Chapter II

Distributions and Sobolev Spaces

1 Distributions

1.1

We shall begin with some elementary results concerning the approximation of functions by very smooth functions. For each $\varepsilon > 0$, let $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be given with the properties

$$\varphi_\varepsilon \geq 0 \quad , \quad \mathrm{supp}(\varphi_\varepsilon) \subset \{x \in \mathbb{R}^n : |x| \leq \varepsilon\} \quad , \quad \int \varphi_\varepsilon = 1 \ .$$

Such functions are called *mollifiers* and can be constructed, for example, by taking an appropriate multiple of

$$\psi_\varepsilon(x) = \begin{cases} \exp(|x|^2 - \varepsilon^2)^{-1} \ , & |x| < \varepsilon \ , \\ 0 \ , & |x| \geq \varepsilon \ . \end{cases}$$

Let $f \in L^1(G)$, where G is open in $\mathbb{R}^n$, and suppose that the support of f satisfies $\mathrm{supp}(f) \subset\subset G$. Then the distance from $\mathrm{supp}(f)$ to ∂G is a positive number δ. We extend f as zero on the complement of G and denote the extension in $L^1(\mathbb{R}^n)$ also by f. Define for each $\varepsilon > 0$ the mollified function

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)\,dy \ , \qquad x \in \mathbb{R}^n \ . \tag{1.1}$$

Lemma 1.1 *For each $\varepsilon > 0$, $\mathrm{supp}(f_\varepsilon) \subset \mathrm{supp}(f) + \{y : |y| \leq \varepsilon\}$ and $f_\varepsilon \in C^\infty(\mathbb{R}^n)$.*

Proof: The second result follows from Leibnitz' rule and the representation

$$f_\varepsilon(x) = \int f(s)\varphi_\varepsilon(x-s)\,ds\ .$$

The first follows from the observation that $f_\varepsilon(x) \neq 0$ only if $x \in \mathrm{supp}(f) + \{y : |y| \leq \varepsilon\}$. Since $\mathrm{supp}(f)$ is closed and $\{y : |y| \leq \varepsilon\}$ is compact, it follows that the indicated set sum is closed and, hence, contains $\mathrm{supp}(f_\varepsilon)$.

Lemma 1.2 *If $f \in C_0(G)$, then $f_\varepsilon \to f$ uniformly on G. If $f \in L^p(G)$, $1 \leq p < \infty$, then $\|f_\varepsilon\|_{L^p(G)} \leq \|f\|_{L^p(G)}$ and $f_\varepsilon \to f$ in $L^p(G)$.*

Proof: The first result follows from the estimate

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &\leq \int |f(x-y) - f(x)|\varphi_\varepsilon(y)\,dy \\ &\leq \sup\{|f(x-y) - f(x)| : x \in \mathrm{supp}(f)\ ,\ |y| \leq \varepsilon\}\end{aligned}$$

and the uniform continuity of f on its support. For the case $p = 1$ we obtain

$$\|f_\varepsilon\|_{L^1(G)} \leq \int\!\!\int |f(x-y)|\varphi_\varepsilon(y)\,dy\,dx = \int \varphi_\varepsilon \cdot \int |f|$$

by Fubini's theorem, since $\int |f(x-y)|\,dx = \int |f|$ for each $y \in \mathbb{R}^n$ and this gives the desired estimate. If $p = 2$ we have for each $\psi \in C_0(G)$

$$\begin{aligned}\Big|\int f_\varepsilon(x)\psi(x)\,dx\Big| &\leq \int\!\!\int |f(x-y)\psi(x)|\,dx\ \varphi_\varepsilon(y)\,dy \\ &\leq \int \|f\|_{L^2(G)}\|\psi\|_{L^2(G)}\varphi_\varepsilon(y)\,dy = \|f\|_{L^2(G)}\|\psi\|_{L^2(G)}\end{aligned}$$

by computations similar to the above, and the result follows since $C_0(G)$ is dense in $L^2(G)$. (We shall not use the result for $p \neq 1$ or 2, but the corresponding result is proved as above but using the Hölder inequality in place of Cauchy-Schwarz.)

Finally we verify the claim of convergence in $L^p(G)$. If $\eta > 0$ we have a $g \in C_0(G)$ with $\|f - g\|_{L^p} \leq \eta/3$. The above shows $\|f_\varepsilon - g_\varepsilon\|_{L^p} \leq \eta/3$ and we obtain

$$\begin{aligned}\|f_\varepsilon - f\|_{L^p} &\leq \|f_\varepsilon - g_\varepsilon\|_{L^p} + \|g_\varepsilon - g\|_{L^p} + \|g - f\|_{L^p} \\ &\leq 2\eta/3 + \|g_\varepsilon - g\|_{L^p}\ .\end{aligned}$$

For ε sufficiently small, the support of $g_\varepsilon - g$ is bounded (uniformly) and $g_\varepsilon \to g$ uniformly, so the last term converges to zero as $\varepsilon \to 0$.

The preceding results imply the following.

Theorem 1.3 *$C_0^\infty(G)$ is dense in $L^p(G)$.*

Theorem 1.4 *For every $K \subset\subset G$ there is a $\varphi \in C_0^\infty(G)$ such that $0 \le \varphi(x) \le 1$, $x \in G$, and $\varphi(x) = 1$ for all x in some neighborhood of K.*

Proof: Let δ be the distance from K to ∂G and $0 < \varepsilon < \varepsilon + \varepsilon' < \delta$. Let $f(x) = 1$ if $\operatorname{dist}(x, K) \le \varepsilon'$ and $f(x) = 0$ otherwise. Then f_ε has its support within $\{x : \operatorname{dist}(x, K) \le \varepsilon + \varepsilon'\}$ and it equals 1 on $\{x : \operatorname{dist}(x, K) \le \varepsilon' - \varepsilon\}$, so the result follows if $\varepsilon < \varepsilon'$.

1.2

A *distribution* on G is defined to be a conjugate-linear functional on $C_0^\infty(G)$. That is, $C_0^\infty(G)^*$ is the linear space of distributions on G, and we also denote it by $\mathcal{D}^*(G)$.

Example. The space $L^1_{\text{loc}}(G) = \cap\{L^1(K) : K \subset\subset G\}$ of locally integrable functions on G can be identified with a subspace of distributions on G as in the Example of I.1.5. That is, $f \in L^1_{\text{loc}}(G)$ is assigned the distribution $T_f \in C_0^\infty(G)^*$ defined by

$$T_f(\varphi) = \int_G f\bar{\varphi} \ , \qquad \varphi \in C_0^\infty(G) \ , \tag{1.2}$$

where the Lebesgue integral (over the support of φ) is used. Theorem 1.3 shows that $T : L^1_{\text{loc}}(G) \to C_0^\infty(G)^*$ is an injection. In particular, the (equivalence classes of) functions in either of $L^1(G)$ or $L^2(G)$ will be identified with a subspace of $\mathcal{D}^*(G)$.

1.3

We shall define the derivative of a distribution in such a way that it agrees with the usual notion of derivative on those distributions which arise from continuously differentiable functions. That is, we want to define $\partial^\alpha : \mathcal{D}^*(G) \to \mathcal{D}^*(G)$ so that

$$\partial^\alpha(T_f) = T_{D^\alpha f} \quad , \quad |\alpha| \le m \quad , \quad f \in C^m(G) \ .$$

But a computation with integration-by-parts gives

$$T_{D^\alpha f}(\varphi) = (-1)^{|\alpha|} T_f(D^\alpha \varphi) \ , \qquad \varphi \in C_0^\infty(G) \ ,$$

and this identity suggests the following.

Definition. The α^{th} partial derivative of the distribution T is the distribution $\partial^\alpha T$ defined by

$$\partial^\alpha T(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi) \ , \qquad \varphi \in C_0^\infty(G) \ . \tag{1.3}$$

Since $D^\alpha \in L(C_0^\infty(G), C_0^\infty(G))$, it follows that $\partial^\alpha T$ is linear. Every distribution has derivatives of all orders and so also then does every function, e.g., in $L^1_{\text{loc}}(G)$, when it is identified as a distribution. Furthermore, by the very definition of the derivative ∂^α it is clear that ∂^α and D^α are compatible with the identification of $C^\infty(G)$ in $\mathcal{D}^*(G)$.

1.4

We give some examples of distributions on $\mathbb{R}$. Since we do not distinguish the function $f \in L^1_{\text{loc}}(\mathbb{R})$ from the functional T_f, we have the identity

$$f(\varphi) = \int_{-\infty}^{\infty} f(x)\overline{\varphi(x)}\, dx \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ .$$

(a) If $f \in C^1(\mathbb{R})$, then

$$\partial f(\varphi) = -f(D\varphi) = -\int f(D\bar{\varphi}) = \int (Df)\bar{\varphi} = Df(\varphi) \ , \tag{1.4}$$

where the third equality follows by an integration-by-parts and all others are definitions. Thus, $\partial f = Df$, which is no surprise since the definition of derivative of distributions was rigged to make this so.

(b) Let the ramp and Heaviside functions be given respectively by

$$r(x) = \begin{cases} x \ , & x > 0 \\ 0 \ , & x \le 0 \ , \end{cases} \qquad H(x) = \begin{cases} 1 \ , & x > 0 \\ 0 \ , & x < 0 \ . \end{cases}$$

Then we have

$$\partial r(\varphi) = -\int_0^\infty x D\bar{\varphi}(x)\, dx = \int_{-\infty}^\infty H(x)\bar{\varphi}(x)\, dx = H(\varphi) \ , \qquad \varphi \in C_0^\infty(G) \ ,$$

so we have $\partial r = H$, although $Dr(0)$ does not exist.

(c) The derivative of the non-continuous H is given by

$$\partial H(\varphi) = -\int_0^\infty D\bar{\varphi} = \bar{\varphi}(0) = \delta(\varphi) \ , \qquad \varphi \in C_0^\infty(G) \ ;$$

that is, $\partial H = \delta$, the Dirac functional. Also, it follows directly from the definition of derivative that

$$\partial^m \delta(\varphi) = (-1)^m \overline{(D^m \varphi)}(0) \ , \qquad m \geq 1 \ .$$

(d) Letting $A(x) = |x|$ and $I(x) = x$, $x \in \mathbb{R}$, we observe that $A = 2r - I$ and then from above obtain by linearity

$$\partial A = 2H - 1 \quad , \quad \partial^2 A = 2\delta \ . \tag{1.5}$$

Of course, these could be computed directly from definitions.

(e) For our final example, let $f : \mathbb{R} \to \mathbb{K}$ satisfy $f|_{\mathbb{R}^-} \in C^\infty(-\infty, 0]$, $f|_{\mathbb{R}^+} \in C^\infty[0, \infty)$, and denote the jump in the various derivatives at 0 by

$$\sigma_m(f) = D^m f(0^+) - D^m f(0^-) \ , \qquad m \geq 0 \ .$$

Then we obtain

$$\begin{aligned} \partial f(\varphi) &= -\int_0^\infty f\overline{(D\varphi)} - \int_{-\infty}^0 f\overline{(D\varphi)} \\ &= \int_0^\infty (Df)\bar{\varphi} + f(0^+)\overline{\varphi(0)} + \int_{-\infty}^0 (Df)\bar{\varphi} - f(0^-)\overline{\varphi(0)} \\ &= Df(\varphi) + \sigma_0(f)\delta(\varphi) \ , \qquad \varphi \in C_0^\infty(G) \ . \end{aligned} \tag{1.6}$$

That is, $\partial f = Df + \sigma_0(f)\delta$, and the derivatives of higher order can be computed from this formula, e.g.,

$$\begin{aligned} \partial^2 f &= D^2 f + \sigma_1(f)\delta + \sigma_0(f)\partial\delta \ , \\ \partial^3 f &= D^3 f + \sigma_2(f)\delta + \sigma_1(f)\partial\delta + \sigma_0(f)\partial^2\delta \ . \end{aligned}$$

For example, we have

$$\begin{aligned} \partial(H \cdot \sin) &= H \cdot \cos \ , \\ \partial(H \cdot \cos) &= -H \cdot \sin + \delta \ , \end{aligned}$$

so $H \cdot \sin$ is a solution (generalized) of the ordinary differential equation

$$(\partial^2 + 1)y = \delta \ .$$

1.5

Before discussing further the interplay between ∂ and D we remark that to claim a distribution T is "constant" on $\mathbb{R}$, means that there is a number $c \in \mathbb{K}$ such that $T = T_c$, i.e., T arises from the locally integrable function whose value everywhere is c:

$$T(\varphi) = c \int_{\mathbb{R}} \bar{\varphi} \ , \qquad \varphi \in C_0^\infty(\mathbb{R}) \ .$$

Hence a distribution is constant if and only if it depends only on the mean value of each φ. This observation is the key to the proof of our next result.

Theorem 1.5 (a) *If S is a distribution on $\mathbb{R}$, then there exists another distribution T such that $\partial T = S$.*

(b) *If T_1 and T_2 are distributions on $\mathbb{R}$ with $\partial T_1 = \partial T_2$, then $T_1 - T_2$ is constant.*

Proof: First note that $\partial T = S$ if and only if

$$T(\psi') = -S(\psi) \ , \qquad \psi \in C_0^\infty(\mathbb{R}) \ .$$

This suggests we consider $H = \{\psi' : \psi \in C_0^\infty(\mathbb{R})\}$. H is a subspace of $C_0^\infty(\mathbb{R})$. Furthermore, if $\zeta \in C_0^\infty(\mathbb{R})$, it follows that $\zeta \in H$ if and only if $\int \zeta = 0$. In that case we have $\zeta = \psi'$, where

$$\psi(x) = \int_{-\infty}^{x} \zeta \ , \qquad x \in \mathbb{R} \ .$$

Thus $H = \{\zeta \in C_0^\infty(\mathbb{R}) : \int \zeta = 0\}$ and this equality shows H is the kernel of the functional $\varphi \mapsto \int \varphi$ on $C_0^\infty(\mathbb{R})$. (This implies H is a hyperplane, but we shall prove this directly.)

Choose $\varphi_0 \in C_0^\infty(\mathbb{R})$ with mean value unity:

$$\int_{\mathbb{R}} \varphi_0 = 1 \ .$$

We shall show $C_0^\infty(\mathbb{R}) = H \oplus \mathbb{K} \cdot \varphi_0$, that is, each φ can be written in exactly one way as the sum of a $\zeta \in H$ and a constant multiple of φ_0. To check the uniqueness of such a sum, let $\zeta_1 + c_1\varphi_0 = \zeta_2 + c_2\varphi_0$ with the $\zeta_1, \zeta_2 \in H$. Integrating both sides gives $c_1 = c_2$ and, hence, $\zeta_1 = \zeta_2$. To verify the existence of such a representation, for each $\varphi \in C_0^\infty(G)$ choose $c = \int \varphi$ and define $\zeta = \varphi - c\varphi_0$. Then $\zeta \in H$ follows easily and we are done.

To finish the proof of (a), it suffices by our remark above to define T on H, for then we can extend it to all of $C_0^\infty(\mathbb{R})$ by linearity after choosing, e.g., $T\varphi_0 = 0$. But for $\zeta \in H$ we can define

$$T(\zeta) = -S(\psi) , \qquad \psi(x) = \int_{-\infty}^{x} \zeta ,$$

since $\psi \in C_0^\infty(\mathbb{R})$ when $\zeta \in H$.

Finally, (b) follows by linearity and the observation that $\partial T = 0$ if and only if T vanishes on H. But then we have

$$T(\varphi) = T(c\varphi_0 + \zeta) = T(\varphi_0)\bar{c} = T(\varphi_0) \int \bar{\varphi}$$

and this says T is the constant $T(\varphi_0) \in \mathbb{K}$.

Theorem 1.6 *If $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then $g = Df$ defines $g(x)$ for almost every $x \in \mathbb{R}$, $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, and $\partial f = g$ in $\mathcal{D}^*(\mathbb{R})$. Conversely, if T is a distribution on $\mathbb{R}$ with $\partial T \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $T(= T_f) = f$ for some absolutely continuous f, and $\partial T = Df$.*

Proof: With f and g as indicated, we have $f(x) = \int_0^x g + f(0)$. Then an integration by parts shows that

$$\int f(D\bar{\varphi}) = -\int g\bar{\varphi} , \qquad \varphi \in C_0^\infty(\mathbb{R}) ,$$

so $\partial f = g$. (This is a trivial extension of (1.4).) Conversely, let $g = \partial T \in L^1_{\mathrm{loc}}(\mathbb{R})$ and define $h(x) = \int_0^x g$, $x \in \mathbb{R}$. Then h is absolutely continuous and from the above we have $\partial h = g$. But $\partial(T - h) = 0$, so Theorem 1.5 implies that $T = h + c$ for some constant $c \in \mathbb{K}$, and we have the desired result with $f(x) = h(x) + c$, $x \in \mathbb{R}$.

1.6

Finally, we give some examples of distributions on $\mathbb{R}^n$ and their derivatives.

(a) If $f \in C^m(\mathbb{R}^n)$ and $|\alpha| \le m$, we have

$$\partial^\alpha f(\varphi) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f D^\alpha \bar{\varphi} = \int_{\mathbb{R}^n} D^\alpha f \cdot \bar{\varphi} = (D^\alpha f)(\varphi) , \qquad \varphi \in C_0^\infty(\mathbb{R}^n) .$$

(The first and last equalities follow from definitions, and the middle one is a computation.) Thus $\partial^\alpha f = D^\alpha f$ essentially because of our definition of ∂^α.

(b) Let

$$r(x) = \begin{cases} x_1 x_2 \dots x_n \ , & \text{if all } x_j \geq 0 \ , \\ 0 \ , & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \partial_1 r(\varphi) = -r(D_1\varphi) &= -\int_0^\infty \dots \int_0^\infty (x_1 \dots x_n) D_1 \varphi \, dx_1 \dots dx_n \\ &= \int_0^\infty \dots \int_0^\infty x_2 \dots x_n \, \overline{\varphi(x)} \, dx_1 \dots dx_n \ . \end{aligned}$$

Similarly,

$$\partial_2 \partial_1 r(\varphi) = \int_0^\infty \dots \int_0^\infty x_3 \dots x_n \, \overline{\varphi(x)} \, dx \ ,$$

and

$$\partial^{(1,1,\dots,1)} r(\varphi) = \int_{\mathbb{R}^n} H(x) \overline{\varphi(x)} \, dx = H(\varphi) \ ,$$

where H is the Heaviside function (= functional)

$$H(x) = \begin{cases} 1 \ , & \text{if all } x_j \geq 0 \ , \\ 0 \ , & \text{otherwise.} \end{cases}$$

(c) The derivatives of the Heaviside functional will appear as distributions given by integrals over subspaces of $\mathbb{R}^n$. In particular, we have

$$\partial_1 H(\varphi) = -\int_0^\infty \dots \int_0^\infty D_1 \overline{\varphi(x)} \, dx = \int_0^\infty \dots \int_0^\infty \bar{\varphi}(0, x_2, \dots, x_n) \, dx_2 \dots dx_n \ ,$$

a distribution whose value is determined by the restriction of φ to $\{0\} \times \mathbb{R}^{n-1}$,

$$\partial_2 \partial_1 H(\varphi) = \int_0^\infty \dots \int_0^\infty \bar{\varphi}(0, 0, x_3, \dots, x_n) \, dx_3 \dots dx_n \ ,$$

a distribution whose value is determined by the restriction of φ to $\{0\} \times \{0\} \times \mathbb{R}^{n-2}$, and, similarly,

$$\partial^{(1,1,\dots,1)} H(\varphi) = \overline{\varphi(0)} = \delta(\varphi) \ ,$$

where δ is the Dirac functional which evaluates at the origin.

(d) Let S be an $(n-1)$-dimensional C^1 manifold (cf. Section 2.3) in $\mathbb{R}^n$ and suppose $f \in C^\infty(\mathbb{R}^n \sim S)$ with f having at each point of S a limit from each side of S. For each j, $1 \leq j \leq n$, we denote by $\sigma_j(f)$ the jump in f at the surface S in the direction of increasing x_j. (Note that $\sigma_j(f)$ is then a function on S.) Then we have

$$\begin{aligned}\partial_1 f(\varphi) &= -f(D_1\varphi) = -\int_{\mathbb{R}^n} f(x) D_1\varphi(x)\,dx \\ &= \int_{\mathbb{R}^n} (D_1 f)\overline{(\varphi)}(x)\,dx + \int \cdots \int \sigma_1(f)\overline{\varphi(s)}\,dx_2 \ldots dx_n\end{aligned}$$

where $s = s(x_2, \ldots, x_n)$ is the point on S which (locally) projects onto $(0, x_2, \ldots, x_n)$. Recall that a surface integral over S is given by

$$\int_S F\,ds = \int_A F \cdot \sec(\theta_1)\,dA$$

when S projects (injectively) onto a region A in $\{0\} \times \mathbb{R}^{n-1}$ and θ_1 is the angle between the x_1-axis and the unit normal ν to S. Thus we can write the above as

$$\partial_1 f(\varphi) = D_1 f(\varphi) + \int_S \sigma_1(f)\cos(\theta_1)\bar{\varphi}\,dS\ .$$

However, in this representation it is clear that the integral is independent of the direction in which S is crossed, since both $\sigma_1(f)$ and $\cos(\theta_1)$ change sign when the direction is reversed. We need only to check that $\sigma_1(f)$ is evaluated in the same direction as the normal $\nu = (\cos(\theta_1), \cos(\theta_2), \ldots, \cos(\theta_n))$. Finally, our assumption on f shows that $\sigma_1(f) = \sigma_2(f) = \cdots = \sigma_n(f)$, and we denote this common value by $\sigma(f)$ in the formulas

$$\partial_j f(\varphi) = (D_j f)(\varphi) + \int_S \sigma(f)\cos(\theta_j)\bar{\varphi}\,dS\ .$$

These generalize the formula (1.6).

(e) Suppose G is an open, bounded and connected set in $\mathbb{R}^n$ whose boundary ∂G is a C^1 manifold of dimension $n-1$. At each point $s \in \partial G$ there is a unit *normal vector* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ whose components are direction cosines, i.e., $\nu_j = \cos(\theta_j)$, where θ_j is the angle between ν and the x_j axis. Suppose $f \in C^\infty(\bar{G})$ is given. Extend f to $\mathbb{R}^n$ by setting $f(x) = 0$ for $x \notin \bar{G}$. In $C_0^\infty(\mathbb{R}^n)^*$ we have by Green's second identity (cf. Exercise 1.6)

$$\begin{aligned}\Big(\sum_{j=1}^n \partial_j^2 f\Big)(\varphi) &= \int_G f\Big(\sum_{j=1}^n D_j^2\bar{\varphi}\Big) = \int_G \sum_{j=1}^n (D_j^2 f)\bar{\varphi} \\ &\quad + \int_{\partial G}\left(f\frac{\partial\bar{\varphi}}{\partial\nu} - \bar{\varphi}\frac{\partial f}{\partial\nu}\right)dS\ , \qquad \varphi \in C_0^\infty(\mathbb{R}^n)\ ,\end{aligned}$$

so the indicated distribution differs from the pointwise derivative by the functional

$$\varphi \mapsto \int_{\partial G} \left(f \frac{\partial \bar{\varphi}}{\partial \nu} - \bar{\varphi} \frac{\partial f}{\partial \nu} \right) dS \ ,$$

where $\frac{\partial f}{\partial \nu} = \nabla f \cdot \nu$ is the indicated (directional) *normal derivative* and $\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_n f)$ denotes the *gradient* of f. Hereafter we shall also let

$$\Delta_n = \sum_{j=1}^{n} \partial_j^2$$

denote the *Laplace* differential operator in $\mathcal{D}^*(\mathbb{R}^n)$.

2 Sobolev Spaces

2.1

Let G be an open set in $\mathbb{R}^n$ and $m \geq 0$ an integer. Recall that $C^m(\bar{G})$ is the linear space of restrictions to $\bar{G}$ of functions in $C_0^m(\mathbb{R}^n)$. On $C^m(\bar{G})$ we define a scalar product by

$$(f, g)_{H^m(G)} = \sum \left\{ \int_G D^\alpha f \cdot \overline{D^\alpha g} : |\alpha| \leq m \right\}$$

and denote the corresponding norm by $\|f\|_{H^m(G)}$.

Define $H^m(G)$ to be the completion of the linear space $C^m(\bar{G})$ with the norm $\|\cdot\|_{H^m(G)}$. $H^m(G)$ is a Hilbert space which is important for much of our following work on boundary value problems. We note that the $H^0(G)$ norm and $L^2(G)$ norm coincide on $C(\bar{G})$, and that we have the inclusions

$$C_0(G) \subset C(\bar{G}) \subset L^2(G) \ .$$

Since we have identified $L^2(G)$ as the completion of $C_0(G)$ it follows that we must likewise identify $H^0(G)$ with $L^2(G)$. Thus $f \in H^0(G)$ if and only if there is a sequence $\{f_n\}$ in $C(\bar{G})$ (or $C_0(G)$) which is Cauchy in the $L^2(G)$ norm and $f_n \to f$ in that norm.

Let $m \geq 1$ and $f \in H^m(G)$. Then there is a sequence $\{f_n\}$ in $C^m(\bar{G})$ such that $f_n \to f$ in $H^m(G)$, hence $\{D^\alpha f_n\}$ is Cauchy in $L^2(G)$ for each multi-index α of order $\leq m$. For each such α, there is a unique $g_\alpha \in L^2(G)$ such that $D^\alpha f_n \to g_\alpha$ in $L^2(G)$. As indicated above, f is the limit of f_n, so, $f = g_\theta$, $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^n$. Furthermore, if $|\alpha| \le m$ we have from an integration-by-parts

$$(D^\alpha f_n, \varphi)_{L^2(G)} = (-1)^{|\alpha|}(f_n, D^\alpha \varphi)_{L^2(G)} , \qquad \varphi \in C_0^\infty(G) .$$

Taking the limit as $n \to \infty$, we obtain

$$(g_\alpha, \varphi)_{L^2(G)} = (-1)^{|\alpha|}(f, D^\alpha \varphi)_{L^2(G)} , \qquad \varphi \in C_0^\infty(G) ,$$

so $g_\alpha = \partial^\alpha f$. That is, each $g_\alpha \in L^2(G)$ is uniquely determined as the α^{th} partial derivative of f in the sense of distribution on G. These remarks prove the following characterization.

Theorem 2.1 *Let G be open in $\mathbb{R}^n$ and $m \ge 0$. Then $f \in H^m(G)$ if and only if there is a sequence $\{f_n\}$ in $C^m(\bar{G})$ such that, for each α with $|\alpha| \le m$, the sequence $\{D^\alpha f_n\}$ is $L^2(G)$-Cauchy and $f_n \to f$ in $L^2(G)$. In that case we have $D^\alpha f_n \to \partial^\alpha f$ in $L^2(G)$.*

Corollary *$H^m(G) \subset H^k(G) \subset L^2(G)$ when $m \ge k \ge 0$, and if $f \in H^m(G)$ then $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \le m$.*

We shall later find that $f \in H^m(G)$ if $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \le m$ (cf. Section 5.1).

2.2

We define $H_0^m(G)$ to be the closure in $H^m(G)$ of $C_0^\infty(G)$. Generally, $H_0^m(G)$ is a proper subspace of $H^m(G)$. Note that for any $f \in H^m(G)$ we have

$$(\partial^\alpha f, \varphi)_{L^2(G)} = (-1)^{|\alpha|}(f, D^\alpha \varphi)_{L^2(G)} , \qquad |\alpha| \le m \ , \ \varphi \in C_0^\infty(G) .$$

We can extend this result by continuity to obtain the generalized integration-by-parts formula

$$(\partial^\alpha f, g)_{L^2(G)} = (-1)^{|\alpha|}(f, \partial^\alpha g)_{L^2(G)} , \quad f \in H^m(G) \ , \ g \in H_0^m(G) \ , \ |\alpha| \le m .$$

This formula suggests that $H_0^m(G)$ consists of functions in $H^m(G)$ which vanish on ∂G together with their derivatives through order $m-1$. We shall make this precise in the following (cf. Theorem 3.4).

Since $C_0^\infty(G)$ is dense in $H_0^m(G)$, each element of $H_0^m(G)'$ determines (by restriction to $C_0^\infty(G)$) a distribution on G and this correspondence is an injection. Thus we can identify $H_0^m(G)'$ with a space of distributions on G, and those distributions are characterized as follows.

Theorem 2.2 *$H_0^m(G)'$ is (identified with) the space of distributions on G which are the linear span of the set*

$$\{\partial^\alpha f : |\alpha| \le m\ ,\ f \in L^2(G)\}\ .$$

Proof: If $f \in L^2(G)$ and $|\alpha| \le m$, then

$$|\partial^\alpha f(\varphi)| \le \|f\|_{L^2(G)}\|\varphi\|_{H_0^m(G)}\ , \qquad \varphi \in C_0^\infty(G)\ ,$$

so $\partial^\alpha f$ has a (unique) continuous extension to $H_0^m(G)$. Conversely, if $T \in H_0^m(G)'$, there is an $h \in H_0^m(G)$ such that

$$T(\varphi) = (h, \varphi)_{H^m(G)}\ , \qquad \varphi \in C_0^\infty(G)\ .$$

But this implies $T = \sum_{|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha(\partial^\alpha h)$ and, hence, the desired result, since each $\partial^\alpha h \in L^2(G)$.

We shall have occasion to use the two following results, each of which suggests further that $H_0^m(G)$ is distinguished from $H^m(G)$ by boundary values.

Theorem 2.3 $H_0^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)$. *(Note that the boundary of $\mathbb{R}^n$ is empty.)*

Proof: Let $\tau \in C_0^\infty(\mathbb{R}^n)$ with $\tau(x) = 1$ when $|x| \le 1$, $\tau(x) = 0$ when $|x| \ge 2$, and $0 \le \tau(x) \le 1$ for all $x \in \mathbb{R}^n$. For each integer $k \ge 1$, define $\tau_k(x) = \tau(x/k)$, $x \in \mathbb{R}^n$. Then for any $u \in H^m(\mathbb{R}^n)$ we have $\tau_k \cdot u \in H^m(\mathbb{R}^n)$ and (exercise) $\tau_k \cdot u \to u$ in $H^m(\mathbb{R}^n)$ as $k \to \infty$. Thus we may assume u has compact support. Letting G denote a sphere in $\mathbb{R}^n$ which contains the support of u, we have from Lemma 1.2 of Section 1.1 that the mollified functions $u_\varepsilon \to u$ in $L^2(G)$ and that $(D^\alpha u)_\varepsilon = D^\alpha(u_\varepsilon) \to \partial^\alpha u$ in $L^2(G)$ for each α with $|\alpha| \le m$. That is, $u_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ and $u_\varepsilon \to u$ in $H^m(\mathbb{R}^n)$.

Theorem 2.4 *Suppose G is an open set in $\mathbb{R}^n$ with $\sup\{|x_1| : (x_1, x_2, \ldots, x_n) \in G\} = K < \infty$. Then*

$$\|\varphi\|_{L^2(G)} \le 2K\|\partial_1\varphi\|_{L^2(G)}\ , \qquad \varphi \in H_0^1(G)\ .$$

Proof: We may assume $\varphi \in C_0^\infty(G)$, since this set is dense in $H_0^1(G)$. Then integrating the identity

$$D_1(x_1 \cdot |\varphi(x)|^2) = |\varphi(x)|^2 + x_1 \cdot D_1(|\varphi(x)|^2)$$

over G by the divergence theorem gives

$$\int_G |\varphi(x)|^2 = -\int_G x_1(D_1\varphi(x) \cdot \bar{\varphi}(x) + \varphi(x) \cdot D_1\bar{\varphi}(x))\, dx\ .$$

The right side is bounded by $2K\|D_1\varphi\|_{L^2(G)}\|\varphi\|_{L^2(G)}$, and this gives the result.

2.3

We describe a technique by which certain properties of $H^m(G)$ can be deduced from the corresponding property for $H_0^m(G)$ or $H^m(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$ has a considerably simpler boundary. This technique is appropriate when, e.g., G is open and bounded in $\mathbb{R}^n$ and lies (locally) on one side of its boundary ∂G which we assume is a C^m-*manifold* of dimension $n-1$. Letting $Q = \{y \in \mathbb{R}^n : |y_j| \le 1,\ 1 \le j \le n\}$, $Q_0 = \{y \in Q : y_n = 0\}$, and $Q_+ = \{y \in Q : y_n > 0\}$, we can formulate this last condition as follows:

There is a collection $\{G_j : 1 \le j \le N\}$ of open bounded sets in $\mathbb{R}^n$ for which $\partial G \subset \cup\{G_j : 1 \le j \le N\}$ and a corresponding collection of functions $\varphi_j \in C^m(Q, G_j)$ with positive Jacobian $J(\varphi_j)$, $1 \le j \le N$, and φ_j is a bijection of Q, Q_+ and Q_0 onto G_j, $G_j \cap G$, and $G_j \cap \partial G$, respectively. For each j, the pair (φ_j, G_j) is a *coordinate patch* for the boundary.

Given the collection $\{(\varphi_j, G_j) : 1 \le j \le N\}$ of coordinate patches as above, we construct a corresponding collection of open sets F_j in $\mathbb{R}^n$ for which each $\bar{F}_j \subset G_j$ and $\cup\{F_j : 1 \le j \le N\} \supset \partial G$. Define $G_0 = G$ and $F_0 = G \sim \cup\{\bar{F}_j : 1 \le j \le N\}$, so $\bar{F}_0 \subset G_0$. Note also that $\bar{G} \subset G \cup \bigcup\{F_j : 1 \le j \le N\}$ and $G \subset \cup\{\bar{F}_j : 0 \le j \le N\}$. For each j, $0 \le j \le N$, let $\alpha_j \in C_0^\infty(\mathbb{R}^n)$ be chosen so that $0 \le \alpha_j(x) \le 1$ for all $x \in \mathbb{R}^n$, $\operatorname{supp}(\alpha_j) \subset G_j$, and $\alpha_j(x) = 1$ for $x \in \bar{F}_j$. Let $\alpha \in C_0^\infty(\mathbb{R}^n)$ be chosen with $0 \le \alpha(x) \le 1$ for all $x \in \mathbb{R}^n$, $\operatorname{supp}(\alpha) \subset G \cup \bigcup\{F_j : 1 \le j \le N\}$, and $\alpha(x) = 1$ for $x \in \bar{G}$. Finally, for each j, $0 \le j \le N$, we define $\beta_j(x) = \alpha_j(x)\alpha(x)/\sum_{k=0}^N \alpha_k(x)$ for $x \in \cup\{\bar{F}_j : 0 \le j \le N\}$ and $\beta_j(x) = 0$ for $x \in \mathbb{R}^n \sim \cup\{\bar{F}_j : 1 \le j \le N\}$. Then we have $\beta_j \in C_0^\infty(\mathbb{R}^n)$, β_j has support in G_j, $\beta_j(x) \ge 0$, $x \in \mathbb{R}^n$ and

$\sum\{\beta_j(x) : 0 \le j \le N\} = 1$ for each $x \in \bar{G}$. That is, $\{\beta_j : 0 \le j \le N\}$ is a *partition-of-unity* subordinate to the open cover $\{G_j : 0 \le j \le N\}$ of $\bar{G}$ and $\{\beta_j : 1 \le j \le N\}$ is a partition-of-unity subordinate to the open cover $\{G_j : 1 \le j \le N\}$ of ∂G.

Suppose we are given a $u \in H^m(G)$. Then we have $u = \sum_{j=0}^{N}\{\beta_j u\}$ on G and we can show that each pointwise product $\beta_j u$ is in $H^m(G \cap G_j)$ with support in G_j. This defines a function $H^m(G) \to H_0^m(G) \times \prod\{H^m(G \cap G_j) : 1 \le j \le N\}$, where $u \mapsto (\beta_0 u, \beta_1 u, \ldots, \beta_N u)$. This function is clearly linear, and from $\sum \beta_j = 1$ it follows that it is an injection. Also, since each $\beta_j u$ belongs to $H^m(G \cap G_j)$ with support in G_j for each $1 \le j \le N$, it follows that the composite function $(\beta_j u) \circ \varphi_j$ belongs to $H^m(Q^+)$ with support in Q. Thus, we have defined a linear injection

$$\begin{aligned} \Lambda : H^m(G) &\longrightarrow H_0^m(G) \times [H^m(Q^+)]^N \ , \\ u &\longmapsto (\beta_0 u, (\beta_1 u) \circ \varphi_1, \ldots, (\beta_N u) \circ \varphi_N) \ . \end{aligned}$$

Moreover, we can show that the product norm on Λu is equivalent to the norm of u in $H^m(G)$, so Λ is a continuous linear injection of $H^m(G)$ onto a closed subspace of the indicated product, and its inverse in continuous.

In a similar manner we can localize the discussion of functions on the boundary. In particular, $C^m(\partial G)$, the space of m times continuously differentiable functions on ∂G, is the set of all functions $f : \partial G \to \mathbb{R}$ such that $(\beta_j f) \circ \varphi_j \in C^m(Q_0)$ for each j, $1 \le j \le N$. The manifold ∂G has an intrinsic measure denoted by "ds" for which integrals are given by

$$\int_{\partial G} f\, ds = \sum_{j=1}^{N} \int_{\partial G \cap G_j} (\beta_j f)\, ds = \sum_{j=1}^{N} \int_{Q_0} (\beta_j f) \circ \varphi_j(y') J(\varphi_j)\, dy' \ ,$$

where $J(\varphi_j)$ is the indicated Jacobian and dy' denotes the usual (Lebesgue) measure on $Q_0 \subset \mathbb{R}^{n-1}$. Thus, we obtain a norm on $C(\partial G) = C^0(\partial G)$ given by $\|f\|_{L^2(\partial G)} = (\int_{\partial G} |f|^2\, ds)^{1/2}$, and the completion is the Hilbert space $L^2(\partial G)$ with the obvious scalar-product. We have a linear injection

$$\begin{aligned} \lambda : L^2(\partial G) &\longrightarrow [L^2(Q_0)]^N \\ f &\longmapsto ((\beta_1 f) \circ \varphi_1, \ldots, (\beta_N f) \circ \varphi_N) \end{aligned}$$

onto a closed subspace of the product, and both λ and its inverse are continuous.

3 Trace

We shall describe the sense in which functions in $H^m(G)$ have "boundary values" on ∂G when $m \geq 1$. Note that this is impossible in $L^2(G)$ since ∂G is a set of measure zero in $\mathbb{R}^n$. First, we consider the situation where G is the half-space $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$, for then $\partial G = \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$ is the simplest possible (without being trivial). Also, the general case can be localized as in Section 2.3 to this case, and we shall use this in our final discussion of this section.

3.1

We shall define the (first) trace operator γ_0 when $G = \mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, where we let x' denote the $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1})$. For any $\varphi \in C^1(\bar{G})$ and $x' \in \mathbb{R}^{n-1}$ we have

$$|\varphi(x', 0)|^2 = -\int_0^\infty D_n(|\varphi(x', x_n)|^2)\, dx_n \ .$$

Integrating this identity over $\mathbb{R}^{n-1}$ gives

$$\begin{aligned} \|\varphi(\cdot, 0)\|^2_{L^2(\mathbb{R}^{n-1})} &\leq \int_{\mathbb{R}^n_+} [(D_n\varphi \cdot \bar{\varphi} + \varphi \cdot D_n\bar{\varphi}_n\,)]\, dx \\ &\leq 2\|D_n\varphi\|_{L^2(\mathbb{R}^n_+)}\|\varphi\|_{L^2(\mathbb{R}^n_+)} \ . \end{aligned}$$

The inequality $2ab \leq a^2 + b^2$ then gives us the estimate

$$\|\varphi(\cdot, 0)\|^2_{L^2(\mathbb{R}^{n-1})} \leq \|\varphi\|^2_{L^2(\mathbb{R}^n_+)} + \|D_n\varphi\|^2_{L^2(\mathbb{R}^n_+)} \ .$$

Since $C^1\overline{(\mathbb{R}^n_+)}$ is dense in $H^1(\mathbb{R}^n_+)$, we have proved the essential part of the following result.

Theorem 3.1 *The* trace *function* $\gamma_0 : C^1(\bar{G}) \to C^0(\partial G)$ *defined by*

$$\gamma_0(\varphi)(x') = \varphi(x', 0) \ , \qquad \varphi \in C^1(\bar{G}) \ , \ x' \in \partial G \ ,$$

(where $G = \mathbb{R}^n_+$*) has a unique extension to a continuous linear operator* $\gamma_0 \in \mathcal{L}(H^1(G), L^2(\partial G))$ *whose range is dense in* $L^2(\partial G)$*, and it satisfies*

$$\gamma_0(\beta \cdot u) = \gamma_0(\beta) \cdot \gamma_0(u) \ , \qquad \beta \in C^1(\bar{G}) \ , \ u \in H^1(G) \ .$$

Proof: The first part follows from the preceding inequality and Theorem I.3.1. If $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ and τ is the truncation function defined in the proof of Theorem 2.3, then

$$\varphi(x) = \psi(x')\tau(x_n) \ , \qquad x = (x', x_n) \in \mathbb{R}^n_+$$

defines $\varphi \in C^1(\bar{G})$ and $\gamma_0(\varphi) = \psi$. Thus the range of γ_0 contains $C_0^\infty(\mathbb{R}^{n-1})$. The last identity follows by the continuity of γ_0 and the observation that it holds for $u \in C^1(\bar{G})$.

Theorem 3.2 *Let $u \in H^1(\mathbb{R}^n_+)$. Then $u \in H_0^1(\mathbb{R}^n_+)$ if and only if $\gamma_0(u) = 0$.*

Proof: If $\{u_n\}$ is a sequence in $C_0^\infty(\mathbb{R}^n_+)$ converging to u in $H^1(\mathbb{R}^n_+)$, then $\gamma_0(u) = \lim \gamma_0(u_n) = 0$ by Theorem 3.1.

Let $u \in H^1(\mathbb{R}^n_+)$ with $\gamma_0 u = 0$. If $\{\tau_j : j \geq 1\}$ denotes the sequence of truncating functions defined in the proof of Theorem 2.3, then $\tau_j u \to u$ in $H^1(\mathbb{R}^n_+)$ and we have $\gamma_0(\tau_j u) = \gamma_0(\tau_j)\gamma_0(u) = 0$, so we may assume that u has compact support in $\mathbb{R}^n$.

Let $\theta_j \in C^1(\mathbb{R}_+)$ be chosen such that $\theta_j(s) = 0$ if $0 < s \leq 1/j$, $\theta_j(s) = 1$ if $s \geq 2/j$, and $0 \leq \theta_j'(s) \leq 2j$ if $(1/j) \leq s \leq (2/j)$. Then the extension of $x \mapsto \theta_j(x_n)u(x', x_n)$ to all of $\mathbb{R}^n$ as 0 on $\mathbb{R}^n_-$ is a function in $H^1(\mathbb{R}^n)$ with support in $\{x : x_n \geq 1/j\}$, and (the proof of) Theorem 2.3 shows we may approximate such a function from $C_0^\infty(\mathbb{R}^n_+)$. Hence, we need only to show that $\theta_j u \to u$ in $H^1(\mathbb{R}^n_+)$.

It is an easy consequence of the Lebesgue dominated convergence theorem that $\theta_j u \to u$ in $L^2(\mathbb{R}^n_+)$ and for each k, $1 \leq k \leq n-1$, that $\partial_k(\theta_j u) = \theta_j(\partial_k u) \to \partial_k u$ in $L^2(\mathbb{R}^n_+)$ as $j \to \infty$. Similarly, $\theta_j(\partial_n u) \to \partial_n u$ and we have $\partial_n(\theta_j u) = \theta_j(\partial_n u) + \theta_j' u$, so we need only to show that $\theta_j' u \to 0$ in $L^2(\mathbb{R}^n_+)$ as $j \to \infty$.

Since $\gamma_0(u) = 0$ we have $u(x', s) = \int_0^s \partial_n u(x', t)\, dt$ for $x' \in \mathbb{R}^{n-1}$ and $s \geq 0$. From this follows the estimate

$$|u(x', s)|^2 \leq s \int_0^s |\partial_n u(x', t)|^2 \, dt \ .$$

Thus, we obtain for each $x' \in \mathbb{R}^{n-1}$

$$\begin{aligned} \int_0^\infty |\theta_j'(s) u(x', s)|^2 \, ds &\leq \int_0^{2/j} (2j)^2 s \int_0^s |\partial_n u(x', t)|^2 \, dt \, ds \\ &\leq 8j \int_0^{2/j} \int_0^s |\partial_n u(x', t)|^2 \, dt \, ds \ . \end{aligned}$$

Interchanging the order of integration gives

$$\begin{aligned}\int_0^\infty |\theta_j'(s)u(x',s)|^2\,ds &\le 8j\int_0^{2/j}\int_t^{2/j}|\partial_n u(x',t)|^2\,ds\,dt\\ &\le 16\int_0^{2/j}|\partial_n u(x',t)|^2\,dt\ .\end{aligned}$$

Integration of this inequality over $\mathbb{R}^{n-1}$ gives us

$$\|\theta_j' u\|^2_{L^2(\mathbb{R}^n_+)} \le 16\int_{\mathbb{R}^{n-1}\times[0,2/j]}|\partial_n u|^2\,dx$$

and this last term converges to zero as $j\to\infty$ since $\partial_n u$ is square-summable.

3.2

We can extend the preceding results to the case where G is a sufficiently smooth region in $\mathbb{R}^n$. Suppose G is given as in Section 2.3 and denote by $\{G_j : 0\le j\le N\}$, $\{\varphi_j : 1\le j\le N\}$, and $\{\beta_j : 0\le j\le N\}$ the open cover, corresponding local maps, and the partition-of-unity, respectively. Recalling the linear injections Λ and λ constructed in Section 2.3, we are led to consider function $\gamma_0 : H^1(G)\to L^2(\partial G)$ defined by

$$\begin{aligned}\gamma_0(u) &= \sum_{j=1}^N \Big(\gamma_0((\beta_j u)\circ\varphi_j)\Big)\circ\varphi_j^{-1}\\ &= \sum_{j=1}^N \gamma_0(\beta_j)\cdot(\gamma_0(u\circ\varphi_j)\varphi_j^{-1})\end{aligned}$$

where the equality follows from Theorem 3.1. This formula is precisely what is necessary in order that the following diagram commutes.

$$\begin{array}{ccc} H^1(G) & \xrightarrow{\Lambda} & H^1_0(G)\times H^1(Q^+)\ \times\cdots\times\ H^1(Q^+)\\ \big\downarrow\gamma_0 & & \big\downarrow \qquad\qquad\qquad \big\downarrow\gamma_0\\ L^2(\partial G) & \xrightarrow{\lambda} & L^2(Q^0)\ \times\cdots\times\ L^2(Q^0)\end{array}$$

Also, if $u\in C^1(\bar G)$ we see that $\gamma_0(u)$ is the restriction of u to ∂G. These remarks and Theorems 3.1 and 3.2 provide a proof of the following result.

Theorem 3.3 *Let G be a bounded open set in $\mathbb{R}^n$ which lies on one side of its boundary, ∂G, which we assume is a C^1-manifold. Then there exists a unique continuous and linear function $\gamma_0 : H^1(G) \to L^2(\partial G)$ such that for each $u \in C^1(\bar{G})$, $\gamma_0(u)$ is the restriction of u to ∂G. The kernel of γ_0 is $H_0^1(G)$ and its range is dense in $L^2(\partial G)$.*

This result is a special case of the trace theorem which we briefly discuss. For a function $u \in C^m(\bar{G})$ we define the various traces of normal derivatives given by

$$\gamma_j(u) = \left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial G} , \qquad 0 \le j \le m-1 .$$

Here ν denotes the unit outward normal on the boundary of G. When $G = \mathbb{R}^n_+$ (or G is localized as above), these are given by $\partial u/\partial \nu = -\partial_n u|_{x_n=0}$. Each γ_j can be extended by continuity to all of $H^m(G)$ and we obtain the following.

Theorem 3.4 *Let G be an open bounded set in $\mathbb{R}^n$ which lies on one side of its boundary, ∂G, which we assume is a C^m-manifold. Then there is a unique continuous linear function γ from $H^m(G)$ into $\prod_{j=0}^{m-1} H^{m-1-j}(\partial G)$ such that*

$$\gamma(u) = (\gamma_0 u, \gamma_1 u, \ldots, \gamma_{m-1}(u)) , \qquad u \in C^m(\bar{G}) .$$

The kernel of γ is $H_0^m(G)$ and its range is dense in the indicated product.

The Sobolev spaces over ∂G which appear in Theorem 3.4 can be defined locally. The range of the trace operator can be characterized by Sobolev spaces of fractional order and then one obtains a space of boundary values which is isomorphic to the quotient space $H^m(G)/H_0^m(G)$. Such characterizations are particularly useful when considering non-homogeneous boundary value problems and certain applications, but the preceding results will be sufficient for our needs.

4 Sobolev's Lemma and Imbedding

We obtained the spaces $H^m(G)$ by completing a class of functions with continuous derivatives. Our objective here is to show that each element of $H^m(G)$ is (represented by) a function with continuous derivatives up to a certain order which depends on m.

Let G be bounded and open in $\mathbb{R}^n$. We say G satisfies a *cone condition* if there is a $\rho > 0$ and $\gamma > 0$ such that each point $y \in \bar{G}$ is the vertex of a cone $K(y)$ of radius ρ and volume $\gamma\rho^n$ with $K(y) \subset \bar{G}$. Thus, γ is a measure of the angle of the cone. To be precise, a ball of radius ρ has volume $\omega_n\rho^n/n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, and the *angle* of the cone $K(y)$ is the ratio of these volumes given by $\gamma n/\omega_n$.

We shall derive an estimate on the value of a smooth function at a point $y \in \bar{G}$ in terms of the norm of $H^m(G)$ for some $m \geq 0$. Let $g \in C_0^\infty(\mathbb{R})$ satisfy $g \geq 0$, $g(t) = 1$ for $|t| \leq 1/2$, and $g(t) = 0$ for $|t| \geq 1$. Define $\tau(t) = g(t/\rho)$ and note that there are constants $A_k > 0$ such that

$$\left|\frac{d^k}{dt^k}\,\tau(t)\right| \leq \frac{A_k}{\rho^k}\ , \qquad \rho > 0\ . \tag{4.1}$$

Let $u \in C^m(\bar{G})$ and assume $2m > n$. If $y \in \bar{G}$ and $K(y)$ is the indicated cone, we integrate along these points $x \in K(y)$ on a given ray from the vertex y and obtain

$$\int_0^\rho D_r(\tau(r)u(x))\,dr = -u(y)\ ,$$

where $r = |x - y|$ for each such x. Thus, we obtain an integral over $K(y)$ in spherical coordinates given by

$$\int_\Omega \int_0^\rho D_r(\tau(r)u(x))\,dr\,d\omega = -u(y)\int_\Omega d\omega = -u(y)\gamma n/\omega_n$$

where ω is spherical angle and $\Omega = \gamma n/\omega_n$ is the total angle of the cone $K(y)$. We integrate by parts $m-1$ times and thereby obtain

$$u(y) = \frac{(-1)^m\omega_n}{\gamma n(m-1)!}\int_\Omega\int_0^\rho D_r^m(\tau u)r^{m-1}\,dr\,d\omega\ .$$

Changing this to Euclidean coordinates with volume element $dx = r^{n-1}\,dr\,d\omega$ gives

$$|u(y)| = \frac{\omega_n}{\gamma n(m-1)!}\int_{K(y)} D_r^m(\tau u)r^{m-n}\,dx\ .$$

The Cauchy-Schwartz inequality gives the estimate

$$|u(y)|^2 \leq \left(\frac{\omega_n}{\gamma n(m-1)!}\right)^2 \int_{K(y)} |D_r^m(\tau u)|^2\,dx\ \int_{K(y)} r^{2(m-n)}\,dx\ ,$$

and we use spherical coordinates to evaluate the last term as follows:

$$\int_{K(y)} r^{2(m-n)}\, dx = \int_{\Omega}\int_0^{\rho} r^{2m-n-1}\, dr\, d\omega = \frac{\gamma n \rho^{2m-n}}{\omega_n(2m-n)} \ .$$

Thus we have

$$|u(y)|^2 \le C_{(m,n)}\rho^{2m-n}\int_{K(y)} |D_r^m(\tau u)|^2\, dx \tag{4.2}$$

where $C_{(m,n)}$ is a constant depending only on m and n. From the estimate (4.1) and the formulas for derivatives of a product we obtain

$$\begin{aligned} |D_r^m(\tau u)| &= \Big|\sum_{k=0}^{m}\binom{n}{k} D_r^{m-k}\tau\cdot D_r^k u\Big| \\ &\le \sum_{k=0}^{m}\binom{n}{k}\frac{A_{m-k}}{\rho^{m-k}}|D_r^k u| \ , \end{aligned}$$

hence,

$$|D_r^m(\tau u)|^2 \le C'\sum_{k=0}^{m}\frac{1}{\rho^{2(m-k)}}|D_r^k u|^2 \ .$$

This gives with (4.2) the estimate

$$|u(y)|^2 \le C(m,n)C'\sum_{k=0}^{m}\rho^{2k-n}\int_{K(y)} |D_r^k u|^2\, dx \ . \tag{4.3}$$

By the chain rule we have

$$|D_r^k u|^2 \le C''\sum_{|\alpha|\le k}|D^\alpha u(x)|^2 \ ,$$

so by extending the integral in (4.3) to all of G we obtain

$$\sup_{y\in G}|u(y)| \le C\|u\|_m \ . \tag{4.4}$$

This proves the following.

Theorem 4.1 *Let G be a bounded open set in $\mathbb{R}^n$ and assume G satisfies the cone condition. Then for every $u \in C^m(\bar{G})$ with $m > n/2$ the estimate (4.4) holds.*

The inequality (4.4) gives us an imbedding theorem. We let $C_u(G)$ denote the linear space of all uniformly continuous functions on G. Then

$$\|u\|_{\infty,0} \equiv \sup\{|u(x)| : x \in G\}$$

is a norm on $C_u(G)$ for which it is a Banach space, i.e., complete. Similarly,

$$\|u\|_{\infty,k} \equiv \sup\{|D^\alpha u(x)| : x \in G\ ,\ |\alpha| \le k\}$$

is a norm on the linear space $C_u^k(G) = \{u \in C_u(G) : D^\alpha \in C_u(G)$ for $|\alpha| \le k\}$ and the resulting normed linear space is complete.

Theorem 4.2 *Let G be a bounded open set in $\mathbb{R}^n$ and assume G satisfies the cone condition. Then $H^m(G) \subset C_u^k(G)$ where m and k are integers with $m > k + n/2$. That is, each $u \in H^m(G)$ is equal a.e. to a unique function in $C_u^k(G)$ and this identification is continuous.*

Proof: By applying (4.4) to $D^\alpha u$ for $|\alpha| \le k$ we obtain

$$\|u\|_{\infty,k} \le C\|u\|_m\ , \qquad u \in C^m(\bar{G})\ . \tag{4.5}$$

Thus, the identity is continuous from the dense subset $C^m(\bar{G})$ of $H^m(G)$ into the Banach space $C_u^k(G)$. The desired result follows from Theorem I.3.1 and the identification of $H^m(G)$ in $L^2(G)$ (cf. Theorem 2.1).

5 Density and Compactness

The complementary results on Sobolev spaces that we obtain below will be used in later sections. We first show that if $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \le m$, and if ∂G is sufficiently smooth, then $f \in H^m(G)$. The second result is that the injection $H^{m+1}(G) \to H^m(G)$ is a compact mapping.

5.1

We first consider the set $\mathcal{H}^m(G)$ of all $f \in L^2(G)$ for which $\partial^\alpha f \in L^2(G)$ for all α with $|\alpha| \le m$. It follows easily that $\mathcal{H}^m(G)$ is a Hilbert space with the scalar product and norm as defined on $H^m(G)$ and that $H^m(G) \le \mathcal{H}^m(G)$. Our plan is to show equality holds when G has a smooth boundary. The case of empty ∂G is easy.

Lemma 5.1 *$C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{H}^m(\mathbb{R}^n)$.*

The proof of this is similar to that of Theorem 2.3 and we leave it as an exercise. Next we obtain our desired result for the case of ∂G being a hyperplane.

Lemma 5.2 $H^m(\mathbb{R}^n_+) = \mathcal{H}^m(\mathbb{R}^n_+)$.

Proof: We need to show each $u \in \mathcal{H}^m(\mathbb{R}^n_+)$ can be approximated from $C^m\overline{(\mathbb{R}^n_+)}$. Let $\varepsilon > 0$ and define $u_\varepsilon(x) = u(x', x_n + \varepsilon)$ for $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n > -\varepsilon$. We have $u_\varepsilon \to u$ in $\mathcal{H}^m(\mathbb{R}^n_+)$ as $\varepsilon \to 0$, so it suffices to show $u_\varepsilon \in H^m(\mathbb{R}^n_+)$. Let $\theta \in C^\infty(\mathbb{R})$ be monotone with $\theta(x) = 0$ for $x \le -\varepsilon$ and $\theta(x) = 1$ for $x > 0$. Then the function θu_ε given by $\theta(x_n)u_\varepsilon(x)$ for $x_n > -\varepsilon$ and by 0 for $x_n \le -\varepsilon$, belongs to $\mathcal{H}^m(\mathbb{R}^n)$ and clearly $\theta u_\varepsilon = u_\varepsilon$ on $\mathbb{R}^n_+$. Now use Lemma 5.1 to obtain a sequence $\{\varphi_n\}$ from $C_0^\infty(\mathbb{R}^n)$ converging to θu_ε in $\mathcal{H}^m(\mathbb{R}^n)$. The restrictions $\{\varphi_n|_{\mathbb{R}^n_+}\}$ belong to $C^\infty\overline{(\mathbb{R}^n_+)}$ and converge to θu_ε in $\mathcal{H}^m(\mathbb{R}^n_+)$.

Lemma 5.3 *There exists an operator $\mathcal{P} \in \mathcal{L}(\mathcal{H}^m(\mathbb{R}^n_+), \mathcal{H}^m(\mathbb{R}^n))$ such that $(\mathcal{P}u)(x) = u(x)$ for a.e. $x \in \mathbb{R}^n_+$.*

Proof: By Lemma 5.2 it suffices to define such a $\mathcal{P}$ on $C^m\overline{(\mathbb{R}^n_+)}$. Let the numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the solution of the system

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_m = 1 \\ -(\lambda_1 + \lambda_2/2 + \cdots + \lambda_m/m) = 1 \\ \quad \cdots\cdots \\ (-1)^{m-1}(\lambda_1 + \lambda_2/2^{m-1} + \cdots + \lambda_m/m^{m-1}) = 1 \end{cases} \quad . \tag{5.1}$$

For each $u \in C^m\overline{(\mathbb{R}^n_+)}$ we define

$$\mathcal{P}u(x) = \begin{cases} u(x)\ , & x_n \ge 0 \\ \lambda_1 u(x', -x_n) + \lambda_2 u\left(x', -\dfrac{x_n}{2}\right) + \cdots + \lambda_m u\left(x', -\dfrac{x_n}{m}\right)\ , & x_n < 0\ . \end{cases}$$

The equations (5.1) are precisely the conditions that $\partial_n^j(\mathcal{P}u)$ is continuous at $x_n = 0$ for $j = 0, 1, \ldots, m-1$. From this follows $\mathcal{P}u \in \mathcal{H}^m(\mathbb{R}^n)$; $\mathcal{P}$ is clearly linear and continuous.

Theorem 5.4 *Let G be a bounded open set in $\mathbb{R}^n$ which lies on one side of its boundary, ∂G, which is a C^m-manifold. Then there exists an operator $\mathcal{P}_G \in \mathcal{L}(\mathcal{H}^m(G), \mathcal{H}^m(\mathbb{R}^n))$ such that $(\mathcal{P}_G u)|_G = u$ for every $u \in \mathcal{H}^m(G)$.*

Proof: Let $\{(\varphi_k, G_k) : 1 \leq k \leq N\}$ be coordinate patches on ∂G and let $\{\beta_k : 0 \leq k \leq N\}$ be the partition-of-unity constructed in Section 2.3. Thus for each $u \in \mathcal{H}^m(G)$ we have $u = \sum_{j=0}^N (\beta_j u)$. The first term $\beta_0 u$ has a trivial extension to an element of $\mathcal{H}^m(\mathbb{R}^n)$. Let $1 \leq k \leq N$ and consider $\beta_k u$. The coordinate map $\varphi_k : Q \to G_k$ induces an isomorphism $\varphi_k^* : \mathcal{H}^m(G_k \cap G) \to \mathcal{H}^m(Q_+)$ by $\varphi_k^*(v) = v \circ \varphi_k$. The support of $\varphi_k^*(\beta_k u)$ is inside Q so we can extend it as zero in $\mathbb{R}^n_+ \sim Q$ to obtain an element of $\mathcal{H}^m(\mathbb{R}^n_+)$. By Lemma 5.3 this can be extended to an element $\mathcal{P}(\varphi_k^*(\beta_k u))$ of $\mathcal{H}^m(\mathbb{R}^n)$ with support in Q. (Check the proof of Lemma 5.3 for this last claim.) The desired extension of $\beta_k u$ is given by $\mathcal{P}(\varphi_k^*(\beta_k u)) \circ \varphi_k^{-1}$ extended as zero off of G_k. Thus we have the desired operator given by

$$\mathcal{P}_G u = \beta_0 u + \sum_{k=1}^N (\mathcal{P}(\beta_k u) \circ \varphi_k) \circ \varphi_i^{-1}$$

where each term is extended as zero as indicated above.

Theorem 5.5 *Let G be given as in Theorem 5.4. Then $H^m(G) = \mathcal{H}^m(G)$.*

Proof: Let $u \in \mathcal{H}^m(G)$. Then $\mathcal{P}_G u \in \mathcal{H}^m(\mathbb{R}^n)$ and Lemma 5.1 gives a sequence $\{\varphi_n\}$ in $C_0^\infty(\mathbb{R}^n)$ which converges to $\mathcal{P}_G u$. Thus, $\{\varphi_n|_G\}$ converges to u in $\mathcal{H}^m(G)$.

5.2

We recall from Section I.7 that a linear function T from one Hilbert space to another is called *compact* if it is continuous and if the image of any bounded set contains a convergent sequence. The following results will be used in Section III.6 and Theorem III.7.7.

Lemma 5.6 *Let Q be a cube in $\mathbb{R}^n$ with edges of length $d > 0$. If $u \in C^1(\bar{Q})$, then*

$$\|u\|^2_{L^2(Q)} \leq d^{-n}\left(\int_Q u\right)^2 + (nd^2/2)\sum_{j=1}^n \|D_j u\|^2_{L^2(Q)} \ . \qquad (5.2)$$

Proof: For $x, y \in Q$ we have

$$u(x) - u(y) = \sum_{j=1}^{n} \int_{x_j}^{y_j} D_j u(y_1, \dots, y_{j-1}, s, x_{j+1}, \dots, x_n)\, ds \ .$$

Square this identity and use Theorem I.4.1(a) to obtain

$$u^2(x)+u^2(y)-2u(x)u(y) \le nd \sum_{j=1}^{n} \int_{a_j}^{b_j} (D_j u)^2(y_1, \dots, y_{j-1}, s, x_{j+1}, \dots, x_n)\, ds$$

where $Q = \{x : a_j \le x_j \le b_j\}$ and $b_k - a_k = d$ for each $k = 1, 2, \dots, n$. Integrate the preceding inequality with respect to $x_1, \dots, x_n$, $y_1, \dots, y_n$, and we have

$$2d^n \|u\|^2_{L^2(Q)} \le 2\Big(\int_Q u\Big)^2 + nd^{n+2} \sum_{j=1}^{n} \|D_j u\|^2_{L^2(Q)} \ .$$

The desired estimate (5.2) follows.

Theorem 5.7 *Let G be bounded in $\mathbb{R}^n$. If the sequence $\{u_k\}$ in $H_0^1(G)$ is bounded, then there is a subsequence which converges in $L^2(G)$. That is, the injection $H_0^1(G) \to L^2(G)$ is compact.*

Proof: We may assume each $u_k \in C_0^\infty(G)$; set $M = \sup\{\|u_k\|_{H_0^1}\}$. Enclose G in a cube Q; we may assume the edges of Q are of unit length. Extend each u_k as zero on $Q \sim G$, so each $u_k \in C_0^\infty(Q)$ with $\|u_k\|_{H_0^1(Q)} \le M$.

Let $\varepsilon > 0$. Choose integer N so large that $2nM^2/N^2 < \varepsilon$. Decompose Q into equal cubes Q_j, $j = 1, 2, \dots, N^n$, with edges of length $1/N$. Since $\{u_k\}$ is bounded in $L^2(Q)$, it follows from Theorem I.6.2 that there is a subsequence (denoted hereafter by $\{u_k\}$) which is weakly convergent in $L^2(Q)$. Thus, there is an integer K such that

$$\Big| \int_{Q_j} (u_k - u_\ell) \Big|^2 < \varepsilon/2N^{2n} \ , \qquad j = 1, 2, \dots, N^n \ ; \ k, \ell \ge K \ .$$

If we apply (5.2) on each Q_j with $u = u_k - u_\ell$ and sum over all j's, we obtain for $k, \ell \ge K$

$$\|u_k - u_\ell\|^2_{L^2(Q)} \le N^n \Big(\sum_{j=1}^{N^n} \varepsilon/2N^{2n} \Big) + (n/2N^2)(2M^2) < \varepsilon \ .$$

Thus, $\{u_k\}$ is a Cauchy sequence in $L^2(Q)$.

Corollary *Let G be bounded in $\mathbb{R}^n$ and let $m \geq 1$. Then the injection $H_0^m(G) \to H_0^{m-1}(G)$ is compact.*

Theorem 5.8 *Let G be given as in Theorem 5.4 and let $m \geq 1$. Then the injection $H^m(G) \to H^{m-1}(G)$ is compact.*

Proof: Let $\{u_k\}$ be bounded in $H^m(G)$. Then the sequence of extensions $\{\mathcal{P}_G(u_k)\}$ is bounded in $H^1(\mathbb{R}^n)$. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta \equiv 1$ on G and let Ω be an open bounded set in $\mathbb{R}^n$ containing the support of θ. The sequence $\{\theta \cdot \mathcal{P}_G(u_k)\}$ is bounded in $H_0^m(\Omega)$, hence, has a subsequence (denoted by $\{\theta \cdot \mathcal{P}_G(u_{k'})\}$) which is convergent in $\mathcal{H}_0^{m-1}(\Omega)$. The corresponding subsequence of restrictions to G is just $\{u_{k'}\}$ and is convergent in $H^{m-1}(G)$.

Exercises

1.1. Evaluate $(\partial - \lambda)(H(x)e^{\lambda x})$ and $(\partial^2 + \lambda^2)(\lambda^{-1}H(x)\sin(\lambda x))$ for $\lambda \neq 0$.

1.2. Find all distributions of the form $F(t) = H(t)f(t)$ where $f \in C^2(\mathbb{R})$ such that

$$(\partial^2 + 4)F = c_1\delta + c_2\partial\delta \ .$$

1.3. Let K be the square in $\mathbb{R}^2$ with corners at (1,1), (2,0), (3,1), (2,2), and let T_K be the function equal to 1 on K and 0 elsewhere. Evaluate $(\partial_1^2 - \partial_2^2)T_K$.

1.4. Obtain the results of Section 1.6(e) from those of Section 1.6(d).

1.5. Evaluate $\Delta_n(1/|x|^{n-2})$.

1.6. (a) Let G be given as in Section 1.6(e). Show that for each function $f \in C^1(\bar{G})$ the identity

$$\int_G \partial_j f(x)\,dx = \int_{\partial G} f(s)\nu_j(s)\,ds \ , \qquad 1 \leq j \leq n \ ,$$

follows from the fundamental theorem of calculus.

(b) Show that Green's first identity

$$\int_G (\nabla u \cdot \nabla v + (\Delta_n u)v)\,dx = \int_{\partial G} \frac{\partial u}{\partial v}\, v\, ds$$

follows from above for $u \in C^2(\bar{G})$ and $v \in C^1(\bar{G})$. Hint: Take $f_j = (\partial_j u)v$ and add.

(c) Obtain Green's second identity from above.

2.1. In the Hilbert space $H^1(G)$ show the orthogonal complement of $H_0^1(G)$ is the subspace of those $\varphi \in H^1(G)$ for which $\Delta_n \varphi = \varphi$. Find a basis for $H_0^1(G)^\perp$ in each of the three cases $G = (0,1)$, $G = (0,\infty)$, $G = \mathbb{R}$.

2.2. If $G = (0,1)$, show $H^1(G) \subset C(\bar{G})$.

2.3. Show that $H_0^1(G)$ is a Hilbert space with the scalar product

$$(f,g) = \int_G \nabla f(x) \cdot \overline{\nabla g}(x)\,dx\ .$$

If $F \in L^2(G)$, show $T(v) = (F,v)_{L^2(G)}$ defines $T \in H_0^1(G)'$. Use the second part of the proof of Theorem 2.2 to show that there is a unique $u \in H_0^1(G)$ with $\Delta_n u = F$.

2.4. If $G_1 \subset G_2$, show $H_0^m(G_1)$ is naturally identified with a closed subspace of $H_0^m(G_2)$.

2.5. If $u \in H^m(G)$, then $\beta \in C^\infty(\bar{G})$ implies $\beta u \in H^m(G)$, and $\beta \in C_0^\infty(G)$ implies $\beta u \in H_0^m(G)$.

2.6. In the situation of Section 2.3, show that $\|u\|_{H^m(G)}$ is equivalent to $(\sum_{j=0}^N \|\beta_j u\|^2_{H^m(G\cap G_j)})^{1/2}$ and that $\|u\|_{L^2(\partial G)}$ is equivalent to $(\sum_{j=1}^N \|\beta_j u\|^2_{L^2(\partial G\cap G_j)})^{1/2}$.

3.1. In the proof of Theorem 3.2, explain why $\gamma_0(u) = 0$ implies $u(x',s) = \int_0^s \partial_n u(x',t)\,dt$ for a.e. $x' \in \mathbb{R}^{n-1}$.

3.2. Provide all remaining details in the proof of Theorem 3.3.

3.3. Extend the first and second Green's identities to pairs of functions from appropriate Sobolev spaces. (Cf. Section 1.6(e) and Exercise 1.6).

4.1. Show that G satisfies the cone condition if ∂G is a C^1-manifold of dimension $n-1$.

4.2. Show that G satisfies the cone condition if it is convex.

4.3. Show $H^m(G) \subset C^k(G)$ for any open set in $\mathbb{R}^n$ so long as $m > k + n/2$. If $x_0 \in G$, show that $\delta(\varphi) = \overline{\varphi(x_0)}$ defines $\delta \in H^m(G)'$ for $m > n/2$.

4.4. Let Γ be a subset of ∂G in the situation of Theorem 3.3. Show that $\varphi \to \int_\Gamma g(s)\varphi(s)\, ds$ defines an element of $H^1(G)'$ for each $g \in L^2(\Gamma)$. Repeat the above for an $(n-1)$-dimensional C^1-manifold in $\bar{G}$, not necessarily in ∂G.

5.1. Verify that $\mathcal{H}^m(G)$ is a Hilbert space.

5.2. Prove Lemma 5.1.