Subdivision Schemes for Geometric Modelling

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Outline

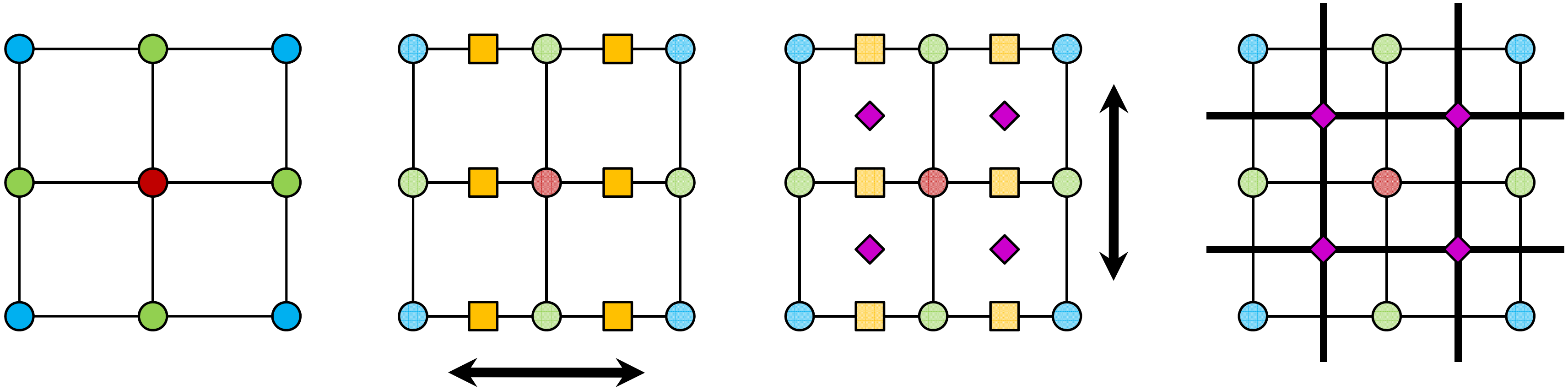
- Sep 5 – Subdivision as a linear process
  - basic concepts, notation, subdivision matrix
- Sep 6 – The Laurent polynomial formalism
  - algebraic approach, polynomial reproduction
- Sep 7 – Smoothness analysis
  - Hölder regularity of limit by spectral radius method
- ***Sep 8 – Subdivision surfaces***
  - *overview of most important schemes & properties*

Lane–Riesenfeld algorithm

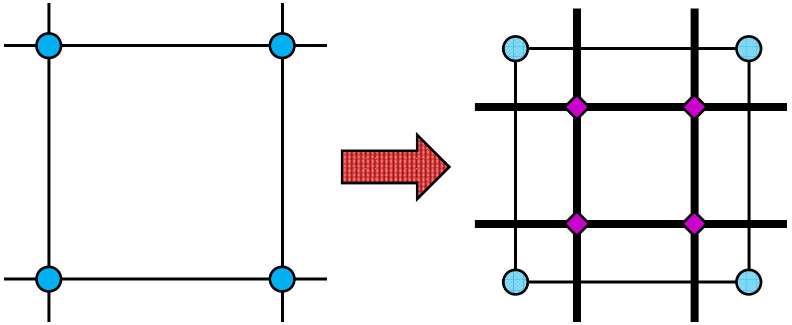
- multiplying the symbol by $(1+z)/2$ increases the smoothness of the limit curves by 1
  - geometrically, this averages the control points
- **Lane–Riesenfeld algorithm**
  - each refinement step first inserts edge midpoints
  - then applies $m-1$ averaging steps
  - symbol for these schemes: $a(z) = \left(\frac{1+z}{2}\right)^{m-1}\frac{(1+z)^2}{2}$
  - regularity of the limit curves: $r = m+1-\log_2(2) = m$
  - limit curves are uniform B-splines of degree m

Tensor-product schemes

- extend this idea to quadrilateral meshes
  - first insert edge and face midpoints
    - splits the quadrilateral into four new quadrilaterals
  - then apply averaging steps
    - each averaging step averages in two directions



Doo–Sabin subdivision



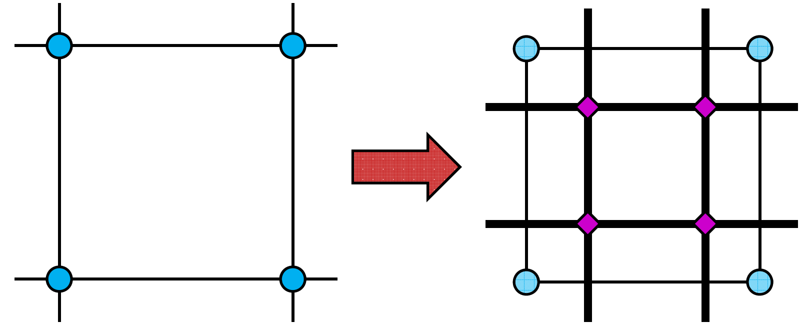
- **Example**
  - 1 averaging step
  - dual scheme
    - 4 new points per face, discard old points
  - 4 stencils for the new points

$$\frac{1}{16}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}, \quad \frac{1}{16}\begin{bmatrix}3 & 9\\1 & 3\end{bmatrix}, \quad \frac{1}{16}\begin{bmatrix}3 & 1\\9 & 3\end{bmatrix}, \quad \frac{1}{16}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}$$

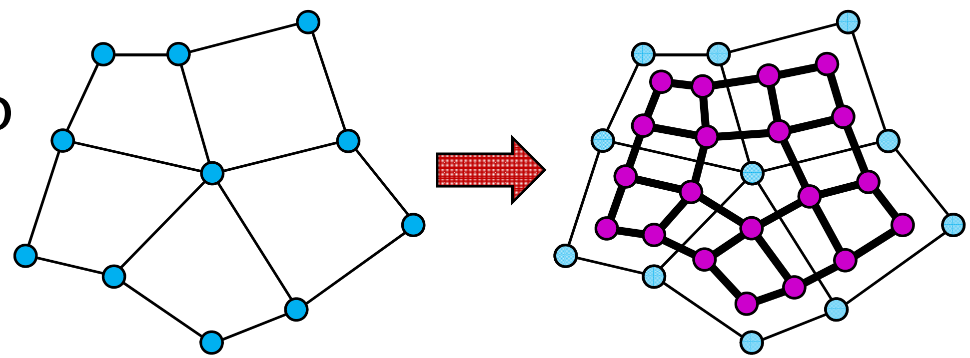
  - tensor products of the stencils [3,1]/4 and [1,3]/4 from Chaikin's scheme, e.g. $\frac{1}{16}\begin{bmatrix}9 & 3\\3 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}3\\1\end{bmatrix} \otimes [3,1]/4$

Doo–Sabin subdivision

- **Example**
  - for a *regular* quad mesh, this gives tensor-product quadratic B-splines
    - C^1 limit surfaces

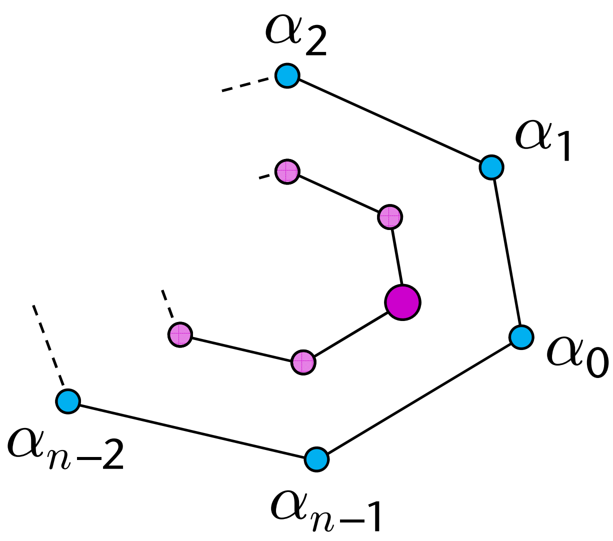


  - a general quad mesh has *extraordinary* vertices
    - where not 4, but 3, 5, or even more faces meet
    - non-quadrilateral faces after one refinement step



  - special rules and analysis needed

Doo–Sabin subdivision



Example

- general stencil for a new point in a face with n vertices

$$\alpha_0 = \frac{1}{4} + \frac{5}{4n}$$

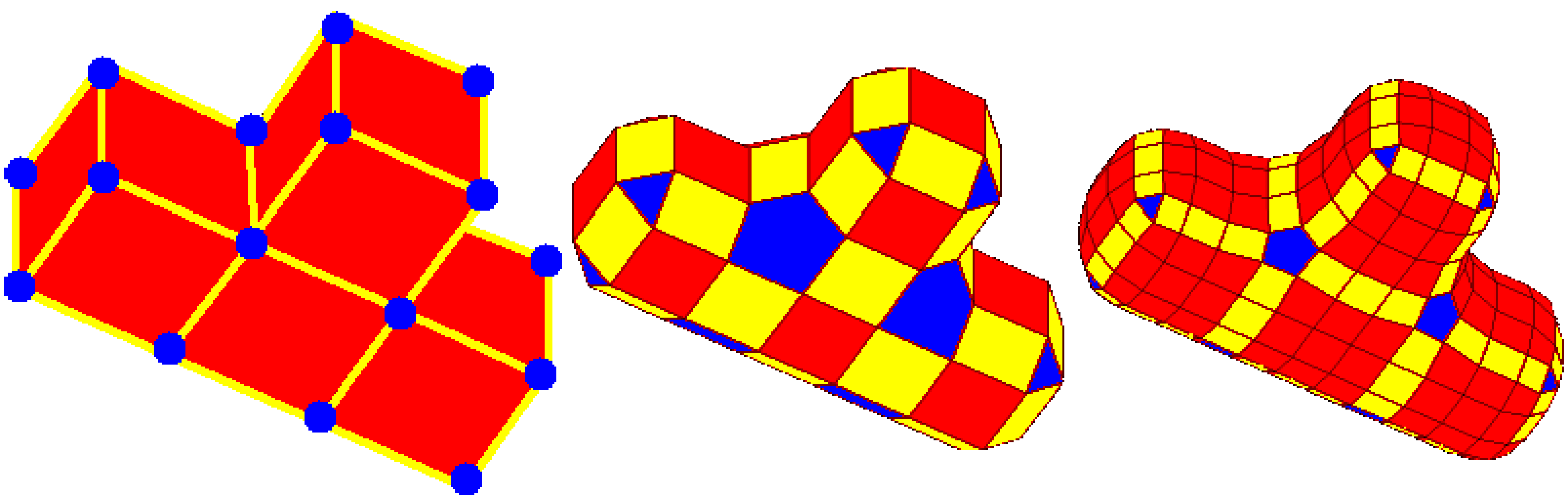
$$\alpha_i = \frac{3+2\cos(2\pi i/n)}{4n}, \quad i = 1,\ldots,n-1$$

  - note: stencil coefficients sum to 1
  - reduces to the regular stencil above for $n=4$

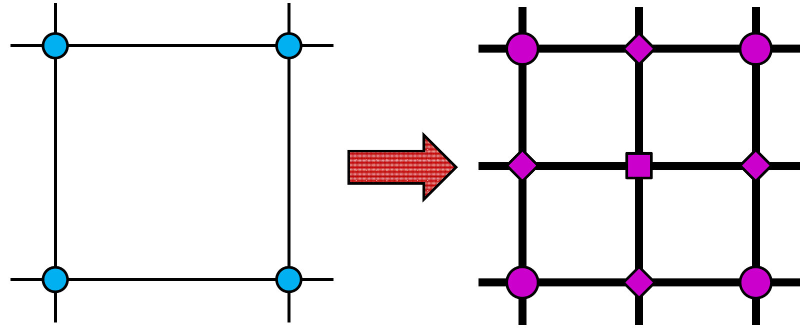
- limit surfaces are C^1-continuous

Doo–Sabin subdivision

- **Example**
  - duality of the scheme
    - valence-n vertex $\to$ n-gon, edge $\to$ 4-gon, n-gon $\to$ n-gon
    - all vertices are regular (valence 4) after first refinement
    - number of irregular faces remains the same

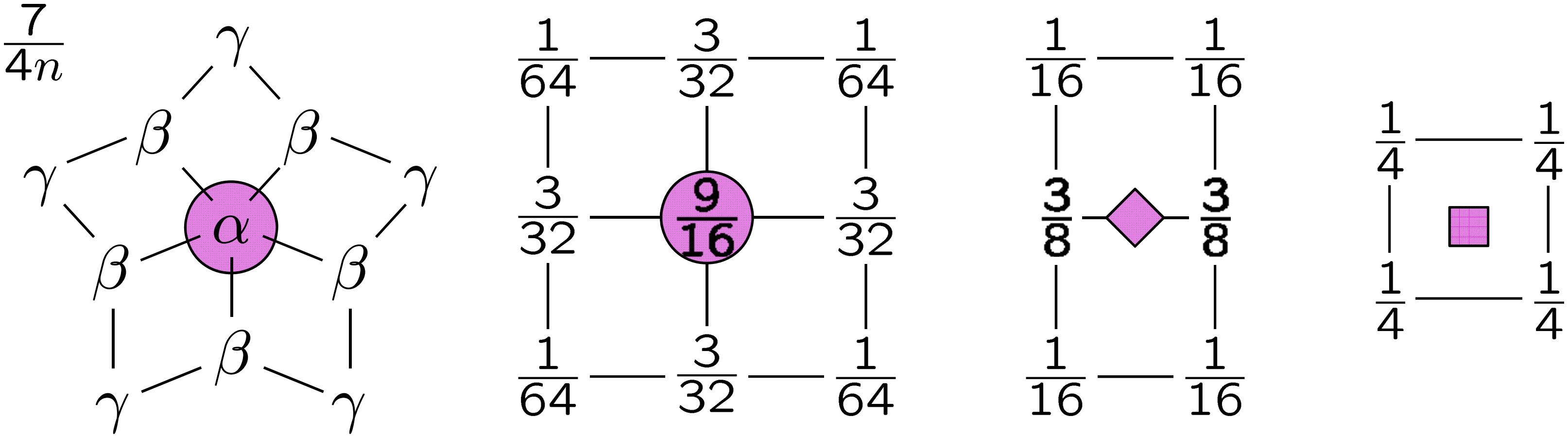


Catmull–Clark subdivision

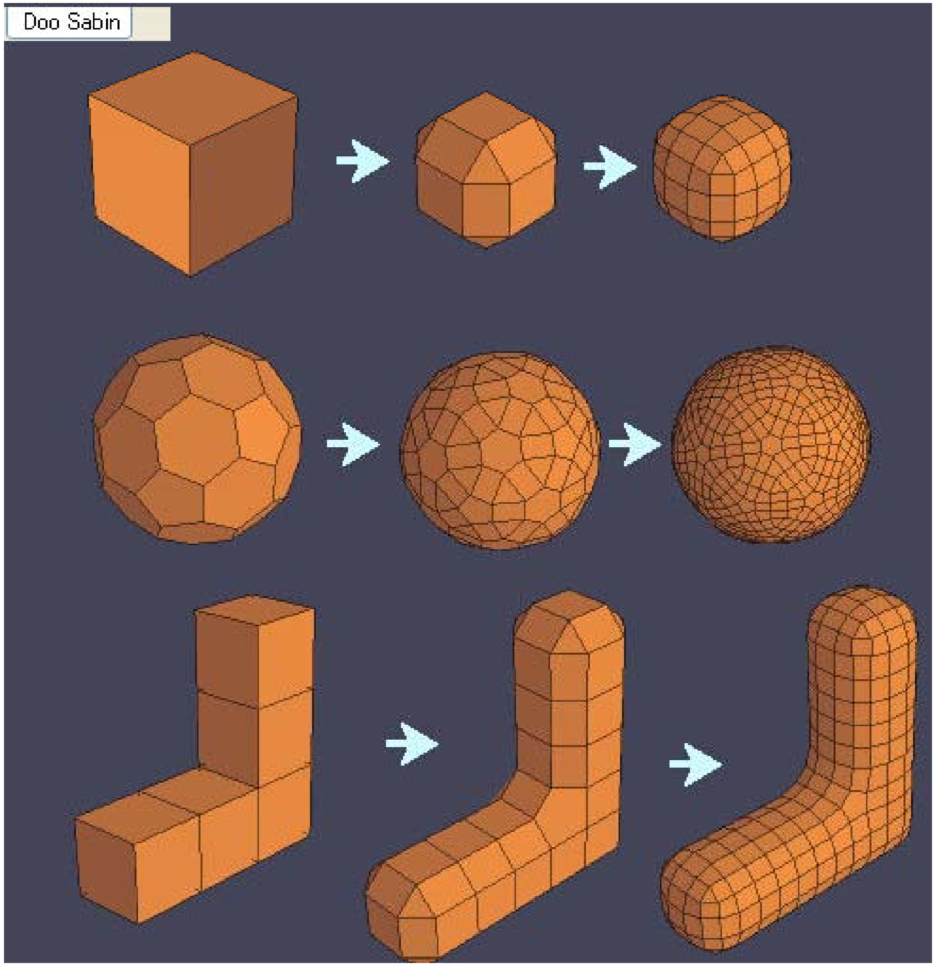


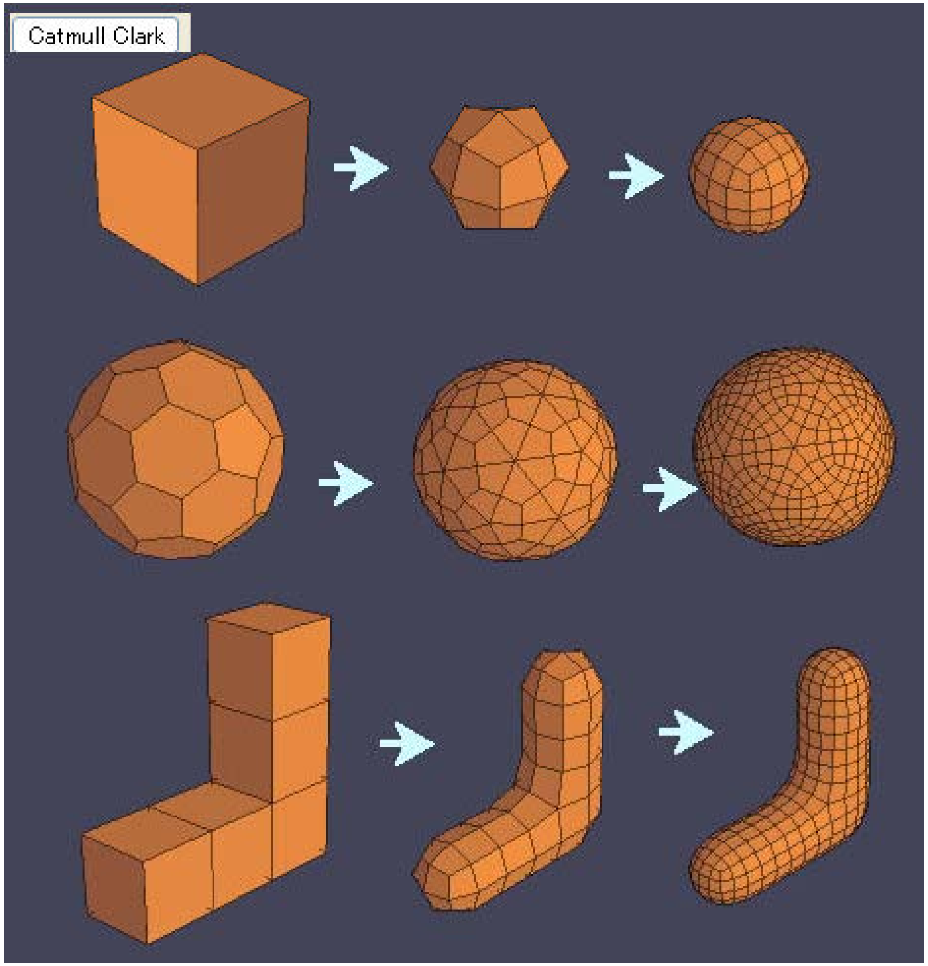
- **Example**
  - 2 averaging steps
  - primal scheme
    - new vertex (●), edge (◆), and face (■) points
  - stencils
    - regular setting: tensor product of cubic B-spline stencils

$$\alpha = 1 - \frac{7}{4n}$$
$$\beta = \frac{6}{4n^2}$$
$$\gamma = \frac{1}{4n^2}$$



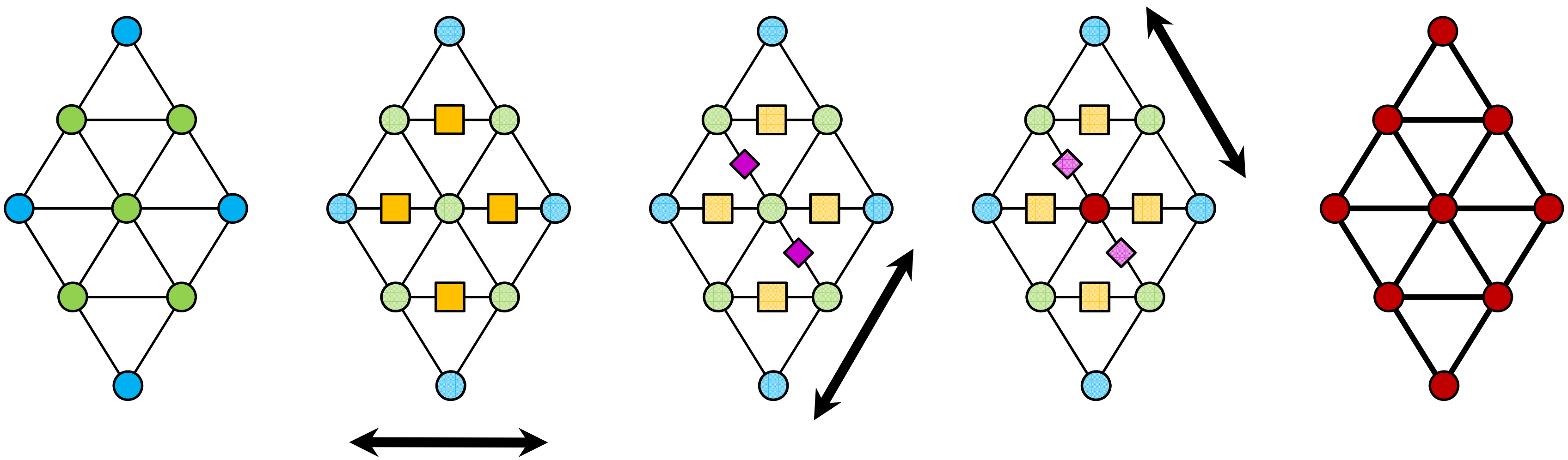
Comparison



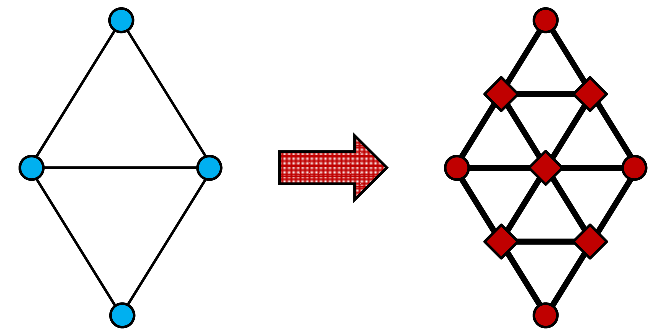


Triangle meshes

- extend Lane–Riesenfeld to triangle meshes
  - first insert edge midpoints
    - splits the triangles into four new triangles
  - then apply averaging steps
    - each averaging step averages in three directions



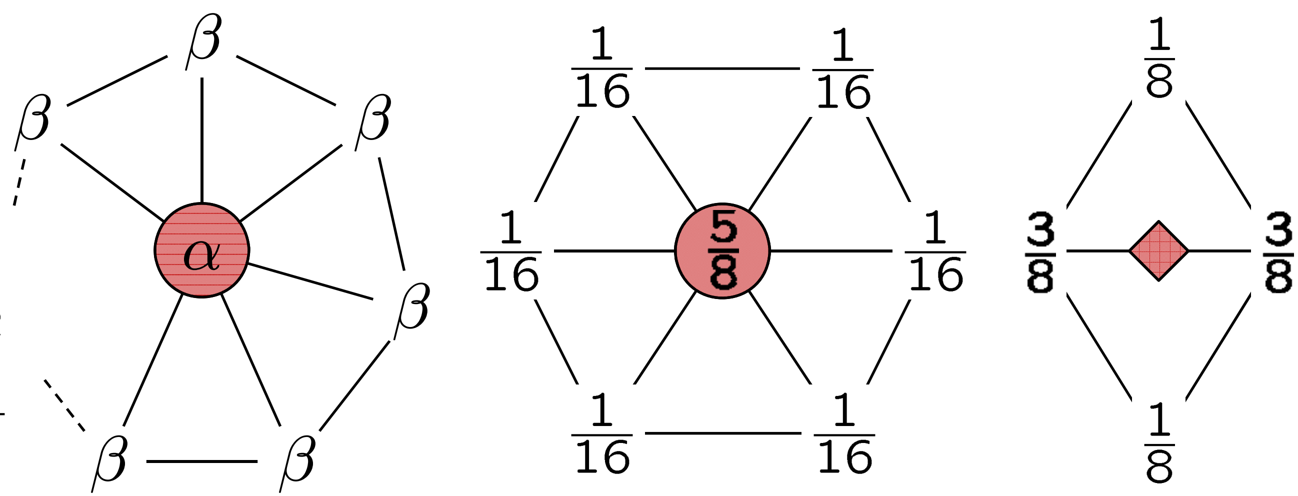
Loop subdivision



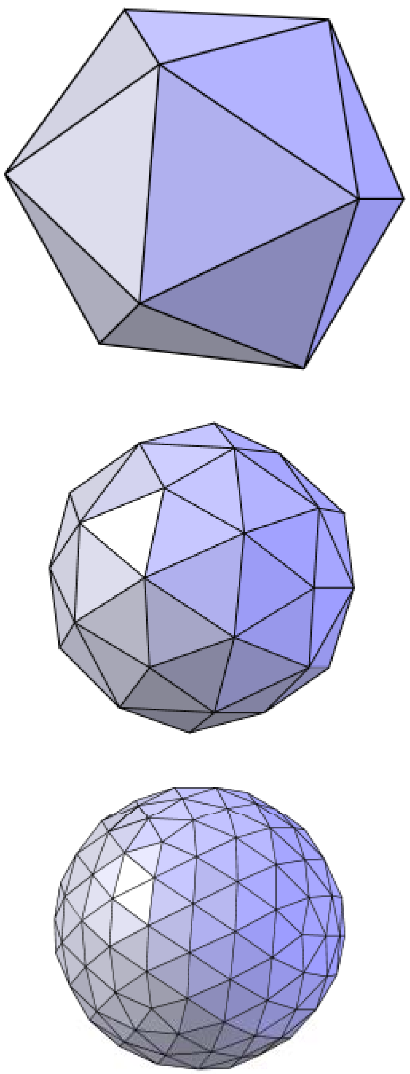
- **Example**
  - 1 averaging step
  - primal scheme
    - new vertex (●) and edge (◆) points
  - stencils

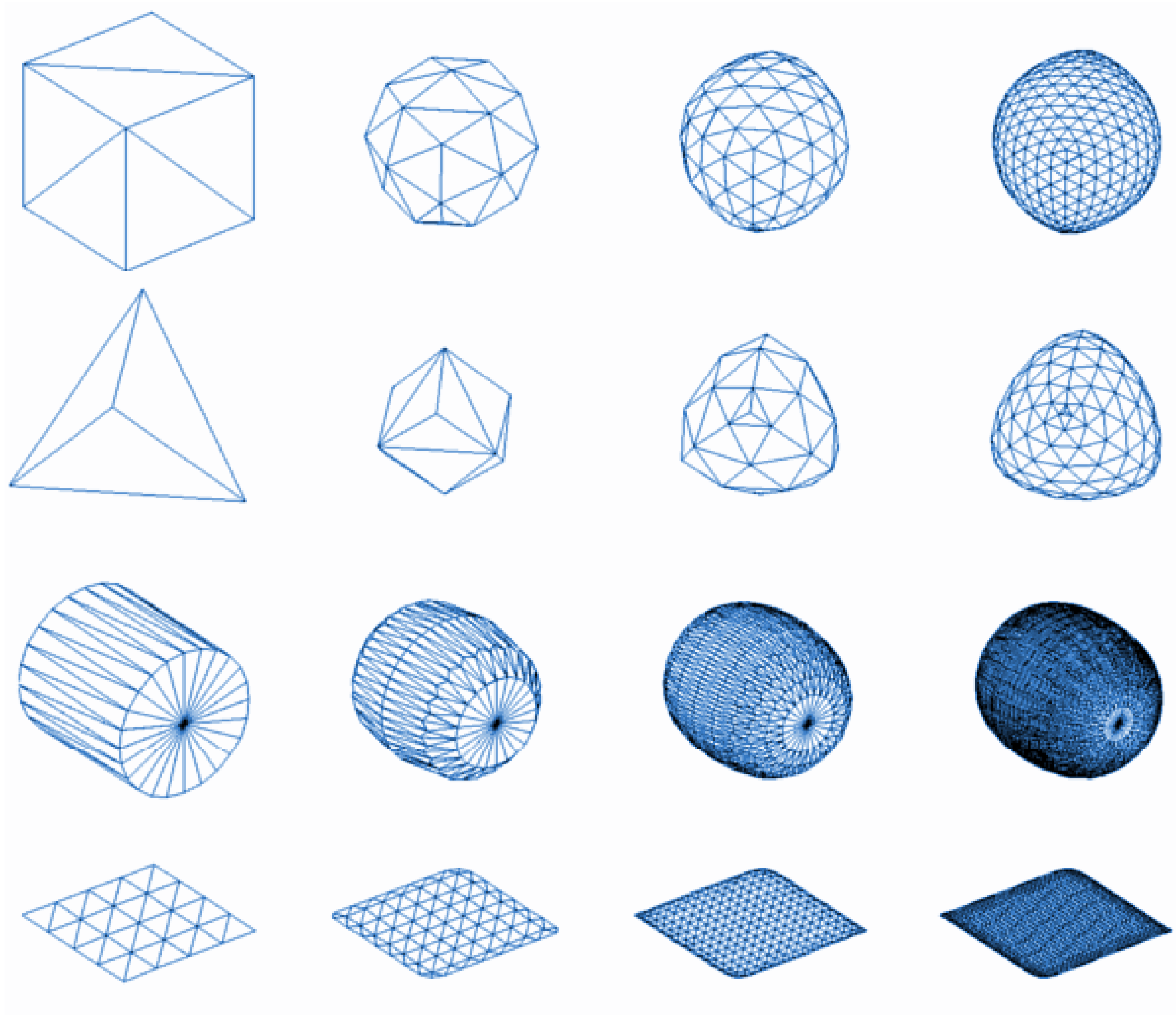
$$\alpha = 1 - n\beta$$

$$\beta = \frac{\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2}{n}$$



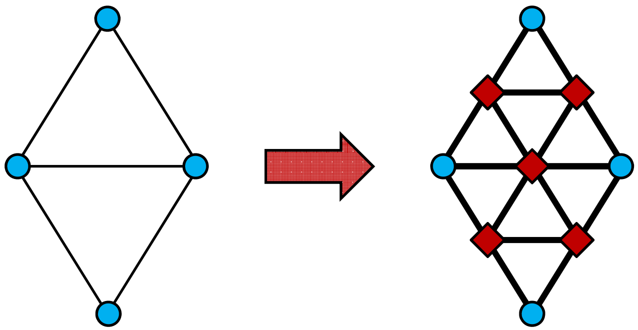
Loop subdivision

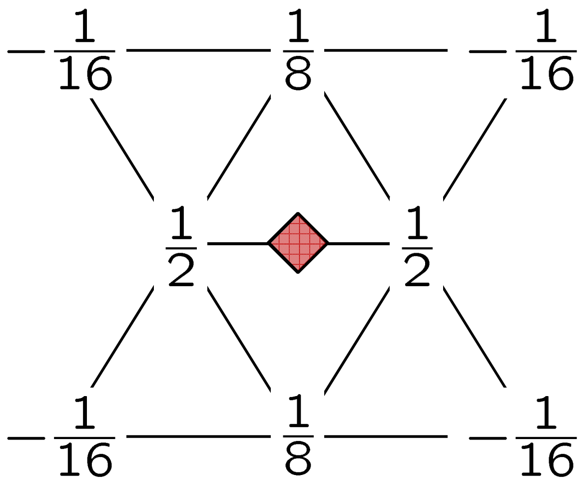




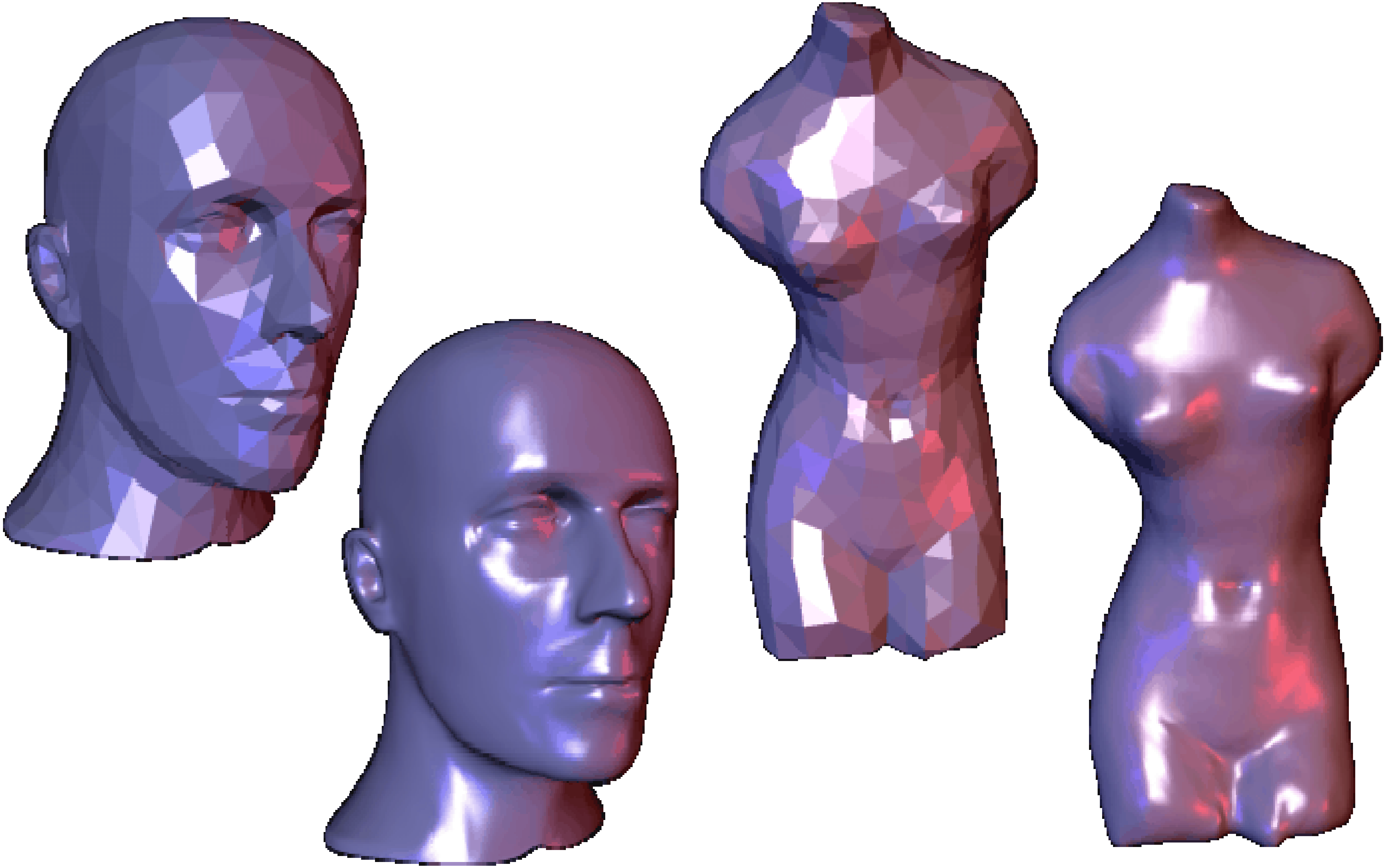
Butterfly subdivision

- extend idea of the 4-point scheme to meshes
  - keep old points, insert a new point for each edge
- **Example**
  - Butterfly scheme
  - stencil for the new point
    - derived from fitting a local interpolating bivariate cubic polynomial (only 8 instead of 10 degrees of freedom because of symmetry)





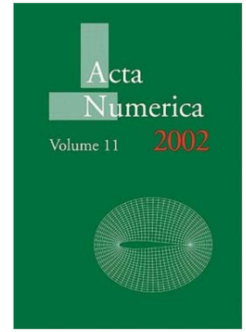
Butterfly scheme



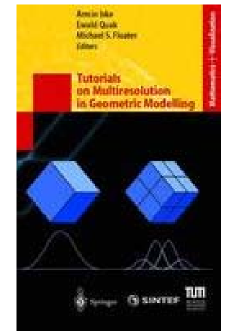
Summary

- "refine-and-smooth" algorithm by Lane and Riesenfeld gives the B-spline curve schemes
- idea can be extended to surface schemes
  - Doo–Sabin $\to C^1$ surfaces
  - Catmull–Clark $\to C^1$ surfaces (C^2 in regular region)
  - Loop $\to C^1$ surfaces (C^2 in regular region)
- interpolatory schemes, based on local polynomial interpolation and evaluation
  - Butterfly $\to C^1$ surfaces (after minor modification)

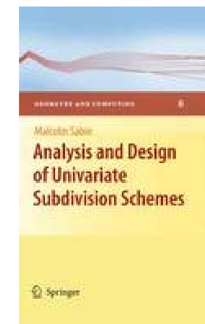
Literature



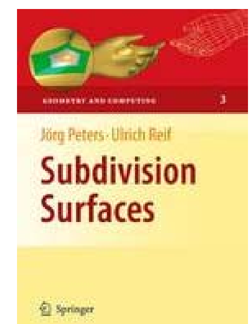
- N. Dyn, D. Levin
  Subdivision Schemes in Geometric Modelling
  Acta Numerica, Vol. 11, 2002, pp. 73–144



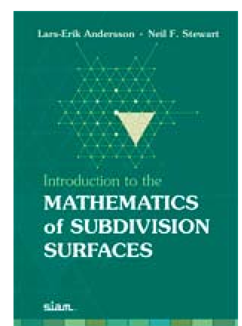
- A. Iske, E. Quak, M.S. Floater
  Tutorials on Multiresolution in Geometric Modelling
  Chapters 1–4 (N. Dyn, M. Sabin), Springer 2002



- M. Sabin
  Analysis and Design of Univariate Subdivision Schemes
  Springer 2010



- J. Peters, U. Reif
  Subdivision Surfaces
  Springer 2008



- L.-E. Andersson, N. F. Stewart
  Mathematics of Subdivision Surfaces
  SIAM 2010