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Geometric conditions for the reconstruction of a holomorphic function by an interpolation formula

Amadeo Irigoyen[a]

Abstract

We give here some precisions and improvements about the validity of the explicit reconstruction of any holomorphic function on a ball of $\mathbb{C}^2$ from its restrictions on a family of complex lines. Such validity depends on the mutual distribution of the lines. This condition can be geometrically described and is equivalent to a stronger stability of the reconstruction formula in terms of permutations and subfamilies of these lines. The motivation of this problem comes from possible applications in mathematical economics and medical imaging.

1 Introduction

1.1 Setting of the problem and some reminders

In this paper we give some answers and improvements of the results from [9], where we deal with a special case of the general problem of reconstruction of a holomorphic function from its restrictions on a family of analytic submanifolds. Here the setting is the following: on the one hand, we consider for the analytic submanifolds any family of complex lines in $\mathbb{C}^2$ that cross the origin. Such a family can be written as

$$\left\{z\in\mathbb{C}^2,\, z_1-\eta_j z_2=0\right\}_{j\geq 1}\,, \tag{1}$$

where the directions $\eta_j\in\mathbb{C}$ are all different (we omit the special line $\{z_2=0\}$). On the other hand, let be $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ (resp. $f\in\mathcal{O}(B_2(0,r_0))$ where for any fixed $r_0>0$, $B_2(0,r_0)\subset\mathbb{C}^2$ is the complex ball defined as

$$B_2(0,r_0) \quad=\quad \left\{z\in\mathbb{C}^2\,,\ |z_1|^2+|z_2|^2<r_0^2\right\}).$$

We then want to give an effective reconstruction of f from its restrictions on these complex lines. An application of some methods from [2] yields the following interpolation formula, that we remind from [9]:

$$\begin{aligned} E_N(f;\eta)(z) \;:=\; &\sum_{p=1}^{N}\left(\prod_{j=p+1}^{N}(z_1-\eta_j z_2)\right)\sum_{q=p}^{N}\frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2}\,\frac{1}{\prod_{j=p,j\neq q}^{N}(\eta_q-\eta_j)}\,\times \\ &\times\sum_{m\geq N-p}\left(\frac{z_2+\overline{\eta_q}z_1}{1+|\eta_q|^2}\right)^{m-N+p}\frac{1}{m!}\frac{\partial^m}{\partial v^m}|_{v=0}[f(\eta_q v,v)]\,, \end{aligned} \tag{2}$$

where $N\geq 1$ and $z=(z_1,z_2)\in\mathbb{C}^2$. We know (see Proposition 3 from [9]) that for all $N\geq 1$ and $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ (resp. $f\in\mathcal{O}(B_2(0,r_0))$), $E_N(f;\eta)$ is well-defined and satisfies the following properties:

- $E_N(f;\eta)\in\mathcal{O}\left(\mathbb{C}^2\right)$ (resp. $E_N(f;\eta)\in\mathcal{O}(B_2(0,r_0))$);
- $E_N(f;\eta)$ is an explicit formula that is constructed with the data

$$\left\{f_{|\{z_1=\eta_j z_2\}}\right\}_{1\leq j\leq N}\;;$$

- $\forall\, j=1,\ldots,N$, $E_N(f;\eta)_{|\{z_1=\eta_j z_2\}}=f_{|\{z_1=\eta_j z_2\}}$;
- $\forall\, P\in\mathbb{C}[z_1,z_2]$ with $\deg P\leq N-1$, $E_N(P;\eta)\equiv P$.

One reason for the choice of a family of lines (1) is that it is well suited for the methods in [2], which readily produce formula (2). But the essential reason comes from possible applications to the real Radon transform theory, that may have consequences in mathematical economics and medical imaging. Indeed, let μ be a measure with compact support $K\subset\mathbb{R}^2$ (w.l.o.g. one can assume that $0\in K$). We want to reconstruct it from its Radon transforms on a finite number of directions, i.e. from $(\mathcal{R}\mu)\left(\theta^{(j)},s\right)$ with $\left(\theta^{(j)},s\right)\in\mathbb{S}^1\times\mathbb{R}$ and $j=1,\ldots,N$, where $\mathbb{S}^1$ is the unit sphere of $\mathbb{R}^2$ and

$$(\mathcal{R}\mu)\left(\theta^{(j)},s\right) \quad:=\quad \frac{\partial}{\partial s}\int_{\{x\in\mathbb{R}^2,\,\theta_1^{(j)}x_1+\theta_2^{(j)}x_2\leq s\}}\mu(dx)\,. \tag{3}$$

[a] Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Spain

As it was explained at the Introduction of [9], we consider the Fantappie transform Φ_μ of μ, that is defined on the dual space $K^\star := \left\{\xi = [\xi_0 : \xi_1 : \xi_2] \in \mathbb{C}\mathbf{P}^2,\ <\xi, x> \neq 0,\ \forall\, x \in K\right\}$ and is holomorphic there. Explicitly,

$$\Phi_\mu \ :\ \xi \in K^\star \quad \mapsto \quad <\mu, \frac{\xi_0}{<\xi, x>}> := \int_{x\in K} \frac{\xi_0}{<\xi, x>}\mu(dx)$$

(see [11]). In addition, we know that there is $r_K > 0$ such that for all $\theta \in \mathbb{S}^1$ and all $u \in \mathbb{C}$ with $|u| < r_K$ (so that $[1 : u\theta_1 : u\theta_2] \in K^\star$),

$$\Phi_\mu([1 : u\theta_1 : u\theta_2]) \quad = \quad \int_{-\infty}^{+\infty} \frac{(\mathcal{R}\mu)(\theta, s)}{1 + su}\, ds\,,$$

i.e. the knowledge of $(\mathcal{R}\mu)(\theta^{(j)}, s)$, $j = 1, \ldots, N$, $s \in \mathbb{R}$, allows to know the restriction of $\Phi_\mu \in \mathcal{O}(B_2(0, r_K))$ on every line $L_{\theta^{(j)}} = \{(u\theta_1, u\theta_2),\ u \in \mathbb{C}\} = \{z \in \mathbb{C}^2,\ z_1 = \eta_j z_2\}$ where

$$\eta_j \quad = \quad \theta_1^{(j)}/\theta_2^{(j)} \in \mathbb{R},\ j = 1, \ldots, N \tag{4}$$

(w.l.o.g. one can assume that $\theta_2^{(j)} \neq 0$ for all $j = 1, \ldots, N$).

The family of measures defined for $N \geq 1$ by $\mu_N := \Phi^{-1}\left[E_N(\Phi_\mu; \eta)\right]$ (where $E_N(\cdot; \eta)$ is the above fomula (2), and Φ^{-1} is the reciprocal isomorphism, whose existence is guaranteed by [11]), is interpolating in the meaning that $<\mu_N, x_1^k x_2^l> = <\mu, x_1^k x_2^l>$ for all $N \geq 1$ and $k, l \geq 0$ with $k + l \leq N$. Since by (4) the set $\left\{\eta_j\right\}_{j\geq 1}$ is a subset of $\mathbb{R}$, by an application of Theorem 1.1 below, we will conclude that the family $\{\mu_N\}_{N\geq 1}$ will approximate μ in an appropriate topology. In addition, an application of some results of Henkin and Shananin from [8] will allow to compute the reconstruction with good estimates. These expected results are handled in [10], that is currently in progress.

1.2 Essential results

The essential problem is that there is no guarantee that, as $N \to \infty$, $E_N(f; \eta)$ will converge to f (although it coincides with f on an increasing number of lines). We know from [9] that in general it is not the case, i.e. there are families of lines with (at least) an associated holomorphic function f such that $E_N(f; \eta)$ will not converge. Since we are interested in a reconstruction formula whose convergence is guaranted for every holomorphic function f, we want to determine all the *good* families of lines $\eta = \left(\eta_j\right)_{j\geq 1}$ for which the convergence of the associated interpolation formula $E_N(\cdot; \eta)$ is guaranteed for every holomorphic function. Theorems 1 and 2 from [9] give equivalent criteria for the validity of such a reconstruction: roughly speaking, the sequence of the directions $\left(\eta_j\right)_{j\geq 1}$ of the lines (1) must satisfy an exponential estimate of their divided differences (an operator of successive discrete derivatives, see for example [5], [7] and [12] for the definition and essential results).

This study is related to a classical problem that is the reconstruction of a holomorphic function on a homogeneous domain of $\mathbb{C}^n$ from some partial data on its boundary (see [1], [3] and [4]). Here we deal with a particularly delicate case since it brings us back to some one-dimensional sets on the boundary (the union of circles that are the intersections of the complex lines in (1) with any sphere). The restriction of the function on these circles determines the Taylor coefficients of its corresponding restrictions to the complex lines. It follows that the above criterion is related to the configuration of the associated circles on the boundary (i.e., a countable union of one-dimensional real sets). That is the essential difference with the case from Theorems 21.1 and 21.4 in [1], where the associated set has a topological closure of real dimension greater than 1.

Nevertheless, the difficulty to check the above condition on divided differences gives us the motivation to find a criterion that is easier to understand. This leads to the following definition:

Definition 1.1. The set $\{\eta_j\}_{j\geq 1}$ is *locally interpolable by real-analytic curves* if, for all $\zeta \in \overline{\{\eta_j\}_{j\geq 1}}$ (the topological closure of $\{\eta_j\}_{j\geq 1}$ in $\mathbb{C}\mathbf{P}^1$), there exist a neighborhood V of ζ and a smooth real-analytic curve $\mathcal{C}$ such that $\zeta \in \mathcal{C}$ and

$$V \bigcap \left\{\eta_j\right\}_{j\geq 1} \quad \subset \quad \mathcal{C}\,. \tag{5}$$

This new geometric condition is a sufficient criterion for the convergence of the interpolation formula $E_N(\cdot; \eta)$ and yields the following result, given as Theorem 3 from [9].

Theorem 1.1. *If $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves, then the interpolation formula $E_N(f, \eta)$ converges to f uniformly on any compact $K \subset \mathbb{C}^2$ and for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$.*

Similarly, r_0 being given, there is $\varepsilon_\eta > 0$ such that, for all $f \in \mathcal{O}(B_2(0, r_0))$, $E_N(f; \eta)$ converges to f uniformly on any compact subset $K \subset B_2(0, \varepsilon_\eta r_0)$.

Nevertheless, this new criterion is not equivalent. Indeed, as it has been suspected in the Introduction of [9], there are sequences of lines that are not locally interpolable by real-analytic curves and whose associated formula $E_N(\cdot; \eta)$ converges. Proposition 1.2 below is the first result of this paper and gives an explicit example of such a family: it consists on constructing a sequence $(\eta_j)_{j\geq 1}$ as the increasing union of $1/2^r$-nets, $r \geq 0$, of the square $[0, 1] + i[0, 1] = \{z \in \mathbb{C},\ 0 \leq \Re(z),\ \Im(z) \leq 1\}$ (so that for all $N \geq 1$, the first N points η_j's are the *most* separated possible from each other).

Proposition 1.2. *There exists (at least) one sequence $\left(\eta_j\right)_{j\geq 1}$ that is not locally interpolable by real-analytic curves but whose associated interpolation formula $E_N(\cdot; \eta)$ converges, i.e. for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $E_N(\cdot; f)$ converges to f uniformly on any compact subset $K \subset \mathbb{C}^2$ (similarly, $r_0 > 0$ being fixed, there is $\varepsilon_\eta > 0$ such that for all $f \in \mathcal{O}(B_2(0, r_0))$, $E_N(f; \eta)$ converges to f uniformly on any compact subset $K \subset B_2(0, \varepsilon_\eta r_0)$).*

This first conclusion leads to the following question: why is this geometric criterion not (always) necessary? On the other hand, the expression (2) of $E_N(\cdot;\eta)$ clearly involves the enumeration of the lines η_j's. Since Definition 1.1 is a condition about sets that does not depend on any of its enumerations, one is tempted into considerating the action of the group of permutations $\mathfrak{S}_{\mathbb{N}}$ and check the validity of the convergence of $E_N(\cdot;\sigma(\eta))$, where the sequence $\sigma(\eta)$ is defined from $\eta = \left(\eta_j\right)_{j\geq 1}$ by

$$\sigma(\eta) \quad = \quad \left(\eta_{\sigma(j)}\right)_{j\geq 1} \tag{6}$$

(in order to simplify the notation, $\mathfrak{S}_{\mathbb{N}}$ will mean $\mathfrak{S}_{\mathbb{N}\setminus\{0\}}$ since all the considered sequences in the paper start by $j=1$).

Now $\sigma \in \mathfrak{S}_{\mathbb{N}}$ being given, one could first think that $E_N(f;\sigma(\eta))$ and $E_N(f;\eta)$ are essentially the *same*. Indeed, if $M_N := \max\{N, \sigma(1), \ldots, \sigma(N)\}$, then $E_{M_N}(f;\eta)$ and $E_{M_N}(f;\sigma(\eta))$ both interpolate f on the N first lines corresponding to $\eta_1, \ldots, \eta_N$ and $\eta_{\sigma(1)}, \ldots, \eta_{\sigma(N)}$. Nevertheless, if we change the order of the sequence of the square from Proposition 1.2 above, the associated interpolation formula may not converge anymore. This leads to the following question: a given sequence $\left(\eta_j\right)_{j\geq 1}$ whose associated interpolation formula $E_N(\cdot;\sigma(\eta))$ always converges under the action of any permutation σ, should be locally interpolable by real-analytical curves? We will see that the answer is affirmative as claimed by Theorem 1.3 below.

In order to deal with this problem, we need to consider the following question: the sequence $\eta = \left(\eta_j\right)_{j\geq 1}$ being fixed, if the formula $E_N(\cdot;\eta)$ converges, what about $E_N(\cdot;\eta')$, where $\eta' := \left(\eta_{j_k}\right)_{k\geq 1}$ is any given (infinite) subsequence of η? On a first sight, the answer looks positive because of the following intuitive argument: if $E_N(f;\eta)$ can interpolate f on *more* lines than $E_{N'}(f;\eta')$ does (where N' is the number of $k \geq 1$ such that $j_k \leq N$) and $E_N(f;\eta)$ converges, then why should not $E_N(f;\eta')$ too? The true answer is that this heuristic argument is false. Indeed, it is also a strong condition that is equivalent to the geometric criterion (5). This claim and the above one are specified by the following result, that is the main theorem of this paper.

Theorem 1.3. *Let $\eta = \left(\eta_j\right)_{j\geq 1}$ be any sequence defined as in (1). The following conditions are equivalent:*

1. *the set $\left\{\eta_j\right\}_{j\geq 1}$ is locally interpolable by real-analytic curves;*
2. *for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and all $\sigma \in \mathfrak{S}_{\mathbb{N}}$, $E_N(f;\sigma(\eta))$ converges to f uniformly on any compact subset $K \subset \mathbb{C}^2$;*
3. *for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and all subsequence $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$, $E_N\left(f;\eta'\right)$ converges to f uniformly on any compact subset $K \subset \mathbb{C}^2$.*

First, this result finally gives an equivalence, for a given sequence $\left(\eta_j\right)_{j\geq 1}$, between the strong geometric hypothesis (5) and sharper conditions in terms of the validity of the convergence of the associated interpolation formula $E_N(\cdot;\eta)$. In particular, it clarifies in which sense this geometric condition is sufficient.

Next, this results only deals with the convergence of $E_N(f;\eta)$ for any $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ and we would like to know what happens if we consider the same assertion with any $f \in \mathcal{O}(B_2(0,r_0))$ for any fixed $r_0 > 0$. One of the applications of Theorem 1.3 is its generalization to the case of every complex ball $B_2(0,r_0)$, as specified by the following result.

Corollary 1.4. *Let $\eta = \left(\eta_j\right)_{j\geq 1}$ be any sequence defined as in (1) and let be $r_0 > 0$. The following conditions are equivalent:*

1. *the set $\left\{\eta_j\right\}_{j\geq 1}$ is locally interpolable by real-analytic curves;*
2. *there is $\varepsilon_\eta > 0$ such that for all $f \in \mathcal{O}(B_2(0,r_0))$ and all $\sigma \in \mathfrak{S}_{\mathbb{N}}$, $E_N(f;\sigma(\eta))$ converges to f uniformly on any compact subset $K \subset B_2\left(0,\varepsilon_\eta r_0\right)$;*
3. *there is $\varepsilon_\eta > 0$ such that for all $f \in \mathcal{O}(B_2(0,r_0))$ and all subsequence $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$, $E_N\left(f;\eta'\right)$ converges to f uniformly on any compact subset $K \subset B_2\left(0,\varepsilon_\eta r_0\right)$.*

As it can be noticed, these results are equivalences between a geometric condition and the validity of the convergence of its associated interpolation formula $E_N(\cdot;\eta)$ (i.e. in terms of functional approximation theory).

Moreover, we have another consequence that gives some precision on the speed of convergence of $E_N(f;\eta)$ to f.

Corollary 1.5. *When any of the equivalent conditions from Theorem 1.3 or Corollary 1.4 is fulfilled, one has in addition the following estimate: for all $\mathcal{K} \subset \mathcal{O}\left(\mathbb{C}^2\right)$ (resp. $\mathcal{K} \subset \mathcal{O}(B_2(0,r_0))$) and $K \subset \mathbb{C}^2$ (resp. $K \subset B_2(0,\varepsilon_\eta r_0)$) compact subsets, there are $C_{\mathcal{K},K}$ and $\varepsilon_K > 0$ such that for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$, for all $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$ and all $N \geq 1$,*

$$\sup_{f\in\mathcal{K}} \sup_{z\in K} |f(z) - E_N(f;\sigma(\eta))(z)| \quad \leq \quad C_{\mathcal{K},K}\left(1-\varepsilon_K\right)^N$$

and

$$\sup_{f\in\mathcal{K}} \sup_{z\in K} \left|f(z) - E_N\left(f;\eta'\right)(z)\right| \quad \leq \quad C_{\mathcal{K},K}\left(1-\varepsilon_K\right)^N .$$

In particular, as it has been pointed out in [9], a simple convergence of $E_N(\cdot;\eta)$ (i.e. convergence of $E_N(f;\eta)$ for every fixed holomorphic function f) implies a uniform one. This can be interpreted as a Banach-Steinhaus property for the family of operators $\{E_N(\cdot;\eta)\}_{N\geq 1}$ in the canonical topology for the holomorphic functions (i.e. the topology of uniform convergence on any compact subset).

Finally, the essential argument for the proof of Theorem 1.3 follows from the following result, whose proof is given in Section 3.

Proposition 1.6. *Let $\left(\eta_j\right)_{j\geq 1}$ be any sequence such that the set $\left\{\eta_j\right\}_{j\geq 1}$ is not locally interpolable by real-analytic curves. Then there exists a subsequence $\left(\eta_{j_k}\right)_{k\geq 1}$ of $\left(\eta_j\right)_{j\geq 1}$ that satisfies the following conditions:*

- *the sequence $\left(\eta_{j_k}\right)_{k\geq 1}$ is convergent in $\mathbb{C}\mathbf{P}^1$;*
- *the set $\left\{\eta_{j_k}\right\}_{k\geq 1}$ is not locally interpolable by real-analytic curves.*

We know that if $\left\{\eta_j\right\}_{j\geq 1}$ (coming from any sequence $\left(\eta_j\right)_{j\geq 1}$) is locally interpolable by real-analytic curves, then so will be any of its subsets (finite or infinite), in particular if it comes from a convergent subsequence $\left(\eta_{j_k}\right)_{k\geq 1}$. Conversely, we may ask what happens if $\left\{\eta_j\right\}_{j\geq 1}$ is not locally interpolable by real-analytic curves. Proposition 1.6 gives an affirmative answer, in the sense that one can extract convergent subsequences that are still not locally interpolable by real-analytic curves.

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2 On the non-equivalence of the geometric criterion

In this section we deal with the proof of Proposition 1.2. We first need some reminders and preliminary results.

2.1 Some reminders

First, the following result is given as Lemma 20 from [9] and is a necessary condition for a set to satisfy the geometric condition (5).

Lemma 2.1. *The topological closure of a set that is locally interpolable by real-analytic curves, has empty interior.*

Next, we remind Theorem 1 as one of the essential results from [9] and that gives an equivalent criterion for a bounded sequence $\left(\eta_j\right)_{j\geq 1}$ to make converge its associated interpolation formula $E_N(\cdot;\eta)$.

Theorem 2.2. *Let $\left(\eta_j\right)_{j\geq 1}$ be bounded and fix any $r_0>0$. The following conditions are equivalent:*

1. *there is $\varepsilon_\eta>0$ such that, for all $f\in\mathcal{O}(B_2(0,r_0))$, the interpolation formula $E_N(f;\eta)$ converges to f, uniformly on any compact subset of $B_2(0,\varepsilon_\eta r_0)$;*
2. *for all $g\in\mathcal{O}\left(\mathbb{C}^2\right)$, the interpolation formula $E_N(g;\eta)$ converges to g, uniformly on any compact subset of $\mathbb{C}^2$;*
3. *$\exists R_\eta\geq 1$, $\forall\, p,q\geq 0$,*

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right]\left(\eta_{p+1}\right)\right| \quad\leq\quad R_\eta^{p+q}\,. \tag{7}$$

The operator Δ_p is called *divided differences* and is defined as follows (for any application h that is defined at the η_j's):

$$\begin{cases}\Delta_0[h](\eta_1)=h(\eta_1)\,;\\[2ex] \text{for all } p\geq 1\,,\ \Delta_{p,(\eta_p,\ldots,\eta_1)}[h](\eta_{p+1}) \ = \ \dfrac{\Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}[h](\eta_{p+1})-\Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}[h](\eta_p)}{\eta_{p+1}-\eta_p}\,.\end{cases} \tag{8}$$

$\Delta_p[h]$ can be seen as the discrete derivative of order p of the function h. A lot its properties can be found in the references, in particular the following one (see for example [6], Chapter 4, 7 (7.7)).

Lemma 2.3. *Let $\{\eta_j\}_{j\geq 1}$ be any set of different points and h any function defined on them. One has for all $p\geq 0$,*

$$\Delta_{p,(\eta_p,\ldots,\eta_1)}[h]\left(\eta_{p+1}\right) \quad=\quad \sum_{q=1}^{p+1}\frac{h\left(\eta_q\right)}{\prod_{j=1,j\neq q}^{p+1}\left(\eta_q-\eta_j\right)}\,.$$

We also deduce as an application the following result that will be useful for the proof of Theorem 1.3.

Corollary 2.4. *For all p, $q\geq 0$,*

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\zeta\mapsto\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right]\left(\eta_{p+1}\right)\right| \quad\leq\quad \sum_{l=1}^{p+1}\frac{1}{\prod_{j=1,j\neq l}^{p+1}\left|\eta_l-\eta_j\right|}\,.$$

In particular, the bound does not depend on $q\geq 0$.

Proof. The proof immediately follows by Lemma 2.3 with the particular choice of $h(\zeta) = \left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q$ since for all $\zeta \in \mathbb{C}$, one has that

$$\left|\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right| \;=\; \left(\frac{|\zeta|}{1+|\zeta|^2}\right)^q = \left(\sqrt{\frac{|\zeta|^2}{1+|\zeta|^2}}\right)^q \times \frac{1}{\left(\sqrt{1+|\zeta|^2}\right)^q} \;\leq\; 1\,.$$

□

2.2 Construction of a counterexample

In this subsection, we construct the explicit sequence $(\eta_j)_{j\geq 1} \subset \mathcal{Q}$, where $\mathcal{Q}$ is the closed square

$$\mathcal{Q} \;=\; [0,1] + i[0,1] \;=\; \{z \in \mathbb{C}\,,\; 0 \leq \Re(z),\, \Im(z) \leq 1\}\,. \tag{9}$$

Its essential required property is that the η_j's must be the *most* separated possible from each other. We start by setting $\eta_1 = 0$, $\eta_2 = 1$, $\eta_3 = 1+i$, $\eta_4 = i$. We find the maximal number of points of $\mathcal{Q}$ whose mutual distance is not smaller than 1. When it is not possible anymore, we add the maximal number of points whose mutual distance is at least $1/2$, then $\eta_5 = 1/2$, $\eta_6 = i/2$, $\eta_7 = (1+i)/2$, $\eta_8 = 1+i/2$, $\eta_9 = 1/2+i$. More generally, we will choose by induction on $r \geq 0$ the maximal number of points whose mutual distance is at least $1/2^r$.

Let fix $r \geq 0$ and let $\mathcal{A}_r$ be an $1/2^r$-net of $\mathcal{Q}$, i.e. a set of points that are at least at a distance of $1/2^r$ from each other. One can choose

$$\mathcal{A}_r \;=\; \left\{\frac{s+it}{2^r}\,,\,(s,t) \in \mathbb{N}^2\,,\, 0 \leq s,t \leq 2^r\right\}\,, \tag{10}$$

whose cardinal is $(1+2^r)^2$ (one can check that $\mathcal{A}_0 = \{0,1,i,1+i\} = \{\eta_1,\eta_2,\eta_3,\eta_4\}$). In addition, one has the sequence of inclusions

$$\mathcal{A}_0 \;\subset\; \mathcal{A}_1 \;\subset\; \cdots \;\subset\; \mathcal{A}_r \;\subset\; \cdots\,. \tag{11}$$

The sequence $\eta = (\eta_j)_{j\geq 1}$ will be defined by induction on $r \geq 0$ as follows: we first choose η_1, η_2, η_3 and η_4 for the first set $\mathcal{A}_0$ (notice that we do not specify any enumeration for these first η_j's); next, if we assume having constructed $\eta_1, \ldots, \eta_{N_r}$ with

$$N_r \;=\; (1+2^r)^2\,, \tag{12}$$

we define $\eta_{N_r+1}, \ldots, \eta_{N_{r+1}}$ so that

$$\left\{\eta_{N_r+1}, \ldots, \eta_{N_{r+1}}\right\} \;=\; \mathcal{A}_{r+1} \setminus \mathcal{A}_r\,. \tag{13}$$

Again, the enumeration for these η_j's does not matter. The only important fact is that $\eta_j \in \mathcal{A}_{r_j}$ for all $j \geq 1$, where r_j is the first $r \geq 0$ such that $j \leq N_r$.

The sequence $(\eta_j)_{j\geq 1}$ can be defined by induction on $r \geq 0$ and one has

$$\left\{\eta_j\right\}_{j\geq 1} \;=\; \mathcal{A}_\infty \;:=\; \bigcup_{r\geq 0} \mathcal{A}_r\,.$$

As it has been specified, the enumeration of $(\eta_j)_{j\geq 1}$ does not matter as long as one has the following important condition: for all $r \geq 0$ and all $j, k \geq 1$ such that $\eta_j \in \mathcal{A}_r$ and $\eta_k \in \mathcal{A}_{r+1} \setminus \mathcal{A}_r$, then one necessarily has $j < k$. Equivalently, for all $r \geq 0$, the first N_r points η_j's belong to $\mathcal{A}_r$.

We can deduce the following preliminar result.

Lemma 2.5. *The sequence $(\eta_j)_{j\geq 1}$ is well-defined and is dense in $\mathcal{Q}$. In addition, $(\eta_j)_{j\geq 1}$ satisfies the following condition:*

$$\forall\, r \geq 0\,,\; \forall\, j \leq N_r = (1+2^r)^2\,,\; \eta_j \in \mathcal{A}_r\,. \tag{14}$$

Proof. The last assertion immediately follows from (12) and (13). In order to prove the density, let consider $z \in \mathcal{Q}$, $\varepsilon > 0$ and let be $r \geq 0$ such that $1/2^r \leq \varepsilon$. There is $\eta_{j_z} \in \mathcal{A}_{r+1}$ such that $\left|\Re\left(\eta_{j_z}\right) - \Re(z)\right| \leq 1/2^{r+1}$ and $\left|\Im\left(\eta_{j_z}\right) - \Im(z)\right| \leq 1/2^{r+1}$, then $\left|\eta_{j_z} - z\right| \leq \sqrt{2}/2^{r+1} < 1/2^r \leq \varepsilon$.

□

In order to prove Proposition 1.2, we first need to give an estimate of the divided differences $\left\{\Delta_p\right\}_{p\geq 1}$ associated with $(\eta_j)_{j\geq 1}$.

2.3 A bound for the associated divided differences

We start by the following preliminar result that is a lower bound for the products that appear on the expression of the Δ_p's given by Lemma 2.3.

Lemma 2.6. *Let consider the sequence $(\eta_j)_{j\geq 1}$ from Lemma 2.5. There is $P_\eta \geq 2$ such that, for all $p \geq P_\eta$ and all $q = 1,\ldots,p+1$, one has*

$$\prod_{j=1, j\neq q}^{p+1} \left|\eta_q - \eta_j\right| \quad \geq \quad \exp(-9p)\,.$$

Proof. Let fix any $p \geq 2$ and let consider the unique $r \geq 0$ such that

$$\left(1+2^{r-1}\right)^2 \;<\; p+1 \;\leq\; (1+2^r)^2\,. \tag{15}$$

Now let fix η_q with $q = 1,\ldots,p+1$, i.e.

$$\eta_q \;=\; \frac{s_q + it_q}{2^r} \quad \text{where} \quad 0 \;\leq\; s_q,\, t_q \;\leq\; 2^r\,.$$

Similarly, for all $\eta_j \neq \eta_q$ with $j = 1,\ldots,p+1$, one has $\eta_j = \dfrac{s_j + it_j}{2^r}$ with $0 \leq s_j,\, t_j \leq 2^r$ and $(s_j,t_j) \neq (s_q,t_q)$. Since $\left|s_j - s_q\right| \leq 2^r$ (resp. $\left|t_j - t_q\right| \leq 2^r$), then

$$k_{q,j} \;:=\; \max\left\{\left|s_j - s_q\right|, \left|t_j - t_q\right|\right\} \quad\in\quad \{1,\ldots,2^r\}\,. \tag{16}$$

It follows that $\eta_j \in \mathcal{D}_{k_{q,j}}(\eta_q)$, where $\mathcal{D}_k(\eta_q)$ is defined for all $k \in \mathbb{N}$ by

$$\mathcal{D}_k(\eta_q) \quad:=\quad \left\{z = \frac{s+it}{2^r}\,,\ (s,t)\in\mathbb{Z}^2\,,\ \max\left\{\left|s-s_q\right|, \left|t-t_q\right|\right\} = k\right\}.$$

We first want to estimate $\operatorname{card}\left[\mathcal{D}_k(\eta_q) \cap \{\eta_j,\ 1\leq j\leq p+1,\ j\neq q\}\right]$ for all $k = 1,\ldots,2^r$. We start by noticing that

$$\mathcal{D}_k(\eta_q) \quad=\quad \mathcal{S}_k(\eta_q) \setminus \mathcal{S}_{k-1}(\eta_q)\,, \tag{17}$$

where $\mathcal{S}_k(\eta_q)$ is defined for all $k \in \mathbb{N}$ by

$$\mathcal{S}_k(\eta_q) \quad:=\quad \left\{z = \frac{s+it}{2^r}\,,\ (s,t)\in\mathbb{Z}^2\,,\ \max\left\{\left|s-s_q\right|, \left|t-t_q\right|\right\} \leq k\right\}.$$

Since on the one hand, one has for all $k \geq 0$, that

$$\operatorname{card}\left[\mathcal{S}_k(\eta_q)\right] \quad=\quad \operatorname{card}\left[\mathcal{S}_k(0)\right] \;=\; \operatorname{card}\left\{z = \frac{s+it}{2^r}\,,\ (s,t)\in\mathbb{Z}^2\,,\ -k\leq s,\, t\leq k\right\} \;=\; (2k+1)^2\,,$$

and on the other hand, $\mathcal{S}_{k-1}(\eta_q) \subset \mathcal{S}_k(\eta_q)$ for all $k \geq 1$, it follows by (17) that

$$\begin{aligned}
\operatorname{card}\left[\mathcal{D}_k(\eta_q) \cap \{\eta_j,\ 1\leq j\leq p+1,\ j\neq q\}\right] \quad &\leq \quad \operatorname{card}\left[\mathcal{D}_k(\eta_q)\right] \\
&= \quad \operatorname{card}\left[\mathcal{S}_k(\eta_q)\right] - \operatorname{card}\left[\mathcal{S}_{k-1}(\eta_q)\right] \\
&= \quad (2k+1)^2 - (2k-1)^2 \;=\; 8k\,.
\end{aligned} \tag{18}$$

Next, one has for all $k = 1,\ldots,2^r$, and all $\eta_j \in \mathcal{D}_k(\eta_q)$ (with $\eta_j = \dfrac{s_j+it_j}{2^r}$),

$$\left|\eta_j - \eta_q\right| \quad\geq\quad \max\left\{\frac{\left|s_j - s_q\right|}{2^r}, \frac{\left|t_j - t_q\right|}{2^r}\right\} \;=\; \frac{k}{2^r}\,. \tag{19}$$

Finally, the estimates (19) and (18) together yield for all $k = 1,\ldots,2^r$,

$$\prod_{\mathcal{D}_k(\eta_q)\cap\{\eta_j,\,1\leq j\leq p+1,\,j\neq q\}} \left|\eta_q - \eta_j\right| \;\geq\; \left(\frac{k}{2^r}\right)^{\operatorname{card}\left[\mathcal{D}_k(\eta_q)\cap\{\eta_j,\,1\leq j\leq p+1,\,j\neq q\}\right]} \;\geq\; \left(\frac{k}{2^r}\right)^{8k},$$

the second inequality being valid since $0 < k/2^r \leq 1$. By applying the following partition (justified by (15) and Lemma 2.5),

$$\left\{\eta_j\,,\ j = 1,\ldots,p+1\,,\ j\neq q\right\} \quad=\quad \bigcup_{k=1}^{2^r}\left[\mathcal{D}_k(\eta_q) \cap \{\eta_j,\ 1\leq j\leq p+1,\ j\neq q\}\right]$$

(one indeed has $1 \le k \le 2^r$ by (16)), we can deduce that

$$\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| = \prod_{k=1}^{2^r} \left[\prod_{\mathcal{D}_k(\eta_q)\cap\{\eta_j, 1\le j\le p+1, j\neq q\}} |\eta_q - \eta_j| \right] \ge \prod_{k=1}^{2^r} \left(\frac{k}{2^r}\right)^{8k}$$
$$= \exp\left[\sum_{k=1}^{2^r} 8k \ln(k/2^r)\right] = \exp\left[2^{2r+3} \times \frac{1}{2^r}\sum_{k=1}^{2^r} \frac{k}{2^r}\ln\left(\frac{k}{2^r}\right)\right]. \tag{20}$$

The last expression involves the Riemann's sum of the continuous function $t \in]0,1] \mapsto t\ln t$, $0 \mapsto 0$, whose integral is $\int_0^1 t\ln t\, dt = \left[\frac{t^2}{2}\ln t\right]_0^1 - \int_0^1 \frac{t}{2}dt = -\frac{1}{4}$. Then

$$\frac{2^{2r+3}}{2^r}\sum_{k=1}^{2^r}\frac{k}{2^r}\ln\left(\frac{k}{2^r}\right) = 2^{2r+3}(-1/4 + \varepsilon(1/r)) = \frac{2^{2r}}{4}(-8 + \varepsilon(1/r)), \tag{21}$$

where $\varepsilon(1/r) \to 0$ as $1/r \to 0$. On the other hand, one has by (15) that

$$p \ge (1 + 2^{r-1})^2 - 1 = 2^{2r-2} + 2^r \ge \frac{2^{2r}}{4}. \tag{22}$$

In addition, (15) also gives that $\varepsilon(1/r) = \varepsilon(1/p)$. It follows by applying (20), (21) and (22) that

$$\prod_{j=1, j\neq q}^{p+1} |\eta_q - \eta_j| \ge \exp[p \times (-8 + \varepsilon(1/p))] \ge \exp(-9p),$$

for all $p \ge P_\eta$ (≥ 2) so that $|\varepsilon(1/p)| \le 1$ (notice that P_η does not depend on $q = 1, \ldots, p+1$). The estimate being true for all $q = 1, \ldots, p+1$, the proof of the lemma is achieved.

□

This allows us to prove the following result that will be useful for the proof of Proposition 1.2.

Lemma 2.7. *Let h be any function defined on the set $\{\eta_j\}_{j\ge1}$ (coming from the sequence $(\eta_j)_{j\ge1}$ of Lemma 2.5) and that is bounded:*

$$\|h\|_\infty := \sup_{j\ge1} |h(\eta_j)| < +\infty.$$

Then there is $R_\eta \ge 1$ such that for all $p \ge 0$,

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h](\eta_{p+1})\right| \le \|h\|_\infty R_\eta^p.$$

Proof. Let be $p \ge 0$. One has by Lemma 2.3 that

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h](\eta_{p+1})\right| \le \sum_{q=1}^{p+1} \frac{|h(\eta_q)|}{\prod_{j=1,j\neq q}^{p+1} |\eta_q - \eta_j|} \le \frac{(p+1)\|h\|_\infty}{\min_{1\le q\le p+1}\left(\prod_{j=1,j\neq q}^{p+1}|\eta_q - \eta_j|\right)}.$$

If $p \le P_\eta - 1$, then

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h](\eta_{p+1})\right| \le C_\eta \|h\|_\infty,$$

where

$$C_\eta := \frac{P_\eta}{\min_{1\le p\le P_\eta - 1}\left(\min_{1\le q\le p+1}\prod_{j=1,j\neq q}^{p+1}|\eta_q - \eta_j|\right)}.$$

Otherwise $p \ge P_\eta$ (≥ 2) then one has by Lemma 2.6 that

$$\min_{1\le q\le p+1}\left(\prod_{j=1,j\neq q}^{p+1}|\eta_q - \eta_j|\right) \ge 1/\exp(9p),$$

thus

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h](\eta_{p+1})\right| \le (p+1)\|h\|_\infty \exp(9p) \le \exp(p)\|h\|_\infty \exp(9p) = \|h\|_\infty \exp(10p)$$

(the second estimate being justified by the classical one: $1 + t \le \exp(t)$, $\forall\, t \in \mathbb{R}$).

It follows that for all $p \ge 1$, one has

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}[h](\eta_{p+1})\right| \le \|h\|_\infty \times \max\left[C_\eta, \exp(10p)\right]$$
$$\le \|h\|_\infty (1 + C_\eta)\exp(10p) \le \|h\|_\infty R_\eta^p,$$

where $R_\eta := (1 + C_\eta)e^{10}$ (for $p = 0$, one just has that $|\Delta_0(h)(\eta_1)| = |h(\eta_1)| \le \|h\|_\infty \times R_\eta^0$) and the proof is achieved.

□

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Remark 1. Notice that we do not need to assume any kind of regularity for the function h, else that it is bounded on the set $\{\eta_j\}_{j\geq 1}$.

2.4 Proof of Proposition 1.2

Now we can give the proof of the proposition.

Proof. First, the set $\{\eta_j\}_{j\geq 1}$ is dense in the square $\mathcal{Q}$ by Lemma 2.5 then its topological closure $\overline{\{\eta_j\}_{j\geq 1}}$ has nonempty interior. It follows by Lemma 2.1 that $\{\eta_j\}_{j\geq 1}$ cannot be locally interpolable by real-analytic curves.

Next, the sequence $(\eta_j)_{j\geq 1}$ being bounded, in order to prove that $E_N(\cdot;\eta)$ converges (i.e. for entire functions as well as for holomorphic functions on any fixed ball $B_2(0,r_0)$), it suffices to show that $(\eta_j)_{j\geq 1}$ satisfies the estimate (7) in Theorem 2.2. For all $q\geq 0$, one has with the choice of $h(\zeta)=\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q$ that

$$\left\|\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right\|_\infty \leq \left\|\left(\sqrt{\frac{\left|\overline{\zeta}\right|^2}{1+|\zeta|^2}}\right)^q\right\|_\infty \times \left\|\frac{1}{\left(\sqrt{1+|\zeta|^2}\right)^q}\right\|_\infty \leq \left\|\frac{\sqrt{1+|\zeta|^2}}{1+|\zeta|^2}\right\|_\infty^q \times 1 \leq 1$$

(in particular h is bounded on $\mathbb{C}$). It follows by Lemma 2.7 that for all $p,q\geq 0$,

$$\left|\Delta_{p,(\eta_p,\ldots,\eta_1)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right](\eta_{p+1})\right| \leq \left\|\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^q\right\|_\infty \times R_\eta^p \leq R_\eta^p \leq R_\eta^{p+q},$$

i.e. $(\eta_j)_{j\geq 1}$ satisfies condition (7) from Theorem 2.2 and this completes the proof of Proposition 1.2.

□

3 An essential result on the extraction of subsequences

In this part we give the proof of Proposition 1.6 that will be useful in order to prove Theorem 1.3. We first need a couple of preliminar results.

3.1 Some reminders and preliminar results

Let fix $\eta=(\eta_j)_{j\geq 1}$. We first remind the following identity that is justified by Proposition 3 from [9] and that involves another analogous formula $R_N(\cdot;\eta)$, that is the essential remainder part of $f-E_N(f;\eta)$: $f\in\mathcal{O}(B_2(0,r_0))$ (resp. $f\in\mathcal{O}(\mathbb{C}^2)$) being given, one has for all $N\geq 1$ and all $z\in B_2(0,r_0)$ (resp. $z\in\mathbb{C}^2$),

$$f(z) = E_N(f;\eta)(z)-R_N(f;\eta)(z)+\sum_{k+l\geq N} a_{k,l}z_1^k z_2^l\,; \tag{23}$$

here,

$$f(z) = \sum_{k,l\geq 0} a_{k,l}z_1^k z_2^l$$

is the Taylor expansion of f, and

$$R_N(f;\eta)(z):=\sum_{p=1}^N\left(\prod_{j=1,j\neq p}^N\frac{z_1-\eta_j z_2}{\eta_p-\eta_j}\right)\sum_{k+l\geq N}a_{k,l}\eta_p^k\left(\frac{z_2+\overline{\eta_p}z_1}{1+|\eta_p|^2}\right)^{k+l-N+1} \tag{24}$$

is well-defined and belongs to $\mathcal{O}(B_2(0,r_0))$ (resp. $\mathcal{O}(\mathbb{C}^2)$).

Since the remainder part of the Taylor expansion of $f\in\mathcal{O}(B_2(0,r_0))$ (resp. $f\in\mathcal{O}(\mathbb{C}^2)$) always converges to 0 uniformly on any compact subset $K\subset B_2(0,r_0)$ (resp. $K\subset\mathbb{C}^2$), this gives an equivalence between the convergence of $E_N(\cdot;\eta)$ and the one of $R_N(\cdot;\eta)$. More precisely, we have the following result that is Lemma 7 from [9].

Lemma 3.1. *$r_0>0$ being fixed, let consider $f\in\mathcal{O}(B_2(0,r_0))$ (resp. $f\in\mathcal{O}(\mathbb{C}^2)$) and K any compact subset of $B_2(0,r_0)$ (resp. $\mathbb{C}^2$). Then for all $N\geq 1$, one has*

$$\sup_{z\in K}|f(z)-E_N(f;\eta)(z)| \leq \sup_{z\in K}|R_N(f;\eta)(z)|+C_K(N+2)\sup_{\|z\|\leq r_K}|f(z)|\,(1-\varepsilon_K)^N,$$

where $\|z\|=\sqrt{|z_1|^2+|z_2|^2}$ is the usual norm on $\mathbb{C}^2$ and C_K, r_K depend only on K.

In particular, $E_N(f;\eta)$ converges to f (uniformly on any compact subset) if and only if so does $R_N(f;\eta)$ to 0.

On the other hand, we will also deal with the action of some homographic transformations on $\left(\eta_j\right)_{j\geq 1}$. We remind some notations and results from [9] (beginning of Section 4): let fix any $\eta^c \notin \left\{\eta_j\right\}_{j\geq 1} \cup \{\infty\}$ and let consider the unitary matrix $U_{\eta^c} \in \mathcal{U}(2,\mathbb{C})$ defined by

$$U_{\eta^c} \quad := \quad \frac{1}{\sqrt{1+|\eta^c|^2}} \begin{pmatrix} \overline{\eta^c} & 1 \\ 1 & -\eta^c \end{pmatrix};$$

let also consider the following homographic application

$$\begin{aligned} h_{\eta^c} \,:\, \mathbb{C}\mathbf{P}^1 \quad &\rightarrow \quad \mathbb{C}\mathbf{P}^1 \\ \zeta \quad &\mapsto \quad \frac{1+\overline{\eta^c}\,\zeta}{\zeta-\eta^c} \end{aligned} \tag{25}$$

(where $\mathbb{C}\mathbf{P}^1 = \mathbb{C}\cup\{\infty\}$) and the new sequence

$$\theta \;=\; \left(\theta_j\right)_{j\geq 1} \quad := \quad \left(h_{\eta^c}\left(\eta_j\right)\right)_{j\geq 1}. \tag{26}$$

Then the set $\left\{\theta_j\right\}_{j\geq 1}$ is well-defined as a subset of $\mathbb{C}$ and one has the following result that is Lemma 16 from [9].

Lemma 3.2. *Let be $f \in \mathcal{O}(B_2(0,r_0))$ (resp. $f \in \mathcal{O}\left(\mathbb{C}^2\right)$). For all $N\geq 1$ and $z \in B_2(0,r_0)$ (resp. $z\in\mathbb{C}^2$),*

$$R_N(f;\eta)(z) \quad = \quad R_N\left(f\circ U_{\eta^c}^{-1};\theta\right)\left(U_{\eta^c}z\right).$$

These lemmas yield the following consequence.

Corollary 3.3. *$\eta = \left(\eta_j\right)_{j\geq 1}$ being any sequence, $\eta^c \notin \left\{\eta_j\right\}_{j\geq 1} \cup \{\infty\}$ being fixed and h_{η^c} (resp. θ) being defined by (25) (resp. (26)), the formula $E_N(f;\eta)$ converges to f uniformly on any compact subset $K\subset\mathbb{C}^2$ and for every function $f\in\mathcal{O}\left(\mathbb{C}^2\right)$, if and only if so does $E_N(f;\theta)$.*

Proof. $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ being given, one has by Lemma 3.1 that the formula $E_N(f;\eta)$ converges to f if and only if $R_N(f;\eta)$ converges to 0 (uniformly on any compact subset). U_{η^c} being an isometry, it follows by Lemma 3.2 that $R_N(f;\eta)$ converges to 0 uniformly on any compact subset $K\subset\mathbb{C}^2$ and for every function $f\in\mathcal{O}\left(\mathbb{C}^2\right)$, if and only if so does $R_N\left(f\circ U_{\eta^c}^{-1};\theta\right)$, thus if and only if so does $R_N(f;\theta)$ for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$. Finally, by applying Lemma 3.1 again, it is true if and only if $E_N(f;\theta)$ converges to f (uniformly on any compact subset) for all $f\in\mathcal{O}\left(\mathbb{C}^2\right)$. □

We also prove the following preliminar result about the homographic transformations defined by (25).

Lemma 3.4. *For all $\eta^c \notin \left\{\eta_j\right\}_{j\geq 1}\cup\{\infty\}$, one has $h_{\eta^c}^{-1} = h_{\overline{\eta^c}}$ where*

$$\begin{aligned} h_{\overline{\eta^c}} \,:\, \mathbb{C}\mathbf{P}^1 \quad &\rightarrow \quad \mathbb{C}\mathbf{P}^1 \\ \zeta \quad &\mapsto \quad \frac{1+\eta^c\,\zeta}{\zeta-\overline{\eta^c}}. \end{aligned}$$

In addition, one also has that $\overline{\eta^c} \notin \left\{h_{\eta^c}\left(\eta_j\right)\right\}_{j\geq 1}\bigcup\{\infty\}$, i.e. $h_{\eta^c}^{-1} = h_{\overline{\eta^c}}$ is of the same kind (25) for the associated set $\left\{h_{\eta^c}\left(\eta_j\right)\right\}_{j\geq 1} = \left\{\theta_j\right\}_{j\geq 1}$.

Proof. Indeed, for all $\zeta\in\mathbb{C}\setminus\left\{\overline{\eta^c}\right\}$, one has that

$$\left(h_{\eta^c}\circ h_{\overline{\eta^c}}\right)(\zeta) \quad = \quad \frac{1+\overline{\eta^c}\,\dfrac{1+\eta^c\,\zeta}{\zeta-\overline{\eta^c}}}{\dfrac{1+\eta^c\,\zeta}{\zeta-\overline{\eta^c}}-\eta^c} \;=\; \frac{\zeta+\eta^c\overline{\eta^c}\,\zeta}{1+\eta^c\overline{\eta^c}} \;=\; \zeta\,,$$

then the equality holds for all $\zeta\in\mathbb{C}\mathbf{P}^1$. The second assertion follows by (25) since $h_{\eta^c}(\infty)=\overline{\eta^c}$, then $h_{\eta^c}\left(\eta_j\right)\neq\overline{\eta^c}$ for all $j\geq 1$. □

We finish the subsection with the following result reminded as Lemma 18 from [9], and that gives an equivalent definition for the geometric criterion (5).

Lemma 3.5. *The set $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves if and only if it can locally holomorphically interpolate the conjugate function, i.e. for all $\zeta\in\overline{\{\eta_j\}_{j\geq 1}}$ (the topological closure of $\{\eta_j\}_{j\geq 1}$ in $\mathbb{C}\mathbf{P}^1$), there are a neighborhood V of ζ and $g\in\mathcal{O}(V)$ such that*

$$\overline{\eta_j} \quad = \quad g(\eta_j)\,,\ \forall\,\eta_j\in V\,. \tag{27}$$

Remark 2. In [9], Lemma 18 was proved in $\mathbb{C}$ whereas the above statement is stronger since its deals with $\mathbb{C}\mathbf{P}^1$. Nevertheless, there is no apparent contradiction because the geometric criterions (5) and (27) are both invariant under the action of any biholomorphic function (then in particular for any homographic transformation). This allows us to extend the claimed equivalence to $\mathbb{C}\mathbf{P}^1$.

3.2 On the extraction of certain subsequences

Now we can give the proof of Proposition 1.6.

Proof. Since the set $\{\eta_j\}_{j\geq 1}$ is not locally interpolable by real-analytic curves, it follows by Lemma 3.5 that there is $\zeta_0 \in \overline{\{\eta_j\}_{j\geq 1}}$ without any neighborhood $V \in \mathcal{V}(\zeta_0)$ and holomorphic function $g \in \mathcal{O}(V_{\zeta_0})$ that can interpolate the conjugate function on $\{\eta_j\}_{j\geq 1} \bigcap V$, i.e.

$$\forall\, V \in \mathcal{V}(\zeta_0),\, \forall\, g \in \mathcal{O}(V),\, \exists\, \eta_j \in V,\; g(\eta_j) \neq \overline{\eta_j}\,. \tag{28}$$

In particular, ζ_0 cannot be isolated in $\{\eta_j\}_{j\geq 1}$. Otherwise, if $\zeta_0 \neq \infty$ (resp. $\zeta_0 = \infty$), then by taking $V_{\zeta_0} \subset \mathbb{C}$ such that $\{\eta_j\}_{j\geq 1} \bigcap V_{\zeta_0} = \{\zeta_0\}$ (resp. $V_\infty = \mathbb{C}\mathbf{P}^1 \setminus K$, where the compact subset K is big enough so that $\{\eta_j\}_{j\geq 1} \setminus K \subset \{\infty\}$) and

$$\begin{array}{rcl} g_{\zeta_0} : V_{\zeta_0} & \to & \mathbb{C} \\ \zeta & \mapsto & g_{\zeta_0}(\zeta) \equiv \overline{\zeta_0} \end{array}$$

(resp.

$$\begin{array}{rcl} g_\infty : V_\infty & \to & \mathbb{C}\mathbf{P}^1 \\ \zeta & \mapsto & g_\infty(\zeta) \equiv \infty\;), \end{array}$$

we would get a contradiction with (28).

As a consequence, there is a subsequence $(\eta_{j_k})_{k\geq 1} \subset (\eta_j)_{j\geq 1}$ that satisfies:

$$\begin{cases} (\eta_{j_k})_{k\geq 1} \text{ converges to } \zeta_0; \\ \eta_{j_k} \neq \zeta_0 \text{ for all } k \geq 1. \end{cases} \tag{29}$$

We will then deal with the cases $\zeta_0 \in \mathbb{C}$ and $\zeta_0 = \infty$ respectively.

$\underline{\zeta_0 \in \mathbb{C}}$

We start by setting $S_0 := (\eta_{j_k})_{k\geq 1}$ and

$$S_1 \;:=\; S_0 \bigcap D(\zeta_0, 1)\,, \tag{30}$$

where $D(\zeta_0,1) = \{\zeta \in \mathbb{C},\, |\zeta - \zeta_0| < 1\}$. By construction, S_1 gives a (nonempty and infinite) sequence that converges to ζ_0. If S_1 (as a set) is not locally interpolable by real-analytic curves, the proposition is proved. Otherwise (because ζ_0 is a limit point of S_1), there are $V_1 \in \mathcal{V}(\zeta_0)$ and $g_1 \in \mathcal{O}(V_1)$ such that

$$\overline{\eta_j} \;=\; g_1(\eta_j) \text{ for all } \eta_j \in S_1 \cap V_1\,.$$

By reducing V_1 if necessary, we can assume that $V_1 \subset D(\zeta_0, 1)$ and V_1 is connected. Since ζ_0 satisfies (28), it follows that $g_1|_{V_1}$ cannot interpolate the conjugate function on $V_1 \bigcap \{\eta_j\}_{j\geq 1}$, i.e. there is $\eta_{s_1} \in V_1 \bigcap \{\eta_j\}_{j\geq 1}$ such that $g_1(\eta_{s_1}) \neq \overline{\eta_{s_1}}$. We set

$$S_2 \;:=\; S_1 \bigcup \{\eta_{s_1}\}$$

and S_2 (with any enumeration) still gives a sequence that converges to ζ_0.

Let fix $m \geq 1$ and let assume having constructed $\eta_{s_1}, \ldots, \eta_{s_m}, S_1, \ldots, S_m, V_1, \ldots, V_m$ and $g_1, \ldots, g_m$ such that for all $q = 1, \ldots, m$, one has the following properties:

$$V_q \in \mathcal{V}(\zeta_0) \text{ and } V_q \text{ is connected;} \tag{31}$$

$$\eta_{s_q} \in V_q \subset D(\zeta_0, 1/2^{q-1})\,; \tag{32}$$

$$g_q \in \mathcal{O}(V_q) \text{ and } g_q(\eta_{s_q}) \neq \overline{\eta_{s_q}}\,; \tag{33}$$

$$g_q(\eta_j) = \overline{\eta_j} \text{ for all } \eta_j \in S_q \cap V_q\,, \tag{34}$$

where

$$S_q \;=\; S_1 \bigcup \{\eta_{s_1}, \ldots, \eta_{s_{q-1}}\} \text{ for all } q = 2, \ldots, m\,. \tag{35}$$

We first consider the set

$$S_{m+1} \;:=\; S_1 \bigcup \{\eta_{s_1}, \ldots, \eta_{s_m}\}$$

and this satisfies (35) for all $q = 2, \ldots, m+1$. Next, S_{m+1} (with any enumeration) will give a sequence that still converges to ζ_0 as the union of S_1 (that converges to ζ_0 by (30) and (29)) and the finite set $\{\eta_{s_1}, \ldots, \eta_{s_m}\}$. If S_{m+1} is not locally interpolable by real-analytic curves, the proposition is proved. Otherwise (because ζ_0 is a limit point of S_{m+1}), there are $V_{m+1} \in \mathcal{V}(\zeta_0)$ and $g_{m+1} \in \mathcal{O}(V_{m+1})$ such that

$$\overline{\eta_j} \quad = \quad g_{m+1}\left(\eta_j\right) \text{ for all } \eta_j \in S_{m+1} \cap V_{m+1}\,.$$

By reducing V_{m+1} if necessary, we can assume that $V_{m+1} \subset D(\zeta_0, 1/2^m)$ and V_{m+1} is connected. On the other hand, since ζ_0 satisfies (28), it follows that there is $\eta_{s_{m+1}} \in V_{m+1}$ such that $g_{m+1}\left(\eta_{s_{m+1}}\right) \neq \overline{\eta_{s_{m+1}}}$. This proves (31), (32), (33) and (34) for $q = m+1$, and completes the induction.

Now if there is $m \geq 1$ such that the set S_m defined by (35) is not locally interpolable by real-analytic curves, then the proposition is proved (since any enumeration of S_m will give a sequence that converges to ζ_0).

Otherwise, we can construct for all $m \geq 1$, such η_{s_m}, S_m, V_m and g_m that fulfill (31), (32), (33) and (34) for all $q = 1, \ldots, m$, and consider the following set

$$S_\infty \quad := \quad \bigcup_{m \geq 1} S_m \;=\; S_1 \bigcup \left\{\eta_{s_m},\, m \geq 1\right\}\,.$$

Then any enumeration of S_∞ will give a sequence that converges to ζ_0 as the union of S_1 (that converges to ζ_0 by (30) and (29)) and the convergent sequence $\left(\eta_{s_m}\right)_{m\geq 1}$ (since by (32), one has $\eta_{s_m} \in D\left(\zeta_0, 1/2^{m-1}\right)$). If we prove that S_∞ is not locally interpolable by real-analytic curves, the proof of the proposition will be achieved in the case for which $\zeta_0 \in \mathbb{C}$.

Let assume on the contrary that S_∞ is, i.e. (since ζ_0 is a limit point of S_∞) there are $V_\infty \in \mathcal{V}(\zeta_0)$ and $g_\infty \in \mathcal{O}(V_\infty)$ such that $g_\infty\left(\eta_j\right) = \overline{\eta_j}$ for all $\eta_j \in S_\infty \bigcap V_\infty$. In particular, this yields for all $m \geq 1$ (since $S_m \subset S_\infty$),

$$g_\infty\left(\eta_j\right) \quad = \quad \overline{\eta_j}\,, \quad \forall\, \eta_j \in S_m \bigcap V_\infty\,. \tag{36}$$

On the other hand, by (32) there is $m_0 \geq 1$ such that $V_{m_0} \subset D\left(\zeta_0, 1/2^{m_0-1}\right) \subset V_\infty$. In addition, one has by (34) for $q = m_0$, that

$$g_{m_0}\left(\eta_j\right) \quad = \quad \overline{\eta_j} \text{ for all } \eta_j \in S_{m_0} \cap V_{m_0}\,.$$

Hence g_{m_0} and $g_\infty|_{V_{m_0}}$ are both holomorphic functions on the domain V_{m_0}, that coincide on the set $S_{m_0} \cap V_{m_0}$. Since by (35), $S_{m_0} \cap V_{m_0} \supset S_1 \cap V_{m_0}$ that is infinite with limit point $\zeta_0 \in V_{m_0}$ by (30), (29) and (31), it follows that

$$g_\infty|_{V_{m_0}} \quad \equiv \quad g_{m_0}\,. \tag{37}$$

But an application of (36) for $m = m_0 + 1$ yields $g_\infty\left(\eta_j\right) = \overline{\eta_j}$ for all $\eta_j \in S_{m_0+1} \cap V_\infty$. In particular, since $\eta_{s_{m_0}} \in V_{m_0} \subset D\left(\zeta_0, 1/2^{m_0-1}\right) \subset V_\infty$ (by (32) for $q = m_0$) and $\eta_{s_{m_0}} \in S_{m_0+1}$ by (35) for $m = m_0+1$, one has $\eta_{s_{m_0}} \in S_{m_0+1} \cap V_\infty$ then

$$g_\infty\left(\eta_{s_{m_0}}\right) \quad = \quad \overline{\eta_{s_{m_0}}}\,. \tag{38}$$

Moreover, an application of (33) for $q = m_0$, also yields

$$g_{m_0}\left(\eta_{s_{m_0}}\right) \neq \overline{\eta_{s_{m_0}}}\,. \tag{39}$$

Finally, (37), (38) and (39) together lead to (since $\eta_{s_{m_0}} \in V_{m_0}$)

$$\overline{\eta_{s_{m_0}}} \;=\; g_\infty\left(\eta_{s_{m_0}}\right) \;=\; g_\infty|_{V_{m_0}}\left(\eta_{s_{m_0}}\right) \;=\; g_{m_0}\left(\eta_{s_{m_0}}\right) \;\neq\; \overline{\eta_{s_{m_0}}}\,,$$

and this is impossible. Necessarily, S_∞ cannot be locally interpolable by real-analytic curves and the proposition is proved in the case for which $\zeta_0 \in \mathbb{C}$.

$\underline{\zeta_0 = \infty}$

First, by removing 0 from $\left\{\eta_j\right\}_{j\geq 1}$ if necessary, we can assume that $\eta_j \neq 0$, $\forall\, j \geq 1$ (as well as $\eta_{j_k} \neq 0$, $\forall\, k \geq 1$). Indeed, since the sequence $\left(\eta_{j_k}\right)_{k\geq 1}$ converges to ∞ by (29), it follows that the subset $\left\{\eta_{j_k}\right\}_{k\geq 1} \setminus \{0\}$ is infinite, then so is the set $\left\{\eta_j\right\}_{j\geq 1} \setminus \{0\} \supset \left\{\eta_{j_k}\right\}_{k\geq 1} \setminus \{0\}$. In addition, the new subset $\left\{\eta_{j_k}\right\}_{k\geq 1} \setminus \{0\}$ gives a new sequence that still satisfies (29).

Now let consider the sequence $\left(\theta_j\right)_{j\geq 1}$ where

$$\theta_j \;:=\; \frac{1}{\eta_j} \quad \text{for all } \; j \geq 1\,.$$

First, $\left(\theta_j\right)_{j\geq 1}$ is well-defined. Next, since $\left(\eta_{j_k}\right)_{k\geq 1}$ satisfies (29) with $\zeta_0 = \infty$, it follows that so does the subsequence $\left(\theta_{j_k}\right)_{k\geq 1}$ with the choice of $\zeta_0' := 0$, i.e.

$$\begin{cases} \left(\theta_{j_k}\right)_{k\geq 1} \text{ converges to } 0\,; \\ \theta_{j_k} \neq 0 \text{ for all } k \geq 1. \end{cases} \tag{40}$$

Lastly, we claim that $\zeta_0' = 0$ satisfies (28) as well. Indeed, let be $V \in \mathcal{V}(0)$ and $g \in \mathcal{O}(V)$. We want to prove that there exists $\theta_j \in V$ such that $g\left(\theta_j\right) \neq \overline{\theta_j}$.

If $g(0) \neq 0$, then by (40), $\overline{\theta_{j_k}} \longrightarrow 0$ and $g\left(\theta_{j_k}\right) \longrightarrow g(0) \neq 0$ as $k \to +\infty$. It follows that $g\left(\theta_{j_k}\right) \neq \overline{\theta_{j_k}}$ for all k large enough and the claim is proved in this case.

Otherwise, $g(0) = 0$. Let consider

$$W \quad := \quad \left\{\frac{1}{\zeta}\,,\, \zeta \in V \setminus \{0\}\right\} \bigcup \{\infty\}$$

and

$$\begin{array}{rcl} h \,:\, W & \longrightarrow & \mathbb{C}\mathbf{P}^1 \\ \infty & \longmapsto & \infty\,, \\ \zeta \in \mathbb{C} \cap W & \longmapsto & \begin{cases} \dfrac{1}{g(1/\zeta)} & \text{if } g(1/\zeta) \neq 0\,, \\ \infty & \text{otherwise.} \end{cases} \end{array}$$

Then $W \in \mathcal{V}(\infty)$, h is well-defined and $h \in \mathcal{O}(W)$. It follows by (28) that there is $\eta_j \in W$ such that $h\left(\eta_j\right) \neq \overline{\eta_j}$. If $g(1/\eta_j) = 0$, i.e. $g\left(\theta_j\right) = 0$, then $\overline{\theta_j} = 1/\overline{\eta_j} \neq 0 = g\left(\theta_j\right)$, and this proves the claim in that case. Otherwise, $g\left(1/\eta_j\right) \neq 0$, i.e. $g\left(\theta_j\right) \neq 0$ then

$$\frac{1}{g\left(\theta_j\right)} = \frac{1}{g\left(1/\eta_j\right)} = h\left(\eta_j\right) \quad \neq \quad \overline{\eta_j} = \frac{1}{\overline{\theta_j}}\,,$$

hence $g\left(\theta_j\right) \neq \overline{\theta_j}$ and the claim is proved in this last case.

We can now apply the previous case of the proposition with the choice of $\left(\theta_j\right)_{j\geq 1}$ and $\zeta_0' = 0$ to get a subsequence $\left(\theta_{j_k'}\right)_{k\geq 1}$ (maybe different from $\left(\theta_{j_k}\right)_{k\geq 1}$) that converges to 0 and that is not locally interpolable by real-analytic curves. It follows that the sequence $\left(\eta_{j_k'}\right)_{k\geq 1} = \left(1/\theta_{j_k'}\right)_{k\geq 1}$ converges to ∞. On the other hand, the inverse function $\zeta \mapsto 1/\zeta$ being a homographic transformation, it is in particular a biholomorphic application of $\mathbb{C}\mathbf{P}^1$. Hence the subset $\left\{\eta_{j_k'}\right\}_{k\geq 1} = \left\{1/\theta_{j_k'}\right\}_{k\geq 1}$ cannot be either locally interpolable by real-analytic curves. Finally, one also has that $\eta_{j_k'} \neq 0$ for all $k \geq 1$. This proves the proposition in this second case and completes its whole proof.

□

Remark 3. As we have seen in the above proof, we know in addition that in the case for which $\zeta_0 = \infty$, we also have that $\eta_{j_k} \neq 0$ for all $k \geq 1$.

4 Proof of the main theorem

In the first subsection, we deal with the proof of the equivalence between (1) and (3) in the statement of Theorem 1.3.

4.1 On the stability by extraction of subsequences

Before giving the proof of this part, we remind the following result as Proposition 2 from [9], and that is a special case of equivalence for the geometric criterion (5), i.e. in the particular case when $\left(\eta_j\right)_{j\geq 1}$ is a convergent sequence, condition (5) also becomes necessary.

Proposition 4.1. *Let $\left(\eta_j\right)_{j\geq 1}$ be any convergent sequence (in $\mathbb{C}$). If the interpolation formula $E_N(f\,;\eta)$ converges to f (uniformly on any compact subset) for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, then $\{\eta_j\}_{j\geq 1}$ is locally interpolable by real-analytic curves.*

Remark 4. To be rigorous, in order to apply Proposition 4.1, we should also assume that $E_N(f\,;\eta)$ converges to f for all $f \in \mathcal{O}(B_2(0,r_0))$. But as specified by Remark 5.2 from [9], it is sufficient to assume the convergence of $E_N(f\,;\eta)$ for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$.

Proof. First, if $\left\{\eta_j\right\}_{j\geq 1}$ is locally interpolable by real-analytic curves, then so is any (infinite) subset $\left\{\eta_{j_k}\right\}_{k\geq 1}$. The implication (1) $\Longrightarrow$ (3) then follows by Theorem 1.1.

Conversely, let assume that $\left\{\eta_j\right\}_{j\geq 1}$ is not locally interpolable by real-analytic curves. By Proposition 1.6, there is a subsequence $\left(\eta_{j_k}\right)_{k\geq 1}$ that is not locally interpolable by real-analytic curves and that is convergent (in $\mathbb{C}\mathbf{P}^1$). In order to get the converse implication (3) $\Longrightarrow$ (1), we want to prove that $\eta' := \left(\eta_{j_k}\right)_{k\geq 1}$ does not make converge its associated interpolation formula $E_N\left(\cdot\,;\eta'\right)$ for entire functions, i.e. there exists (at least) one function $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ such that $E_N\left(f\,;\eta'\right)$ does not converge to f (uniformly on any compact subset $K \subset \mathbb{C}^2$).

Let be $\zeta_0 = \lim_{k\to+\infty} \eta_{j_k}$. If ζ_0 is finite, the required assertion follows by Proposition 4.1. Otherwise, $\zeta_0 = \infty$ and by Remark 3, one also has that $\eta_{j_k} \neq 0$ for all $k \geq 1$. It follows that the sequence $\theta' := \left(\theta_{j_k}\right)_{k\geq 1}$ where

$$\theta_{j_k} \quad := \quad \frac{1}{\eta_{j_k}} \quad \text{for all } k \geq 1\,,$$

is well-defined (as a subset of $\mathbb{C}$), bounded and converges to 0. On the other hand, $\theta_{j_k} = h_0\left(\eta_{j_k}\right)$ for all $k \geq 1$, where h_0 is the homographic transformation defined as $h_0(\zeta) = 1/\zeta$ (see (25) with the choice of $\eta^c = 0$). Thus $\left\{\theta_{j_k}\right\}_{k\geq 1}$ is not locally interpolable by real-analytic curves (because any homographic transformation is in particular biholomorphic). Again, by Proposition 4.1, the sequence $\theta' = \left(\theta_{j_k}\right)_{k\geq 1}$ does not make converge its associated interpolation formula $E_N(\cdot;\theta')$ for entire functions, i.e. there exists $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ such that $E_N(f;\theta')$ does not converge to f (uniformly on any compact subset). Finally, since $E_N(\cdot;\theta') = E_N(\cdot;h_0(\eta'))$, it follows by Corollary 3.3 that neither can do the sequence $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$ for the formula $E_N(\cdot;\eta')$ for entire functions, and this proves the implication (3) $\Longrightarrow$ (1).

□

4.2 On the action by permutations

Now we can give the proof of the second part of Theorem 1.3 that is the equivalence between (1) and (2), and achieve its whole proof. We first need a specific result that is a part of the proof for Theorem 1 from [9] (reminded above as Theorem 2.2).

Lemma 4.2. *Let be $\left(\eta_j\right)_{j\geq 1}$ such that, for all $f \in \mathcal{O}\left(\mathbb{C}^2\right)$, $R_N(f;\eta)$ is uniformly bounded on any compact subset of $\mathbb{C}^2$. Then the estimate (7) from Theorem 2.2 is satisfied.*

This result is Lemma 11 from [9] and yields the part (2) $\Longrightarrow$ (3) in the statement of Theorem 2.2. In particular, the important fact is that no one condition is needed for the set $\left\{\eta_j\right\}_{j\geq 1}$ (like boundedness, see Remark 3.1 from [9]). This will be useful in order to prove the implication (2) $\Longrightarrow$ (1) in the statement of Theorem 1.3.

Proof. The implication (1) $\Longrightarrow$ (2) immediately follows by Theorem 1.1 since the property of being locally interpolable by real-analytic curves is a condition about sets, then it does not depend on any enumeration of $\{\eta_j\}_{j\geq 1}$.

Conversely, let assume that $\{\eta_j\}_{j\geq 1}$ (coming from the sequence $\eta = \left(\eta_j\right)_{j\geq 1}$) is not locally interpolable by real-analytic curves. We want to find a permutation σ of $\mathbb{N}\setminus\{0\}$ such that $E_N(\cdot;\sigma(\eta))$ does not converge for entire functions (where $\sigma(\eta) := \left(\eta_{\sigma(j)}\right)_{j\geq 1}$). We know by Proposition 1.6 that there are $\zeta_0 \in \mathbb{C}\mathbf{P}^1$ and a subsequence $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$ of $\left(\eta_j\right)_{j\geq 1}$ that satisfy the following conditions:

$$\begin{cases} \text{the sequence } \left(\eta_{j_k}\right)_{k\geq 1} \text{ converges to } \zeta_0; \\ \text{the set } \left\{\eta_{j_k}\right\}_{k\geq 1} \text{ is not locally interpolable by real-analytic curves.} \end{cases} \tag{41}$$

First, we claim that can w.l.o.g. assume that ζ_0 is finite. Indeed, if $\zeta_0 = \infty$, let consider $\eta^c \notin \left\{\eta_j\right\}_{j\geq 1} \bigcup \{\infty\}$, $h_{\eta^c} \in \mathcal{O}\left(\mathbb{C}\mathbf{P}^1\right)$ defined by (25) and the associated sequence $\theta = \left(\theta_j\right)_{j\geq 1} = \left(h_{\eta^c}\left(\eta_j\right)\right)_{j\geq 1}$ (that is well-defined by (26)). Then the subsequence $\left(\theta_{j_k}\right)_{k\geq 1} = \left(h_{\eta^c}\left(\eta_{j_k}\right)\right)_{k\geq 1}$ satisfies (41) with $\zeta_0' = h_{\eta^c}(\infty) = \overline{\eta^c} \in \mathbb{C}$ (because $\left(\eta_{j_k}\right)_{k\geq 1}$ does and h_{η^c} is biholomorphic). It will follow that there will be a permutation σ such that the sequence $\sigma(\theta) = \left(\theta_{\sigma(j)}\right)_{j\geq 1}$ does not make converge its associated interpolation formula $E_N(\cdot;\sigma(\theta))$ for entire functions. Since

$$h_{\eta^c}^{-1}(\sigma(\theta)) \quad = \quad h_{\eta^c}^{-1}\left[\left(\theta_{\sigma(j)}\right)_{j\geq 1}\right] = \left(h_{\eta^c}^{-1}\left(\theta_{\sigma(j)}\right)\right)_{j\geq 1} = \left(h_{\eta^c}^{-1}\left[h_{\eta^c}\left(\eta_{\sigma(j)}\right)\right]\right)_{j\geq 1} = \left(\eta_{\sigma(j)}\right)_{j\geq 1} = \sigma(\eta)$$

(notice that $\left\{\left(h_{\eta^c}^{-1}(\sigma(\theta))\right)_j\right\}_{j\geq 1} = \left\{h_{\eta^c}^{-1}\left(\theta_j\right)\right\}_{j\geq 1}$ is well-defined as a subset of $\mathbb{C}$ by Lemma 3.4), an application of Corollary 3.3 (which is possible because $h_{\eta^c}^{-1} = h_{\overline{\eta^c}}$ by Lemma 3.4) will allow us to deduce that neither will do the sequence $h_{\eta^c}^{-1}(\sigma(\theta)) = \sigma(\eta)$ for $E_N(\cdot;\sigma(\eta))$ for entire functions, i.e. there will exist (at least) one function $f \in \mathcal{O}\left(\mathbb{C}^2\right)$ such that $E_N(f;\sigma(\eta))$ will not converge to f (uniformly on any compact subset $K \subset \mathbb{C}^2$). This will prove the required implication (2) $\Longrightarrow$ (1) of the theorem for the case $\zeta_0 = \infty$ and complete the whole proof of the equivalence between (1) and (2) in the general case.

We can then assume that $\zeta_0 \in \mathbb{C}$ in (41). Let fix the enumeration of the associated subsequence $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$ as well as the canonical one for the complementary subsequence

$$\eta'' \;=\; \left(\eta_{r_m}\right)_{m\geq 1} \;:=\; \left(\eta_j\right)_{j\geq 1} \setminus \left(\eta_{j_k}\right)_{k\geq 1}. \tag{42}$$

Since $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$ is a (bounded) convergent sequence that is not locally interpolable by real-analytic curves, it follows by Proposition 4.1 that $E_N(\cdot;\eta')$ cannot converge for entire functions. By an application of (2)$\Longleftrightarrow$(3) in Theorem 2.2, it follows that the sequence of the associated divided differences is not exponentially bounded, i.e. $\forall R \geq 1$, $\exists p_R, q_R \geq 0$ such that

$$\left|\Delta_{p_R,\left(\eta_{j_{p_R}},\ldots,\eta_{j_1}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_R}\right]\left(\eta_{j_{p_R+1}}\right)\right| \quad > \quad R^{p_R+q_R}. \tag{43}$$

In particular, there are $p_1, q_1 \geq 0$ such that

$$\left|\Delta_{p_1,\left(\eta_{j_{p_1}},\ldots,\eta_{j_1}\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_1}\right]\left(\eta_{j_{p_1+1}}\right)\right| \quad \geq \quad 1. \tag{44}$$

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We set

$$\sigma(k) \;=\; j_k \;\text{ for all }\; k=1,\dots,p_1+1 \quad\text{ and }\quad \sigma(p_1+2) \;=\; r_1\,. \tag{45}$$

Then σ is injective on the first (p_1+2) indices, 1 is attained since $1\in\{j_1\,,r_1\}\subset\sigma(\{1,\dots,p_1+1,p_1+2\})$ and (44) can be rewritten as

$$\left|\Delta_{p_1,(\eta_{\sigma(p_1)},\dots,\eta_{\sigma(1)})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_1}\right](\eta_{\sigma(p_1+1)})\right| \;\;\geq\;\; 1\,. \tag{46}$$

The permutation σ will be constructed by induction on $m\geq 1$. We first set

$$p_0 \;\;:=\;\; -2\,, \tag{47}$$

and we assume having defined σ on $\{1,\dots,p_m+2\}$ where

$$p_{l-1}+2 \;\leq\; p_l \quad\text{ for all }\quad l=1,\dots,m\,, \tag{48}$$

as follows: for all $l=1,\dots,m$,

$$\sigma(k) \;\;=\;\; \begin{cases} j_{k-l+1} \;\text{ for all }\; k=p_{l-1}+3,\dots,p_l+1\,,\\ r_l \;\text{ if }\; k=p_l+2\,.\end{cases} \tag{49}$$

We also assume that for all $l=1,\dots,m$,

$$\left|\Delta_{p_l,(\eta_{\sigma(p_l)},\dots,\eta_{\sigma(1)})}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_l}\right](\eta_{\sigma(p_l+1)})\right| \;\;\geq\;\; l^{p_l+q_l}\,. \tag{50}$$

We indeed check that (48) is fulfilled for $m=1$ since $p_1\geq 0$ and $p_0=-2$ by (47). Similarly, (49) (resp. (50)) is satisfied for $m=1$ by (47) and (45) (resp. by (46)).

Now let consider the sequence $\eta^{(m)}=\left(\eta_k^{(m)}\right)_{k\geq 1}$ defined as follows:

$$\eta_k^{(m)} \;\;:=\;\; \begin{cases}\eta_{\sigma(k)} \;\text{ for all }\; k=1,\dots,p_m+2\,,\\ \eta_{j_{k-m}} \;\text{ for all }\; k\geq p_m+3\,.\end{cases} \tag{51}$$

Since $\left\{\eta^{(m)}\right\}$ (as a set) is the union of $\left\{\eta_{j_k}\right\}_{k\geq 1}$ and the finite set $\left\{\eta_{r_1},\dots,\eta_{r_m}\right\}$ by (49), the sequence $\eta^{(m)}$ is bounded and satisfies (41) as well (with the same limit point ζ_0). Again, by successive applications of Proposition 4.1 and (2)$\Longleftrightarrow$(3) from Theorem 2.2, it follows that $\eta^{(m)}$ satisfies (43). In particular, with the choice of $R=m+1$, there are p_{m+1}, $q_{m+1}\geq 0$ such that

$$\left|\Delta_{p_{m+1},\left(\eta^{(m)}_{p_{m+1}},\dots,\eta^{(m)}_1\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q_{m+1}}\right]\left(\eta^{(m)}_{p_{m+1}+1}\right)\right| \geq (m+1)^{p_{m+1}+q_{m+1}}\,. \tag{52}$$

In addition, we can choose $p_{m+1}\geq p_m+2$ (this will satisfy (48) for all $l=1,\dots,m+1$). Indeed, if it were not possible, this would mean that for all $R\geq m+1$, the associated p_R should be bounded. By Corollary 2.4, so would be all the terms $\left|\Delta_{p_R,\left(\eta^{(m)}_{p_R},\dots,\eta^{(m)}_1\right)}\left[\left(\frac{\overline{\zeta}}{1+|\zeta|^2}\right)^{q}\right]\left(\eta^{(m)}_{p_R+1}\right)\right|$ for all $q\geq 0$, and this would contradict (43) for $\eta^{(m)}$. We can then extend σ to $\{1,\dots,p_{m+1}+2\}$ as follows:

$$\sigma(k) \;\;=\;\; \begin{cases} j_{k-m} \;\text{ for all }\; k=p_m+3,\dots,p_{m+1}+1\,,\\ r_{m+1} \;\text{ if }\; k=p_{m+1}+2\,.\end{cases} \tag{53}$$

The induction hypothesis (49) and (53) show that σ is well-defined on $\{1,\dots,p_{m+1}+2\}$. Moreover, one has by (51) and (53) that $\eta_{\sigma(k)}=\eta_k^{(m)}$ for all $k=1,\dots,p_{m+1}+1$, then it follows by (52) that (50) is still satisfied for $l=m+1$. This last assertion with the induction hypotheses (49) and (50) complete the case for $m+1$, i.e. (49) and (50) are still satisfied for all $l=1,\dots,m+1$.

The sequence $(p_m)_{m\geq 1}$ constructed above allows us to define σ for all $k\geq 1$ by (49) since we have the following partition from (47) and (48),

$$\mathbb{N}\setminus\{0\} \;\;=\;\; \bigcup_{m\geq 1}\{k\,,\; p_{m-1}+3\;\leq\;k\;\leq\;p_m+2\}\,.$$

Next, σ is a permutation of $\mathbb{N}\setminus\{0\}$: indeed, it follows from (48) that every set $\{k\,,\;p_{m-1}+3\leq k\leq p_m+2\}$ contains at least two elements, i.e. σ attains by (49) exactly one of the type r_m and at least one of the type j_k as well. On the other hand, one has by (49) again that for all $m\geq 1$, $\sigma(p_m+1)=j_{p_m-m+2}$ and $\sigma(p_m+3)=j_{p_m+3-(m+1)+1}=j_{p_m-m+3}$. This last assertion and (45) together show that all the j_k's (resp. r_m's) are reached exactly once.

Finally, the estimate (50) being satisfied for all $m\geq 1$ (i.e. this contradicts the estimate (7) from Theorem 2.2), it follows by an application of Lemma 4.2 that there is $f\in\mathcal{O}\left(\mathbb{C}^2\right)$ such that $R_N(f\,;\sigma(\eta))$ cannot be uniformly bounded (on any compact subset $K\subset\mathbb{C}^2$). In particular, $R_N(f\,;\sigma(\eta))$ cannot even converge to 0, then by (23), $E_N(f\,;\sigma(\eta))$ does not converge to f (uniformly on any compact subset $K\subset\mathbb{C}^2$). This achieves the implication (2) $\Longrightarrow$ (1) from Theorem 1.3 and completes its whole proof.

□

4.3 Proof of Corollaries 1.4 and 1.5

In order to prove Corollary 1.4, we first remind the following auxiliary result that is Lemma 8 from [9].

Lemma 4.3. *Let $r_0 > 0$ be fixed. If there is $\varepsilon_\eta > 0$ such that, $\forall f \in \mathcal{O}(B_2(0, r_0))$, $R_N(f;\eta)$ converges to 0 uniformly on any compact subset of $B_2(0, \varepsilon_\eta r_0)$, then $\forall g \in \mathcal{O}\left(\mathbb{C}^2\right)$, $R_N(g;\eta)$ converges to 0 uniformly on any compact subset of $\mathbb{C}^2$.*

We can then give the proof of Corollary 1.4.

Proof. First, as for the proof of Theorem 1.3, the implication (1) $\Longrightarrow$ (2) (resp. (1) $\Longrightarrow$ (3)) from the statement of the corollary immediately follows by Theorem 1.1 since, if $\left\{\eta_j\right\}_{j\geq 1}$ is locally interpolable by real-analytic curve, then so will be the (same) set $\{\sigma(\eta)\} = \left\{\eta_{\sigma(j)}\right\}_{j\geq 1}$ for all $\sigma \in \mathfrak{S}_{\mathbb{N}}$ (resp. the subset $\left\{\eta_{j_k}\right\}_{k\geq 1}$ coming from any subsequence $\left(\eta_{j_k}\right)_{k\geq 1}$).

Conversely, in order to prove the implication (2) $\Longrightarrow$ (1) (resp. (3) $\Longrightarrow$ (1)), let fix $\sigma \in \mathfrak{S}_{\mathbb{N}}$ (resp. $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$) and $g \in \mathcal{O}\left(\mathbb{C}^2\right)$. The hypothesis (2) (resp. (3)) and Lemma 3.1 imply that for all $f \in \mathcal{O}(B_2(0, r_0))$, $R_N(f;\sigma(\eta))$ (resp. $R_N(f;\eta')$) converges to 0 uniformly on any compact subset $K \subset B_2\left(0, \varepsilon_\eta r_0\right)$. It follows by Lemma 4.3 that in particular $R_N(g;\sigma(\eta))$ (resp. $R_N(g;\eta')$) converges to 0 uniformly on any compact subset $K \subset \mathbb{C}^2$. Again, by an application of Lemma 3.1, one can deduce that $E_N(g;\sigma(\eta))$ (resp. $E_N(g;\eta')$) converges to g uniformly on any compact subset $K \subset \mathbb{C}^2$.

Finally, $\sigma \in \mathfrak{S}_{\mathbb{N}}$ (resp. $\eta' = \left(\eta_{j_k}\right)_{k\geq 1}$) and $g \in \mathcal{O}\left(\mathbb{C}^2\right)$ being arbitrary, the condition (2) (resp. (3)) from the statement of Theorem 1.3 is satisfied, whose application yields the required assertion (1). □

Lastly, the proof of Corollary 1.5 immediately follows by an application of Theorem 1.1 (or Theorem 3 from [9]) and Corollary 1 from [9] since the set $\left\{\eta_{\sigma(j)}\right\}_{j\geq 1}$ (resp. $\left\{\eta_{j_k}\right\}_{k\geq 1}$) is still locally interpolable by real-analytic curves.

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