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Iterative Collocation Method for Solving a class of Nonlinear Weakly Singular Volterra Integral Equations

Khedidja Kherchouche[a] · Azzeddine Bellour[b] · Pedro Lima[c]

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Abstract

In this paper, an iterative collocation method based on the use of Lagrange polynomials is developed for the numerical solution of a class of nonlinear weakly singular Volterra integral equations. The error analysis of the proposed numerical method is studied theoretically. Numerical illustrations confirm our theoretical analysis.

1 Introduction

In this paper, we develop an approximation based on iterative collocation method to obtain numerical solutions of the following nonlinear weakly singular Volterra integral equations,

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, t \in I = [0,T], \tag{1}$$

where the functions g, k are sufficiently smooth and $p(t,s) = \frac{s^{\mu-1}}{t^{\mu}}$, $\mu > 1$.
Equations with this kind of kernel have a weak singularity at $t = 0$ and they are a particular case of the cordial equations, studied by G. Vainikko in [12, 13, 14, 15]. Actually, as shown in [13], if the core function of a cordial operator is $\phi(s) = s^{\mu-1}$, then its kernel is $s^{\mu-1}t^{-\mu}k(t,s)$, which is the kind of kernel we are concerned with. Equations of this type are also the subject of the article [6].

The cordial integral operators have the interesting property that they are bounded but non-compact, which implies that some of the classical results for Volterra integral equations (for example, about existence and uniqueness of solution) are not applicable in this case. However an existence and uniqueness result in $C^m([0,T])$ was obtained in [12], provided that the core function satisfies $\phi(x) \in L^1([0,1])$, which is the case of our equation, when $\mu > 0$.

The application of polynomial and spline collocation methods to cordial equations was studied in [12, 14] and [15], respectively, where sufficient conditions for convergence were obtained and error estimates were derived. Superconvergence results for collocation methods were obtained in [6].

Equations of this type arise from heat conduction problems. As it was shown in [5], they may result from boundary value problems for partial differential equations with mixed-type boundary conditions.

In [4] and [6] the authors were concerned with the numerical solution of linear cordial equations. Here we propose a computational method for a nonlinear Volterra integral equation with a weakly singular kernel of the same type.

In [3] a similar approach was proposed for nonlinear Volterra integral equations with regular kernels (when $p(t,s) \equiv 1$). This case was also well studied in the literature. In particular, Babolian and his co-authors [2] have proposed a Chebyshev approximation. In [1] and [8] numerical algorithms based on the Adomian's method were developed. In [16] an approach was proposed, based on Taylor polynomial approximation, while the homotopy perturbation method was applied to the same equation in [7]. The authors of [9] have introduced a scheme based on the fixed point method. Finally, the Haar wavelet method and the Haar rationalized functions method were proposed in [10] and [11], respectively.

In Section 2 of the present work we describe a numerical scheme for the solution of equation 1. In Section 3 we analyze the convergence and obtain error estimates. Numerical examples that illustrate the performance of the method are presented in Section 4 and the paper finishes with conclusions in Section 5.

2 Description of the collocation method

Let Π_N be a uniform partition of the interval $I = [0,T]$ defined by $t_n = nh, \quad n = 0,...,N-1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 < c_1 < ...... < c_m \leq 1$ and the collocation points be $t_{n,j} = t_n + c_j h$, $j = 1,...,m, n = 0,...,N-1$.

[a]Laboratoire de Mathématiques appliquées et didactique, Ecole Normale Supérieure de Constantine, Constantine, Algeria, email: kherchouchekhedidja@gmail.com
[b]Laboratoire de Mathématiques appliquées et didactique, Ecole Normale Supérieure de Constantine, Constantine, Algeria, email: bellourazze123@yahoo.com
[c]Department of Mathematics, Instituto Superior Técnico, University of Lisbon, Lisboa, Portugal, email: plima@math.tecnico.ulisboa.pt

Define the subintervals $\sigma_n = [t_n, t_{n+1}[$, and $\sigma_{N-1} = [t_{N-1}, t_N]$.
Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.
We define the real polynomial spline space of degree $m-1$ as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1}, n = 0, .., N-1\}.$$

This is the space of piecewise polynomials of degree at most $m-1$. Its dimension is Nm. We consider the space $L^\infty(I)$ with the norm

$$\|\varphi\| = \inf\{C \in \mathbb{R} : |\varphi(t)| \leq C \ \text{ for a.e. } t \in I\} < \infty.$$

It holds for any $y \in C^m([0, T])$ that

$$y(t_n + \tau h) = \sum_{l=1}^{m} \lambda_l(\tau) y(t_{n,l}) + \epsilon_n(\tau), \ \epsilon_n(\tau) = h^m \frac{y^{(m)}(\zeta_n(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j), \tag{2}$$

where $\tau \in [0, 1]$ and $\lambda_j(\tau) = \prod_{l \neq j}^{m} \frac{\tau - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters $c_j, j = 1, ..., m$.
Let $\Gamma_m = \|\sum_{j=1}^{m} |\lambda_j|\|$ be the Lebesgue constants, such that

$$\|\sum_{j=1}^{m} |\lambda_j|\| = \max\left\{\sum_{j=1}^{m} |\lambda_j(s)|, s \in [0, 1]\right\}.$$

We have from (1) for each $j = 1, ..., m, n = 0, ..., N-1$

$$\begin{aligned} x(t_{nj}) &= g(t_{nj}) + \int_0^{t_{nj}} p(t_{nj}, s) k(t_{nj}, s, x(s)) ds \\ &= g(t_{nj}) + \int_0^{t_n} p(t_{nj}, s) k(t_{nj}, s, x(s)) ds + \int_{t_n}^{t_{nj}} p(t_{nj}, s) k(t_{nj}, s, x(s)) ds \\ &= g(t_{nj}) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} p(t_{nj}, s) k(t_{nj}, s, x(s)) ds + \int_{t_n}^{t_{nj}} p(t_{nj}, s) k(t_{nj}, s, x(s)) ds \end{aligned} \tag{3}$$

Now, for $s \in [t_i, t_{i+1}]$, we use the following change of variable: $s = t_i + \tau h$ with $\tau \in [0, 1]$, and for $s \in [t_n, t_{nj}]$, we use the following change of variable: $s = t_n + \tau h$ with $\tau \in [0, c_j]$. Then, from (3), we have

$$\begin{aligned} x(t_{nj}) = g(t_{nj}) &+ \sum_{i=0}^{n-1} \int_0^1 h p(t_n + c_j h, t_i + \tau h) k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h)) d\tau \\ &+ \int_0^{c_j} h p(t_n + c_j h, t_n + \tau h) k(t_n + c_j h, t_n + \tau h, x(t_n + \tau h)) d\tau \end{aligned} \tag{4}$$

By substituting the expression of the function p into (4), we obtain

$$\begin{aligned} x(t_{nj}) = g(t_{nj}) &+ \sum_{i=0}^{n-1} \int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^\mu} k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h)) d\tau \\ &+ \int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^\mu} k(t_n + c_j h, t_n + \tau h, x(t_n + \tau h)) d\tau \end{aligned} \tag{5}$$

Now, for $j = 1, ..., m$, by apply the formula (2) for the function
$y_i(\tau) = k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h))$, we have

$$k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h)) = \sum_{l=1}^{m} \lambda_l(\tau) k(t_n + c_j h, t_{n,l}, x(t_{n,l})) + \epsilon_i(\tau) \tag{6}$$

where $\epsilon_i(\tau) = h^m \frac{y_i^{(m)}(\eta_i)}{m!} \prod_{j=1}^{m} (\tau - c_j)$.
Inserting (6) into (5), we obtain for each $j = 1, ..., m, n = 0, ..., N-1$

$$\begin{aligned} x(t_{nj}) = g(t_{nj}) &+ \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^\mu} k(t_n + c_j h, t_n + c_l h, x(t_{nl})) \lambda_l(\tau) d\tau \right) + \\ &\sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^\mu} k(t_n + c_j h, t_i + c_l h, x(t_{il})) \lambda_l(\tau) d\tau \right) + o(h^m), \end{aligned} \tag{7}$$

where,

$$o(h^m) = \int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} \epsilon_n(\tau) d\tau + \sum_{i=0}^{n-1} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} \epsilon_i(\tau) d\tau \right).$$

Since the function k is smooth, then there exists $\alpha_1 > 0$, such that for $i = 0, ..., N-1$, we have $\|y_i^{(m)}\| \leq \alpha_1$, which implies that

$$\| o(h^m) \| \leq h^m \frac{\alpha_1}{m!} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} d\tau + \sum_{i=0}^{n-1} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} d\tau \right) \right)$$

Since $i + \tau \leq n + c_j$ for all $i = 0, ..., n-1$, then for all $n = 0, ..., N-1$

$$\begin{aligned} \| o(h^m) \| &\leq h^m \frac{\alpha_1}{m!} \left(\frac{1}{(n+c_j)} + \sum_{i=0}^{n-1} \left(\frac{1}{(n+c_j)} \right) \right) \\ &\leq h^m \frac{\alpha_1}{m!} \left(\frac{1}{c_1} + \frac{n}{(n+c_j)} \right) \\ &\leq h^m \underbrace{\frac{\alpha_1}{m!} \left(\frac{1}{c_1} + 1 \right)}_{=\alpha}. \end{aligned}$$

It holds for any $u \in S_{m-1}^{(-1)}(I, \Pi_N)$ that

$$u(t_n + \tau h) = \sum_{l=1}^{m} \lambda_l(\tau) u(t_{n,l}), \tau \in [0,1]. \tag{8}$$

Now, we approximate the exact solution x by $u \in S_{m-1}^{(-1)}(I, \Pi_N)$ such that $u(t_{n,j})$ satisfy the following nonlinear system,

$$\begin{aligned} u(t_{n,j}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u(t_{nl})) \lambda_l(\tau) d\tau \right) + \\ \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u(t_{il})) \lambda_l(\tau) d\tau \right). \end{aligned} \tag{9}$$

for $j = 1, ..., m$, $n = 0, ..., N-1$.

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_{m-1}^{(-1)}(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1) such that

$$u^q(t_n + \tau h) = \sum_{j=1}^{m} \lambda_j(\tau) u^q(t_{n,j}), \tau \in [0,1] \tag{10}$$

where the coefficients $u^q(t_{n,j})$ are given by the following formula:

$$\begin{aligned} u^q(t_{n,j}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u^{q-1}(t_{nl})) \lambda_l(\tau) d\tau \right) + \\ \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u^q(t_{il})) \lambda_l(\tau) d\tau \right). \end{aligned} \tag{11}$$

such that the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).

The above formula is explicit and the approximate solution u^q is obtained without solving any algebraic system.

In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (1).

3 Convergence analysis

In this section, we assume that the function k satisfies the Lipschitz condition with respect to the third variable: there exists $L \geq 0$ such that

$$|k(t,s,y_1) - k(t,s,y_2)| \leq L|y_1 - y_2|, \tag{12}$$

for all $t, s \in I$, where L is independent of t and s.

The following result gives the existence and the uniqueness of a solution for (1).

Lemma 3.1. *Let $g \in C([0,T])$, $k(t,s,u) \in C(\Delta T \times \mathbb{R})$, where $\Delta T = \{(t,s) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq s \leq t\}$. Let*

$$\frac{\partial k}{\partial u} \in C(\Delta T \times \mathbb{R}).$$

Assume that equation

$$\xi = k(0,0,\xi) \frac{1}{\mu} + g(0) \tag{13}$$

has a unique solution $\xi^* \in \mathbb{R}$*, and that*

$$1 \neq \frac{a^*(0,0)}{\lambda+\mu}, \forall \lambda : Re(\lambda) \geq 0, \tag{14}$$

where

$$a^*(0,0) = \frac{\partial k}{\partial u}(0,0,u)\Big|_{u=\xi^*}.$$

Moreover, let k satisfy

$$k(t,s,u)| \leq c_0 + c_1|u|, \tag{15}$$

with $\frac{c_1}{\mu} < 1$.

Then there is a unique solution $x^* \in C([0,T])$ *of* (1)*, such that* $x^*(0) = \xi^*$.

Proof. The result follows from Theorems 7.1 and 7.5 of [12], taking into account that in our case $\phi(x) = x^{\mu-1}$,with $\mu > 1$, and therefore the linear integral operator V_ϕ (using the same notation as in [12]) is defined by

$$V_\phi u(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds;$$

hence the spectrum of this operator is

$$\sigma_0(V_\phi) = \{0\} \cup \{\frac{1}{\lambda+\mu} : \lambda \in \mathbb{C}, Re(\lambda) \geq 0\},$$

which is used to obtain condition 14. □

Lemma 3.2. *Let the conditions of Lemma 3.1 be satisfied and let* x^* *be a solution of* (1)*. Moreover, let* $g \in C^m([0,T])$ *and* $k \in C^m(\Delta T \times \mathbb{R})$*, for some natural m.*

Then $x^* \in C^m([0,T])$*.*

Proof. The result follows from Theorem 8.1 of [12]. □

The following result gives the existence and the uniqueness of a solution for the nonlinear system (9).

Lemma 3.3. *If* $\frac{L\Gamma_m}{\mu} < 1$*, then the nonlinear system* (9) *has a unique solution* $u \in S^{(-1)}_{m-1}(I, \Pi_N)$*. Moreover, the function u is bounded.*

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (9) becomes

$$u(t_{0,j}) = g(t_{0,j}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(\tau)^{\mu-1}}{(c_j)^\mu} k(t_0 + c_j h, t_0 + c_l h, u(t_{0l}))\lambda_l(\tau)d\tau. \right)$$

We consider the operator Ψ defined by

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = g(t_{0,j}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(\tau)^{\mu-1}}{(c_j)^\mu} k(t_0 + c_j h, t_0 + c_l h, x_l)\lambda_l(\tau)d\tau \right).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq \frac{L\Gamma_m}{\mu} \|x - y\|,$$

Since $\frac{L\Gamma_m}{\mu} < 1$, then by Banach fixed point theorem, the nonlinear system (9) has a unique solution u on the interval σ_0.

(ii) Suppose that u exists and is unique on the intervals $\sigma_i, i = 0, ..., n-1$ for $n \geq 1$, we show now that u exists and is unique on the interval σ_n.

On the interval σ_n, the nonlinear system (9) becomes

$$u(t_{n,j}) = G(t_{n,j}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^\mu} k(t_n + c_j h, t_n + c_l h, u(t_{nl}))\lambda_l(\tau)d\tau \right) \tag{16}$$

where,

$G(t_{n,j}) = g(t_{n,j}) + \sum_{i=0}^{n-1}\sum_{l=1}^{m} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^\mu} k(t_n + c_j h, t_i + c_l h, u(t_{il}))\lambda_l(\tau)d\tau \right).$

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j=1,\dots,m$, we have

$$\Psi_j(x)=G(t_{n,j})+\sum_{l=1}^{m}\left(\int_0^{c_j}\frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}}k(t_n+c_jh,t_n+c_lh,x_l)\lambda_l(\tau)d\tau\right).$$

Hence, for all $x,y\in\mathbb{R}^m$, we have

$$\|\Psi(x)-\Psi(y)\|\leq\frac{L\Gamma_m}{\mu}\|x-y\|.$$

Since $\frac{L\Gamma_m}{\mu}<1$, then by Banach fixed point theorem, the nonlinear system (16) has a unique solution u on the interval σ_n. ☐

Corollary 3.4. *Under the condition $\frac{L\Gamma_m}{\mu}<1$, the following conditions of Lemma 3.1 are fulfilled:*

1. *Equation (13) has a unique solution $\xi^*\in\mathbb{R}$.*
2. *Inequality (15) is satisfied, moreover $\frac{c_1}{\mu}<1$.*

Proof. 1. We consider the operator Ψ defined by

$$\Psi:\mathbb{R}\longrightarrow\mathbb{R}$$
$$\xi\longmapsto\Psi(x)=k(0,0,\xi)\frac{1}{\mu}+g(0),$$

Hence, for all $\xi_1,\xi_2\in\mathbb{R}$, we have

$$|\Psi(\xi_1)-\Psi(\xi_2)|\leq\frac{L}{\mu}|\xi_1-\xi_2|,$$

Since $\frac{L}{\mu}\leq\frac{L\Gamma_m}{\mu}<1$, then by Banach fixed point theorem, Equation (13) has a unique solution $\xi^*\in\mathbb{R}$.

2. We have,

$$|k(t,s,u)|\leq|k(t,s,u)-k(t,s,0)|+|k(t,s,0)|\leq\underbrace{L}_{=c_1}|u|+c_0,$$

such that $c_0=\max\{|k(t,s,0)|,(t,s)\in I\times I\}$.
Hence the inequality (15) is satisfied, moreover $\frac{c_1}{\mu}=\frac{L}{\mu}\leq\frac{L\Gamma_m}{\mu}<1$. ☐

Remark 1. Under our assumptions and by Lemma 3.1, Lemma 3.2 and Corollary 3.4, to prove the existence and uniqueness solution for Equation (1), we need only to show the condition (14).

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 3.5. *Let f,k be m times continuously differentiable on their respective domains. If $\frac{L\Gamma_m}{\mu}<\frac{1}{2}$, then the collocation solution u converges to the exact solution x, and the resulting error function $e:=x-u$ satisfies:*

$$\|e\|\leq Ch^m,$$

where C is a finite constant independent of h.

Proof. From (9) and (7), using (12), we obtain

$$|e(t_{nj})|\leq\alpha h^m+\frac{L\Gamma_m}{\mu}e_n+\frac{L\Gamma_m}{\mu n^{\mu}}\sum_{i=0}^{n-1}((i+1)^{\mu}-i^{\mu})e_i \tag{17}$$

where α is a positive number and $e_n=\max\{|e(t_{n,l})|,l=1,\dots,m\}$, $n=0,\dots,N-1$.
Then, from (17), e_n satisfies for $n=0,\dots,N-1$,

$$e_n\leq\alpha h^m+\frac{L\Gamma_m}{\mu}e_n+\frac{L\Gamma_m}{\mu n^{\mu}}\sum_{i=0}^{n-1}((i+1)^{\mu}-i^{\mu})e_i,$$

which implies that,

$$e_n\leq\frac{\alpha}{1-\frac{L\Gamma_m}{\mu}}h^m+\frac{L\Gamma_m}{(1-\frac{L\Gamma_m}{\mu})\mu n^{\mu}}\sum_{i=0}^{n-1}((i+1)^{\mu}-i^{\mu})e_i.$$

Let $C_1=\frac{\alpha}{1-\frac{L\Gamma_m}{\mu}}$ and $C_2=\frac{L\Gamma_m}{\mu(1-\frac{L\Gamma_m}{\mu})}$, it follows that

$$e_n\leq C_1h^m+\frac{C_2}{n^{\mu}}\sum_{i=0}^{n-1}((i+1)^{\mu}-i^{\mu})e_i.$$

Hence, for $\xi = \max\{e_n, n = 0, ..., N-1\}$, we deduce that

$$\xi \leq C_1 h^m + C_2 \xi.$$

Since $C_2 < 1$, we obtain

$$\xi \leq \frac{C_1}{1-C_2} h^m.$$

Which implies, from (2) and (8), that there exists $C > 0$ such that

$$\begin{aligned} \|e\| &\leq \Gamma_m \xi + h^m \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1-c_j) \\ &\leq \Gamma_m \frac{C_1}{1-C_2} h^m + h^m \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1-c_j). \end{aligned}$$

Thus, the proof is completed by setting $C = \Gamma_m \frac{C_1}{1-C_2} + \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1-c_j)$. $\square$

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 3.6. *Consider the iterative collocation solution $u^q, q \geq 1$ defined by (10) and (11). If $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then for any initial condition $u^0(t_{n,j}) \in J$ (bounded interval), the iterative collocation solution $u^q, q \geq 1$ converges to the exact solution x. Moreover, the following error estimate holds*

$$\|u^q - x\| \leq d\rho^q + Ch^m$$

where d, C are finite constants independent of h and $\rho < 1$.

Proof. We define the error e^q and ξ^q by $e^q(t) = u^q(t) - x(t)$ and $\xi^q(t) = u^q(t) - u(t)$, where u is defined by lemma 3.3. It follows that

$$e^q = \xi^q + u - x. \tag{18}$$

We have, from (9) and (11), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$|\xi^q(t_{n,j})| \leq \frac{L\Gamma_m}{n^\mu \mu} \sum_{i=0}^{n-1} [(i+1)^\mu - i^\mu] \xi_i^q + \frac{L\Gamma_m}{\mu} \xi_n^{q-1}, \tag{19}$$

where $\xi_n^q = \max\{\left|\xi^q(t_{n,l})\right|, l = 1....m\}$ for $n = 0, ..., N-1$, it follows from (19) that,

$$\xi_n^q \leq \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} [(i+1)^\mu - i^\mu] \xi_i^q + \frac{L\Gamma_m}{\mu} \xi_n^{q-1}.$$

We consider the sequence $\eta^q = \max\{\xi_n^q, n = 0, ..., N-1\}$ for $q \geq 1$.
Then, η^q satisfies,

$$\begin{aligned} \eta^q &\leq \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} [(i+1)^\mu - i^\mu] \eta^q + \frac{L\Gamma_m}{\mu} \eta^{q-1} \\ &\leq \frac{L\Gamma_m}{\mu} \eta^q + \frac{L\Gamma_m}{\mu} \eta^{q-1}. \end{aligned}$$

Hence,

$$\eta^q \leq \rho \eta^{q-1}, \tag{20}$$

where $\rho = \dfrac{\frac{L\Gamma_m}{\mu}}{1 - \frac{L\Gamma_m}{\mu}}$, since $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then $\rho < 1$.
Which implies, from (20), that for all $q \geq 1$, that

$$\eta^q \leq \rho \eta^{q-1} \leq \rho^2 \eta^{q-2} \leq ... \leq \rho^q \eta^0 \leq \rho^q \|\xi^0\|. \tag{21}$$

Since, $u^0(t_{n,j}) \in J$, the function u^0 is bounded.
Hence, there exists $M > 0$ such that

$$\|\xi^0\| = \|u^0 - u\| \leq \|u^0 - x\| + \|u - x\| \leq M. \tag{22}$$

From (21) and (22), we conclude that

$$\|\xi^q\| \leq \Gamma_m \eta^q \leq \underbrace{\Gamma_m M}_{d} \rho^q.$$

On the other hand, from Theorem (3.5), we have $\|u - x\| \leq Ch^m$ and therefore by (18) we obtain

$$\|e^q\| \leq \|\xi^q\| + \|u - x\| \leq d\rho^q + Ch^m.$$

Thus, the proof is completed. $\square$

4 Numerical Examples

To illustrate the theoretical results obtained in the previous section, we present the following examples with $T = 1$. All the exact solutions x are already known.
In all the examples, we have $a^*(0,0) = 0$, hence the condition (14) is satisfied.
In each example, we calculate the error between x and the iterative collocation solution u^q for $N = 10, 20$ and $m = 2, 3, 5$ at $t = 0, 0.1, ..., 1$. In all the examples, we choose, $q = 5$, $u^0(t_{nj}) = 1$, and we use the collocation parameters $c_j = \frac{j}{m+1}, j = 1, ..., m$. Since the condition $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$ is essential to guarantee the convergence of the numerical method, we checked that it is satisfied in all the numerical examples. Moreover, $\Gamma_2 = 3$, $\Gamma_3 = 7$ and $\Gamma_5 = 31$.

Example 4.1. Consider the following integral equation

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \ t \in [0,1].$$

with $k(t,s,z) = \frac{st\exp(z)}{40(1+\exp(z))}$, $\mu = 2$ and $g(t)$ is chosen such that the exact solution of this equation is $x(t) = \ln(1+t^2)$. The absolute errors are presented in Table 1. The experimental orders of convergence (EOC) by using the maximum error $\|e_N\| = \max\{|x(t_i) - u^q(t_i)|, i = 0, ..., N\}$ for $N = 5, 10, 15, 20$ and $m = 1, 2, 3, 4$ are given in Table 3.

Table 1: Absolute errors for Example 4.1

	$N=10$	$N=10$	$N=10$	$N=20$	$N=20$	$N=20$
t	$m=2$	$m=3$	$m=5$	$m=2$	$m=3$	$m=5$
0	2.21×10^{-3}	6.98×10^{-6}	1.66×10^{-8}	5.55×10^{-4}	4.38×10^{-7}	3.56×10^{-9}
0.1	2.10×10^{-3}	2.41×10^{-5}	3.39×10^{-8}	5.33×10^{-4}	2.65×10^{-6}	2.47×10^{-10}
0.2	1.89×10^{-3}	3.70×10^{-5}	5.27×10^{-8}	4.83×10^{-4}	4.38×10^{-6}	9.09×10^{-9}
0.3	1.60×10^{-3}	4.40×10^{-5}	5.01×10^{-8}	4.13×10^{-4}	5.39×10^{-6}	2.44×10^{-9}
0.4	1.28×10^{-3}	4.53×10^{-5}	3.02×10^{-8}	3.33×10^{-4}	5.69×10^{-6}	2×10^{-10}
0.5	9.65×10^{-4}	4.25×10^{-5}	2.23×10^{-8}	2.54×10^{-4}	5.41×10^{-6}	6.6×10^{-9}
0.6	6.79×10^{-4}	3.71×10^{-5}	9.3×10^{-9}	1.81×10^{-4}	4.78×10^{-6}	2.3×10^{-9}
0.7	4.36×10^{-4}	3.07×10^{-5}	4.5×10^{-9}	1.18×10^{-4}	3.99×10^{-6}	6×10^{-10}
0.8	2.38×10^{-4}	2.44×10^{-5}	2.6×10^{-9}	6.69×10^{-5}	3.19×10^{-6}	3.20×10^{-9}
0.9	8.37×10^{-5}	1.87×10^{-5}	2.3×10^{-9}	2.66×10^{-5}	2.46×10^{-6}	2.50×10^{-9}
1	4.06×10^{-5}	1.74×10^{-5}	1.77×10^{-7}	5.13×10^{-6}	2.05×10^{-6}	2.42×10^{-8}

Example 4.2. Consider the following integral equation

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \ t \in [0,1].$$

with $k(t,s,z) = \frac{ts}{40(2+z^2)}$, $\mu = 2$ and $g(t)$ is chosen so that the exact solution of this equation is $x(t) = \frac{1}{5(t^3+1)}$. The absolute errors are presented in Table 2. The experimental orders of convergence (EOC) by using the maximum error $\|e_N\| = \max\{|x(t_i) - u^q(t_i)|, i = 0, ..., N\}$ for $N = 5, 10, 15, 20$ and $m = 1, 2, 3, 4$ are given in Table 3.

Table 2: Absolute errors for Example 4.2

	$N=10$	$N=10$	$N=10$	$N=20$	$N=20$	$N=20$
t	$m=2$	$m=3$	$m=5$	$m=2$	$m=3$	$m=5$
0	4.44×10^{-5}	1.87×10^{-5}	5.30×10^{-9}	5.55×10^{-6}	2.34×10^{-6}	3.10×10^{-9}
0.1	1.75×10^{-4}	1.77×10^{-5}	2.39×10^{-8}	3.85×10^{-5}	2.26×10^{-6}	1.60×10^{-9}
0.2	2.91×10^{-4}	1.39×10^{-5}	3.29×10^{-8}	6.86×10^{-5}	1.87×10^{-6}	1.70×10^{-9}
0.3	3.68×10^{-4}	6.42×10^{-6}	2.22×10^{-8}	8.99×10^{-5}	1.01×10^{-6}	5.00×10^{-10}
0.4	3.81×10^{-4}	3.66×10^{-6}	5.70×10^{-9}	9.62×10^{-5}	2.13×10^{-7}	9.00×10^{-10}
0.5	3.23×10^{-4}	1.30×10^{-5}	3.50×10^{-8}	8.45×10^{-5}	1.44×10^{-6}	1.00×10^{-9}
0.6	2.10×10^{-4}	1.84×10^{-5}	4.57×10^{-8}	5.79×10^{-5}	2.23×10^{-6}	5.00×10^{-10}
0.7	7.62×10^{-5}	1.84×10^{-5}	3.15×10^{-8}	2.46×10^{-5}	2.36×10^{-6}	9.00×10^{-10}
0.8	4.41×10^{-5}	1.45×10^{-5}	1.31×10^{-8}	6.59×10^{-6}	1.94×10^{-6}	2.20×10^{-9}
0.9	1.30×10^{-4}	9.14×10^{-6}	2.90×10^{-9}	2.97×10^{-5}	1.27×10^{-6}	3.00×10^{-10}
1	1.50×10^{-4}	7.79×10^{-6}	1.52×10^{-8}	3.97×10^{-5}	8.50×10^{-7}	7.00×10^{-9}

Example 4.3. Consider the following integral equation

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \ t \in [0,1].$$

Table 3: Experimental ordres of convergence (EOC) of Examples 4.1-4.3

N	$m=1$	$m=2$	$m=3$	$m=4$
5				
10	0.99	1.98	2.97	3.92
15	0.99	1.98	2.98	3.93
20	0.99	1.98	2.98	3.95

EOC of Example4.1

N	$m=1$	$m=2$	$m=3$	$m=4$
5				
10	0.98	1.94	3.01	4.05
15	0.99	1.96	2.99	3.92
20	0.99	1.96	2.99	3.94

EOC of Example4.2

with $k(t,s,z) = \frac{t\cos(s+z)}{65}$, $\mu = 1.03$ and $g(t)$ is chosen such that the exact solution of this equation is $x(t) = \frac{t}{10}$. The absolute errors are presented in Table 4.

Table 4: Absolute errors for Example 4.3

	$N=10$	$N=10$	$N=10$	$N=20$	$N=20$	$N=20$
t	$m=2$	$m=3$	$m=5$	$m=2$	$m=3$	$m=5$
0	3.35×10^{-7}	1.81×10^{-10}	7×10^{-12}	4.20×10^{-8}	7×10^{-12}	2.00×10^{-12}
0.1	8.33×10^{-7}	5.40×10^{-10}	4×10^{-11}	1.67×10^{-7}	2×10^{-11}	3.00×10^{-11}
0.2	1.32×10^{-6}	9.30×10^{-10}	1.7×10^{-10}	2.89×10^{-7}	4×10^{-11}	2.3×10^{-10}
0.3	1.79×10^{-6}	1.28×10^{-9}	7.00×10^{-11}	4.08×10^{-7}	5×10^{-11}	1.3×10^{-10}
0.4	2.23×10^{-6}	1.59×10^{-9}	2.4×10^{-10}	5.22×10^{-7}	6×10^{-11}	1.3×10^{-10}
0.5	2.65×10^{-6}	1.90×10^{-9}	1.3×10^{-10}	6.30×10^{-7}	8×10^{-11}	3×10^{-11}
0.6	3.04×10^{-6}	2.21×10^{-9}	3×10^{-11}	7.29×10^{-7}	2×10^{-10}	2×10^{-11}
0.7	3.39×10^{-6}	2.39×10^{-9}	6×10^{-11}	8.19×10^{-7}	1.1×10^{-10}	3.00×10^{-11}
0.8	3.69×10^{-6}	2.65×10^{-9}	1.4×10^{-10}	8.99×10^{-7}	1.00×10^{-10}	3.00×10^{-11}
0.9	3.95×10^{-6}	2.88×10^{-9}	8×10^{-11}	9.68×10^{-7}	1.2×10^{-10}	8×10^{-11}
1	3.85×10^{-6}	2.90×10^{-9}	1.99×10^{-8}	9.85×10^{-7}	2×10^{-10}	5.12×10^{-9}

5 Conclusion

In this paper, we have used an iterative collocation method based on the Lagrange polynomials for solving a class of nonlinear weakly singular Volterra integral equations (1) in the spline space $S_{m-1}^{(-1)}(\Pi_N)$. The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of approximate solution are determined by using iterative formulas without solving any system of algebraic equations. The numerical examples confirm that the method is convergent with a good accuracy.

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References

[1] E. Babolian, H. Sadeghi Goghary, SH. Javadi and M. Ghasemi. Restarted Adomian method for nonlinear differential equations. *Int. J. Comput. Math.*, 82(1):97-102, 2005.

[2] E. Babolian, F. Fattahzadeh and E. Golpar Raboky. A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type. *Appl. Math. Comput.*, 189:641-646, 2007.

[3] A. Bellour, K. Rouibah, P.M. Lima and E. Rawashdeh. Iterative continuous collocation method for solving nonlinear Volterra integral equations. *Kragujevac J. Math.*, 46(4):635-648, 2022.

[4] T. Diogo. Collocation and iterated collocation methods for a class of weakly singular Volterra integral equations. *J. Comput. Appl. Math.*, 229(2):363-372, 2009.

[5] T. Diogo, N. Franco and P.M. Lima. High-horder product integration methods for a Volterra integral equation with logarithmic singular kernel. *Commun. Pure Appl. Anal.*, 3(2):217-235, 2004.

[6] T. Diogo and P. Lima. Superconvergence of collocation methods for a class of weakly singular Volterra integral equations. *J. Comput. Appl. Math.*, 218:307-316, 2008.

[7] M. Ghasemia, M. Tavassoli Kajani and E. Babolian. Application of He's homotopy perturbation method to nonlinear integro-differential equations. *Appl. Math. Comput.*, 188:538-548, 2007.

[8] N.M. Madbouly, D.F. McGhee and G.F. Roach. Adomian's method for Hammerstein integral equations arising from chemical reactor theory. *Appl. Math. Comput.*, 117:241-249, 2001.

[9] K. Maleknejad, P. Torabi and R. Mollapourasl. Fixed point method for solving nonlinear quadratic Volterra integral equations. *Comp. Math. Appl.*, 62:2555-2566, 2011.

[10] K. Maleknejad and F. Mirzaee. Using rationalized Haar wavelet for solving linear integral equations. *Appl. Math. Comput.*, 160:579-587, 2005.

[11] M.H. Reihani and Z. Abadi. Rationalized Haar functions method for solving Fredholm and Volterra integral equations. *J. Comput. Appl. Math.*, 200:12-20, 2007.

[12] G. Vainikko. Spline collocation unterpolation method for linear and nonlinear cordial Volterra integral equations. *Numer. Funct. Anal. Optim.*, 32(1):83-109, 2011.

[13] G. Vainikko. Cordial Volterra integral equations 1. *Numer. Funct. Anal. Optim.*, 30:1145-1172, 2009.

[14] G. Vainikko. Cordial Volterra integral equations 2. *Numer. Funct. Anal. Optim.*, 31(1):191-219, 2010.

[15] G. Vainikko. Spline collocation for cordial Volterra integral equations. *Numer. Funct. Anal. Optim..*, 31:313-338, 2010.

[16] S. Yalçinbaş. Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations. *Appl. Math. Comput.*, 127:195-206, 2002.