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On generalized least square approximation

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Abstract

We study generalized least square approximation polynomials which are built from sets of functionals. We construct sets of functionals for bivariate harmonic functions, univariate holomorphic functions and sufficiently smooth functions on curves such that the sequences of the generalized least square approximation polynomials converge uniformly.

Keywords: Least square approximation, Admissible meshes, Radon projections.
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1 Introduction

Let $\mathbb{K}$ be the real or complex field. Consider the space $\mathcal{P}(\mathbb{K}^d)$ of all algebraic polynomials in $\mathbb{R}^d$. Let $\mathcal{P}_n(\mathbb{K}^d)$ be the subspace consisting of all polynomials of total degree at most n. Given a compact subset $K \subset \mathbb{K}^d$ and $f : K \to \mathbb{K}$ we denote by $\|f\|_K = \sup_{\mathbf{x}\in K} |f(\mathbf{x})|$ the usual supremum norm on K.

A set $A \subset \mathbb{K}^d$ is said to be *determining* for the space of functions $\mathcal{F}$, or, for short, $\mathcal{F}$-determining, if $p \in \mathcal{F}$ and $p|_A = 0$ force $p \equiv 0$. Here $p|_A$ is restriction of p to A.

Let A be $\mathcal{P}_n(\mathbb{K}^d)$-determining and $f : A \to \mathbb{K}$. In [13, Theorem 1], Calvi and Levenberg showed that there exists a unique polynomial $p \in \mathcal{P}_n(\mathbb{K}^d)$ which minimizes the quantity

$$\Phi_{f,A}(q) := \sum_{\mathbf{a}\in A} |q(\mathbf{a}) - f(\mathbf{a})|^2, \quad q \in \mathcal{P}_n(\mathbb{K}^d). \tag{1}$$

The polynomial p is denoted by $\Lambda(A, f)$ and is called the discrete least square approximation polynomial. The authors also proved a Lebesgue type inequality

$$\|f - \Lambda(A, f)\|_K \le \Big(1 + C(A,K)(1 + \sqrt{\#A})\Big)\operatorname{dist}_K(f, \mathcal{P}_n(\mathbb{K}^d)), \tag{2}$$

where K is a compact set containing A, $\operatorname{dist}_K(f, \mathcal{P}_n(\mathbb{K}^d))$ denotes the uniform distance from f to $\mathcal{P}_n(\mathbb{K}^d)$ and the constant $C(A,K)$ is defined by

$$\|q\|_K \le C(A,K)\|q\|_A, \quad q \in \mathcal{P}_n(\mathbb{K}^d).$$

Examining (2) we see that the uniform error between f and $\Lambda(A, f)$ is controlled by the quantity $C(A,K)\sqrt{\#A}$ which depends only on A and K. This fact had suggested Calvi and Levenberg to give the theory of admissible meshes. It is defined as follows. A sequence of discrete sets $\mathbf{A} = \{A_n \subset K : n \in \mathbb{N}^*\}$ is called an *admissible mesh* in K if there exist constants c_1 and c_2 depending only on K such that

$$\|p\|_K \le c_1\|p\|_{A_n}, \quad p \in \mathcal{P}_n(\mathbb{K}^d), \quad n \ge 1, \tag{3}$$

where the cardinality of A_n grows at most polynomially on n, i.e., $\#A_n \le c_2 n^m$ for some fixed $m \in \mathbb{N}^*$ depending only on K.

For such a compact set K, if $\mathbf{A} = \{A_n \subset K : n \in \mathbb{N}^*\}$ is an admissible mesh, then no non-zero polynomial in $\mathcal{P}_n(\mathbb{K}^d)$ vanishes on A_n. Hence we must have $m \ge d$ since $\dim \mathcal{P}_n(\mathbb{R}^d) = \binom{n+d}{d} \sim n^d$. This leads to the definition of *optimal admissible meshes* introduced in [17]: An admissible mesh $\mathbf{A}$ is optimal if $\#A_n \le c_3 n^d$ for some $c_3 > 0$ depending only on K.

Note that admissible meshes are preserved by the operations of taking unions, product and transformation of sets under affine automorphisms. They are also stable under small pertubation and analytic transformations, see [24, 25]. From computational point of views, admissible meshes are very useful. It was proved by Calvi and Levenberg that the least square approximation polynomials based on admissible meshes approximate the smooth functions or holomorphic functions uniformly. Moreover, in [8, 9, 10], the authors showed that discrete extremal sets of Fekete and Leja types can be exacted from admissible meshes. For standard compact sets, for instance triangles, quadrangles, disks, cylinders, the authors constructed in [11, 14] low cardinality admissible meshes. General results on the construction of admissible meshes in compact sets in $\mathbb{R}^d$ are recently given by Kroo [17, 18, 19]. He used constructive methods

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to get optimal or near optimal admissible meshes on compact sets which are graph domains of polynomials, differentiable or analytic functions, and on the compact sets which admit Bernstein type or Markov type inequalities. In a recent work [26], Piazzon built optimal admissible meshes on two classes of compact set in $\mathbb{R}^d$.

Observe that the value $p(\mathbf{a})$ is identical with $\delta_{\mathbf{a}}(p)$, where $\delta_{\mathbf{a}}$ is the Dirac functional. Hence we can rewrite (1) as

$$\Phi_{f,A}(q) := \sum_{\mathbf{a}\in A} |\delta_{\mathbf{a}}(q) - \delta_{\mathbf{a}}(f)|^2, \quad q \in \mathcal{P}_n(\mathbb{K}^d). \tag{4}$$

In this note, we replace the $\delta_{\mathbf{a}}$'s and $\mathcal{P}_n(\mathbb{K}^d)$ by functionals on the space of continuous functions $\mathcal{C}(K)$ and a finite dimensional subspace $\mathcal{Q}$ of $\mathcal{P}(\mathbb{K}^d)$ respectively. We show that the corresponding sum of squares also attains its infimum at a unique element $p \in \mathcal{Q}$. The generalized least square approximation polynomial p also admits a Lebesgue type inequality. This leads to the notion of generalized admissible meshes.

In Section 3, we construct good functionals for the spaces of bivariate harmonic polynomials and univariate holomorphic polynomials such that the sequence of the generalized least square approximation polynomials converges uniformly. They consist of nests of points and families of Radon projections.

Section 4 deals with the construction of admissible meshes on smooth curves in $\mathbb{R}^d$. The method relies heavily on Markov type inequalities on curves. Note that if Γ is an algebraic curve in $\mathbb{R}^d$, then the restriction of $\mathcal{P}_n(\mathbb{K}^d)$ to Γ will form a vector space $\mathcal{P}_n(\Gamma)$ whose dimension N_n is smaller than $\dim \mathcal{P}_n(\mathbb{K}^d)$ in general. Hence the (generalized) optimal admissible meshes for a compact set on $\mathcal{P}_n(\Gamma)$ should have cardinality $O(N_n)$ rather than $O(n^d)$. The same situation happens in [12] in which Bos and Vianello constructed Tchakaloff polynomial meshes on algebraic varieties in $\mathbb{R}^d$.

2 Generalized least square approximation

Let $\mathcal{Q}$ be a finite dimensional subspace of $\mathcal{P}(\mathbb{K}^d)$. A set of functionals $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_m\}$ in the dual space $\mathcal{Q}^\sharp$ is said to be determining for $\mathcal{Q}$, or, for short $\mathcal{Q}$-determining, if $p \in \mathcal{Q}$ and $\mu_j(p) = 0$ for all $j = 1, \ldots, m$ force $p \equiv 0$. The cardinality of an $\mathcal{Q}$-determining set $\mathcal{M}$ can not smaller than the dimensional of $\mathcal{Q}$, that is $m \geq \dim \mathcal{Q}$. Note that if $\mu_i = \delta_{\mathbf{a}_i}$ for $i = 1, \ldots, m$, then $\mathcal{M}$ is $\mathcal{Q}$-determining if and only if $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is $\mathcal{Q}$-determining.

Theorem 2.1. *Let $\mathcal{Q}$ be a finite dimensional subspace of $\mathcal{P}(\mathbb{K}^d)$ and let $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_m\} \subset \mathcal{Q}^\sharp$ be $\mathcal{Q}$-determining. Then, for any assigned numbers $c_1, c_2, \ldots, c_m$, there exists a unique polynomial $p \in \mathcal{Q}$ which minimizes the quantity*

$$\Phi(p) := \sum_{j=1}^m |\mu_j(p) - c_j|^2,$$

i.e., $\Phi(p) < \Phi(q)$ for every $q \in \mathcal{Q} \setminus \{p\}$.

Proof. The proof is adapted from the proof of the Hilbert projection theorem. Let us set $\alpha = \inf\{\Phi(p) : p \in \mathcal{Q}\}$. Then there exists a sequence $\{p_n\}$ in $\mathcal{Q}$ such that

$$\alpha \leq \Phi(p_n) \leq \alpha + 1/n, \quad n \geq 1.$$

For $n, k \geq 1$, we have

$$\begin{aligned}
\Phi(p_n) + \Phi(p_k) &= \sum_{j=1}^m \Big(|\mu_j(p_n) - c_j|^2 + |\mu_j(p_k) - c_j|^2\Big) \\
&= \sum_{j=1}^m \frac{1}{2}\Big(|\mu_j(p_n) + \mu_j(p_k) - 2c_j|^2 + |\mu_j(p_n) - \mu_j(p_k)|^2\Big) \\
&= \sum_{j=1}^m \Big(2|\mu_j(\frac{p_n + p_k}{2}) - c_j|^2 + \frac{1}{2}|\mu_j(p_n) - \mu_j(p_k)|^2\Big) \\
&= 2\Phi(\frac{p_n + p_k}{2}) + \frac{1}{2}\sum_{j=1}^m |\mu_j(p_n) - \mu_j(p_k)|^2.
\end{aligned}$$

It follows that

$$\sum_{j=1}^m |\mu_j(p_n) - \mu_j(p_k)|^2 = 2\Phi(p_n) + 2\Phi(p_k) - 4\Phi(\frac{p_n + p_k}{2}) \leq 4\alpha + \frac{2}{n} + \frac{2}{k} - 4\alpha = \frac{2}{n} + \frac{2}{k}.$$

Hence

$$\lim_{n,k\to\infty} \sum_{j=1}^m |\mu_j(p_n) - \mu_j(p_k)|^2 = 0. \tag{5}$$

Consider a sesquilinear form defined on $\mathcal{Q}$ by

$$\langle q, r\rangle_{\mathcal{M}} = \sum_{j=1}^m \mu_j(q)\overline{\mu_j(r)}, \quad q, r \in \mathcal{Q}.$$

By hypothesis that $\mathcal{M}$ is $\mathcal{Q}$-determining, we easily check that $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ defines a Hermitian inner product on $\mathcal{Q}$. Now, since $\mathcal{Q}$ is a finite dimensional space, it becomes a Hilbert space in which the norm induced by the Hermitian inner product is given by

$$\|q\|_{\mathcal{M}} = \sqrt{\sum_{j=1}^{m} |\mu_j(q)|^2}, \quad q \in \mathcal{Q}.$$

Hence relation (5) implies that there exists $\lim_{n\to\infty} p_n = p^* \in \mathcal{Q}$. Equivalently, $\lim_{n\to\infty} \mu_j(p_n) = \mu_j(p^*)$ for $j = 1,\dots,m$, and hence

$$\Phi(p^*) = \sum_{j=1}^{m} |\mu_j(p^*) - c_j|^2 = \lim_{n\to\infty} \sum_{j=1}^{m} |\mu_j(p_n) - c_j|^2 = \alpha.$$

It remains to prove the uniqueness. Assume that $\Phi(\tilde{p}) = \alpha$ with $\tilde{p} \in \mathcal{Q}$. Repeating the above arguments, we can write

$$\begin{aligned} 2\alpha &= \Phi(p^*) + \Phi(\tilde{p}) \\ &= \sum_{j=1}^{m} \Big(|\mu_j(p^*) - c_j|^2 + |\mu_j(\tilde{p}) - c_j|^2 \Big) \\ &= 2\Phi(\frac{p^* + \tilde{p}}{2}) + \frac{1}{2} \sum_{j=1}^{m} |\mu_j(p^*) - \mu_j(\tilde{p})|^2 \\ &\geq 2\alpha + \frac{1}{2}\|p^* - \tilde{p}\|_{\mathcal{M}}^2. \end{aligned}$$

It follows that $\|p^* - \tilde{p}\|_{\mathcal{M}} = 0$, and hence, $p^* = \tilde{p}$. The proof is complete. □

Definition 2.1. Let f be a function such that $\mu_j(f)$ is well-defined for every $j = 1,\dots,m$. Then the unique element $p \in \mathcal{Q}$ which minimizes the quantity

$$\Phi(p) := \sum_{j=1}^{m} |\mu_j(p) - \mu_j(f)|^2$$

is denoted by $\mathrm{S}(\mathcal{M}; f)$.

In the special case where $\mu_j = \delta_{\mathbf{a}_j}$ for $j = 1,\dots,m$ and $\mathcal{Q} = \mathcal{P}_n(\mathbb{R}^d)$, then $\mathrm{S}(\mathcal{M}; f)$ becomes $\Lambda(A, f)$ where $A = \{\mathbf{a}_1,\dots,\mathbf{a}_m\}$.

Starting from any ordered basis for $\mathcal{Q}$, we can use the Gram-Schmidt algorithm to construct an orthonormal basis $\{r_j\}_{j=1}^{m}$ of $\mathcal{Q}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{M}}$. From the projection theorem we have

$$\mathrm{S}(\mathcal{M}; f) = \sum_{j=1}^{m} \langle f, r_j\rangle_{\mathcal{M}} r_j.$$

It is interesting if we can find an orthonormal basis corresponding to the Radon projections in Section 3.

Let K be a $\mathcal{Q}$-determining compact set. Then the map $q \mapsto \|q\|_K := \sup_{\mathbf{x}\in K} |q(\mathbf{x})|$ defines a norm on $\mathcal{Q}$. Since $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{Q}$, so is the map $q \mapsto \max_{1\le j\le m} |\mu_j(q)|$. From the hypothesis that $\mathcal{Q}$ is a finite dimensional space, we can find a positive constant $C_1 = C_1(\mathcal{M}, \mathcal{Q}, K)$ such that

$$\|q\|_K \leq C_1 \max_{1\le j\le m} |\mu_j(q)|, \quad q \in \mathcal{Q}. \tag{6}$$

The following Lebesgue type inequality is similar to (2).

Theorem 2.2. *Let $\mathcal{Q}$ be a finite dimensional subspace of $\mathcal{P}(\mathbb{K}^d)$ and let K be an $\mathcal{Q}$-determining compact set. Let $\mathcal{F}$ be a subspace of $\mathcal{C}(K)$ that contains $\mathcal{Q}$ and $\mathcal{M} = \{\mu_1, \mu_2, \dots, \mu_m\} \subset \mathcal{F}^\sharp$ such that $\mathcal{M}$ is $\mathcal{Q}$-determining and there exists a positive constant $C_2 = C_2(\mathcal{M}, \mathcal{F}, K)$ satisfying*

$$\max_{1\le j\le m} |\mu_j(f)| \leq C_2 \|f\|_K, \quad f \in \mathcal{F}. \tag{7}$$

Then, for each $f \in \mathcal{F}$, we have

$$\|f - \mathrm{S}(\mathcal{M}; f)\|_K \leq \Big(1 + 2C_1C_2\sqrt{m}\Big)\, \mathrm{dist}_K(f, \mathcal{Q}), \tag{8}$$

where $\mathrm{dist}_K(f, \mathcal{Q}) = \inf\{\|f - q\|_K : q \in \mathcal{Q}\}$.

Proof. For simplicity of notation, we let S stand for $\mathrm{S}(\mathcal{M}; f)$. We choose $h \in \mathcal{Q}$ such that $\|f - h\|_K = \mathrm{dist}_K(f, \mathcal{Q})$. Observe that

$$\|f - \mathrm{S}\|_K \leq \|f - h\|_K + \|\mathrm{S} - h\|_K.$$

Since $\mathrm{S} - h \in \mathcal{Q}$, relation (6) gives

$$\|\mathrm{S} - h\|_K \leq C_1 \max_{1\le j\le m} |\mu_j(\mathrm{S} - h)| = C_1 \max_{1\le j\le m} |\mu_j(\mathrm{S}) - \mu_j(h)|.$$

On the other hand, for each $1 \le j \le m$, we can write

$$\begin{aligned}
|\mu_j(\mathrm{S}) - \mu_j(h)| &\le \sqrt{\sum_{k=1}^{m} |\mu_k(\mathrm{S}) - \mu_k(h)|^2} \\
&\le \sqrt{\sum_{k=1}^{m} |\mu_k(f) - \mu_k(h)|^2} + \sqrt{\sum_{k=1}^{m} |\mu_k(f) - \mu_k(\mathrm{S})|^2} \\
&\le 2\sqrt{\sum_{k=1}^{m} |\mu_k(f) - \mu_k(h)|^2} \\
&\le 2\sqrt{m} \max_{1\le k\le m} |\mu_k(f) - \mu_k(h)| \\
&\le 2\sqrt{m} C_2 \|f - h\|_K,
\end{aligned}$$

where we use the minimal property of S in the third relation and (7) in the last one.

Combining the above estimates we finally obtain

$$\|f - \mathrm{S}\|_K \le \left(1 + 2C_1C_2\sqrt{m}\right)\|f - h\|_K = \left(1 + 2C_1C_2\sqrt{m}\right)\mathrm{dist}_K(f, \mathcal{Q}).$$

The proof is complete. □

Evidently, if $\mathcal{M} = \{\delta_{\mathbf{a}_1}, \ldots, \delta_{\mathbf{a}_m}\}$ with $\mathbf{a}_j \in K$, then we can take $C_2 = 1$. It is known that, by Jackson and Bernstein types theorem, the distance $\mathrm{dist}_K(f, \mathcal{Q})$ will tend to 0 rapidly as $\dim \mathcal{Q} \to \infty$ when f is sufficiently smooth or holomorphic. Hence, Theorem 2.2 gives good approximation if the quantity $C_1C_2\sqrt{m}$ is not too big. This leads to the following definition.

Definition 2.2. Let $\mathcal{Q}_n$ be finite dimensional subspaces of $\mathcal{P}(\mathbb{K}^d)$ such that $\lim_{n\to\infty} m_n = \infty$, $m_n = \dim \mathcal{Q}_n$. Let K be an $\left(\cup_{n=1}^{\infty} \mathcal{Q}_n\right)$-determining compact set. A sequence of discrete sets $\mathbf{A} = \{A_n \subset K : n \in \mathbb{N}^*\}$ is called an admissible mesh in K for $\{\mathcal{Q}_n\}$ if there exist constants c_4 and c_5 depending only on K such that

$$\|p\|_K \le c_4 \|p\|_{A_n}, \quad p \in \mathcal{Q}_n, \quad n \ge 1, \tag{9}$$

where the cardinality of A_n grows at most polynomially on m_n, i.e., $\#A_n \le c_2(m_n)^\alpha$ for some fixed $\alpha > 0$ depending only on K. In the case where $\alpha = 1$, $\mathbf{A}$ is called an optimal admissible mesh.

3 Construction of good functionals for harmonic and holomorphic polynomials

Throughout this section, $D(\mathbf{a}, r)$ denotes the open disk of center $\mathbf{a}$ and radius $r > 0$. The unit disk is denoted by $\mathbb{D}$. We write $\mathcal{H}_n(\mathbb{R}^2)$ for the space of all bivariate harmonic polynomials of total degree at most n. It is well known that

$$\mathcal{H}_n(\mathbb{R}^2) = \mathrm{span}\{1, \Re(x+yi), \Im(x+yi), \ldots, \Re(x+yi)^n, \Im(x+yi)^n\}, \quad \mathbf{x} = x + iy.$$

Hence $\dim \mathcal{H}_n(\mathbb{R}^2) = 2n+1$. Next we recall the definition of Radon projections.

For any given pair $(\theta, t) \in \mathbb{R} \times [0, 1)$, we denote by $I(\theta, t)$ the line segment of $\mathbb{D}$, where the line passes through the point $(t\cos\theta, t\sin\theta)$ and is perpendicular to the vector $(\cos\theta, \sin\theta)$.

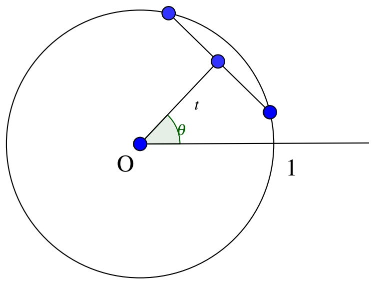

Figure 1: The chord $I(\theta, t)$ of the unit circle

The Radon projection $\mathcal{R}_\theta(f; t)$ of a real (or complex) valued function f defined on $\overline{\mathbb{D}}$ is the line integral of f over $I(\theta, t)$. More precisely

$$\mathcal{R}_\theta(f; t) = \int_{I(\theta,t)} f\,ds = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds.$$

In [15, 16], Georgieva and Hofreither studied interpolation by harmonic polynomials using Radon projections. They found sets of chords called regular sets which determine harmonic polynomials of total degree at most n uniquely. The authors also investigated the convergence of interpolation polynomial of harmonic functions based on Radon projections in the norms $L^2(\partial\mathbb{D})$, $L^2(\mathbb{D})$, $H^s(\partial\mathbb{D})$. In [21, 22, 23], we generalized results of Georgieva and Hofreither and pointed out that the interpolation polynomials based on Radon projections are continuous with respect to the chords. In a forthcoming paper we are going to construct regular sets of chords of Hermite type for $\mathcal{P}_n(\mathbb{C})$ and study the continuity and convergence properties of corresponding interpolation polynomials. This section is intended to construct certain sets of functionals such that the generalized least square approximation polynomials converge uniformly to harmonic functions and holomorphic functions.

Proposition 3.1. *If $m \geq 7n$, then there exists a set A of m points on $\overline{D}(\mathbf{a},r)$ such that*

$$\big(1-\frac{2n\pi}{m}\big)\|p\|_{\overline{D}(\mathbf{a},r)} \leq \|p\|_A, \quad p \in \mathcal{H}_n(\mathbb{R}^2).$$

Proof. Without loss of generality we assume that $\overline{D}(\mathbf{a},r)$ is the closed unit disk $\overline{\mathbb{D}}$. For each $0 \leq k \leq m-1$ we take a point $\mathbf{a}_{k+1}$ on the arc $\widehat{e^{i\theta_k}e^{i\theta_{k+1}}}$ of the unit circle joining $e^{i\theta_k}$ and $e^{i\theta_{k+1}}$ where $\theta_k = \dfrac{2k\pi}{m}$ for $k = 0,\dots,m$. Let us set $A = \{\mathbf{a}_1,\dots,\mathbf{a}_m\}$.

By the maximum principle we can find a point $\mathbf{a}^*$ with $|\mathbf{a}^*| = 1$ such that $|p(\mathbf{a}^*)| = \|p\|_{\overline{\mathbb{D}}}$. The point $\mathbf{a}^*$ must lie on one arc, say $\widehat{e^{i\theta_k}e^{i\theta_{k+1}}}$ with $0 \leq k \leq m-1$. Hence, if we write $\mathbf{a}_{k+1} = e^{i\varphi_{k+1}}$ and $\mathbf{a}^* = e^{i\varphi^*}$ with $\theta_k \leq \varphi_{k+1}, \varphi^* \leq \theta_{k+1}$ then $|\varphi_{k+1} - \varphi^*| \leq \dfrac{2\pi}{m}$. Since $p \in \mathcal{H}_n(\mathbb{R}^2)$ we can find a trigonometric polynomial T_n of degree at most n such that $p(e^{i\theta}) = T(\theta)$, $\theta \in \mathbb{R}$. Using the Markov inequality for trigonometric polynomial we have

$$\begin{aligned} |p(\mathbf{a}^*) - p(\mathbf{a}_{k+1})| &= |T(\varphi^*) - T(\varphi_{k+1})| \leq \|T'\|_{[0,2\pi]}|\varphi^* - \varphi_{k+1}| \\ &\leq n\|T\|_{[0,2\pi]}|\varphi^* - \varphi_{k+1}| = n\|p\|_{\partial\mathbb{D}}|\varphi^* - \varphi_{k+1}| \\ &\leq \frac{2n\pi}{m}\|p\|_{\overline{\mathbb{D}}} \end{aligned}$$

It follows that

$$\|p\|_A \geq |p(\mathbf{a}_{k+1})| \geq |p(\mathbf{a}^*)| - \frac{2n\pi}{m}\|p\|_{\overline{\mathbb{D}}} = \big(1-\frac{2n\pi}{m}\big)\|p\|_{\overline{\mathbb{D}}}.$$

Note that the condition $m \geq 7n$ gives $1 - \dfrac{2n\pi}{m} > 0$ which makes the estimate meaningful. The proof is complete. □

Proposition 3.2. *If $m \geq 7n$, then there exists a set $\mathcal{M}$ of m Radon projections*

$$\mathcal{M} = \{\mathcal{R}_{\theta_k}(\cdot;t_k) : k = 0,\dots,m-1\}$$

such that

$$\max_{0\leq j\leq m-1} |\mathcal{R}_{\theta_j}(p;t_j)| \geq \frac{2}{m}\big(1-\frac{2\pi n}{m}\big)\|p\|_{\overline{\mathbb{D}}}, \quad p \in \mathcal{H}_n(\mathbb{R}^2)$$

and

$$\max_{0\leq j\leq m-1} |\mathcal{R}_{\theta_j}(f;t_j)| \leq \frac{2\pi}{m}\|f\|_{\overline{\mathbb{D}}}, \quad f \in \mathcal{C}(\overline{\mathbb{D}}).$$

Proof. We set $\theta_j = \dfrac{2j\pi}{m}$ for $j = 0,\dots,m-1$ and

$$\varphi_0 = \frac{\theta_{m-1}+\theta_0}{2}, \quad \varphi_{j+1} = \frac{\theta_j + \theta_{j+1}}{2}, \quad 0 \leq j \leq m-2, \quad \varphi_m = \varphi_0.$$

We choose $t_j \in [\cos\dfrac{\pi}{m}, \cos\dfrac{\pi}{2m}]$ arbitrarily. Consider the set of Radon projections

$$\mathcal{M} = \{\mathcal{R}_{\theta_j}(\cdot;t_j) : j = 0,\dots,m-1\}.$$

Let $p \in \mathcal{H}_n$. By the maximum principle we can find a point $\mathbf{a}^* = e^{i\varphi^*}$ with $\varphi_k \leq \varphi^* \leq \varphi_{k+1}$ such that $|p(\mathbf{a}^*)| = \|p\|_{\overline{\mathbb{D}}}$. From the mean-valued theorem for integration, there exists a point $\mathbf{x}^* \in I(\theta_k,t_k)$ such that

$$\mathcal{R}_{\theta_k}(p;t_k) = 2\sqrt{1-t_k^2}\,p(\mathbf{x}^*). \tag{10}$$

It is easily seen that

$$\sqrt{1-t_k^2} \geq \sqrt{1-\cos^2\frac{\pi}{2m}} = \sin\frac{\pi}{2m} \geq \frac{2}{\pi}\frac{\pi}{2m} = \frac{1}{m}. \tag{11}$$

Since $t_k \in [\cos\dfrac{\pi}{m}, \cos\dfrac{\pi}{2m}]$, the chord $I(\theta_k,t_k)$ is contained in the circular segment which is cut off from $\mathbb{D}$ by the chord joining $e^{i\varphi_k}$ and $e^{i\varphi_{k+1}}$. Hence $\mathbf{a}^*, \mathbf{x}^*$ are both in this circular segment. It follows that

$$\|\mathbf{a}^* - \mathbf{x}^*\| \leq |e^{i\varphi_{k+1}} - e^{i\varphi_k}| = 2\sin\frac{\pi}{m} \leq \frac{2\pi}{m}.$$

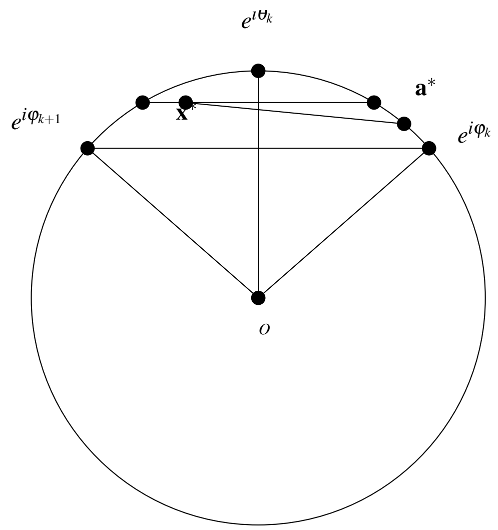

Figure 2: An illustration of objects in the proof of Proposition 3.2

From Markov inequality for bivariate harmonic polynomial polynomials in [28] we can write

$$|p(\mathbf{a}^*) - p(\mathbf{x}^*)| \le \|\nabla p\|_{\overline{\mathbb{D}}} \|\mathbf{a}^* - \mathbf{x}^*\| \le n\|p\|_{\overline{\mathbb{D}}} \|\mathbf{a}^* - \mathbf{x}^*\| \le \frac{2\pi n}{m}\|p\|_{\overline{\mathbb{D}}}.$$

Hence

$$|p(\mathbf{x}^*)| \ge |p(\mathbf{a}^*)| - \frac{2n\pi}{m}\|p\|_{\overline{\mathbb{D}}} = \big(1 - \frac{2n\pi}{m}\big)\|p\|_{\overline{\mathbb{D}}}. \tag{12}$$

Combining (10), (11) and (12) we obtain

$$\max_{0\le j\le m-1} |\mathcal{R}_{\theta_j}(p;t_j)| \ge |\mathcal{R}_{\theta_k}(p;t_k)| \ge \frac{2}{m}\big(1 - \frac{2n\pi}{m}\big)\|p\|_{\overline{\mathbb{D}}}.$$

It remains to show the inequality for $f \in \mathcal{C}(\overline{\mathbb{D}})$. We use the mean-value theorem again to get

$$|\mathcal{R}_{\theta_j}(f;t_j)| \le 2\sqrt{1-t_j^2} \sup_{\mathbf{x}\in I(\theta_j,t_j)} |f(\mathbf{x})| \le 2\sqrt{1-t_j^2}\|f\|_{\overline{\mathbb{D}}}.$$

By construction, we have

$$\sqrt{1-t_j^2} \le \sqrt{1-\cos^2\frac{\pi}{m}} = \sin\frac{\pi}{m} \le \frac{\pi}{m}.$$

Hence

$$|\mathcal{R}_{\theta_j}(f;t_j)| \le \frac{2\pi}{m}\|f\|_{\overline{\mathbb{D}}}, \quad 0 \le j \le m-1.$$

This completes the proof. □

To get uniform approximation, we need a version of Jackson-type theorem for harmonic polynomials. The following result probably known. For completeness we give a proof.

Lemma 3.3. *Let f be harmonic in $D(\mathbf{a},r)$ and continuous in $\overline{D}(\mathbf{a},r)$ such that $f|_{\partial D(\mathbf{a},r)}$ is of class $\mathcal{C}^p$ with $p \in \mathbb{N}^*$. Then*

$$\mathrm{dist}_{\overline{D}(\mathbf{a},r)}(f, \mathcal{H}_n(\mathbb{R}^2)) = o(\frac{1}{n^p}).$$

Proof. Without loss of generality we assume that $D(\mathbf{a},r)$ is the unit disk. Since $f|_{\partial\mathbb{D}}$ is of class $\mathcal{C}^p$, Theorem 3 in [20, p. 57] enables us to find a trigonometric polynomial T_n of degree at most n such that

$$\|f(e^{i\theta}) - T_n(\theta)\|_{[0,2\pi]} = o(\frac{1}{n^p}).$$

We write $T_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ with $c_{-k} = \bar{c}_k$ for $0 \le k \le n$. Define

$$p_n(r\cos\theta, r\sin\theta) = \sum_{k=-n}^{n} c_k r^{|k|} e^{ik\theta}.$$

Then p_n is an element of $\mathcal{H}_n(\mathbb{R}^2)$ and $p_n|_{\partial\mathbb{D}} = T_n$. Hence, by the maximum principle, we have

$$\|f - p_n\|_{\overline{\mathbb{D}}} = \|f(e^{i\theta}) - T_n(\theta)\|_{[0,2\pi]} = o(\frac{1}{n^p}).$$

□

Proposition 3.4. *Let $\mathbf{A} = \{A_n \subset \overline{D}(\mathbf{a}, r) : n \in \mathbb{N}^*\}$ be the optimal admissible mesh for $\mathcal{H}_n(\mathbb{R}^2)$ constructed in Proposition 3.1 such that $\#A_n = O(n)$ and $\#A_n \geq 7n$. Let f be harmonic in $D(\mathbf{a}, r)$ and continuous in $\overline{D}(\mathbf{a}, r)$ such that $f|_{\partial D(\mathbf{a},r)}$ is of class $\mathcal{C}^p$ with $p \in \mathbb{N}^*$. Then*

$$\|f - \mathrm{S}[A_n; f]\|_{\overline{D}(\mathbf{a},r)} = o\Big(\frac{1}{n^{p-1/2}}\Big).$$

Moreover, if f is a harmonic function in a neighborhood of $\overline{D}(\mathbf{a}, r)$, then there exists $\rho \in (0, 1)$ such that

$$\limsup_{n\to\infty} \Big(\|f - \mathrm{S}[A_n; f]\|_{\overline{D}(\mathbf{a},r)}\Big)^{\frac{1}{n}} \leq \rho.$$

Proof. We set $m_n = \#A_n$. Then $7n \leq m_n \leq Cn$ where C is a positive constant. By construction we have

$$\|p\|_{A_n} \geq (1 - \frac{2\pi n}{m_n})\|p\|_{\overline{D}(\mathbf{a},r)}, \quad p \in \mathcal{H}_n(\mathbb{R}^2).$$

Evidently, $\|g\|_{A_n} \leq \|g\|_{\overline{D}(\mathbf{a},r)}$ for any $g \in \mathcal{C}(\overline{D}(\mathbf{a}, r))$. Hence, applying Theorem 2.2 we obtain

$$\|f - \mathrm{S}[A_n; f]\|_{\overline{D}(\mathbf{a},r)} \leq \Big(1 + \frac{2m_n}{m_n - 2\pi n}\sqrt{m_n}\Big)\mathrm{dist}_{\overline{D}(\mathbf{a},r)}(f, \mathcal{H}_n(\mathbb{R}^2)). \tag{13}$$

Since $7n \leq m_n \leq Cn$, we have

$$(1 + \frac{2m_n}{m_n - 2\pi n})\sqrt{m_n} = (1 + \frac{2}{1 - 2\pi n/m_n})\sqrt{m_n} \leq (1 + \frac{2}{1 - 2\pi/7})\sqrt{Cn} = O(\sqrt{n}).$$

From Lemma 3.3 and (13), we conclude that

$$\|f - \mathrm{S}[A_n; f]\|_{\overline{D}(\mathbf{a},r)} = o\Big(\frac{1}{n^{p-1/2}}\Big).$$

The first part of the proposition is proved.

Under the hypothesis that f is a harmonic function in a neighborhood of $\overline{D}(\mathbf{a}, r)$, the main theorem in [3] guarantees the existence of $\rho \in (0, 1)$ such that

$$\limsup_{n\to\infty} \Big(\mathrm{dist}_{\overline{D}(\mathbf{a},r)}(f, \mathcal{H}_n(\mathbb{R}^2))\Big)^{\frac{1}{n}} \leq \rho. \tag{14}$$

Combining the last relation with (13) we get the desired estimate. The proof is complete. □

Proposition 3.5. *Let $\mathcal{M}_n$ be the set of Radon projections constructed in Proposition 3.2 such that $\#\mathcal{M}_n = O(n)$ and $\#\mathcal{M}_n \geq 7n$. Let f be harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ such that $f|_{\partial\mathbb{D}}$ is of class $\mathcal{C}^p$ with $p \in \mathbb{N}^*$. Then*

$$\|f - \mathrm{S}[\mathcal{M}_n; f]\|_{\overline{\mathbb{D}}} = o\Big(\frac{1}{n^{p-1/2}}\Big).$$

Furthermore if f is a harmonic function on a neighborhood of $\overline{\mathbb{D}}$ then there exists $\rho \in (0, 1)$ such that

$$\limsup_{n\to\infty} \Big(\|f - \mathrm{S}[\mathcal{M}_n; f]\|_{\overline{\mathbb{D}}}\Big)^{\frac{1}{n}} \leq \rho.$$

Proof. As in the proof of Proposition 3.4, we can use Theorem 2.2 and Proposition 3.2 to write

$$\begin{aligned}
\|f - \mathrm{S}[\mathcal{M}_n; f]\|_{\overline{\mathbb{D}}} &\leq \Big(1 + \frac{2m_n^2}{2(m_n - 2\pi n)}\frac{2\pi}{m_n}\sqrt{m_n}\Big)\mathrm{dist}_{\overline{\mathbb{D}}}(f, \mathcal{H}_n(\mathbb{R}^2)) \\
&= \Big(1 + \frac{2\pi m_n}{m_n - 2\pi n}\sqrt{m_n}\Big)\mathrm{dist}_{\overline{\mathbb{D}}}(f, \mathcal{H}_n(\mathbb{R}^2)) \\
&= O(\sqrt{n})\mathrm{dist}_{\overline{\mathbb{D}}}(f, \mathcal{H}_n(\mathbb{R}^2)),
\end{aligned}$$

where $m_n = \#\mathcal{M}_n$ with $7n \leq m_n \leq Cn$. Hence the desired estimates follows directly from Lemma 3.3 and relation (14), and the proof is complete. □

In the proof of Proposition 3.2, we use the maximum principle, mean-value theorem for integration and the Markov inequality for harmonic polynomials. These three properties also hold for homomorphic polynomials in $\mathbb{C}$. Hence we can repeat the proof of Proposition 3.2 to obtain the following result. We states it without proof.

Proposition 3.6. *Let $m \geq 7n$ and*

$$\mathcal{M} = \{\mathcal{R}_{\theta_k}(\cdot; t_k) : k = 0, \ldots, m-1\}$$

be the set of m Radon projections constructed in Proposition 3.2. Then

$$\max_{0\leq j\leq m-1} |\mathcal{R}_{\theta_j}(p; t_j)| \geq \frac{2}{m}(1 - \frac{2\pi n}{m})\|p\|_{\overline{\mathbb{D}}}, \quad p \in \mathcal{P}_n(\mathbb{C})$$

and

$$\max_{0\leq j\leq m-1} |\mathcal{R}_{\theta_j}(f; t_j)| \leq \frac{2\pi}{m}\|f\|_{\overline{\mathbb{D}}}, \quad f \in \mathcal{C}(\overline{\mathbb{D}}).$$

Proposition 3.7. *Let $\mathcal{M}_n$ be the set of Radon projections constructed in Proposition 3.6 such that $\#\mathcal{M}_n = O(n)$ and $\#\mathcal{M}_n \geq 7n$. Let f be a holomorphic function in a neighborhood of $\overline{\mathbb{D}}$ then there exists $\rho \in (0,1)$ such that*

$$\limsup_{n\to\infty}\left(\|f - \mathrm{S}[\mathcal{M}_n; f]\|_{\overline{\mathbb{D}}}\right)^{\frac{1}{n}} \leq \rho.$$

Proof. Repeating the arguments in the proof of Proposition 3.5, we use Theorem 2.2 and Proposition 3.6 to get

$$\|f - \mathrm{S}[\mathcal{M}_n; f]\|_{\overline{\mathbb{D}}} \leq O(\sqrt{n})\mathrm{dist}_{\overline{\mathbb{D}}}(f, \mathcal{P}_n(\mathbb{C})). \tag{15}$$

Suppose that f is holomorphic in a neighborhood of $\overline{D}(0,r)$ with $r > 1$. From the classical Bernstein theorem in [1] we have

$$\limsup_{n\to\infty}\left(\mathrm{dist}_{\overline{D}(0,r)}(f, \mathcal{P}_n(\mathbb{C}))\right)^{\frac{1}{n}} \leq \frac{1}{r}.$$

Combining the last estimate with (15) we get the desired relation. The proof is complete. □

4 Admissible meshes on curves

This section is devoted the study of admissible meshes on curves in $\mathbb{R}^d$. Using the classical method, we show that curves that admitting tangential Markov inequality always contain admissible meshes.

Proposition 4.1. *Let Γ be a smooth curve in $\mathbb{R}^d$. If Γ admits a tangential Markov inequality of the form*

$$\|D_T p\|_\Gamma \leq C(\deg p)^\ell \|p\|_\Gamma, \quad p \in \mathcal{P}(\mathbb{R}^d), \tag{16}$$

or

$$\|(p\circ\rho)'\|_{[a,b]} \leq C(\deg p)^\ell \|p\|_\Gamma, \quad p \in \mathcal{P}(\mathbb{R}^d), \tag{17}$$

where $\rho : [a,b] \to \mathbb{R}^d$ is a parameterization of Γ, then there exists an admissible mesh $\mathbf{A} = \{A_n\}$ on Γ for $\mathcal{P}_n(\mathbb{R}^d)$ with $\#A_n = O(n^\ell)$.

Proof. We first assume that (17) holds. We set $m = 2([C(b-a)n^\ell] + 1)$ and

$$A_n = \{\rho(t_j) : t_j = a + \frac{(b-a)j}{m}, j = 0, \ldots, m\}.$$

For arbitrary $p \in \mathcal{P}_n(\mathbb{R}^d)$, let $\|p\|_\Gamma = p(\rho(t^*))$ with $t^* \in [a,b]$. We can find j such that $|t^* - t_j| \leq \dfrac{b-a}{m}$. It follows that

$$\begin{aligned} |p(\rho(t_j)) - p(\rho(t^*))| &\leq \|(p\circ\rho)'\|_{[a,b]}|t_j - t^*| \\ &\leq \frac{C(b-a)n^\ell\|p\|_\Gamma}{m} \leq \frac{\|p\|_\Gamma}{2}. \end{aligned}$$

Hence

$$\|p\|_{A_n} \geq |p(\rho(t_j))| \geq |p(\rho(t^*))| - \frac{\|p\|_\Gamma}{2} = \frac{\|p\|_\Gamma}{2}.$$

If (16) holds, then we choose a parameterization $\rho : [a,b] \to \mathbb{R}^d$ of Γ. By definition we have

$$|D_T p(\rho(t))| = \frac{|\big(p\circ\rho(t)\big)'|}{\|\rho'(t)\|} \geq \frac{|\big(p\circ\rho(t)\big)'|}{M}, \quad t \in [a,b],$$

where $M = \sup\{\|\rho'(t)\| : a \leq t \leq b\}$. It follows that

$$\|(p\circ\rho)'\|_{[a,b]} \leq CM(\deg p)^\ell \|p\|_\Gamma, \quad p \in \mathcal{P}(\mathbb{R}^N). \tag{18}$$

The last equality is analogous of (17). Using (18) we can construct admissible meshes. The proof is complete. □

Corollary 4.2. *Let $P_1, P_2, \ldots, P_d$ be univariate real polynomials and*

$$\Gamma = \{(t, e^{P_1(t)}, \ldots, e^{P_d(t)}) : t \in [a,b]\}.$$

Then there exists an admissible mesh $\mathbf{A} = \{A_n\}$ on Γ for $\mathcal{P}_n(\mathbb{R}^{d+1})$ with $\#A_n = O(n^{6(d+1)})$.

Proof. We set

$$\rho(t) = (t, e^{P_1(t)}, \ldots, e^{P_d(t)}), \quad t \in \mathbb{R}.$$

By Theorem 3.2 in [5], there is a constant $C = C(a,b,\Gamma)$ such that, for all $P \in \mathcal{P}(\mathbb{R}^{d+1})$,

$$\|g'(t)\|_{[a,b]} \leq C(\deg P)^{6(d+1)}\|g\|_{[a,b]}, \quad g = P\circ\rho. \tag{19}$$

Applying Proposition 4.1 we can construct an admissible mesh $\mathbf{A} = \{A_n : n \in \mathbb{N}^*\}$ on Γ with $\#A_n = O(n^{6(d+1)})$.

□

A sharp estimate is obtained in [4]: If Q is a univariate real polynomial, then there is a constant $C = C(a,b,Q)$ such that for all $P \in \mathcal{P}(\mathbb{R}^2)$ we have

$$\|\big(P(t,e^{Q(t)})\big)'\|_{[a,b]} \le C(\deg P)^4 \|P(t,e^{Q(t)})\|_{[a,b]}. \tag{20}$$

Using Proposition 4.1 we can construct an admissible mesh $\mathbf{B} = \{B_n : n \in \mathbb{N}^*\}$ on $\gamma = \{(t,e^{Q(t)}) : t \in [a,b]\}$ with $\#B_n = O(n^4)$. Here we note that $\dim \mathcal{P}_n(\gamma) = \dim \mathcal{P}_n(\mathbb{R}^2) = O(n^2)$.

Corollary 4.3. *Let Q a univariate real polynomial and $\gamma = \{(t,e^{Q(t)}) : t \in [a,b]\}$. Then there exists an admissible mesh $\mathbf{B} = \{B_n : n \in \mathbb{N}^*\}$ on γ for $\mathcal{P}_n(\mathbb{R}^2)$ with $\#B_n = O(n^4)$.*

Corollary 4.4. *Let K be a smooth compact algebraic curve in $\mathbb{R}^d$ without boundary. Then there exists an optimal admissible mesh $\mathbf{A} = \{A_n \subset K : n \in \mathbb{N}^*\}$.*

Proof. It is known that $\dim \mathcal{P}_n(K) = O(n)$ when n is sufficiently large (see [2]). The main theorem in [6] asserts that K admits the tangential Markov inequality of the form

$$\|D_T p\|_K \le M(\deg p)\|p\|_K, \quad p \in \mathcal{P}(\mathbb{R}^d).$$

By Proposition 4.1, we can construct an optimal mesh $\mathbf{A} = \{A_n \subset K : n \ge 1\}$ such that $\#A_n = O(n)$. □

On the other hand, if K is an arc of a smooth algebraic curve $\widetilde{K}$ in $\mathbb{R}^2$ in which we allow end points of K, then K behave just as the Markov inequality of $[-1,1]$. The following inequality was proved in [6, Proposition 6.1],

$$\|D_T p\|_K \le M(\deg p)^2\|p\|_K, \quad p \in \mathcal{P}(\mathbb{R}^2).$$

By Proposition 4.1, we have the following corollary.

Corollary 4.5. *Let K be an arc of a smooth algebraic curve $\widetilde{K}$ in $\mathbb{R}^2$. Then there exists an admissible mesh $\mathbf{A} = \{A_n \subset K : n \ge 1\}$ such that $\#A_n = O(n^2)$.*

Finally, when the algebraic curves are not smooth, it can admit the tangential Markov inequality, but the exponent is bigger. In [7, Theorem 3] the authors showed that if $\gamma = \{(x,x^r) : x \in [0,1]\}$ where $r = l/m$, $l \ge m$ positive integers, is in the lowest term, then

$$\|D_T p\|_\gamma \le C(\deg p)^{2m}\|p\|_\gamma, \quad p \in \mathcal{P}_d.$$

Proposition 4.1 guarantees the existence of an admissible mesh $\mathbf{A} = \{A_n \subset \gamma : n \in \mathbb{N}^*\}$ such that $\#A_n = O(n^{2m})$.

In the above results, we have constructed admissible meshes on some kind of curves. If the compact curves admit Jackson-type theorems, then we can use Theorem 2.2 to obtain uniform approximation results. Here we only deal with smooth compact algebraic curves in $\mathbb{R}^d$ without boundary. We recall the definition of Ragozin [27]. Let K be such a curve. For $0 < \alpha \le 1$ and $k \ge 0$. We say that a function $f : K \to \mathbb{R}$ belongs to $\mathcal{C}^{k,(\alpha)}(K)$ if for each $x \in K$ there exists a chart $\varphi : (-1,1) \to K$ with $x \in \mathrm{int}\varphi((-1,1))$ and $f \circ \varphi \in \mathcal{C}^{k,(\alpha)}(-1,1)$, i.e.,

$$\sup_{s,t \in (-1,1), s \ne t} \frac{|(f\circ\varphi)^{(k)}(s) - (f\circ\varphi)^{(k)}(t)|}{|s-t|^\alpha} < \infty.$$

In [27, Section 2], Ragozin proved the following approximation property

$$\mathrm{dist}_K(f, \mathcal{P}_n(\mathbb{R}^d)) = O\Big(\frac{1}{n^{k+\alpha}}\Big). \tag{21}$$

Corollary 4.6. *Let K be a smooth compact algebraic curve in $\mathbb{R}^d$ without boundary. Let $\mathbf{A} = \{A_n\}$ be the optimal admissible meshes constructed in Corollary 4.4. Let $f \in C^{k,(\alpha)}(K)$ with $0 < \alpha \le 1$. Then*

$$\|f - \mathrm{S}(A_n; f)\|_K = O(\frac{1}{n^{k+\alpha-1/2}}).$$

Proof. From Theorem 2.2 we get

$$\|f - \mathrm{S}(A_n; f)\|_K \le \Big(1 + 2C\sqrt{\#A_n}\Big)\,\mathrm{dist}_K(f, \mathcal{P}_n(\mathbb{R}^d)).$$

Since $\#A_n = O(n)$, relation (21) implies the desired relation. The proof is complete. □

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