A classification of Ramanujan unitary Cayley graphs

Andrew Droll

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Abstract

The unitary Cayley graph on n vertices, X_n, has vertex set $\frac{\mathbb{Z}}{n\mathbb{Z}}$, and two vertices a and b are connected by an edge if and only if they differ by a multiplicative unit modulo n, i.e. $\gcd(a-b,n)=1$. A k-regular graph X is Ramanujan if and only if $\lambda(X) \leqslant 2\sqrt{k-1}$ where $\lambda(X)$ is the second largest absolute value of the eigenvalues of the adjacency matrix of X. We obtain a complete characterization of the cases in which the unitary Cayley graph X_n is a Ramanujan graph.

1 Unitary Cayley graphs

Given a finite additive abelian group G and a symmetric subset S of G, we define the Cayley graph $X(G,S)$ to be the graph whose vertex set is G, and in which two vertices v and w in G are connected by an edge if and only if $v-w$ is in S. A Cayley graph of the form $X(G,S)$ with $G=\frac{\mathbb{Z}}{n\mathbb{Z}}$ is called a *circulant* graph.

The *unitary* Cayley graph on n vertices, X_n, is defined to be the undirected graph whose vertex set is $\frac{\mathbb{Z}}{n\mathbb{Z}}$, and in which two vertices a and b are connected by an edge if and only if $\gcd(\mathrm{a}-\mathrm{b},\mathrm{n})=1$. This can also be stated as $X_n = X\left(\frac{\mathbb{Z}}{n\mathbb{Z}}, \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^*\right)$, where $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is the additive group of integers modulo n and $\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^*$ is the set of multiplicative units modulo n. It is easy to see that X_n is a simple, $\varphi(n)$-regular graph, where φ is the Euler totient function. Here $\varphi(n)$ is defined by $\varphi(1)=1$, and for an integer $n>1$ with distinct prime power factorization $p_0^{e_0}p_1^{e_1}\cdots p_t^{e_t}$ for distinct primes $p_0,\cdots,p_t$ and nonnegative integers $e_0,\cdots,e_t$, with $t \geqslant 0$, $\varphi(n) = p_0^{e_0-1}\cdots p_t^{e_t-1}(p_0-1)\cdots(p_t-1)$.

The eigenvalues of the adjacency matrix of $X(G,S)$ for an abelian group G and symmetric subset S are

$$\lambda_m = \sum_{s\in S}\chi_m(s), \tag{1}$$

for $m=0,\cdots,|G|-1$, where $\chi_0,\cdots,\chi_{|G|-1}$ are the irreducible characters of G (see, for example, [Murty (2003)]). We therefore have the following lemma (see [Klotz, W. and Sander, T. (2007)], for example).

Lemma 1.1 *The eigenvalues of any adjacency matrix of X_n are*

$$\lambda_m(n) = \sum_{a,\ (a,n)=1} e^{\frac{2i\pi am}{n}}, \quad m = 0, \ldots, n-1. \tag{2}$$

In fact, these are Ramanujan sums, which are known to have the simpler closed form

$$\lambda_m(n) = \mu\left(\frac{n}{(n,m)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)}, \tag{3}$$

where μ is the Möbius function and φ is the Euler totient function (see [Murty (2003)], for example). Recall that the Möbius function μ is defined for positive integers n by $\mu(n) = 0$ if n is not square-free, $\mu(n) = 1$ if n is square-free and has an even number of distinct prime factors, and $\mu(n) = -1$ if n is square-free and has an odd number of distinct prime factors. Since the Möbius function is zero at non-square-free arguments, the eigenvalue corresponding to m, $0 \leqslant m \leqslant n-1$ is nonzero if and only if $\frac{n}{(n,m)}$ is square-free. When $\frac{n}{(n,m)}$ is square-free,

$$|\lambda_m(n)| = \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)}. \tag{4}$$

Recall that the adjacency matrix of any k-regular graph X has eigenvalues between $-k$ and k, and k is an eigenvalue with multiplicity precisely equal to the number of connected components of X. Furthermore, if $\lambda(X)$ denotes the largest absolute value of the eigenvalues of the adjacency matrix of X, smaller than k, then the graph X is called Ramanujan if and only if

$$\lambda(X) \leqslant 2\sqrt{k-1}. \tag{5}$$

Note that $\lambda(X)$ is only defined for regular graphs X with 3 or more vertices, so when discussing X_n, we always assume $n \geqslant 3$. Writing n in the form $2^s p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ for some distinct odd primes $p_1 < p_2 < \cdots < p_t$, nonnegative integer s, and positive integers $e_1, \cdots, e_t$, we can determine $\lambda(X_n)$ as follows. Since X_n is $\varphi(n)$-regular, we find the maximum absolute value of an eigenvalue $\lambda_m(n)$ of the adjacency matrix of X_n, smaller than $\varphi(n)$. This can be accomplished by looking at (3).

Indeed, we see that if $t = 0$ then $n = 2^s$ and the eigenvalues have absolute value of either 0 or $\varphi(n)$ (since the only values of m, $0 \leqslant m \leqslant n-1$, which make $\frac{n}{(n,m)}$ square-free are $m = 0$ and $m = 2^{s-1}$, resulting in eigenvalues $\varphi(n)$ and $-\varphi(n)$). Thus $\lambda(X_{2^s}) = 0$ and so X_{2^s} satisfies (5) and thus is Ramanujan. To consider the case $t > 0$, we adopt the notation $\max(a, 0) = (a)^+$ for any a. Then, if $t > 0$ and $\frac{n}{(n,m)}$ is to be squarefree, m must clearly be divisible by $2^{(s-1)^+} p_1^{e_1-1} ... p_t^{e_t-1}$. Looking at (4), it is apparent that $m = n/p_1$ maximizes the absolute value of $\lambda_m(n)$ while keeping it smaller than $\varphi(n)$, since this choice minimizes the quantity $\varphi\left(\frac{n}{(n,m)}\right)$ while keeping its value greater than 1, with $\frac{n}{(n,m)}$ square-free (We can not take $m = n/2$, since $\varphi\left(\frac{n}{n/2}\right) = \varphi(2) = 1$.) Thus for

$t > 0$, we have

$$\begin{aligned}
\lambda(X_n) = \quad & \frac{\varphi(n)}{\varphi\left(\frac{n}{(n/p_1)}\right)} \\
= \quad & \frac{\varphi(n)}{p_1 - 1} && (6)\\
= \quad & 2^{(s-1)^+} p_1^{e_1 - 1} \cdots p_t^{e_t - 1} (p_2 - 1) \cdots (p_t - 1). && (7)
\end{aligned}$$

We can use this to restate (5) in a simpler form for X_n when $t \geqslant 1$. X_n is Ramanujan if and only if

$$\lambda(X_n) = \frac{\varphi(n)}{p_1 - 1} \leqslant 2\sqrt{\varphi(n) - 1}. \tag{8}$$

By noting that (8) implies $\frac{\varphi(n)}{p_1-1} < 2\sqrt{\varphi(n)}$, we easily obtain that a necessary condition for X_n to be Ramanujan when $t \geqslant 1$ is

$$\frac{\varphi(n)}{p_1 - 1} < 4(p_1 - 1). \tag{9}$$

We now have the tools to state and prove our main result.

Theorem 1.2 *The graph X_n is Ramanujan if and only if n satisfies one of the following conditions for some distinct odd primes $p < q$ and natural s.*

$$\begin{array}{ll}
(a) & n = 2^s, \ \ s \geqslant 2 \\
(b) & n = p \\
(c) & n = 2^s p \quad \text{with} \quad s \geqslant 1, \ p > 2^{s-3} + 1 \\
(d) & n = p^2, \ 2p^2, \ 4p^2 \\
(e) & n = pq, 2pq \quad \text{with} \quad p < q \leqslant 4p - 5 \\
(f) & n = 4pq \quad \text{with} \quad p < q \leqslant 2p - 3
\end{array}$$

Proof As in our discussion above, $n = 2^s p_1^{e_1} \cdots p_t^{e_t}$. If $t \geqslant 3$ then

$$\frac{\varphi(n)}{p_1 - 1} \geqslant p_1^{e_1 - 1} p_2^{e_2 - 1} p_3^{e_3 - 1} (p_2 - 1)(p_3 - 1) \geqslant 4(p_1 - 1),$$

since $p_1 < p_2 < p_3$ are odd primes, and $p_2 - 1 > p_1 - 1$ and $p_3 - 1 > 4$. This violates the necessary condition (9). Thus we see that if $t \geqslant 3$ (i.e. n has at least three distinct odd prime factors), then X_n is not Ramanujan.

This shows that any n for which X_n is Ramanujan must have at most two distinct odd prime factors, i.e. $t \leqslant 2$. We have already shown in our earlier discussion that if $t = 0$ (i.e. $n = 2^s$, as in case (a)), then X_n is Ramanujan. Next, we consider $t = 2$, i.e. $n = 2^s p^b q^c$ for some distinct odd primes $p < q$, nonnegative integer s, and positive integers b, c. By (9), the graph X_n will not be Ramamujan if

$$\frac{\varphi(n)}{p - 1} = 2^{(s-1)^+} p^{b-1} q^{c-1} (q - 1) \geqslant 4(p - 1).$$

This inequality holds unless $b = c = 1$ and $s \leqslant 2$, since if $s > 2$ we have $2^{(s-1)^+} \geqslant 4$ and $q-1 > p-1$, and if $c > 1$ or $b > 1$, we have $q - 1 \geqslant 4$ and $p^{b-1}q^{c-1} > p-1$. Thus if $t = 2$, X_n is Ramanujan only if n has the form pq, $2pq$, or $4pq$. If $n = pq$ or $n = 2pq$, we have $\frac{\varphi(n)}{p-1} = (q-1)$, and the Ramanujan condition (8) is

$$q - 1 = \frac{\varphi(n)}{p-1} \leqslant 2\sqrt{\varphi(n) - 1} = 2\sqrt{(p-1)(q-1) - 1},$$

which is easily seen to be equivalent to

$$q - 1 \leqslant 4(p-1) - \frac{4}{q-1}.$$

Noting that $p < q$ are odd primes, and in particular $q \geqslant 5$ (so $\frac{4}{q-1} \leqslant 1$), we see that this implies $q \leqslant 4p - 5$, as in case (e) above, and it is straightforward to check that the converse holds as well. On the other hand, if $n = 4pq$, we have $\frac{\varphi(n)}{p-1} = 2(q-1)$, and the Ramanujan condition (8) is

$$\frac{\varphi(n)}{p-1} = 2(q-1) \leqslant 2\sqrt{2(p-1)(q-1) - 1} = 2\sqrt{\varphi(n) - 1},$$

which, similarly, is equivalent to

$$q - 1 \leqslant 2(p-1) - \frac{1}{q-1}.$$

Again, we note that $p < q$ are odd primes, and $q \geqslant 5$ (so $\frac{1}{q-1} < 1$), to see that the above line is equivalent to $q \leqslant 2p - 3$, as in case (f). Thus we see that cases (e) and (f) completely characterize the values of n with exactly two distinct odd prime factors such that X_n is Ramanujan, as we wanted.

Finally we must consider the case $t = 1$, i.e. n has exactly one odd prime factor. Here $n = 2^s p^b$ for some odd prime p, nonnegative integer s, and positive integer b. If n is prime, i.e. $s = 0$, $b = 1$, then the graph X_n is easily seen to be the complete graph on n vertices, which is well-known to be Ramanujan [Murty (2003)]. More generally, with $n = 2^s p^b$, we see that by (9), X_n will not be Ramanujan if

$$\frac{\varphi(n)}{p-1} = 2^{(s-1)^+} p^{b-1} \geqslant 4(p-1).$$

This holds unless $b = 1$ and s is sufficiently small compared to p, or $b = 2$ and $s \leqslant 2$, since otherwise, $p^{b-1} > (p-1)$, and $2^{s-1} \geqslant 4$ for $s \geqslant 3$. Suppose $b = 2$ and $s \leqslant 2$. Since $\varphi(2p^2) = \varphi(p^2) = p(p-1)$, the cases $s = 0$ and $s = 1$ are identical, because the Ramanujan condition (8) depends only on $\varphi(n)$. It is straightforward to check that if $n = p^2$ or $2p^2$, then

$$p = \frac{\varphi(n)}{p-1} \leqslant 2\sqrt{\varphi(n) - 1} = 2\sqrt{p(p-1) - 1},$$

while if $n = 4p^2$, then

$$2p = \frac{\varphi(n)}{p-1} \leqslant 2\sqrt{\varphi(n)-1} = 2\sqrt{2p(p-1)-1},$$

i.e. the Ramanujan condition (8) is satisfied in these three cases, and thus X_{p^2}, X_{2p^2}, and X_{4p^2} are all Ramanujan for any odd prime p, as claimed in case (d). The final case to consider is $b = 1$ and $s \geqslant 1$. In this case $n = 2^s p$ and $\frac{\varphi(n)}{p-1} = 2^{s-1}$. The Ramanujan condition (8) is

$$2^{s-1} = \frac{\varphi(n)}{p-1} \leqslant 2\sqrt{\varphi(n)-1} = 2\sqrt{2^{s-1}(p-1)-1},$$

which is easily rearranged to

$$p \geqslant 2^{s-3} + 2^{1-s} + 1.$$

Thus we see that for $s \geqslant 1$, $X_{2^s p}$ is Ramanujan whenever $p \geqslant 2^{s-3} + 2^{1-s} + 1$. Since $p \geqslant 3$ is an odd prime, it is easily verified that this is equivalent to $p > 2^{s-3} + 1$, as claimed in case (c).

We have now examined all possible cases for the prime decomposition of n, so we are done. ∎

2 Concluding remarks

We have completely characterized which unitary Cayley graphs are Ramanujan. We remark that every case of Theorem 1 gives rise to infinite families of Ramanujan graphs in this form. As shown by Murty in [Murty (2005)], it is impossible to construct an infinite family of k-regular abelian Cayley graphs which are all Ramanujan for any particular k. However, finding examples of Ramanujan graphs in the way that we have presented here is still of some interest. It is also interesting to remark on some other work that has been done on unitary Cayley graphs. Various properties of the graph X_n were determined in [Klotz, W. and Sander, T. (2007)], including the chromatic number, the clique number, the independence number, the diameter, and the vertex connectivity, in addition to some work on the eigenvalues. The energy of X_n was determined and studied independently in [Ilić (2009)] and [Ramaswamy, H.N. and Veena, C.R. (2009)]. It is also interesting to note that [Ramaswamy, H.N. and Veena, C.R. (2009)] hints at a new approach to finding the eigenvalues of X_n using properties of the graph instead of relying on their expression as Ramanujan sums.

Another point of interest arises in noting that unitary Cayley graphs are examples of connected circulant integral graphs. A recent paper by Wasin So (see [So (2005)]) characterizes the family of integral circulant graphs, and it is worth noting that this entire family of graphs may easily be constructed from the graphs X_n which we have discussed in this paper. This fact naturally invites attempting to apply the results of Theorem 1 to larger families of integral circulant graphs.

For further discussion on the topic of Ramanujan graphs, one is invited to look at [Murty (2003)].

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