Classification of (p, q, n)-dipoles on nonorientable surfaces*

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Abstract

A type of rooted map called (p, q, n)-dipole, whose numbers on surfaces have some applications in string theory, are defined and the numbers of (p, q, n)-dipoles on orientable surfaces of genus 1 and 2 are given by Visentin and Wieler (The Electronic Journal of Combinatorics 14 (2007),#R12). In this paper, we study the classification of (p, q, n)-dipoles on nonorientable surfaces and obtain the numbers of (p, q, n)-dipoles on the projective plane and Klein bottle.

1 Introduction

A *surface* is a compact 2-dimensional manifold without boundary. It can be represented by a polygon of even edges in the plane whose edges are pairwise identified and directed clockwise or counterclockwise. Such polygonal representations of surfaces can also be written by words. For example, the sphere is written as $O_0 = aa^-$ where a^- is identified with the opposite direction of a on the boundary of the polygon. In general, $O_p = \prod_{i=1}^{p} a_i b_i a_i^- b_i^-$ and $N_q = \prod_{i=1}^{q} a_i a_i$ denote, respectively, an orientable surface of genus p and a nonorientable surface of genus q. Of course, N_1, O_1 and N_2 are, respectively, the projective plane, the torus and the Klein bottle. Every surface is homeomorphic to precisely one of the surfaces O_p $(p \geqslant 0)$, or N_q $(q \geqslant 1)$ [2,5].

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Let $\mathcal{S}$ be the collection of surfaces and let AB be a surface. The following topological transformations and their inverses do not change the orientability and genus of a surface:

TT 1: $Aaa^-B \Leftrightarrow AB$ where $a \notin AB$, TT 2: $AabBab \Leftrightarrow AcBc$ where $c \notin AB$ and
TT 3: $AB \Leftrightarrow (Aa)(a^-B)$ where $AB \neq \emptyset$.

In fact, what is determined under these transformations is a topological equivalence $\sim$ on $\mathcal{S}$. Suppose $A = a_1a_2\cdots a_t (t \geqslant 1)$ is a word, then $A^- = a_t^- \cdots a_2^- a_1^-$ is called the *inverse* of A. The following relations as shown in, e.g.,[2] can be deduced by using TT 1-3.

Relation 1: $(AxByCx^-Dy^-) \sim ((ADCB)(xyx^-y^-))$,
Relation 2: $(AxBx) \sim ((AB^-)(xx))$,
Relation 3: $(Axxyzy^-z^-) \sim ((A)(xx)(yy)(zz))$.

In TT 1-3 and Relation 1-3, A, B, C, and D are all linear orders of letters and permitted to be empty. The parentheses stand for cyclic order and they are always omitted when unnecessary to distinguish cyclic or linear order. The following two lemmas can both be deduced by Relation 1-3.

Lemma 1.1[3] *An orientable surface S is a surface of orientable genus 0 if and only if there is no form as $AxByCx^-Dy^-$ in it.*

Lemma 1.2[9] *Let S be a nonorientable surface, if there is a form as $AxByCx^-Dy^-$ in S, then the genus of S will be not less than 3; if there is a form as $AxByCx^-Dy$ or $AxByCyDx$ in S, then the genus of S will be not less than 2.*

A *map* is a 2-cell embedding of a graph on a surface. The enumeration of maps on surfaces has been developed and deepened by people, based on the initial works by W.T.Tutte in the 1960s. The reader is referred to the monograph [4] for further background about enumerative theory of maps.

The joint tree model of a graph embedding which was established in [2] by Liu, can be used as a model for constructing an embedding of a graph on surfaces without repetition in sense of topological equivalence. Some works have been done based on the joint tree model, such as [1,7,8,9] etc.

Given a spanning tree T of a graph G, for $1 \leqslant i \leqslant \beta$, we split each cotree edge e_i into two semi-edges and label them by a_i and $a_i^{\varepsilon_i}$ where ε_i is a binary index, it can be $+$(always omit) or $-$, and β is the number of cotree edges of G. The resulting graph consisting of tree edges in T and 2β semi-edges is a tree, denote by $\hat{T}$. A *rotation* at a vertex v, denoted by σ_v, is a cyclic permutation of edges incident with v. Let $\sigma_G = \prod_{v \in V(G)} \sigma_v$ be a rotation system of G.

The tree $\hat{T}$ with a choice of index of each pair of semi-edges labelled by the same letter and a rotation system of G is called a *joint tree* of G, denote by $\hat{T}_\sigma^\varepsilon$. By reading these lettered semi-edges with indices of a $\hat{T}_\sigma^\varepsilon$ in a fixed orientation (clockwise or counterclockwise), we can get an algebraic representation for a surface. It is a cyclic order of 2β letters with indices. Such a surface is called an *associate surface* [3] of G. If two associate surfaces of G have the same cyclic order with the same ε in their algebraic representations, then we say that they are the same; otherwise, distinct. In fact, the edge e_i whose two semi-edges have the distinct indices *i.e.* a_i and a_i^- is the untwisted edge in the embedding;

otherwise twisted.

From [3], there is a 1-to-1 correspondence between associate surfaces and embeddings of a graph, hence the problem of determining the nonequivalent embeddings for a graph on a surface with given genus can be transformed into that of finding the number of distinct associate surfaces in an equivalent class(up to genus).

A type of rooted map called (p, q, n)-dipole is defined in [6]. Let M be a rooted map with 2 vertices of degree n (with no loops) and one other distinguished edge e. If edge e is the pth edge after the root edge in the rotation of the root vertex, but is the qth edge after the root edge in the rotation of the nonroot vertex, then M is a (p, q, n)-dipole. Without the distinguished edge e, the map is a rooted dipole. The numbers of (p, q, n)-dipoles on orientable surfaces of genus 1 and 2 are given by Visentin and Wieler in [6]. Their interest in doing it comes out of an application to string theory. The reader is referred to [6] for more detail about dipoles and (p, q, n)-dipoles. In this paper, the numbers of (p, q, n)-dipoles on the nonorientable surfaces of genus 1 (projective plane) and 2 (Klein bottle) are obtained, by the joint tree method.

2 The number of rooted dipoles on the projective plane and Klein bottle

According to the joint tree method, we can choose the rooted edge as the tree edge and label the $n-1$ cotree edges by $a_1, \ldots, a_{n-1}$, then the associate surfaces of rooted dipoles with n edges are of the form $(a_1 \cdots a_{n-1} A)$, in which $|A| = n - 1$.

Lemma 2.1[8] *The numbers of the spheres, projective planes and Klein bottles in the surface set* $T_1^{n-1} = \{a_1 a_2 \cdots a_{n-1} A \mid |A| = n-1\}$ *are*

$$g_0(T_1^{n-1}) = 1, \quad \tilde{g}_1(T_1^{n-1}) = \frac{(n-1)n}{2} \quad \text{and} \quad \tilde{g}_2(T_1^{n-1}) = \frac{(n-2)(n-1)n^2}{6},$$

respectively.

From Lemma 2.1, the following two theorems follow.

Theorem 2.1 *The number of rooted dipoles with* n *edges on the projective plane is*

$$\frac{(n-1)n}{2}.$$

Theorem 2.2 *The number of rooted dipoles with* n *edges on the Klein bottle is*

$$\frac{(n-2)(n-1)n^2}{6}.$$

3 The number of (p,q,n)-dipoles on the projective plane and Klein bottle

According to [6], we need only calculate the number of (p,q,n)-dipoles for $1 \leqslant p \leqslant q \leqslant n-p$. Suppose a_p is the distinguished edge other than the rooted edge and the rotation of the rooted vertex is $(a_0, a_1, \ldots, a_{n-1})$ where a_0 is the rooted edge, then the associate surfaces of (p,q,n)-dipoles are of the form

$$a_1 \cdots a_{p-1} a_p a_{p+1} \cdots a_{n-1} B_1 a_p^{\varepsilon_p} B_2,$$

in which $|B_1| = q-1$, $|B_2| = n-1-q$ and $\varepsilon_p = \begin{cases} +(\text{always omit}), & e_p \text{ is a twisted edge;} \\ -, & \text{otherwise.} \end{cases}$
In order to get the numbers of (p,q,n)-dipoles on the projective plane and Klein bottle, we need only calculate the numbers of the projective planes and Klein bottles in the surface set $\{a_1 \cdots a_{p-1} a_p a_{p+1} \cdots a_{n-1} B_1 a_p^{\varepsilon_p} B_2\}$, for the joint tree method.

Theorem 3.1 *The number of (p,q,n)-dipoles on the projective plane is*

$$\begin{cases} p & \textit{when } p+q<n; \\ \dfrac{p(p+1)+q(q-1)}{2} & \textit{when } p+q=n. \end{cases}$$

Proof When e_p is a twisted edge, $\varepsilon_p = +$. According to Relation 2,

$$a_1 \cdots a_{p-1} a_p a_{p+1} \cdots a_{n-1} B_1 a_p B_2 \sim a_1 \cdots a_{p-1} B_1^- a_{n-1}^- \cdots a_{p+1}^- B_2 a_p a_p.$$

$$a_1 \cdots a_{p-1} B_1^- a_{n-1}^- \cdots a_{p+1}^- B_2 a_p a_p \sim N_1 \Leftrightarrow a_1 \cdots a_{p-1} B_1^- a_{n-1}^- \cdots a_{p+1}^- B_2 \sim O_0.$$

$|B_1| = q-1 \geqslant p-1$ and for Lemma 1.1, we have

$$B_1^- = a_{p-1}^- \cdots a_j^- a_{n-q-j+p+1} \cdots a_{n-1},\ B_2 = a_{p+1} \cdots a_{n-q-j+p} a_{j-1}^- \cdots a_1^-,\ 1 \leqslant j \leqslant p.$$

Hence the number of (p,q,n)-dipoles on the projective plane in this case is p.

When e_p is an untwisted edge, $\varepsilon_p = -$. According to Lemma 1.2, for $1 \leqslant i \leqslant p-1$, $a_i^{\varepsilon_i} \in B_2$ and for $p+1 \leqslant j \leqslant n-1$, $a_j^{\varepsilon_j} \in B_1$. Hence, $|B_1| = n-p-1 = q-1$, i.e., in this case, $p+q=n$.

$$a_1 \cdots a_{p-1} a_p a_{p+1} \cdots a_{n-1} B_1 a_p^- B_2 \sim N_1 \Leftrightarrow$$

$$a_1 \cdots a_{p-1} B_2 \sim O_0 \text{ and } a_{p+1} \cdots a_{n-1} B_1 \sim N_1;$$

or $$a_1 \cdots a_{p-1} B_2 \sim N_1 \text{ and } a_{p+1} \cdots a_{n-1} B_1 \sim O_0.$$

From Lemma 2.1, the number of (p,q,n)-dipoles on the projective plane in this case is

$$\begin{cases} 0 & \text{when } p+q<n; \\ \dfrac{p(p-1)+q(q-1)}{2} & \text{when } p+q=n. \end{cases}$$

Summarizing the above, the theorem is obtained. □

For convenience, we write $A_1 = a_1 \cdots a_{p-1}$ and $A_2 = a_{p+1} \cdots a_{n-1}$ in the following.

Theorem 3.2 *The number of (p,q,n)-dipoles on the Klein bottle is*

$$\begin{cases} \dfrac{(n-p-1)(n-p)p}{2} + \dfrac{(p-1)p(3n+3q-2p-5)}{6} + pq(n-p-q) & \text{when } p+q<n; \\ \dfrac{(p-1)p\big((p-1)p+6q-5\big)}{6} + \dfrac{(q-1)q\big(2(q-2)q+3p(p+1)\big)}{12} & \text{when } p+q=n. \end{cases}$$

Proof When e_p is an untwisted edge, the associate surfaces of (p,q,n)-dipoles have the form as $A_1 a_p A_2 B_1 a_p^- B_2$.

Case 1 $\forall a_i \in A_1, a_i^{\varepsilon_i} \in B_2$ and $\forall a_j \in B_1, a_j^{\varepsilon_i} \in A_2$.
For $|A_1| + |A_2| = |B_1| + |B_2| = n-2$, we have $|A_1| = |B_2| = p-1, |A_2| = |B_1| = q-1$ and $p+q=n$.

$$A_1 a_p A_2 B_1 a_p^- B_2 \sim N_2 \Leftrightarrow$$

$$A_2 B_1 \sim N_2 \text{ and } A_1 B_2 \sim O_0; \text{ or } A_2 B_1 \sim O_0 \text{ and } A_1 B_2 \sim N_2;$$

or $$A_2 B_1 \sim N_1 \text{ and } A_1 B_2 \sim N_1.$$

From Lemma 2.1, the number of (p,q,n)-dipoles on the Klein bottle in Case 1 is

$$\begin{cases} 0 & \text{when } p+q<n; \\ \dfrac{p^2(p-1)(p-2)+q^2(q-1)(q-2)}{6} + \dfrac{p(p-1)q(q-1)}{4} & \text{when } p+q=n. \end{cases}$$

Case 2 $\forall a_i \in A_1, a_i^{\varepsilon_i} \in B_2$ and $\exists a_j \in A_2, a_j^{\varepsilon_j} \in B_2$.
In this case, $|A_1| < |B_2|$, i.e., $p-1 < n-1-q$, hence $p+q<n$. According to Lemma 1.2, e_j is a twisted edge, let $A_1 a_p A_2 B_1 a_p^- B_2 = A_1 a_p A_{21} a_j A_{22} B_1 a_p^- B_{21} a_j B_{22}$. By using Relation 2 twice, $A_1 a_p A_{21} a_j A_{22} B_1 a_p^- B_{21} a_j B_{22} \sim A_1 B_{21} A_{21}^- B_1^- A_{22}^- B_{22} a_p a_p a_j a_j$.

$$A_1 a_p A_2 B_1 a_p^- B_2 \sim N_2 \Leftrightarrow A_1 B_{21} A_{21}^- B_1^- A_{22}^- B_{22} \sim O_0$$

Let $B_{21} = B_{21}' B_{21}'', B_{22} = B_{22}'' B_{22}'$, for $\forall a_i \in A_1, a_i^{\varepsilon_i} \in B_2$,

$$A_1 B_{21} A_{21}^- B_1^- A_{22}^- B_{22} \sim O_0 \Leftrightarrow A_1 B_{21}' B_{22}' \sim O_0 \text{ and } B_{21}'' A_{21}^- B_1^- A_{22}^- B_{22}'' \sim O_0.$$

For $|A_1| = |B_{21}'| + |B_{22}'| = p-1$, $|B_1| = q-1$ and $|B_{21}''| + |B_{22}''| = n-1-p-q$, the number of (p,q,n)-dipoles on the Klein bottle in Case 2 is

$$\begin{cases} pq(n-p-q) & \text{when } p+q<n; \\ 0 & \text{when } p+q=n. \end{cases}$$

Case 3 $\exists a_i \in A_1, a_i^{\varepsilon_i} \in B_1$.
From Lemma 1.2, e_i is a twisted edge. Let $A_1 a_p A_2 B_1 a_p^- B_2 = A_{11} a_i A_{12} a_p A_2 B_{11} a_i B_{12} a_p^- B_2$, and we can also suppose that $\forall a_k \in A_{11}, a_k^{\varepsilon_k} \in B_2$. By Relation 2, $A_1 a_p A_2 B_1 a_p^- B_2 \sim N_2 \Leftrightarrow A_{11} B_{11}^- A_2^- B_{12}^- A_{12} B_2 \sim O_0$. With a similar argument in Case 2, we can obtain that the number of (p,q,n)-dipoles on the Klein bottle in Case 3 is

$$\frac{(p-1)p(3q-p-1)}{6}.$$

Summarizing Cases 1-3, when e_p is an untwisted edge, the number of (p,q,n)-dipoles on the Klein bottle is

$$\begin{cases} pq(n-p-q)+\dfrac{(p-1)p(3q-p-1)}{6} & \text{when } p+q<n; \\ \dfrac{p^2(p-1)(p-2)+q^2(q-1)(q-2)+(p-1)p(3q-p-1)}{6}+\dfrac{p(p-1)q(q-1)}{4} & \\ & \text{when } p+q=n. \end{cases}$$

In a similar way, we can get that when e_p is a twisted edge, the number of (p,q,n)-dipoles on the Klein bottle is

$$\frac{(n-p-1)(n-p)p}{2}+\frac{(p-1)p(3n-p-4)}{6}.$$

Above all, the theorem is obtained. □

The number of (p,q,n)-dipoles on surfaces of higher genera depends greatly on those of lower genera. The results here and the method we used may be helpful for the further research of (p,q,n)-dipoles on surfaces of higher genera.

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