PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special issue dedicated to Mirosław Baran on the occasion of his 60th birthday, Volume 14 · 2021 · Pages 1–6

My joint research with Mirosław Baran

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Two things have a significant impact on a scientist's research: whether he had a competent teacher and whether he gathered talented students around him. In both cases I was very lucky, my teacher was an outstanding mathematician, Józef Siciak, and I can include Mirosław Baran among my best students. I recently devoted an extensive article to Józef Siciak, in which I discussed his most important results and their impact on the development of complex analysis in the world (*see* [28]). Mirosław's sixtieth birthday is a good opportunity to summarize the results of our joint research. Before I move on to that, I would like to remind you how this cooperation came about. I met Mirosław in 1983, when as a student he asked me to be the tutor of his master thesis. I did not know him before, because at that time I did not teach students of the first three years of studies at the Institute of Mathematics of the Jagiellonian University. However, I agreed, and as it turned out later, it was a very good decision. For the subject of that work, I chose, with some risk, the then current problem related to the famous Siciak extremal function. Let us recall that it can be defined as follows. Given a compact subset E of the space $\mathbb{C}^N$ of N complex numbers, the function

$$\Phi_E(z) := \sup_{n\in\mathbb{N}} \sup\{|p(z)|^{1/n} : \ p \in \mathcal{P}_n(\mathbb{C}^N), \ \|p\|_E \le 1\}, \quad z \in \mathbb{C}^N,$$

where $\mathcal{P}_n(\mathbb{C}^N)$ is the space of polynomials on $\mathbb{C}^N$ of (total) degree at most n, and $\|p\|_E := \sup_E |p|$, is called the (polynomial) extremal function associated with the set E. It was introduced in [32]. Zakharyuta [35] (if Φ_E is continuous) and later Siciak [33] (for any E) showed that $\log\Phi_E$ is equal to the function

$$V_E(z) := \sup\{u(z) : \ u \in \mathcal{L}(\mathbb{C}^N), \ u \le 0 \text{ on } E\},$$

where $\mathcal{L}(\mathbb{C}^N)$ is the Lelong class of all plurisubharmonic functions on $\mathbb{C}^N$ with logarithmic growth as $|z| \to \infty$. By the Bedford-Taylor results on the complex Monge-Ampère operator [11], the function V_E is known to be a multidimensional counterpart of the classical Green function of the unbounded component of $\mathbb{C} \setminus E$ with (logarithmic) pole at ∞. From the definition of Φ_E one can immediately derive the Bernstein-Walsh-Siciak inequality: for each polynomial p on $\mathbb{C}^N$,

$$|p(z)| \le \|p\|_E \, [\Phi_E(z)]^{\deg p} \quad \text{for } z \in \mathbb{C}^N, \tag{B-W-S}$$

which is specially useful if Φ_E is continuous on E. (Then it must be continuous on the whole space $\mathbb{C}^N$ and the set E is said to be L-regular.) If, moreover, $\log\Phi_E$ is Hölder continuous, *i.e.*

$$V_E(z) = \log\Phi_E(z) \le M\,(\operatorname{dist}(z,E))^s \quad \text{for } z \in \mathbb{C}^N \tag{HCP}$$

with some constants $M > 0$ and $s \in (0,1]$ independent of z, then by Cauchy's Integral Formula one easily gets a (multivariate) Markov type inequality: for each polynomial p,

$$\|\operatorname{grad} p\|_E \le M_1 (\deg p)^r \|p\|_E, \tag{M}$$

where M_1 is a positive constant and $r = 1/s$. So far, all known (Andrei) Markov sets have the HCP property. It is a long-standing open problem of whether both properties (M) and (HCP) are equivalent. Surprisingly, however, in the paper [4] Baran and Białas-Cież showed that the HCP condition is equivalent to a Vladimir Markov type inequality

$$\|D^\alpha P\|_E \le M^{|\alpha|} (\deg P)^{m|\alpha|} (|\alpha|!)^{1-m} \|P\|_E,$$

where the constants $m, M > 0$ are independent of the polynomial $P \in \mathbb{C}[z_1,\dots,z_N]$, which is a multivariate counterpart of the classical Vladimir Markov inequality

$$\|P^{(k)}\|_{[-1,1]} \le \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)} \|P\|_{-1,1]}$$

for any polynomial P of degree not greater than n.

Siciak introduced the function Φ_E in order to extend to the multidimensional case the well-known Bernstein-Walsh theorem characterizing analytic functions in a neighbourhood of a (regular) compact set E in $\mathbb{C}$ by uniform polynomial approximation on E with geometric rate. Thus, in particular, he got a far reaching strengthening of Runge's theorem. A crucial role was played there by the (B-W-S) inequality. It has appeared (*see* [27]) that an analogous role but in polynomial approximation of C^∞ functions is played by the Markov inequality (M). Let us also recall a *uniform version* of the Bernstein-Walsh-Siciak theorem which turned out to be very useful in many problems of the constructive function theory. It first appeared in my paper [23].

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Theorem. Let $H^\infty(U)$ be the Banach space of bounded holomorphic functions on an open subset U of $\mathbb{C}^N$ equipped with the uniform norm $\|f\|_U := \sup|f|(U)$. Then for every polynomially convex compact subset E of U there exist constants $C>0$ and $a\in(0,1)$ such that for each function $f\in H^\infty(U)$ and for each $n\in\mathbb{N}$,

$$\mathrm{dist}_E(f,\mathcal{P}_n)\le C\|f\|_U\, a^n.$$

Since the introduction of the extremal function, Siciak has asked for a formula for Φ_E in case E is the unit ball in the space $\mathbb{R}^n$. The problem has been extensively studied starting from the 70s of the last century. An answer was first given by Lundin in [18]. I recommended the reading of this paper to Mirosław. The results of his work on Lundin's article went far beyond the standard master degree paper. Using an original method based on the properties of the Joukovski transformation $J(\zeta)=\frac{1}{2}(\zeta+\zeta^{-1})$ $(\zeta\in\mathbb{C}\setminus\{0\})$ and its generalized form $J_z(\zeta)=\frac{1}{2}(\zeta z+\zeta^{-1}\overline{z})$, where $z\in\mathbb{C}^N$ and $\zeta\in\mathbb{C}\setminus\{0\}$, Baran extended Lundin's ideas and provided formulas for Siciak's function for a wider family of subsets in $\mathbb{R}^N$. In particular, if $E=B:=B(0,1)$ is the unit ball in $\mathbb{R}^N$, he got again Lundin's formula

$$\Phi_B(z)=\sqrt{h(\|z\|^2+|z_1^2+\cdots+z_N^2-1|)},\quad (z\in\mathbb{C}^N),$$

where $h(\zeta)=\zeta+\sqrt{\zeta^2-1}$ $(\zeta\in\mathbb{C}\setminus[-1,1])$ is the inverse of the map J restricted to $\{z\in\mathbb{C}:|z|>1\}$, and the branch of the square root is chosen so that $h(t)>1$ for $t>1$, which also was proved independently by Bedford and Taylor [10] and Sadullaev [30]. If $E=S^N$ is the standard simplex in $\mathbb{R}^N$, i.e. S^N is the convex envelope of the set $\{0,e^1,\ldots,e^N\}$, where $\{e_1,\ldots,e_N\}$ is the standard orthonormal basis in $\mathbb{R}^N$, Baran's method provides the formula

$$\Phi_{S^N}(z_1,\ldots,z_N)=h(|z_1|+\cdots+|z_N|+|z_1+\cdots+z_N-1|)$$

for $(z_1,\ldots,z_N)\in\mathbb{C}^N$, and the same method can be applied to other non-symmetric convex compact sets. The results of Baran's master degree paper were published in the Annales Polonici Mathematici [1], which is not often the case with such works. Moreover, they were also included in Maciej Klimek's well-known monograph on the pluripotential theory [17], section 5.4. To paraphrase a well-known saying about Hitchcock's films, Mirosław's first results were like an earthquake, and the later ones were more and more louder. This is evidenced by the awards he received for them: the award of the Minister of National Education for his doctoral dissertation "*Siciak's extremal functioin and complex equilibrium measure for subsets of the space* $\mathbb{R}^n$" written under my supervision and defended in 1990 at the Jagiellonian University, the prestigious Stanisław Zaremba Prize awarded to him by the Polish Mathematical Society in 1993 for papers on pluripotential theory published in 1992 or the Prime Minister Award for the habilitation thesis "*Conjugate norms in* $\mathbb{C}^n$ *and related geometrical problems*" (*see* [3]) defended at the Institute of Mathematics of the Polish Academy of Sciences in 1998.

It is not my intention, nor would it be possible, to discuss in this short article all the achievements of Mirosław Baran. I will limit myself here to only a few results obtained in cooperation with me. The study of the properties of Siciak's extremal function found a prominent place in both the investigations of Baran and myself. It was also the starting point for our common interest in polynomial inequalities in the space $\mathbb{C}^N$. We focused a lot of attention in these studies on the classical Markov and Bernstein inequalities and their multidimensional generalizations.

Given a non-empty compact set E in $\mathbb{C}^N$ and a number $r\ge 1$, consider the following condition

M(r) there exists a constant $C>0$ such that for every polynomial $P\in\mathcal{P}_n(\mathbb{C}^N)$ $(n=1,2,\ldots)$ one has

$$\|\mathrm{grad}\,P\|_E\le Cn^r\|P\|_E.$$

If E satisfies **M(r)** for some $r\ge1$, we say that E is a Markov set. If $N=1$ and $E=[-1,1]$ then by the classical Markov inequality E satisfies **M(2)** with $C=1$. We define

$$\mu(E):=\inf\{r:\ E \text{ satisfies } \mathbf{M(r)}\}$$

and call this number *Markov's exponent* of E. If E is a continuum in the complex plane $\mathbb{C}$, then by the well-known result of Pommerenke [29], $1\le\mu(E)\le2$. For any compact subset E of $\mathbb{R}^N$, we have $\mu(E)\ge2$, which easily follows from extremal properties of Chebyshev polynomials. If E is a fat (i.e. $E=\overline{\mathrm{int}\,\mathrm{E}}$) convex compact subset of $\mathbb{R}^N$ then $\mu(E)=2$. Goetgheluck [16] showed that if $E=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le1, 0\le y\le x^p\}$ $(p\ge1)$, then $\mu(E)=2p$. This result inspired me to study the L-regularity and Markov's property of semianalytic and more general, subanalytic subsets of $\mathbb{R}^N$ (cf. [26],[20]). If E is an m-UPC subset of $\mathbb{R}^N$ (for the definition of a UPC set *see* [20]), whence in particular, if (for some m) E is a fat subanalytic compact set, then by Baran [2], $\mu(E)\le 2m$. If $E=\{(x,y)\in\mathbb{R}^2:\ 0<x\le1,\ 0<y\le e^{-1/x}\}\cup\{(0,0)\}$, then by Zerner [36], $\mu(E)=\infty$, which means that E is not Markov (although it is L-regular). If E is a Markov set in $\mathbb{C}^N$ $(N>1)$ then E does not have to satisfy **M(r)** with $r=\mu(E)$. Such an example was constructed by Baran in $\mathbb{R}^2$; it can be found in [5].

In constructive function theory, there is an important question concerning the invariance of fundamental inequalities (like the Bernstein-Walsh or Markov ones) under polynomial or more general, holomorphic maps. Such a problem for inequalities related to Siciak's extremal function Φ_E (multidimensional Bernstein-Walsh-Siciak inequality or Hölder property of Φ_E) was studied by myself in [24] and [25]. (The results of the latter paper have recently been been essentially sharpened by Rafał Pierzchała [22].) In [6], there was investigated a corresponding problem for Markov's inequality. In particular we showed the following

Theorem. Let E be a polynomially convex, compact subset of $\mathbb{C}^N$ satisfying **M(r)**. Let f be a holomorphic mapping defined in a neighbourhood U of E, with values in $\mathbb{C}^N$, such that $f(E)$ is not pluripolar and $\mathrm{Jac}\,f(z)\ne0$ for each $z\in E$. Then $f(E)$ satisfies **M(r)** as well.

Here the assumption that $f(E)$ is not pluripolar seems to be too restrictive. If we knew that the Markov sets are not pluripolar, we could replace it by the requirement that f is non-degenerate on at least one of the connected components of U, say V, that meets E at a non-pluripolar set (i.e. $\operatorname{rank}_V f := \sup_{z\in V} \operatorname{rank} f(z) = N$), since then $f(E)$ would also be non-pluripolar. This, however, still seems to be unknown except for $N = 1$ due to a long-awaited result of Leokadia Białas-Cież [12]. The situation is much better if E is a UPC compact set in $\mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a polynomial map such that $\operatorname{Jac} f(x) \neq 0$ for each $x \in \operatorname{int} E$. Then by [6],Theorem 2.8, $f(E)$ is Markov. Actually, in this case, one can also give an estimate for $\mu(f(E))$ in function of $\mu(E)$ and parameters of E and f. Results of [6] have recently been essentially extended by Pierzchała, who succeded in replacing our (rather strong) assumption of non-vanishing of $\operatorname{Jac} f$ on E by a natural one of non-degeneracy of f.

Theorem ([21]). Let $\emptyset \neq E \subset \mathbb{K}^N$ $(\mathbb{K} = \mathbb{R} \vee \mathbb{C})$ be a Markov set and $f : \mathbb{K}^N \to \mathbb{K}^{N'}$ be a polynomial map such that

$$\operatorname{rank} f := \max\{\operatorname{rank} f(\zeta) : \zeta \in \mathbb{K}^N\} = N' \quad (N, N' \in \mathbb{N}).$$

Then $f(E)$ is also a Markov set.

In case $N = N' = 1$ this result was also proved in [5].

Theorem ([22]) Let $f : U \to \mathbb{C}^{N'}$, where $U \subset \mathbb{C}^N$ is an open set, be a non-degenerate holomorphic map $(N, N' \in \mathbb{N})$. Assume that a compact set $E \subset \mathbb{C}^N$ is Markov, the polynomial hull $\hat{K}$ of K is contained in U, and $f(K)$ is a non-pluripolar subset of $\mathbb{C}^{N'}$. Then $h(K)$ is Markov as well.

At the end of the 20th century, many authors dealt with Markov or Bernstein type inequalities on algebraic subvarieties. Such investigations were conducted by L. Bos, N. Levenberg, P. Milman, B.A. Taylor, A. Brudnyi, V. Totik, C. Fefferman, R. Narasimhan, N. Roytvarf, Y. Yomdin among others. Also, Baran and I devoted three papers [7], [8] and [9] to these issues; one can find there references to the papers of the authors cited above.

In 1983, Sadullaev [31] proved an important characterization of algebraic sets in the class of analytic subsets A of the space $\mathbb{C}^N$, viz. A is algebraic if and only if Siciak's extremal function Φ_E is locally bounded in A for some (and hence for each) non-pluripolar compact subset K of A. In other words, he characterized algebraic sets in terms of the Bernstein-Walsh-Siciak inequality for compact subsets of an analytic set. It appears that analogously one can characterize algebraic submanifolds of the space $\mathbb{R}^N$ with the aid of (tangential) Markov or Bernstein type inequalities for derivatives of polynomials on curves. Let us add that such inequalities do not directly follow from their full-dimensional versions. In paper [7], motivated by earlier results of Bos, Levenberg, Milman and Taylor (*see* [14], [15]), we characterized semialgebraic curves in $\mathbb{R}^N$ admitting so called *analytic parameterization* in terms of Bernstein type or van der Corput-Schaake type inequalities. The ingenious concept of analytic parameterization that was introduced by Baran, helped us to overcome problems with the regularity of compact submanifolds of $\mathbb{R}^N$.

Definition. A compact curve K in $\mathbb{R}^N$ is said to admit an analytic parameterization if there exist $r \in \mathbb{N}$, $\alpha > 1$ and $\mathbb{R}$-analytic maps $\varphi_j = (\varphi_{j,1}, \dots, \varphi_{j,N}) : \alpha I \to K$, $j = 1, \dots, r$, where $I = [-1, 1]$, such that each $\varphi_j|_I$ is a bijection onto $\varphi_j(I)$ and $K = \bigcup_{j=1}^r \varphi_j(I)$.

Observe that the natural parameterization $h(t) = t$ of the line segment $I = [-1, 1]$ does not fit the requirements of the above definition, since there is no $\alpha > 1$ such that $h((-\alpha, \alpha)) \subset I$. If we replace h by $\varphi(t) = \sin \frac{\pi}{2} t$ then (K, φ) becomes a curve with an analytic parameterization. A curve may admit an analytic parameterization even without being of class C^1. An example is given by $K = \{(x, y) \in \mathbb{R}^2 : y^2 = (1 - x^2)^3\}$ which can be parameterized by $\varphi(t) = (\cos \pi t, \sin^3 \pi t)$. Any curve K in $\mathbb{R}^N$ such that $K = h(I)$ where h is an analytic map in an open neighbourhood of I, obviously admits an analytic parameterization. By Puiseux's theorem (*see* e.g. [19]) any *semialgebraic curve* in $\mathbb{R}^N$, i.e. a finite union of subsets of $\mathbb{R}^N$ of the form

$$\{x \in \mathbb{R}^N : f_i(x) = 0, \ g_j(x) > 0, \ i = 1, \dots, m, j = 1, \dots, n\},$$

where f_i and g_j are in $\mathbb{R}[x_1, \dots, x_N]$, is piecewise C^1 and moreover, it admits an analytic parameterization. The main result of [7] reads as follows.

Theorem. Let K be a compact curve in $\mathbb{R}^N$ with an analytic parameterization $\{\varphi_j\}$ with parameters r and α. Then the following conditions are equivalent:

(i) K is semialgebraic;

(ii) there exist positive constants M_1 and δ_0 such that

$$V_K(\varphi_j(\zeta)) \le M_1 \delta \quad \text{if } \operatorname{dist}(\zeta, I) \le \delta \le \delta_0, \ j = 1, \dots, r;$$

(iii) there exist positive constants M_2 and C such that for each $j = 1, \dots, r$ and $P \in \mathbb{C}[z_1, \dots, z_N]$,

$$|P(\varphi_j(\zeta))| \le M_2 \|P\|_K \quad \text{if } \operatorname{dist}(\zeta, I) \le C/\deg P;$$

(iv) K admits a Bernstein type inequality: there exists a constant $M_3 > 0$ such that for each $j = 1, \dots, r$ and $P \in \mathbb{C}[x_1, \dots, x_N]$,

$$|(P \circ \varphi_j)'(t)| \le M_3 (\deg P) \|P\|_K, \quad t \in I;$$

(v) K admits a van der Corput-Schaake type inequality: there exists a constant $M_4 > 0$ such that for each $j = 1, \dots, r$ and $P \in \mathbb{R}[x_1, \dots, x_N]$,

$$|(P \circ \varphi_j)'(t)| \le M_4 (\deg P) \left(\|P\|_K^2 - P^2(\varphi_j(t))\right)^{1/2}, \quad t \in I.$$

In the proof of the above theorem we needed, among other things, the estimate

$$\delta_n(E) = \mathrm{O}(n^{\dim X(E)})$$

of the dimension $\delta_n(E)$ of the space $\mathcal{P}_n(E)$ of the restrictions to a compact set $E \subset \mathbb{C}^N$ of all polynomials on $\mathbb{C}^N$ of degree at most n, where $X(E)$ is the Zariski closure of E. Such an estimate is well-known in algebraic geometry. However, we found it interesting to provide its proof by "purely analytic methods". To do this, we combined the uniform version of the Bernstein-Walsh-Siciak theorem with the Krein-Krasnoselski-Milman Lemma from the geometric theory of Banach spaces and Sadullaev's criterion as well. In the proof of the estimates of V_K in (ii) we have used delicate techniques involving properties of the inverse of the Joukovski function which is *spécialité de la maison* of Baran. For the convenience of the reader we cite here *in extenso* the

Krein-Krasnoselski-Milman Lemma (*see* e.g. [34], p. 269). Let X be a normed linear space and G_1, G_2 two linear subspaces of X such that $\dim G_1 < \infty$, $\dim G_1 < \dim G_2$. Then there exists a point $x \in G_2 \setminus \{0\}$ that is orthogonal to G_1 in the sense of Birkhoff, i.e.

$$\|x + y\| \geq \|x\| \quad \text{for every } \; y \in G_1.$$

Paper [8] deals with tangential Markov type inequalities for (the traces of) polynomials on algebraic sets. A crucial role there is played by the invariance of non-pluripolarity of sets under non-degenerate analytic maps defined on open subsets of $\mathbb{C}^{N'}$, with values in a locally analytic subset $\mathbb{M}$ of $\mathbb{C}^N$ of pure dimension $\min(N', N)$ (*see* [8], Lemma 0.1). This fact together with the unifom Bernstein-Walsh-Siciak theorem and again Sadullaev's criterion made it possible to prove the following estimate for the Zakhariuta-Siciak extremal function:

Proposition. Let E be a compact non-pluripolar subset of $\mathbb{C}^{N'}$ and let f be an analytic map defined in an open neighbourhood U of $\hat{E}$, the polynomial hull of E, with values in a $\min(N', N)$-dimensional algebraic set $\mathbb{M}$ in $\mathbb{C}^N$ (where $\mathbb{M} = \mathbb{C}^N$ if $N' \geq N$). Assume that $\mathrm{rank}_E f = \min(N', N)$. Then there exist constants $M > 0$ and $\delta_0 > 0$ such that

$$V_{f(E)}(f(z)) \leq M V_E(z) \quad \text{as } \mathrm{dist}(z, E) \leq \delta \leq \delta_0.$$

The above estimate combined with Cauchy's Integral Formula yield a tangential Markov type inequality, which reads as follows.

Theorem. With the above assumptions on f, if E is an HCP compact subset of $\mathbb{C}^{N'}$ with parameter r, then there exists a constant $C_1 > 0$ such that for any polynomial $Q \in \mathbb{C}[z_1, \ldots, z_N]$ of degree d one has

$$|D_{\mathcal{T}(t,v)} Q(z)| \leq C_1 d^r \|Q\|_{f(E)},$$

where $z = f(t)$ with $t \in E$. Here $\mathcal{T}(t, v) = D_v f(t)$, the derivative at the point t of the map f in direction v.

If $k = 1$ and $E = [0, 1]$, Theorem above covers in particular Proposition 6.1 of [15]. Some special cases have also been handled, namely if E is a UPC subset of the space $\mathbb{R}^N$ or f is a polynomial map from $\mathbb{R}$ to $\mathbb{R}^N$.

In the last joint paper [9] we go back to the problem of the characterization of semialgebraic curves in $\mathbb{R}^N$ in terms of Bernstein and van der Corput-Schaake type inequalities but now in the essentially more difficult setting of semialgebraic sets in $\mathbb{R}^N$ of higher dimensions. For this purpose we had to introduce an analytic parameterization of order m.

Definition. A compact subset K of $\mathbb{R}^N$ is said to admit an analytic parameterization of dimension m, $1 \leq m \leq N$, if there exist $\rho > 1$, $r \in \mathbb{N}$ an real-analytic maps $\varphi_j = (\varphi_{j1}, \ldots, \varphi_{jN}) : \mathbb{B}^m(\rho) \to K$, $j = 1, \ldots, r$, such that for each j we have $\mathrm{rank}\, \varphi_j = m$ and

$$K = \bigcup_{j=1}^{r} \varphi_j(\mathbb{B}^m).$$

It is not difficult to see that in the Definition above, instead of the unit ball $\mathbb{B}^m$, we could be working with the m-dimensional cube $\mathbb{I}^m = [-1, 1]^m$. A large family of compact sets with an analytic parameterization is furnished by the Gabrielov-Hironaka-Łojasiewicz subanalytic geometry (for the rudiments of this theory the reader is referred to [13]). It follows from the famous Hironaka Rectilinearization Theorem that for any compact, subanalytic subset K of an m-dimensional real-analytic manifold $\mathbb{M}$ of pure dimension m there exist a finite number of real-analytic maps $\varphi_k : \mathbb{R}^m \to \mathbb{M}$ such that $\bigcup_k \varphi_k(\mathbb{I}^m) = K$ (*see* [20], Corollary 6.2). Hence such a K admits an analytic parameterization of dimension m. Let us recall that if $\mathbb{M} = \mathbb{R}^N$, then such a K is UPC, whence HCP, and consequently it admits Markov's inequality ([20]). The main result of [9] reads as follows.

Theorem. Let K be a compact subset of $\mathbb{R}^N$ with an analytic parameterization $\{\varphi_j\}_{j=1}^r$ of dimension m, $1 \leq m \leq N$, with parameters $r \in \mathbb{N}$ and $\rho > 1$. Then the following conditions are equivalent:

(a) the Zariski dimension of K is m;

(b) there exist positive constants C_2 and δ_2 such that

$$V_K(\varphi_j(z)) \leq C_2 \delta$$

for $\mathrm{dist}(z, \mathbb{B}^N) \leq \delta \leq \delta_2$, $z \in \mathbb{C}^m$, $j = 1, \ldots, r$;

(c) there exist positive constants C_3 and δ_3 such that for every polynomial $P \in \mathbb{C}[x_1, \ldots, x_N]$ of degree at most k,

$$|P(\varphi_j(z))| \leq C_3 \|P\|_K$$

for $\mathrm{dist}(z, \mathbb{B}^m) \leq \delta_3/k$, $z \in \mathbb{C}^m$, $j = 1, \ldots, r$;

(d) (Bernstein Inequality) there exists a constant $C_4 > 0$ such that for each polynomial $P \in \mathbb{C}[x_1, \dots, x_N]$,

$$D_{\mathcal{T}_j(t,v)}P(x)| \le C_4(\deg P)\|P\|_K,$$

for $x \in K_j := \varphi_j(\mathbb{B}^m)$, $t \in \varphi^{-1}(x) \cap \mathbb{B}^m$ and $v \in \mathbb{S}^{m-1}$, $j = 1, \dots, r$. (here $\mathcal{T}_j(t,v) = D_v\varphi_j(t)$);

(e) (van der Corput-Schaake Inequality) there exists a constant $C_5 > 0$ such that for each polynomial $P \in \mathbb{R}[x_1, \dots, x_N]$,

$$|D_{\mathcal{T}_j(t,v)}P(x)| \le C_5(\deg P)(\|P\|_K^2 - P^2(x))^{1/2},$$

for $x \in K_j = \varphi_j(\mathbb{B}^m)$, where $t \in \varphi_j^{-1}(x) \cap \mathbb{B}^m$, and $v \in \mathbb{S}^{m-1}$,
$j = 1, \dots, r$.

Although the above theorem looks similar to the 1-dimensional counterpart from [7], its proof required much more difficult techniques, developed by Baran in his papers [1] and [2] concerning formulas for Siciak's extremal function for the unit ball in $\mathbb{R}^m$. We also took advantage of the fact that an irreducible closed analytic subset E of $\mathbb{C}^N$ of pure dimension m ($m \le N$)is algebraic if and only if $\dim \mathcal{P}_k(E) = O(k^m)$. Let us end by the remark that since every compact real-analytic manifold admits an analytic parameterization, the equivalence $(a) \Leftrightarrow (d)$ covers the main result of [15].

My cooperation with Mirosław Baran was not limited to joint publications. For a long time, we animated together a seminar on the theory of approximation in the Institute of Mathematics of the Jagiellonian University, which I initiated in the year 1979. Mirosław was its pillar during this period and one of the most frequent speakers. This seminar ended in 2015 when I retired. But that's a quite different story. I also participated in his training of young reserchers, I was a reviewer of doctoral dissertations of three of his students. We participated together in many international conferences on complex analysis and approximation theory. I will mention only two here that were especially important to me: *Journée d'analyse réelle et complexe*, organized on 23th October 2003 by the Université de Toulon et du Var which awarded me an honorary doctorate, and *Conference on Constructive Approximation of Functions*, Będlewo, 30th June - 5th July 2014, dedicated to me on the occasion of the 70th birthday, of which Mirosław was the main organizer. With this article, I would like to thank him for our long-term cooperation. AD MULTOS ANNOS, MIROSŁAW!

Figure 1: Toulon 2003. From the left: Mirosław Baran, Wiesław Pleśniak, Pierre Goetgheluck, Józef Siciak.

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Figure 2: Będlewo 2014. From the left (behind Mirosław Baran): Malgorzata Stawiska-Friedland, Alicja Skiba, Marta Kosek, Ewa Ciechanowicz, Janina Kotus, Leokadia Białas-Cież, Rafał Pierzchała.

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