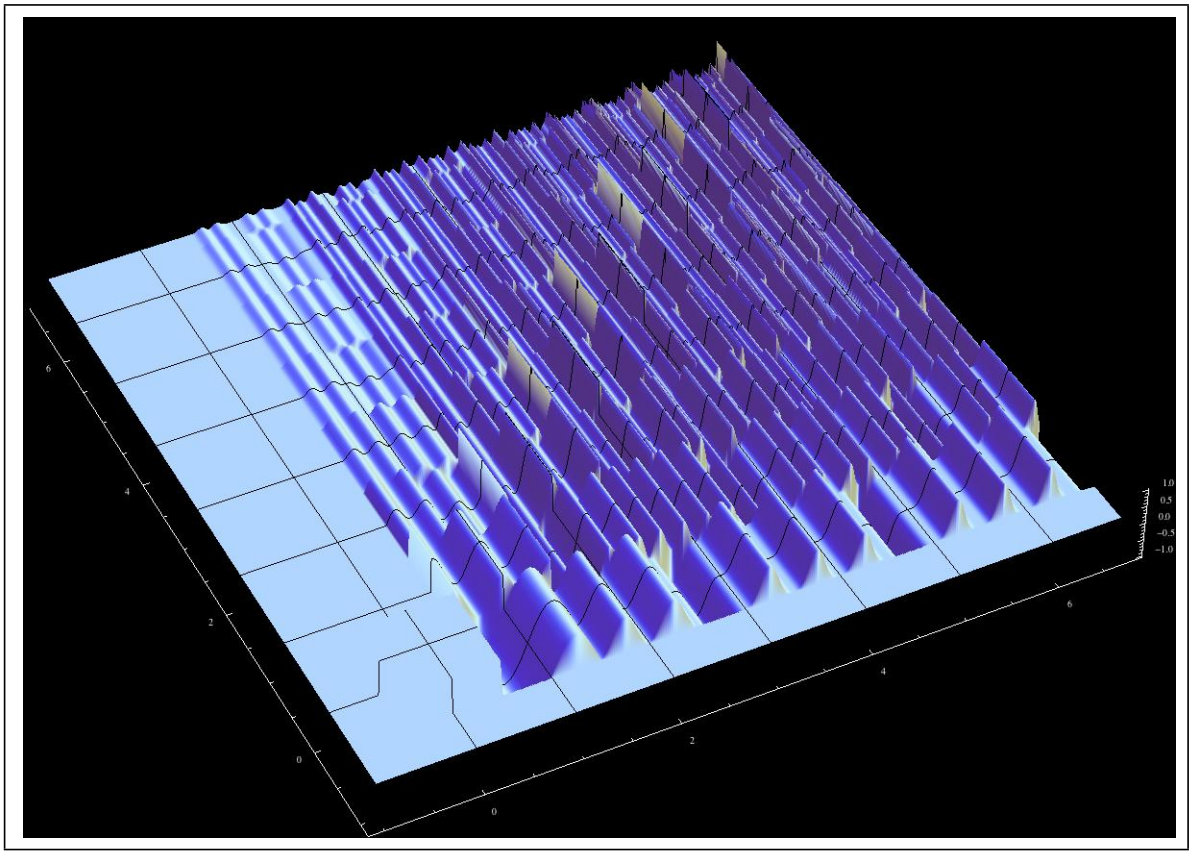

Figure 3: The total amplitude $\psi:\mathbb{R}^2\to[-1,1]$.

Definition 4.2 (Covering amplitude). For $n\ge 1$, the covering amplitude $\mathfrak{a}_n:\mathbb{R}^{n+1}\to[-1,1]$ is defined in terms of the total amplitude factor as follows. For $\xi=(\xi_0,\dots,\xi_n)\in\mathbb{R}^{n+1}$ let $\xi_{n+1}=0$ and set

$$\mathfrak{a}_n(\xi)=\chi_{[\frac{1}{2},\frac{3}{2})}(\xi_0)\prod_{j=0}^{n}\psi(\xi_j,\xi_{j+1}). \tag{4.2}$$

See Figure 4 for an illustration of the covering amplitude.

A key step toward formulating our interpolation problem is to represent the Green's function in terms of the covering amplitude.

Theorem 4.1. Given $(\tau,R)\in\mathbb{R}_+^{n+1}\times[-1,1]^{n+1}$, set

$$\theta=(\theta_0,\dots,\theta_n)=\big(\tfrac{1}{2\pi}\arcsin R_0,\dots,\tfrac{1}{2\pi}\arcsin R_n\big),$$

Then the Green's function is given by the formula

$$G^{(\tau,R)}(t)=\sum_{k\in\mathbb{Z}^{n+1}}\mathfrak{a}_n(\theta+k)\,\delta\big(t-\langle k,\tau\rangle\big). \tag{4.3}$$

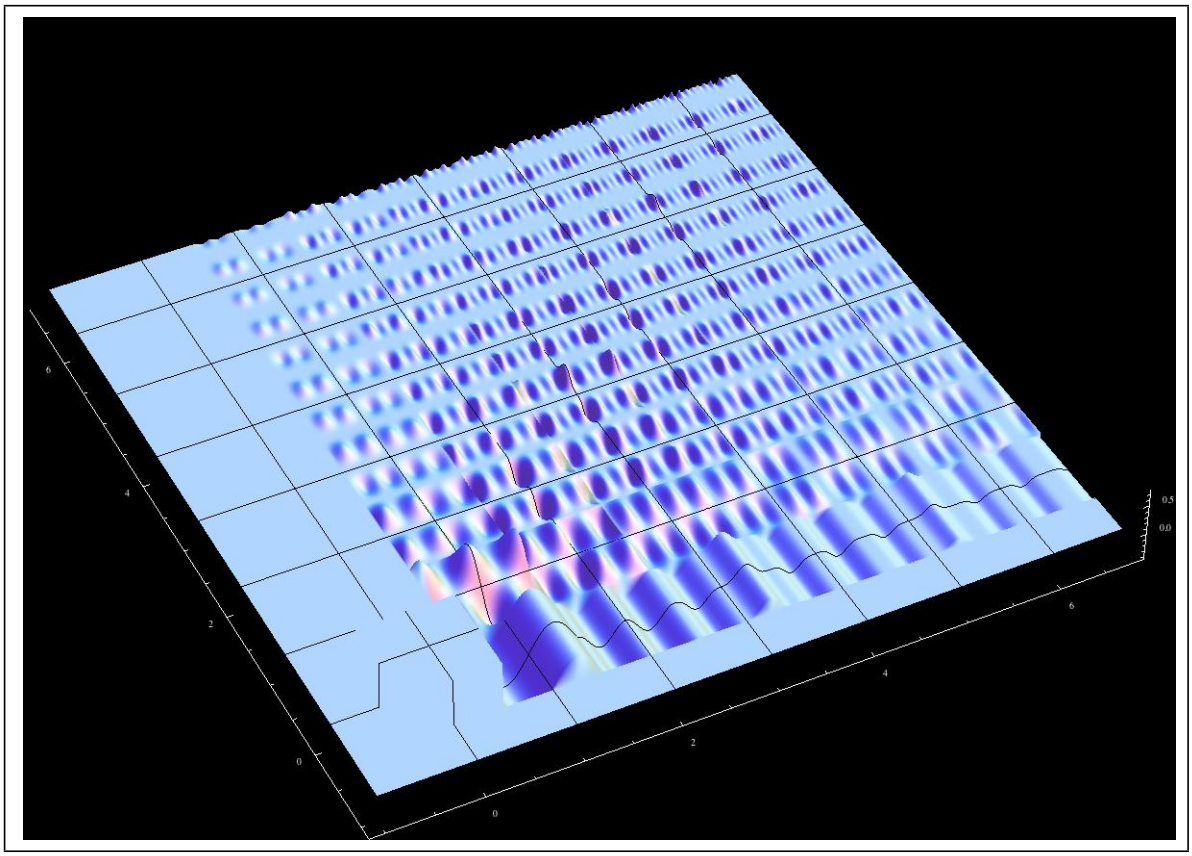

Figure 4: The surface $\mathfrak{a}_2(1.13, x, y)$ for $-1 < x, y < 6.75$.

Proof. It is proved in [4, §4] that

$$\alpha(R,k) = R_n^{k_n} \prod_{j=0}^{n-1} f^{(k_j,k_{j+1})}(R_j), \tag{4.4}$$

where the $f^{(p,q)}$ are defined as follows. Let $(p,q) \in \mathbb{Z}^2$. Set $\alpha = |p-q|$ and $m = \min\{p,q\} - 1$. If $m \geq 0$, then

$$f^{(p,q)}(x) = \begin{cases} (-x)^\alpha (1-x^2) P_m^{(\alpha,1)}(1-2x^2) & \text{if } p \leq q \\ \dfrac{p}{q}\, x^\alpha (1-x^2) P_m^{(\alpha,1)}(1-2x^2) & \text{if } p > q \end{cases}. \tag{4.5}$$

Using (4.1) and Definitions 4.1 and 4.2, it follows directly from (4.4) and (4.5) that

$$\alpha(R,k) = \mathfrak{a}_n(\theta + k).$$

The representation (2.3) then yields the present theorem. ∎

4.2 The interpolation problem

Given the measured data (σ, a), one can generically compute $\tau = (\tau_0, \ldots, \tau_n)$ from σ using Algorithm A.1 in Appendix A.0.2. Note that in particular the algorithm determines the index n. Let $\mathcal{K}$ denote the set of lattice points k such that $\langle k, \tau \rangle$ occurs as an arrival time in the data σ. For each $k \in \mathcal{K}$, let $j(k)$ denote the index such that

$$\langle k, \tau \rangle = \sigma_{j(k)}.$$

By Theorem 4.1, we have that

$$a_{j(k)} = \mathfrak{a}_n(\theta + k), \tag{4.6}$$

where

$$\theta = (\theta_0, \ldots, \theta_n) = \left(\tfrac{1}{2\pi} \arcsin R_0, \ldots, \tfrac{1}{2\pi} \arcsin R_n\right).$$

Recall that the function $\mathfrak{a}_n$ is known explicitly from Defintion 4.2. Thus in order to determine R, and hence the medium (τ, R), it suffices to solve the following interpolation problem.

Interpolation problem.
Determine the translate $\theta = (\theta_0, \dots, \theta_n)$ that minimizes

$$\left|\mathfrak{a}_n(\theta + k) - a_{j(k)}\right| \tag{4.7}$$

over $k \in \mathcal{K}$.

It turns out that R corresponds to the unique zero of the quanties (4.7) and hence the unique minimizer, as can be seen by studying the formulas (4.4) and (4.5). In fact, by restricting attention to lattice points of the form $k^{(j)}$, where

$$k_i^{(j)} = \begin{cases} 1 & \text{if } i \le j \\ 0 & \text{otherwise} \end{cases},$$

one can successively solve for R_0, R_1, and so on, using the formulas (4.4), which are especially simple for these lattice points. But the computation is highly unstable and sensitive to errors in the a_j.

A virtue of above formulation of the interpolation problem is that it is global, involving the full data set. Furthermore, the problem is highly structured in that the interpolating family consists of translates of a single function and the interpolation points lie on a lattice, a scenario for which there is an existing literature (e.g.,[6] and references therein). It remains to be seen which particular implementation is best suited to the problem.

5 Conclusion

We have given a new, conceptually simple, picture of the inverse scattering problem for layered media. This has been made possible by recent results concerning the combinatorics of scattering sequences, [4]. The new picture also offers an approach to treating measured data, by inverse projection, followed by multivariate interpolation—about which some further remarks are in order.

In practice (using a piezoelectric transducer, for example), one does not directly measure the time-limited Green's function $g = \chi_{(0,T]} G^{(\tau,R)}$, but rather a sampled version of $g * w$, where $w(t)$ is compactly supported source waveform. So it is necessary first to extract g from this measured data. While there is no consensus as to how this should be done, recent results on superresolution, [2], suggest that convex optimization with respect to the ℓ_1 norm may give highly accurate results. Thus it is not necessarily unrealistic to expect the g be known, even beyond the sampling rate of a digital instrument.

From the theoretical point of view, it is a remarkable fact that a single bivariate function $\psi : \mathbb{R}^2 \to [-1, 1]$ governs the normal incidence scattering process for all layered media, whatever the physical parameters or number of layers. Moreover, ψ has an explicit formulation (Definition 4.1). This is a new result.

A Appendix

Here we detail a solution to the inverse projection problem stated in Section 3.

A.0.1 Factorization of $\Phi(\tau)$ when L_τ is injective on $\mathcal{L}_\tau$

For $x \in \mathbb{R}^{n+1}$, let $x^\perp$ denote its orthogonal complement, the hyperplane

$$x^\perp = \left\{ y \in \mathbb{R}^{n+1} \,\middle|\, \langle y, x \rangle = 0 \right\}.$$

Let H_x^+ denote the closed half space bounded by $x^\perp$ on which L_x is non-negative,

$$H_x^+ = \left\{ y \in \mathbb{R}^{n+1} \,\middle|\, \langle y, x \rangle \ge 0 \right\}.$$

Given $\tau \in \mathbb{R}^{n+1}_{>0}$ and $k \in \mathcal{L}_n$, observe that $k \in \mathcal{L}^\tau$ if and only if $\tau \in H^+_{\mathbb{1}-k}$. The observation below follows directly from the established notation.

Proposition A.1. Given $\tau \in \mathbb{R}^{n+1}_{>0}$, the linear functional L_τ fails to be injective on $\mathcal{L}^\tau$ if and only if τ belongs to

$$H^+_{\mathbb{1}-k} \cap (k' - k)^\perp \tag{A.1}$$

for some pair of distinct lattice points $k, k' \in \mathcal{L}_n$.

Let $\mathcal{H}_n$ denote the union of all sets of the form $H_k^+ \cap (k - k')^\perp$, where $k \ne k'$ belong to $\mathcal{L}_n$. Each of these sets is either empty, a half hyperplane, or a hyperplane (in the case $k' = \mathbb{1}$). Since $\mathcal{H}_n$ has measure zero in $\mathbb{R}^{n+1}$, it follows that for a generic set of $\tau \in \mathbb{R}^{n+1}_{>0}$, namely

$$\mathcal{G}_n = \mathbb{R}^{n+1}_{>0} \setminus \mathcal{H}_n,$$

the linear functional L_τ is injective on $\mathcal{L}^\tau$. Restricted to

$$\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n,$$

the lattice projection problem can be expressed as a factorization problem, as follows. For every $\tau \in \mathcal{G}_n$ (viewed as a column vector) there is a unique matrix A whose rows belong to $\mathcal{L}_n$ such that

$$A\tau = \Phi(\tau). \tag{A.2}$$

This is a direct consequence of injectivity of L_τ on $\mathcal{L}^\tau$; the rows of A are simply the elements k of $\mathcal{L}^\tau$, ordered from top to bottom according to increasing value of $L_\tau(k)$. Uniqueness of the factorization (A.2) fails precisely when L_τ is not injective on $\mathcal{L}^\tau$, i.e., when $\tau \in \cup_{n=1}^{\infty} \mathcal{H}_n$.

Factorization problem. Let $\sigma \in \Phi(\mathcal{G})$ be given. Determine $N \geq 1$, $\tau \in \mathbb{R}_{>0}^{N+1}$, and an integer matrix A such that $\sigma = A\tau$, where the rows of A belong to $\mathcal{L}_N$.

A.0.2 The solution algorithm

The following algorithm gives a direct method to compute the integer matrix A and the vector τ from their product $\widetilde{\sigma} \in \Phi(\mathcal{G})$, as required in the factorization problem. In fact any *primary subvector* σ of $\widetilde{\sigma} \in \Phi(\mathcal{G})$ suffices, as follows.

Definition A.2 (Primary subvector). Let $\widetilde{\sigma} \in \Phi(\mathcal{G}_n)$ have factorization $A\tau$. Any subvector σ of $\widetilde{\sigma}$ (obtained by deleting entries) that includes all of the numbers $\tau_0 + \cdots + \tau_j$, for $0 \leq j \leq n$, is called a primary subvector of $\widetilde{\sigma}$.

Algorithm A.1. Input:

Let $\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_d \end{pmatrix}$ be a primary subvector of some $\widetilde{\sigma} \in \Phi(\mathcal{G})$.

Initial Step: Set $\tau_0 = \sigma_1$, set $\tau^0 = (\tau_0)$ (viewed as a 1×1 vector), and set

$$\mathcal{L}_0^\tau = \left\{ k \in \mathcal{L}_0 \,\middle|\, \tau_0 \leq k\tau_0 \leq \sigma_d \right\}.$$

Construct σ^0 from σ by deleting the entries of σ belonging to $L_{\tau^0}(\mathcal{L}_0^\tau)$.

Continuing Step: Let $\tau^n = \begin{pmatrix} \tau_0 \\ \vdots \\ \tau_n \end{pmatrix}$, $\mathcal{L}_n^\tau$ and σ^n be given. If $\sigma^n = \emptyset$ then set $N = n$, $\tau = \tau^n$, and go to the output step.

Otherwise, set $\tau_{n+1} = \sigma_1^n - \langle \mathbb{1}, \tau^n \rangle$, set $\tau^{n+1} = \begin{pmatrix} \tau_0 \\ \vdots \\ \tau_{n+1} \end{pmatrix}$, and set

$$\mathcal{L}_{n+1}^\tau = \left\{ k \in \mathcal{L}_{n+1} \,\middle|\, k_{n+1} \geq 1 \text{ and } \langle k, \tau^{n+1} \rangle \leq \sigma_d \right\}.$$

Construct σ^{n+1} from σ^n by deleting the elements of $L_{\tau^{n+1}}(\mathcal{L}_{n+1}^\tau)$ from σ^n.

Output Step: For $0 \leq n \leq N$, extend each element of $\mathcal{L}_n^\tau$ by a string of $N-n$ zeros to form $\widetilde{\mathcal{L}}_n^\tau \subset \mathbb{Z}_+^{N+1}$. Construct A to be the $d \times (N+1)$ array whose rows consist of the elements k of $\bigcup_{n=0}^{N} \widetilde{\mathcal{L}}_n^\tau$, ordered such that $L_\tau(k)$ increases from the top row to the bottom. Output the pair (A, τ).

The fact that the pair (A, τ) solves the factorization problem for the given input σ rests on the following straighforward proposition.

Proposition A.3. The value $\tau_0 + \cdots + \tau_{n+1}$ is the least element of

$$L_\tau(\mathcal{L}^\tau) \setminus \bigcup_{j=0}^{n} L_{\tau^j}(\mathcal{L}_j^\tau),$$

for each n in the range $0 \leq n \leq N-1$.

Theorem A.1. If σ is a primary subvector of $\widetilde{\sigma} \in \Phi(\mathcal{G})$ then the corresponding output (A, τ) of Algorithm A.1 satisfies $\widetilde{\sigma} = A\tau$.

Proof. Let $(\widetilde{A}, \widetilde{\tau})$ be the unique pair such that $\widetilde{\sigma} = \widetilde{A}\widetilde{\tau}$, with $\widetilde{\tau} \in \mathcal{G}$. If

$$(\tau_0, \ldots, \tau_n) = (\widetilde{\tau}_0, \ldots, \widetilde{\tau_n})$$

then Proposition A.3 implies that in the Continuing Step, σ_1^n, which is the least element of σ^n, has the form

$$\sigma_1^n = \widetilde{\tau}_0 + \cdots + \widetilde{\tau}_{n+1},$$

so that $\sigma_1^n - \langle \mathbb{1}, \tau^n \rangle$ is exactly $\widetilde{\tau}_{n+1}$. It follows by induction that $\tau = \widetilde{\tau}$.

Observe that $A\widetilde{\tau} = \widetilde{\sigma}$ by construction. Therefore $\widetilde{A} = A$ since the equation $X\widetilde{\tau} = \widetilde{\sigma}$ has a unique solution by definition of $\mathcal{G}$. ∎

Algorithm A.1 is related to the well-known method of "surface calculations" as described, for example, in [7]. One important difference, however, is that our version is decoupled from amplitudes, and makes clear that the method's validity is restricted to generic travel time vectors.

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