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On the limit points of pseudo Leja sequences

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Abstract

We prove the existence of pseudo Leja sequences with large sets of limit points for many plane compact sets.

Keywords: Leja sequences, Pseudo Leja sequences.

1 Introduction

Let K be a non-empty compact subset of the complex plane and let $A = (a_n : n \in \mathbb{N})$ be a sequence of points in K. We write

$$w\big(A, a_d; z\big) := \prod_{j=0}^{d-1}(z - a_j), \quad d \geq 1. \tag{1.1}$$

One says that A is a Leja sequence for K if, for all $d \geq 1$, the $(d+1)$-st entry a_d maximizes the product of the distances to the d previous ones, that is

$$|w\big(A, a_d; a_d\big)| = \max_{z \in K} |w\big(A, a_d; z\big)|. \tag{1.2}$$

By the maximal principle, all points of A (except perhaps a_0) must lie on the outer boundary $\partial_\infty K$ of K. It is known that every non-polar compact set (that has a positive logarithmic capacity) possesses infinitely many Leja sequences but it is in general impossible to compute them. Recently, Białas-Cież and Calvi [2] described the structure of Leja sequences for the unit disk in $\mathbb{C}$. Also in [2], the authors introduced the concept of pseudo Leja sequence for K, that is a sequence $Z = (z_n : n \in \mathbb{N})$ in K such that the $(d+1)$-st entry z_d satisfies the inequality

$$M_Z(z_d)|w\big(Z, z_d; z_d\big)| \geq \max_{z \in K} |w\big(Z, z_d; z\big)|, \quad d \geq 1, \tag{1.3}$$

where $\big(M_Z(z_d) : d \in \mathbb{N}^*\big)$ is a sequence of positive real numbers greater than or equal to 1 of subexponential growth, i.e., $\lim_{d\to\infty} \big(M_Z(z_d)\big)^{\frac{1}{d}} = 1$. The sequence $M_Z(z_d)$ is called the Edrei growth of the pseudo Leja sequence Z. There are some advantages in working with pseudo Leja sequences. First, unlike Leja sequences, pseudo Leja sequences can be easily computed and are therefore suitable for numerical purposes. For details, we refer the reader to [2]. Second, from a theorical point of view, pseudo Leja sequences also provide excellent points for polynomial interpolation. We shall explain the second point. Suppose that K is a non-polar, polynomially convex, compact set in $\mathbb{C}$. Białas-Cież and Calvi showed that

$$\lim_{d\to\infty} |\mathrm{VDM}(z_0, \ldots, z_{d-1})|^{\frac{2}{(d-1)d}} = C(K), \tag{1.4}$$

where $\mathrm{VDM}(z_0, \ldots, z_{d-1}) = \prod_{0 \leq j < k \leq d-1}(z_k - z_j)$ and $C(K)$ is the logarithmic capacity of K. This asymptotic behavior enables one to use [3, Theorem 1.5] and get the following two properties.

1. $\lim_{d\to\infty}(1/d)\sum_{j=0}^{d-1}[z_j] = \mu_K$, where $[z_j]$ is the Dirac measure at z_j and μ_K is the equilibrium measure of K.
2. Under the additional hypothesis that K is regular in the sense of the potential theory, for every holomorphic function f in a neighborhood of K, the Lagrange interpolation polynomial of f at $z_0, \ldots, z_{d-1}$ converges uniformly to f on K as $d \to \infty$.

According to a remark in [2], the first property implies that every point in the support of μ_K is a limit point of $Z = (z_n : n \in \mathbb{N})$. Since this support lies in $\partial_\infty K$, the number of points lying in any compact subset G of the interior of K is small in the sense that

$$\lim_{n\to\infty} \frac{1}{d} \sharp\big(G \cap \{z_0, \ldots, z_{d-1}\}\big) = 0.$$

There arises a natural problem to decide whether there exists a pseudo Leja sequence of Edrei growth for a compact set with a limit point in the interior of the compact set. The aim of this paper is to give an affirmative answer. It is shown that the sets of limit points of pseudo Leja sequences for many plane compact sets constructed below contain neighborhoods of boundaries of these compact sets, and is even equal to the whole set when the compact set is the unit disk.

Notation. The closed disk of center about $a \in \mathbb{C}$ and radius $r > 0$ is denoted by $D(a,r)$. For simplicity, we write $D := D(0,1)$. Let $Z = (z_n : n \in \mathbb{N})$ be a sequence of distinct complex numbers. The index of $z \in Z$ is denoted by $s_Z(z)$, which shows the position of z in Z, so that $s_Z(z_j) = j+1$ for $j \geq 0$. For each $d \geq 1$, we define $Z_d := (z_0, \ldots, z_{d-1})$. For $T_k = (t_0, \ldots, t_{k-1})$, we write $(Z_d, T_k) := (z_0, \ldots, z_{d-1}, t_0, \ldots, t_{k-1})$.

2 Pseudo Leja sequences for the unit disk

Given a Leja sequence A for D and a sequence B of distinct points in $\mathrm{int}(D)$, we show how to insert the entries of B into A in order that the resulting sequence is a pseudo Leja sequence for D. We exploit the structure of Leja sequences for D that is given in [2, Theorem 5] and that we recall below. Here a d-tuple $A_d := (a_0, \ldots, a_{d-1})$ is called a d-Leja section of the sequence A. The underlying sets of A_d and A are $\{a_0, \ldots, a_{d-1}\}$ and $\{a_n : n \geq 0\}$ respectively.

Theorem 2.1 (Białas-Cież and Calvi). *The structure of a Leja sequence $A = (a_n : n \in \mathbb{N})$ for the unit disk D with $a_0 = 1$ is given by the following rules.*

1. *The underlying set of the 2^n-Leja section A_{2^n} consists of the 2^n-th roots of unity;*
2. *The 2^{n+1}-Leja section $A_{2^{n+1}}$ is $(A_{2^n}, \rho U_{2^n})$, where ρ is a 2^n-th root of -1 and U_{2^n} is the 2^n-Leja section of a Leja sequence $U = (u_n : n \in \mathbb{N})$ for D with $u_0 = 1$.*

Theorem 2.2. *Let $A = (a_n : n \in \mathbb{N})$ be a Leja sequence for D with $a_0 \in \partial D$ and $B = (b_n : n \in \mathbb{N})$ a sequence of distinct points in* $\mathrm{int}(D)$. *Then there exists a pseudo Leja sequence for D whose underlying set is $A \cup B$.*

Proof. Without loss of generality, we assume that $a_0 = 1$. We consider three sequences of positive real numbers whose entries are defined by

$$\alpha_j = \mathrm{dist}(B_j, \partial D) = \inf\{|b_k - a| : 0 \leq k \leq j-1, |a| = 1\}, \quad j \geq 1. \tag{2.1}$$

$$\beta_j = |w(B, b_j; b_j)| \quad \text{and} \quad \gamma_j = \sup_{z\in D} |w(B, b_j; z)|, \quad j \geq 1, \tag{2.2}$$

where $w(B, b_j; z)$ is defined as (1.1). Take a subsequence $4 < n_0 < n_1 < \cdots < n_k < \cdots$ such that

$$\lim_{j\to\infty} \Big(\frac{2}{\alpha_{j+1}}\Big)^{\frac{j+1}{2^{n_j}}} = \lim_{j\to\infty} \Big(\frac{2\gamma_j}{(1-|b_j|)\beta_j}\Big)^{\frac{1}{2^{n_j}}} = 1. \tag{2.3}$$

Let X be a new sequence obtained by inserting the entries of B into A such that b_j is inserted between $a_{2^{n_j}-1}$ and $a_{2^{n_j}}$ for all $j \geq 0$,

$$a_0, a_1 \ldots, a_{2^{n_0}-1}, b_0, a_{2^{n_0}}, \ldots, a_{2^{n_1}-1}, b_1, a_{2^{n_1}}, \ldots, a_{2^{n_j}-1}, b_j, a_{2^{n_j}}, \ldots, a_{2^{n_{j+1}}-1}, b_{j+1}, a_{2^{n_{j+1}}}, \ldots \tag{2.4}$$

We will show that X is a pseudo Leja sequence for D. To do this, we construct a discrete function $M_X : X \longmapsto [1,\infty)$ (of subexponential growth of $s_X(x)$) such that

$$M_X(x)|w(X,x;x)| \geq \sup_{z\in D}|w(X,x;z)|, \quad x \in X. \tag{2.5}$$

We shall define differently M_X on A and on B and prove first for $x = a_d$ and then for $x = b_j$.

Assume first that $x = a_d$. Since the first entries of X play no role in the required property that X is a pseudo Leja sequence, we may assume that $2^{n_j} \leq d \leq 2^{n_{j+1}} - 1$ for $j \geq 1$. Looking at the definition of X in (2.4) and relation (1.1), we have

$$w(X,a_d;z) = w(A,a_d;z)\cdot w(B,b_{j+1};z). \tag{2.6}$$

Since the second factor at the right hand side of (2.6) contains $j+1$ factors $z-b_k$, $0 \leq k \leq j$, and $|z-b_k| \leq 2$ for every $z \in D$, the hypothesis that A is a Leja sequence for D implies

$$\sup_{z\in D}|w(X,a_d;z)| \leq \sup_{z\in D}|w(A,a_d;z)|\cdot \sup_{z\in D}|w(B,b_{j+1};z)| \leq 2^{j+1}|w(A,a_d;a_d)|. \tag{2.7}$$

On the other hand, relation (2.1) gives $|a_d - b_k| \geq \alpha_{j+1}$ for all $0 \leq k \leq j$. Thus

$$|w(X,a_d;a_d)| \geq (\alpha_{j+1})^{j+1}|w(A,a_d;a_d)|. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$M_X(a_d)|w(X,a_d;a_d)| \geq \sup_{z\in D}|w(X,a_d;z)| \quad \text{with} \quad M_X(a_d) = \Big(\frac{2}{\alpha_{j+1}}\Big)^{j+1}. \tag{2.9}$$

Since the index of a_d in X is $s_X(a_d) = d+j+2 > 2^{n_j}$, equation (2.3) yields

$$\lim_{d\to\infty}\Big(M_X(a_d)\Big)^{\frac{1}{s_X(a_d)}} = \lim_{j\to\infty}\Big(\frac{2}{\alpha_{j+1}}\Big)^{\frac{j+1}{2^{n_j}}} = 1. \tag{2.10}$$

Second, we work with $x = b_j$ for $j \geq 1$. We have

$$w(X,b_j;z) = w(A,a_{2^{n_j}};z)\cdot w(B,b_j;z) = (z^{2^{n_j}} - 1)w(B,b_j;z). \tag{2.11}$$

Here we use the fact that $w(A,a_{2^{n_j}};z) = z^{2^{n_j}} - 1$, since the set $\{a_0,\ldots,a_{2^{n_j}-1}\}$ forms a complete set of the 2^{n_j}-roots of unity (see Theorem 2.1). It follows that

$$\sup_{z\in D}|w(X,b_j;z)| \leq 2\sup_{z\in D}|w(B,b_j;z)| = 2\gamma_j, \tag{2.12}$$

and

$$|w(X,b_j;b_j)| = |b_j^{2^{n_j}} - 1|\cdot|w(B,b_j;b_j)| \geq (1-|b_j|)\beta_j. \tag{2.13}$$

We get from (2.12) and (2.13) the following estimate

$$M_X(b_j)|w(X,b_j;b_j)| \geq \sup_{z\in D}|w(X,b_j;z)| \quad \text{with} \quad M_X(b_j) = \frac{2\gamma_j}{(1-|b_j|)\beta_j}. \tag{2.14}$$

Since the index of b_j in X is $s_X(b_j) = 2^{n_j} + j + 1 > 2^{n_j}$, equation (2.3) gives

$$\lim_{j\to\infty}\Big(M_X(b_j)\Big)^{\frac{1}{s_X(b_j)}} = \lim_{j\to\infty}\Big(\frac{2\gamma_j}{(1-|b_j|)\beta_j}\Big)^{\frac{1}{2^{n_j}}} = 1. \tag{2.15}$$

This completes the proof of the theorem. □

If we choose the sequence B in Theorem 2.2 to be dense in D, then we have the following corollary.

Corollary 2.3. *There exist pseudo Leja sequences for D such that the set of their limit points is D.*

3 Pseudo Leja sequences for compact sets in $\mathbb{C}$

This section is devoted to the construction of pseudo Leja sequences for more general compact sets. Our idea is similar in spirit to [2, Theorem 6], but original arguments come from the work of Alper [1, p. 48-49] with a weaker assumption on K.

Let K be a compact set in $\mathbb{C}$ such that ∂K is an analytic Jordan curve. Suppose that $\phi(z)$ is a conformal mapping of $\overline{\mathbb{C}}\setminus D$ onto $\overline{\mathbb{C}}\setminus K$. It is known that

$$\phi(z) = cz + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots \quad \text{with} \quad |c| = C(K). \tag{3.1}$$

Since ∂K is assumed to be an analytic Jordan curve, there exists an analytic and univalent continuation of ϕ to the domain $\overline{\mathbb{C}}\setminus D(0,\rho_0)$ for some $\rho_0 < 1$ (see for instance [4, p.12]). If $1 > \rho_1 > \rho_0$, then the function

$$\psi(t,z) = \begin{cases} \frac{\phi(t)-\phi(z)}{t-z} & \text{if } t \neq z \\ \phi'(z) & \text{if } t = z \end{cases}$$

is continuous and does not vanish when $t,z \in D\setminus \operatorname{int}(D(0,\rho_1))$. Thus, there exist $M_1, M_2 > 0$ such that

$$M_1 \leq \Big|\frac{\phi(t)-\phi(z)}{t-z}\Big| \leq M_2, \quad t,z \in D\setminus \operatorname{int}(D(0,\rho_1)),\ t \neq z. \tag{3.2}$$

Lemma 3.1. *Under the above assumptions. If $\rho_0 < \rho_1 < 1$ and $e_k = \exp(2k\pi i/d)$ for $0 \leq k \leq d-1$, then*

$$\frac{C(K)^d}{V} \leq \prod_{k=0}^{d-1} \frac{|\phi(t)-\phi(e_k)|}{|t-e_k|} \leq VC(K)^d, \quad \rho_1 \leq |t| \leq 1, \tag{3.3}$$

where V is a positive constant independent of d.

Proof. The proof is a slight adaptation of the reasoning used in [2]. Since $z \mapsto \psi(z,t)$ is a nowhere-vanishing holomorphic function on $\overline{\mathbb{C}}\setminus D$ for all $\rho_1 \leq |t| \leq 1$, there exists a branch of $\log \psi(z,t)$ that, as a function of z, is holomorphic on $\overline{\mathbb{C}}\setminus D$ and continuous on $\overline{\mathbb{C}}\setminus \operatorname{int}(D)$. For convenience, we set

$$f_t(z) = \log \frac{\phi(t)-\phi(z)}{t-z}, \quad |z| \geq 1,\ \rho_1 \leq |t| \leq 1. \tag{3.4}$$

The real part of $f_t(z)$, say $\Re f_t(z)$, is harmonic on $\overline{\mathbb{C}}\setminus D$ and continuous on $\overline{\mathbb{C}}\setminus \operatorname{int}(D)$. It is clear that $\Re f_t(z) = \log\big|\frac{\phi(t)-\phi(z)}{t-z}\big|$. By the mean value theorem for harmonic functions we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Re f_t(e^{i\theta})\mathrm{d}\theta = \lim_{z\to\infty} \Re f_t(z) = \log|c| = \log C(K). \tag{3.5}$$

From (3.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta}) = ie^{i\theta}\Big(\frac{\phi'(e^{i\theta})}{\phi(e^{i\theta})-\phi(t)} - \frac{1}{e^{i\theta}-t}\Big), \quad \theta \in [0,2\pi],\ \rho_1 \leq |t| \leq 1,\ t \neq e^{i\theta}. \tag{3.6}$$

The limit $\lim_{t\to e^{i\theta}} \frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta})$ exists and is denoted by $\frac{\mathrm{d}}{\mathrm{d}\theta} f_{e^{i\theta}}(e^{i\theta})$, since

$$\lim_{t\to e^{i\theta}} \frac{\mathrm{d}}{\mathrm{d}\theta} f_t(e^{i\theta}) = ie^{i\theta} \lim_{t\to e^{i\theta}} \frac{\phi'(e^{i\theta}) - \frac{\phi(e^{i\theta})-\phi(t)}{e^{i\theta}-t}}{(e^{i\theta}-t)\frac{\phi(e^{i\theta})-\phi(t)}{e^{i\theta}-t}} = \frac{ie^{i\theta}\phi''(e^{i\theta})}{2\phi'(e^{i\theta})}. \tag{3.7}$$

Here we use the Taylor expansion of the holomorphic function ϕ at $e^{i\theta}$ to get the second equality in (3.7). Thus, the function $(\theta,t)\mapsto \frac{\mathrm{d}}{\mathrm{d}\theta}f_t(e^{i\theta})$ is continuous on $[0,2\pi]\times\{\rho_1\le|t|\le 1\}$. It follows that there exists $V_0<\infty$ such that

$$\int_0^{2\pi}\Big|\frac{\mathrm{d}}{\mathrm{d}\theta}f_t(e^{i\theta})\Big|\mathrm{d}\theta < V_0,\quad \rho_1\le|t|\le 1. \tag{3.8}$$

Consequently, $f_t(e^{i\theta})$ is a function of total variation bounded by V_0 for all $\rho_1\le|t|\le 1$. Therefore, so is its real part $\Re f_t(e^{i\theta})$. Using [2, Lemma 1] for $\Re f_t$, the formula for $\Re f_t(e^{i\theta})$ and relation (3.5), we obtain

$$\Big|\log C(K)-\frac{1}{d}\sum_{k=1}^{d-1}\log\Big|\frac{\phi(t)-\phi(e_k)}{t-e_k}\Big|\Big|\le\frac{V_0}{d},\quad \rho_1\le|t|\le 1. \tag{3.9}$$

This estimate implies the conclusion of the lemma. □

Theorem 3.2. *Let K be a compact set in $\mathbb{C}$ such that ∂K is an analytic Jordan curve. Then there exists a pseudo Leja sequence for K whose set of limit points contains $\{z\in K:\operatorname{dist}(z,\partial K)\le r\}$ for some $r>0$.*

Proof. Let $\phi(z)$ be a conformal mapping of $\overline{\mathbb{C}}\setminus D$ onto $\overline{\mathbb{C}}\setminus K$. Then ϕ admits an analytic and univalent continuation to the domain $\overline{\mathbb{C}}\setminus D(0,\rho_0)$ for $\rho_0<1$. Let $A=(a_n:n\in\mathbb{N})$ be a Leja sequence for D with $a_0=1$. Take $\rho_1\in(\rho_0,1)$ and a sequence $B=(b_n:n\in\mathbb{N})$ of distinct points in the open annulus $\{\rho_1<|z|<1\}$ such that B is dense in the closure of this annulus. Let X be a sequence defined as in (2.4) such that X is a pseudo Leja sequence for D. Since the closure of B is $\{\rho_1\le|z|\le 1\}$, the set of limit points of $\phi(X)$ contains $\phi(\{\rho_1\le|z|\le 1\})$, a compact subset of K containing $\{z\in K:\operatorname{dist}(z,\partial K)\le r\}$ for some $r>0$. Thus, it remains to verify that $\phi(X)$ is a pseudo Leja sequence for K. For simplicity, we write

$$\widetilde{A}:=\phi(A),\quad \widetilde{B}:=\phi(B),\quad \widetilde{X}:=\phi(X),$$
$$\tilde{z}:=\phi(z),\quad \tilde{a}_d:=\phi(a_d),\quad\text{and}\quad \tilde{b}_k:=\phi(b_k),\quad d\ge 0,\quad k\ge 0. \tag{3.10}$$

With this notation, the sequence $\widetilde{X}$ is given by

$$\tilde{a}_0,\tilde{a}_1\dots,\tilde{a}_{2^{n_0}-1},\tilde{b}_0,\tilde{a}_{2^{n_0}},\dots,\tilde{a}_{2^{n_1}-1},\tilde{b}_1,\tilde{a}_{2^{n_1}},\dots,\tilde{a}_{2^{n_j}-1},\tilde{b}_j,\tilde{a}_{2^{n_j}},\dots,\tilde{a}_{2^{n_{j+1}}-1},\tilde{b}_{j+1},\tilde{a}_{2^{n_{j+1}}},\dots \tag{3.11}$$

We also consider two cases as in the proof of Theorem 2.2 and use the formula for the product defined in (1.1).

For $2^{n_j}\le d\le 2^{n_{j+1}}-1$ with $j\ge 1$, we have

$$w\big(\widetilde{X},\tilde{a}_d;\tilde{z}\big)=w\big(\widetilde{A},\tilde{a}_d;\tilde{z}\big)\cdot w\big(\widetilde{B},\tilde{b}_{j+1};\tilde{z}\big). \tag{3.12}$$

Using [2, Lemma 3] we get

$$c_d^{-1}C(K)^d|w(A,a_d;z)|\le\big|w\big(\widetilde{A},\tilde{a}_d;\tilde{z}\big)\big|\le c_dC(K)^d|w(A,a_d;z)|,\quad z\in\partial D, \tag{3.13}$$

where $c_d<(d+1)^{C/\log 2}$ and C is a positive constant depending only on K, see [2, Subsection 3.2] for the precise definition for c_d. On the other hand, inequality (3.2) gives

$$M_1|z-b_k|\le\big|\tilde{z}-\tilde{b}_k\big|\le M_2|z-b_k|,\quad \rho_1\le|z|\le 1,\ k\ge 0. \tag{3.14}$$

Hence

$$M_1^{j+1}|w(B,b_{j+1};z)|\le\big|w\big(\widetilde{B},\tilde{b}_{j+1};\tilde{z}\big)\big|\le M_2^{j+1}|w(B,b_{j+1};z)|,\quad z\in\partial D. \tag{3.15}$$

Multiplying sides by sides of (3.13) and (3.15), and using (3.12), we obtain

$$c_d^{-1}C(K)^d M_1^{j+1}|w(X,a_d;z)| \le |w(\widetilde{X},\tilde{a}_d;\tilde{z})| \le c_d C(K)^d M_2^{j+1}|w(X,a_d;z)|, \quad z\in\partial D \tag{3.16}$$

Since $\phi(\partial D)=\partial K$, the maximum principle and the relations in (3.16) (for $z\in\partial D$ and a_d) now give

$$\begin{aligned}\sup_{t\in K}|w(\widetilde{X},\tilde{a}_d;t)| &= \sup_{\tilde{z}=\phi(z),|z|=1}|w(\widetilde{X},\tilde{a}_d;\tilde{z})|\\ &\le c_d C(K)^d M_2^{j+1}\sup_{|z|=1}|w(X,a_d;z)|\\ &\le c_d C(K)^d M_2^{j+1}M_X(a_d)|w(X,a_d;a_d)|\\ &\le c_d^2\Big(\frac{M_2}{M_1}\Big)^{j+1}M_X(a_d)|w(\widetilde{X},\tilde{a}_d;\tilde{a}_d)|.\end{aligned}$$

Let us set $M_{\widetilde{X}}(\tilde{a}_d)=c_d^2\big(\frac{M_2}{M_1}\big)^{j+1}M_X(a_d)$. Since $s_{\widetilde{X}}(\tilde{a}_d)=s_X(a_d)=d+2+j>2^{n_j}\ge 2^j$, we have

$$\lim_{d\to\infty}c_d^{\frac{2}{s_{\widetilde{X}}(\tilde{a}_d)}}=\lim_{d\to\infty}c_d^{\frac{2}{d}}=1,\quad \lim_{d\to\infty}\Big(\frac{M_2}{M_1}\Big)^{\frac{j+1}{s_{\widetilde{X}}(\tilde{a}_d)}}=\lim_{j\to\infty}\Big(\frac{M_2}{M_1}\Big)^{\frac{j+1}{2^j}}=1,$$
$$\lim_{d\to\infty}\Big(M_X(a_d)\Big)^{\frac{1}{s_{\widetilde{X}}(\tilde{a}_d)}}=\lim_{d\to\infty}\Big(M_X(a_d)\Big)^{\frac{1}{s_X(a_d)}}=1. \tag{3.17}$$

Here we use the fact that c_d grows at most like a polynomial in d and the hypothesis that X is a pseudo Leja sequence for D. Hence

$$\lim_{d\to\infty}\Big(M_{\widetilde{X}}(\tilde{a}_d)\Big)^{\frac{1}{s_{\widetilde{X}}(\tilde{a}_d)}}=1. \tag{3.18}$$

We now turn to $\tilde{b}_j$ for $j\ge 1$. We have

$$w(\widetilde{X},\tilde{b}_j;\tilde{z})=w(\tilde{A},\tilde{a}_{2^{n_j}};\tilde{z})\cdot w(\widetilde{B},\tilde{b}_j;\tilde{z}). \tag{3.19}$$

Using relation (3.13) for $d=2^{n_j}$ and relation (3.14) to estimate the first and the second factor at the right hand side of (3.19) respectively, we obtain

$$|w(\widetilde{X},\tilde{b}_j;\tilde{z})|\le c_{2^{n_j}}C(K)^{2^{n_j}}M_2^j|w(X,b_j;z)|,\quad z\in\partial D. \tag{3.20}$$

On the other hand, since $\{a_0,\dots,a_{2^{n_j}-1}\}$ is a complete set of the 2^{n_j}-th roots of unity, Lemma 3.1 gives

$$|w(\widetilde{A},\tilde{a}_{2^{n_j}};\tilde{b}_j)|\ge\frac{C(K)^{2^{n_j}}}{V}|w(A,a_{2^{n_j}};b_j)|. \tag{3.21}$$

By $|\tilde{b}_j-\tilde{b}_k|\ge M_1|b_j-b_k|$ for all $0\le k\le j-1$, relations (3.19) and (3.21) show that

$$|w(\widetilde{X},\tilde{b}_j;\tilde{b}_j)|\ge\frac{C(K)^{2^{n_j}}}{V}M_1^j|w(X,b_j;b_j)|. \tag{3.22}$$

Combining (3.20) and (3.22), and using the maximum principle again, we have

$$\begin{aligned}\sup_{t\in K}|w(\widetilde{X},\tilde{b}_j;t)| &= \sup_{\tilde{z}=\phi(z),|z|=1}|w(\widetilde{X},\tilde{b}_j;\tilde{z})|\\ &\le c_{2^{n_j}}C(K)^{2^{n_j}}M_2^j\sup_{|z|=1}|w(X,b_j;z)|\\ &\le c_{2^{n_j}}C(K)^{2^{n_j}}M_2^jM_X(b_j)|w(X,b_j;b_j)|\\ &\le c_{2^{n_j}}V\Big(\frac{M_2}{M_1}\Big)^jM_X(b_j)|w(\widetilde{X},\tilde{b}_j;\tilde{b}_j)|.\end{aligned}$$

Set $M_{\widetilde{X}}(\tilde{b}_j) = c_{2^{n_j}} V(\frac{M_2}{M_1})^j M_X(b_j)$. We also have $s_{\widetilde{X}}(\tilde{b}_j) = s_X(b_j) = 2^{n_j} + j + 1 \geq 2^j + j + 1$. A passage to the limit similar to (3.17) implies that

$$\lim_{j\to\infty} \left(M_{\widetilde{X}}(\tilde{b}_j) \right)^{\frac{1}{s_{\widetilde{X}}(\tilde{b}_j)}} = 1. \tag{3.23}$$

This completes the proof of the theorem. □

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