Chapter III

Boundary Value Problems

1 Introduction

We shall recall two classical boundary value problems and show that an appropriate generalized or abstract formulation of each of these is a well-posed problem. This provides a weak global solution to each problem and motivates much of our latter discussion.

1.1

Suppose we are given a subset G of $\mathbb{R}^n$ and a function $F : G \to \mathbb{K}$. We consider two boundary value problems for the partial differential equation

$$-\Delta_n u(x) + u(x) = F(x) \ , \qquad x \in G \ . \tag{1.1}$$

The *Dirichlet problem* is to find a solution of (1.1) for which $u = 0$ on ∂G. The *Neumann problem* is to find a solution of (1.1) for which $(\partial u/\partial \nu) = 0$ on ∂G. In order to formulate these problems in a meaningful way, we recall the first formula of Green

$$\int_G ((\Delta_n u)v + \nabla u \cdot \nabla v) = \int_{\partial G} \frac{\partial u}{\partial \nu} v = \int_{\partial G} \gamma_1 u \cdot \gamma_0 v \tag{1.2}$$

which holds if ∂G is sufficiently smooth and if $u \in H^2(G)$, $v \in H^1(G)$. Thus, if u is a solution of the Dirichlet problem and if $u \in H^2(G)$, then we have $u \in H_0^1(G)$ (since $\gamma_0 u = 0$) and (from (1.1) and (1.2))

$$(u, v)_{H^1(G)} = (F, v)_{L^2(G)} \ , \qquad v \in H_0^1(G) \ . \tag{1.3}$$

Note that the identity (1.3) holds in general only for those $v \in H^1(G)$ for which $\gamma_0 v = 0$. If we drop the requirement that v vanish on ∂G, then there would be a contribution from (1.2) in the form of a boundary integral. Similarly, if u is a solution of the Neumann problem and $u \in H^2(G)$, then (since $\gamma_1 u = 0$) we obtain from (1.1) and (1.2) the identity (1.3) for all $v \in H^1(G)$. That is, $u \in H^2(G)$ and (1.3) holds for all $v \in H^1(G)$.

Conversely, suppose $u \in H^2(G) \cap H_0^1(G)$ and (1.3) holds for all $v \in H_0^1(G)$. Then (1.3) holds for all $v \in C_0^\infty(G)$, so (1.1) is satisfied in the sense of distributions on G, and $\gamma_0 u = 0$ is a boundary condition. Thus, u is a solution of a Dirichlet problem. Similarly, if $u \in H^2(G)$ and (1.3) holds for all $v \in H^1(G)$, then $C_0^\infty(G) \subset H^1(G)$ shows (1.1) is satisfied as before, and substituting (1.1) into (1.3) gives us

$$\int_{\partial G} \gamma_1 u \cdot \gamma_0 v = 0 \ , \qquad v \in H^1(G) \ .$$

Since the range of γ_0 is dense in $L^2(\partial G)$, this implies that $\gamma_1 u = 0$, so u is a solution of a Neumann problem.

1.2

The preceding remarks suggest a weak formulation of the Dirichlet problem as follows:

> Given $F \in L^2(G)$, find $u \in H_0^1(G)$ such that (1.3) holds for all $u \in H_0^1(G)$.

In particular, the condition that $u \in H^2(G)$ is not necessary for this formulation to make sense. A similar formulation of the Neumann problem would be the following:

> Given $F \in L^2(G)$, find $u \in H^1(G)$ such that (1.3) holds for all $v \in H^1(G)$.

This formulation does not require that $u \in H^2(G)$, so we do not necessarily have $\gamma_1 u \in L^2(\partial G)$. However, we can either extend the operator γ_1 so (1.2) holds on a larger class of functions, or we may prove a regularity result to the effect that a solution of the Neumann problem is necessarily in $H^2(G)$. We shall achieve both of these in the following, but for the present we consider the following abstract problem:

Given a Hilbert space V and $f \in V'$, find $u \in V$ such that for all $v \in V$

$$(u, v)_V = f(v) \ .$$

By taking $V = H_0^1(G)$ or $V = H^1(G)$ and defining f to be the functional $f(v) = (F, v)_{L^2(G)}$ of V', we recover the weak formulations of the Dirichlet or Neumann problems, respectively. But Theorem I.4.5 shows that this problem is well-posed.

Theorem 1.1 *For each $f \in V'$, there exists exactly one $u \in V$ such that $(u, v)_V = f(v)$ for all $v \in V$, and we have $\|u\|_V = \|f\|_{V'}$.*

Corollary *If u_1 and u_2 are the solutions corresponding to f_1 and f_2, then*

$$\|u_1 - u_2\|_V = \|f_1 - f_2\|_{V'} \ .$$

Finally, we note that if $V = H_0^1(G)$ or $H^1(G)$, and if $F \in L^2(G)$ then $\|f\|_{V'} \leq \|F\|_{L^2(G)}$ where we identify $L^2(G) \subset V'$ as indicated.

2 Forms, Operators and Green's Formula

2.1

We begin with a generalization of the weak Dirichlet problem and of the weak Neumann problem of Section 1:

Given a Hilbert space V, a continuous sesquilinear form $a(\cdot, \cdot)$ on V, and $f \in V'$, find $u \in V$ such that

$$a(u, v) = f(v) \ , \qquad v \in V \ . \tag{2.1}$$

The sesquilinear form $a(\cdot, \cdot)$ determines a pair of operators $\alpha, \beta \in \mathcal{L}(V)$ by the identities

$$a(u, v) = (\alpha(u), v)_V = (u, \beta(v))_V \ , \qquad u, v \in V \ . \tag{2.2}$$

Theorem I.4.5 is used to construct α and β from $a(\cdot, \cdot)$, and $a(\cdot, \cdot)$ is clearly determined by either of α or β through (2.2). Theorem I.4.5 also defines the bijection $J \in \mathcal{L}(V', V)$ for which

$$f(v) = (J(f), v)_V \ , \qquad f \in V' \ , \ v \in V \ .$$

In fact, J is just the inverse of R_V. It is clear that u is a solution of the "weak" problem associated with (2.1) if and only if $\alpha(u) = J(f)$. Since J is a bijection, the solvability of this functional equation in V depends on the invertibility of the operator α. A useful sufficient condition for α to be a bijection is given in the following.

Definition. The sesquilinear form $a(\cdot,\cdot)$ on the Hilbert space V is V-*coercive* if there is a $c > 0$ such that

$$|a(v,v)| \geq c\|v\|_V^2 \ , \qquad v \in V \ . \tag{2.3}$$

We show that the weak problem associated with a V-coercive form is well-posed.

Theorem 2.1 *Let $a(\cdot,\cdot)$ be a V-coercive continuous sesquilinear form. Then, for every $f \in V'$, there is a unique $u \in V$ for which* (2.1) *is satisfied. Furthermore, $\|u\|_V \leq (1/c)\|f\|_{V'}$.*

Proof: The estimate (2.3) implies that both α and β are injective, and we also obtain

$$\|\alpha(v)\|_V \geq c\|v\|_V \ , \qquad v \in V \ .$$

This estimate implies that the range of α is closed. But β is the adjoint of α in V, so the range of α, $\mathrm{Rg}(\alpha)$, satisfies the orthogonality condition $\mathrm{Rg}(\alpha)^\perp = K(\beta) = \{0\}$. Hence, $\mathrm{Rg}(\alpha)$ is dense in V, and this shows $\mathrm{Rg}(\alpha) = V$. Since J is norm-preserving the stated results follow easily.

2.2

We proceed now to construct some operators which characterize solutions of problem (2.1) as solutions of boundary value problems for certain choices of $a(\cdot,\cdot)$ and V. First, define $\mathcal{A} \in \mathcal{L}(V, V')$ by

$$a(u,v) = \mathcal{A}u(v) \ , \qquad u, v \in V \ . \tag{2.4}$$

There is a one-to-one correspondence between continuous sesquilinear forms on V and linear operators from V to V', and it is given by the identity (2.4). In particular, u is a solution of the weak problem (2.1) if and only if $u \in V$ and $\mathcal{A}u = f$, so the problem is determined by $\mathcal{A}$ when $f \in V'$ is regarded as data. We would like to know that the identity $\mathcal{A}u = f$ implies that u satisfies a partial differential equation. It will not be possible in all of our examples to identify V' with a space of distributions on a domain G in $\mathbb{R}^n$. (For example, we are thinking of $V = H^1(G)$ in a Neumann problem as in (1.1). The difficulty is that the space $C_0^\infty(G)$ is *not dense* in V.)

There are two "natural" ways around this difficulty. First, we assume there is a Hilbert space H such that V is dense and continuously imbedded in H (hence, we may identify $H' \subset V'$) and such that H is identified with H' through the Riesz map. Thus we have the inclusions

$$V \hookrightarrow H = H' \hookrightarrow V'$$

and the identity

$$f(v) = (f, v)_H \ , \qquad f \in H \ , \ v \in V \ . \tag{2.5}$$

We call H the *pivot space* when we identify $H = H'$ as above. (For example, in the Neumann problem of Section 1, we choose $H = L^2(G)$, and for this choice of H, the Riesz map is the identification of functions with functionals which is compatible with the identification of $L^2(G)$ as a space of distributions on G; cf., Section I.5.3.) We define $D = \{u \in V : \mathcal{A}u \in H\}$. In the examples, $\mathcal{A}u = f$, $u \in D$, will imply that a partial differential equation is satisfied, since $C_0^\infty(G)$ will be dense in H. Note that $u \in D$ if and only if $u \in V$ and there is a $K > 0$ such that

$$|a(u, v)| \le K\|v\|_H \ , \qquad v \in V \ .$$

(This follows from Theorem I.4.5.) Finally, we obtain the following result.

Theorem 2.2 *If $a(\cdot,\cdot)$ is V-coercive, then D is dense in V, hence, dense in H.*

Proof: Let $w \in V$ with $(u, w)_V = 0$ for all $u \in D$. Then the operator β from (2.2) being surjective implies $w = \beta(v)$ for some $v \in V$. Hence, we obtain $0 = (u, \beta(v))_V = \mathcal{A}u(v) = (\mathcal{A}u, v)_H$ by (2.5), since $u \in D$. But $\mathcal{A}$ maps D onto H, so $v = 0$, hence, $w = 0$.

A second means of obtaining a partial differential equation from the continuous sesquilinear form $a(\cdot,\cdot)$ on V is to consider a closed subspace V_0 of V, let $i : V_0 \hookrightarrow V$ denote the identity and $\rho = i' : V' \to V_0'$ the restriction to V_0 of functionals on V, and define $A = \rho\mathcal{A} : V \to V_0'$. The operator $A \in \mathcal{L}(V, V_0')$ defined by

$$a(u,v) = Au(v)\ , \qquad u \in V\ ,\ v \in V_0$$

is called the *formal operator* determined by $a(\cdot,\cdot)$, V and V_0. In examples, V_0 will be the closure in V of $C_0^\infty(G)$, so V_0' is a space of distributions on G. Thus, $Au = f \in V_0'$ will imply that a partial differential equation is satisfied.

2.3

We shall compare the operators $\mathcal{A}$ and A. Assume V_0 is a closed subspace of V, H is a Hilbert space identified with its dual, the injection $V \hookrightarrow H$ is continuous, and V_0 is dense in H. Let D be given as above and define $D_0 = \{u \in V : Au \in H\}$, where we identify $H \subset V_0'$. Note that $u \in D_0$ if and only if $u \in V$ and there is a $K > 0$ such that

$$|a(u,v)| \le K\|v\|_H\ , \qquad v \in V_0\ ,$$

so $D \subset D_0$. It is on D_0 that we compare $\mathcal{A}$ and A. So, let $u \in D_0$ be fixed in the following and consider the functional

$$\varphi(v) = \mathcal{A}u(v) - (Au, v)_H\ , \qquad v \in V\ . \tag{2.6}$$

Then we have $\varphi \in V'$ and $\varphi|_{V_0} = 0$. But these are precisely the conditions that characterize those $\varphi \in V'$ which are in the range of $q' : (V/V_0)' \to V'$, the dual of the quotient map $q : V \to V/V_0$. That is, there is a unique $F \in (V/V_0)'$ such that $q'(F) = F \circ q = \varphi$. Thus, (2.6) determines an $F \in (V/V_0)'$ such that $F(q(v)) = \varphi(v)$, $v \in V$. In order to characterize (V/V_0), let V_0 be the kernel of a linear surjection $\gamma : V \to B$ and denote by $\hat\gamma$ the quotient map which is a bijection of V/V_0 onto B. Define a norm on B by $\|\hat\gamma(\hat x)\|_B = \|\hat x\|_{V/V_0}$ so $\hat\gamma$ is bicontinuous. Then the dual operator $\hat\gamma' : B' \to (V/V_0)'$ is a bijection. Given the functional F above, there is a unique $\partial \in B'$ such that $F = \hat\gamma'(\partial)$. That is, $F = \partial \circ \hat\gamma$. We summarize the preceding discussion in the following result.

Theorem 2.3 *Let V and H be Hilbert spaces with V dense and continuously imbedded in H. Let H be identified with its dual H' so* (2.5) *holds. Suppose γ is a linear surjection of V onto a Hilbert space B such that the quotient map $\hat\gamma : V/V_0 \to B$ is norm-preserving, where V_0, the kernel of γ, is dense in H.*

Thus, we have $V_0 \hookrightarrow H \hookrightarrow V_0'$. Let $\mathcal{A} \in \mathcal{L}(V, V')$ and define $A \in \mathcal{L}(V, V_0')$ by $A = \rho\mathcal{A}$, where $\rho : V' \to V_0'$ is restriction to V_0, the dual of the injection $V_0 \hookrightarrow V$. Let $D_0 = \{u \in V : Au \in H\}$. Then, for every $u \in D_0$, there is a unique $\partial(u) \in B'$ such that

$$\mathcal{A}u(v) - (Au, v)_H = \partial(u)(\gamma(v)) \ , \qquad v \in V \ . \tag{2.7}$$

The mapping $\partial : D_0 \to B'$ is linear.

When V_0' is a space of distributions, it is the formal operator A that determines a partial differential equation. When γ is a trace function and V_0 consists of those elements of V which vanish on a boundary, the quotient V/V_0 represents boundary values of elements of V. Thus B is a realization of these abstract boundary values as a function space and (2.7) is an abstract Green's formula. We shall call ∂ the abstract *Green's operator*.

Example. Let $V = H^1(G)$ and $\gamma : H^1(G) \to L^2(\partial G)$ be the trace map constructed in Theorem II.3.1. Then $H_0^1(G) = V_0$ is the kernel of γ and we denote by B the range of γ. Since $\hat{\gamma}$ is norm-preserving, the injection $B \hookrightarrow L^2(\partial G)$ is continuous and, by duality, $L^2(\partial G) \subset B'$, where we identify $L^2(\partial G)$ with its dual space. In particular, B consists of functions on ∂G and $L^2(\partial G)$ is a subspace of B'. Continuing this example, we choose $H = L^2(G)$ and $a(u, v) = (u, v)_{H^1(G)}$, so $Au = -\Delta_n u + u$ and $D_0 = \{u \in H^1(G) : \Delta_n u \in L^2(G)\}$. By comparing (2.7) with (1.2) we find that when ∂G is smooth $\partial : D_0 \to B'$ is an extension of $\partial/\partial\nu = \gamma_1 : H^2(G) \to L^2(\partial G)$.

3 Abstract Boundary Value Problems

3.1

We begin by considering an abstract "weak" problem (2.1) motivated by certain carefully chosen formulations of the Dirichlet and Neumann problems for the Laplace differential operator. The sesquilinear form $a(\cdot, \cdot)$ led to two operators: $\mathcal{A}$, which is equivalent to $a(\cdot, \cdot)$, and the formal operator A, which is determined by the action of $\mathcal{A}$ restricted to a subspace V_0 of V. It is A that will be a partial differential operator in our applications, and its domain will be determined by the space V and the difference of $\mathcal{A}$ and A as characterized by the Green's operator ∂ in Theorem 2.3. If V is prescribed by boundary conditions, then these same boundary conditions will be forced on a solution

u of (2.1). Such boundary conditions are called *stable* or *forced* boundary conditions. A second set of constraints may arise from Theorem 2.3 and these are called *unstable* or *variational* boundary conditions. The complete set of both stable and unstable boundary conditions will be part of the characterization of the domain of the operator A.

We shall elaborate on these remarks by using Theorem 2.3 to characterize solutions of (2.1) in a setting with more structure than assumed before. This additional structure consists essentially of splitting the form $a(\cdot,\cdot)$ into the sum of a spatial part which determines the partial differential equation in the region and a second part which contributes only boundary terms. The functional f is split similarly into a spatial part and a boundary part.

3.2

We assume that we have a Hilbert space V and a linear surjection $\gamma : V \to B$ with kernel V_0 and that B is a Hilbert space isomorphic to V/V_0. Let V be continuously imbedded in a Hilbert space H which is the pivot space identified with its dual, and let V_0 be dense in H. Thus we have the continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$ and the identity (2.5). Let $a_1 : V \times V \to \mathbb{K}$ and $a_2 : B \times B \to \mathbb{K}$ be continuous sesquilinear forms and define

$$a(u,v) = a_1(u,v) + a_2(\gamma u, \gamma v) \ , \qquad u,v \in V \ .$$

Similarly, let $F \in H$, $g \in B'$, and define

$$f(v) = (F,v)_H + g(\gamma v) \ , \qquad v \in V \ .$$

The problem (2.1) is the following: find $u \in V$ such that

$$a_1(u,v) + a_2(\gamma u, \gamma v) = (F,v)_H + g(\gamma v) \ , \qquad v \in V \ . \tag{3.1}$$

We shall use Theorem 2.3 to show that (3.1) is equivalent to an abstract boundary value problem.

Theorem 3.1 *Assume we are given the Hilbert spaces, sesquilinear forms and functionals as above. Let $\mathcal{A}_2 : B \to B'$ be given by*

$$\mathcal{A}_2\varphi(\psi) = a_1(\varphi,\psi) \ , \qquad \varphi,\psi \in B \ ,$$

and $A : V \to V_0'$ by

$$Au(v) = a_1(u,v) \ , \qquad u \in V \ , \ v \in V_0 \ .$$

Let $D_0 = \{u \in V : Au \in H\}$ *and* $\partial_1 \in L(D_0, B')$ *be given by (Theorem 2.3)*

$$a_1(u,v) - (Au,v)_H = \partial_1 u(\gamma v) \ , \qquad u \in D_0 \ , \ v \in V \ . \tag{3.2}$$

Then, u *is a solution of* (3.1) *if and only if*

$$u \in V \ , \quad Au = F \ , \quad \partial_1 u + \mathcal{A}_2(\gamma u) = g \ . \tag{3.3}$$

Proof: Since $a_2(\gamma u, \gamma v) = 0$ for all $v \in V_0$, it follows that the formal operator A and space D_0 (determined above by $a_1(\cdot,\cdot)$) are equal, respectively, to the operator and domain determined by $a(\cdot,\cdot)$ in Section 2.3. Suppose u is a solution of (3.1). Then $u \in V$, and the identity (3.1) for $v \in V_0$ and V_0 being dense in H imply that $Au = F \in H$. This shows $u \in D_0$ and using (3.2) in (3.1) gives

$$\partial_1 u(\gamma v) + a_2(\gamma u, \gamma v) = g(\gamma v) \ , \qquad v \in V \ .$$

Since γ is a surjection, this implies the remaining equation in (3.3). Similarly, (3.3) implies (3.1).

Corollary 3.2 *Let* D *be the space of those* $u \in V$ *such that for some* $F \in H$

$$a(u,v) = (F,v)_H \ , \qquad v \in V \ .$$

Then $u \in D$ *if and only if* u *is a solution of* (3.3) *with* $g = 0$.

Proof: Since V_0 is dense in H, the functional $f \in V'$ defined above is in H if and only if $g = 0$.

4 Examples

We shall illustrate some applications of our preceding results in a variety of examples of boundary value problems. Our intention is to indicate the types of problems which can be described by Theorem 3.1.

4.1

Let there be given a set of (coefficient) functions

$$a_{ij} \in L^\infty(G)\ ,\quad 1 \le i\ ,\ j \le n\ ;\qquad a_j \in L^\infty(G)\ ,\quad 0 \le j \le n\ ,$$

where G is open and connected in $\mathbb{R}^n$, and define

$$a(u,v) = \int_G \Big\{ \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j \overline{v(x)} + \sum_{j=0}^n a_j(x)\partial_j u(x)\overline{v(x)} \Big\}\, dx\ ,$$
$$u,v \in H^1(G)\ , \tag{4.1}$$

where $\partial_0 u = u$. Let $F \in L^2(G) \equiv H$ be given and define $f(v) = (F,v)_H$. Let Γ be a closed subset of ∂G and define

$$V = \{v \in H^1(G) : \gamma_0(v)(s) = 0\ ,\quad \text{a.e.}\quad s \in \Gamma\}\ .$$

V is a closed subspace of $H^1(G)$, hence a Hilbert space. We let $V_0 = H_0^1(G)$ so the formal operator $A : V \to V_0' \subset \mathcal{D}^*(G)$ is given by

$$Au = -\sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) + \sum_{j=0}^n a_j \partial_j u\ .$$

Let γ be the restriction to V of the trace map $H^1(G) \to L^2(\partial G)$, where we assume ∂G is appropriately smooth, and let B be the range of γ, hence $B \hookrightarrow L^2(\partial G \sim \Gamma) \hookrightarrow B'$. If all the $a_{ij} \in C^1(\bar{G})$, then we have from the classical Green's theorem

$$a(u,v) - (Au,v)_H = \int_{\partial G \sim \Gamma} \frac{\partial u}{\partial \nu_A} \cdot \gamma_0(v)\, ds\ ,\qquad u \in H^2(G)\ ,\ v \in V$$

where

$$\frac{\partial u}{\partial \nu_A} = \sum_{i=1}^n \partial_i u(s) \sum_{j=1}^n a_{ij}(s)\nu_j(s)$$

denotes the (weighted) normal derivative on $\partial G \sim \Gamma$. Thus, the operator ∂ is an extension of $\partial/\partial\nu_A$ from $H^2(G)$ to the domain $D_0 = \{u \in V : Au \in L^2(G)\}$. Theorem 3.1 now asserts that u is a solution of the problem (2.1) if and only if $u \in H^1(G)$, $\gamma_0 u = 0$ on Γ, $\partial u = 0$ on $\partial G \sim \Gamma$, and $Au = F$.

That is, u is a generalized solution of the *mixed Dirichlet-Neumann boundary value problem*

$$\left.\begin{array}{ll} Au(x) = F(x)\ , & x \in G\ , \\ u(s) = 0\ , & s \in \Gamma\ , \\ \dfrac{\partial u(s)}{\partial \nu_A} = 0\ , & s \in \partial G \sim \Gamma\ . \end{array}\right\} \tag{4.2}$$

If $\Gamma = \partial G$, this is called the Dirichlet problem or the boundary value problem of *first type*. If $\Gamma = \emptyset$, it is called the Neumann problem or boundary value problem of *second type*.

4.2

We shall simplify the partial differential equation but introduce boundary integrals. Define $H = L^2(G)$, $V_0 = H_0^1(G)$, and

$$a_1(u, v) = \int_G \nabla u \cdot \nabla \bar{v} \qquad u, v \in V \tag{4.3}$$

where V is a subspace of $H^1(G)$ to be chosen below. The corresponding distribution-valued operator is given by $A = -\Delta_n$ and ∂_1 is an extension of the standard normal derivative given by

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu\ .$$

Suppose we are given $F \in L^2(G)$, $g \in L^2(\partial G)$, and $\alpha \in L^\infty(\partial G)$. We define

$$\begin{aligned} a_2(\varphi, \psi) &= \int_{\partial G} \alpha(s)\varphi(s)\bar{\psi}(s)\, ds\ , \qquad \varphi, \psi \in L^2(\partial G) \\ f(v) &= (F, v)_H + (g, \gamma_0 v)_{L^2(\partial G)}\ , \qquad v \in V\ , \end{aligned}$$

and then use Theorem 3.1 to characterize a solution u of (2.1) for different choices of V.

If $V = H^1(G)$, then u is a generalized solution of the boundary value problem

$$\left.\begin{array}{ll} -\Delta_n u(x) = F(x)\ , & x \in G\ , \\ \dfrac{\partial u(s)}{\partial \nu} + \alpha(s)u(s) = g(x)\ , & s \in \partial G\ . \end{array}\right\} \tag{4.4}$$

The boundary condition is said to be of *third type* at those points $s \in \partial G$ where $\alpha(s) \neq 0$.

For an example of *non-local* boundary conditions, choose $V = \{v \in H^1(G) : \gamma_0(v) \text{ is constant}\}$. Let $g(s) = g_0$ and $\alpha(s) = \alpha_0$ be constants, and define $a_2(\cdot,\cdot)$ and f as above. Then u is a solution of the boundary value problem of *fourth type*

$$\left.\begin{aligned} &-\Delta_n u(x) = F(x)\ , && x \in G\ ,\\ &u(s) = u_0 \quad (\text{constant})\ , && s \in \partial G\ ,\\ &\left(\int_{\partial G} \frac{\partial u(s)}{\partial \nu}\, ds \Big/ \int_{\partial G} ds\right) + \alpha_0 \cdot u_0 = g_0\ . && \end{aligned}\right\} \tag{4.5}$$

Note that $B = \mathbb{K}$ in this example and u_0 is not prescribed as data. Also, *periodic* boundary conditions are obtained when G is an interval.

4.3

We consider a problem with a prescribed derivative on the boundary in a direction which is not necessarily normal. For simplicity we assume $n = 2$, let $c \in \mathbb{R}$, and define

$$a(u,v) = \int_G \{\partial_1 u(\partial_1 \bar{v} + c\partial_2 \bar{v}\,) + \partial_2 u(\partial_2 \bar{v} - c\partial_1 \bar{v}\,)\} \tag{4.6}$$

for u, $v \in V = H^1(G)$. Taking $V_0 = H_0^1(G)$ gives $A = -\Delta_2$ and the classical Green's theorem shows that for $u \in H^2(G)$ and $v \in H^1(G)$ we have

$$a(u,v) - (Au,v)_{L^2(G)} = \int_{\partial G} \left(\frac{\partial u}{\partial \nu} + c\frac{\partial u}{\partial \tau}\right) \bar{v}\, ds$$

where

$$\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$$

is the derivative in the direction of the tangent vector $\tau = (\nu_2, -\nu_1)$ on ∂G. Thus ∂ is an extension of the *oblique derivative* in the direction $\nu + c\tau$ on the boundary. If f is chosen as in (4.2), then Theorem 3.1 shows that problem (2.1) is equivalent to a weak form of the boundary value problem

$$\begin{aligned} &-\Delta_2 u(x) = F(x)\ , && x \in G\ ,\\ &\frac{\partial u}{\partial \nu} + c\frac{\partial u}{\partial \tau} = g(s)\ , && s \in \partial G\ . \end{aligned}$$

4.4

Let G_1 and G_2 be disjoint open connected sets with smooth boundaries ∂G_1 and ∂G_2 which intersect in a C^1 manifold Σ of dimension $n-1$. If ν_1 and ν_2 denote the unit outward normals on ∂G_1 and ∂G_2, then $\nu_1(s) = -\nu_2(s)$ for $s \in \Sigma$. Let G be the interior of the closure of $G_1 \cup G_2$, so that

$$\partial G = \partial G_1 \cup \partial G_2 \sim (\Sigma \sim \partial\Sigma) \; .$$

For $k = 1, 2$, let γ_0^k be the trace map $H^1(G_k) \to L^2(\partial G_k)$. Define $V = H^1(G)$ and note that $\gamma_0^1 u_1(s) = \gamma_0^2 u_2(s)$ for a.e. $s \in \Sigma$ when $u \in H^1(G)$ and u_k is the restriction of u to G_k, $k = 1, 2$. Thus we have a natural trace map

$$\begin{aligned} \gamma : H^1(G) &\longrightarrow L^2(\partial G) \times L^2(\Sigma) \\ u &\longmapsto (\gamma_0 u, \gamma_0^1 u_1|_\Sigma) \; , \end{aligned}$$

where $\gamma_0 u(s) = \gamma_0^k u_k(s)$ for $s \in \partial G_k \sim \Sigma$, $k = 1, 2$, and its kernel is given by $V_0 = H_0^1(G_1) \times H_0^1(G_2)$.

Let $a_1 \in C^1(\bar{G}_1)$, $a_2 \in C^1(\bar{G}_2)$ and define

$$a(u, v) = \int_{G_1} a_1 \nabla u \cdot \nabla \bar{v} + \int_{G_2} a_2 \nabla u \cdot \nabla \bar{v} \; , \qquad u, v \in V \; .$$

The operator A takes values in $\mathcal{D}^*(G_1 \cup G_2)$ and is given by

$$Au(x) = \begin{cases} -\sum_{j=1}^{n} \partial_j (a_1(x) \partial_j u(x)) \; , & x \in G_1 \; , \\ -\sum_{j=1}^{n} \partial_j (a_2(x) \partial_j u(x)) \; , & x \in G_2 \; . \end{cases}$$

The classical Green's formula applied to G_1 and G_2 gives

$$a(u, v) - (Au, v)_{L^2(G)} = \int_{\partial G_1} a_1 \frac{\partial u_1}{\partial \nu_1} \, \bar{v}_1 + \int_{\partial G_2} a_2 \frac{\partial u_2}{\partial \nu_2} \, \bar{v}_2$$

for $u \in H^2(G)$ and $v \in H^1(G)$. It follows that the restriction of the operator ∂ to the space $H^2(G)$ is given by $\partial u = (\partial_0 u, \partial_1 u) \in L^2(\partial G) \times L^2(\Sigma)$, where

$$\begin{aligned} \partial_0 u(s) &= a_k(s) \frac{\partial u_k(s)}{\partial \nu_k} \; , \text{ a.e. } \; s \in \partial G_k \sim \Sigma \; , \quad k = 1, 2 \; , \\ \partial_1 u(s) &= a_1(s) \frac{\partial u_1(s)}{\partial \nu_1} + a_2(s) \frac{\partial u_2(s)}{\partial \nu_2} \; , \qquad s \in \Sigma \; . \end{aligned}$$

Let f be given as in Section 4.2. Then a solution of u of (2.1) is characterized by Theorem 3.1 as a weak solution of the boundary value problem

$$\begin{cases} u_1 \in H^1(G_1)\ , \ -\sum_{j=1}^{n} \partial_j a_1(x)\partial_j u_1(x) = F(x)\ , & x \in G_1\ , \\ u_2 \in H^1(G_2)\ , \ -\sum_{j=1}^{n} \partial_j a_2(x)\partial_j u_2(x) = F(x)\ , & x \in G_2\ , \\ a_1(s)\dfrac{\partial u_1(s)}{\partial \nu_1} = g(s)\ , & s \in \partial G_1 \sim \Sigma\ , \\ a_2(s)\dfrac{\partial u_2(s)}{\partial \nu_2} = g(s)\ , & s \in \partial G_2 \sim \Sigma\ , \\ u_1(s) = u_2(s)\ , & \\ a_1(s)\dfrac{\partial u_1(s)}{\partial \nu_1} + a_2(s)\dfrac{\partial u_2(s)}{\partial \nu_2} = 0\ , & s \in \Sigma\ . \end{cases}$$

Since $\nu_1 = -\nu_2$ on Σ, this last condition implies that the normal derivative has a prescribed jump on Σ which is determined by the ratio of $a_1(s)$ to $a_2(s)$. The pair of equations on the interface Σ are known as *transition conditions*.

4.5

Let the sets G_1, G_2 and G be given as in Section 4.4. Suppose Σ_0 is an open subset of the interface Σ which is also contained in the hyperplane $\{x = (x', x_n) : x_n = 0\}$ and define $V = \{v \in H_0^1(G) : \gamma_0^1 u_1|_{\Sigma_0} \in H^1(\Sigma_0)\}$. With the scalar product

$$(u, v)_V \equiv (u, v)_{H_0^1(G)} + (\gamma_0^1 u, \gamma_0^1 v)_{H^1(\Sigma_0)}\ , \qquad u, v \in V\ ,$$

V is a Hilbert space. Let $\gamma(u) = \gamma_0^1(u)|_\Sigma$ be the corresponding trace operator $V \to L^2(\Sigma)$, so $K(\gamma) = H_0^1(G_1) \times H_0^1(G_2)$ contains $C_0^\infty(G_1 \cup G_2)$ as a dense subspace. Let $\alpha \in L^\infty(\Sigma_0)$ and define the sesquilinear form

$$a(u, v) = \int_G \nabla u \cdot \nabla \bar{v} + \int_{\Sigma_0} \alpha \nabla'(\gamma u) \cdot \nabla'\overline{(\gamma v)}\ , \qquad u, v \in V\ . \tag{4.7}$$

Where ∇' denotes the gradient in the first $n-1$ coordinates. Then $A = -\Delta_n$ in $\mathcal{D}^*(G_1 \cup G_2)$ and the classical Green's formula shows that ∂u is given by

$$\partial u(v) = \int_\Sigma \left(\frac{\partial u_1}{\partial \nu_1}\,\bar{v} + \frac{\partial u_2}{\partial \nu_2}\,\bar{v}\right) + \int_{\Sigma_0} \alpha \nabla'(\gamma(u))\nabla'\bar{v}$$

for $u \in H^2(G)$ and $v \in B$. Since the range of γ is dense in $L^2(\Sigma \sim \Sigma_0)$, it follows that if $\partial u = 0$ then

$$\frac{\partial u_1(s)}{\partial \nu_1} + \frac{\partial u_2(s)}{\partial \nu_2} = 0 \ , \qquad s \in \Sigma \sim \Sigma_0 \ .$$

But $\nu_1 = -\nu_2$ on Σ, so the normal derivative of u is continuous across $\Sigma \sim \Sigma_0$. Since the range of γ contains $C_0^\infty(\Sigma_0)$, it follows that if $\partial u = 0$ then we obtain the identity

$$\int_{\Sigma_0} \alpha \nabla'(\gamma u) \nabla' \overline{(\gamma v)} + \int_{\Sigma_0} \frac{\partial u_1}{\partial \nu_1} \overline{(\gamma v)} + \frac{\partial u_2}{\partial \nu_2} \overline{(\gamma v)} = 0 \ , \qquad v \in V \ ,$$

and this shows that $\gamma u|_{\Sigma_0}$ satisfies the abstract boundary value

$$\begin{aligned} &-\Delta_{n-1}(\gamma u)(s) = \frac{\partial u_2(s)}{\partial \nu_1} - \frac{\partial u_1(s)}{\partial \nu_1} \ , && s \in \Sigma_0 \ , \\ &(\gamma u)(s) = 0 \ , && s \in \partial\Sigma_0 \cap \partial G \ , \\ &\frac{\partial (\gamma u)(s)}{\partial \nu_0} = 0 \ , && s \in \partial\Sigma_0 \sim \partial G \ , \end{aligned}$$

where ν_0 is the unit normal on $\partial\Sigma_0$, the $(n-2)$-dimensional boundary of Σ_0.

Let $F \in L^2(G)$ and $f(v) = (F, v)_{L^2(G)}$ for $v \in V$. Then from Corollary 3.2 it follows that (3.3) is a generalized boundary value problem given by

$$\left.\begin{aligned} &-\Delta_n u(x) = F(x) \ , && x \in G_1 \cup G_2 \ , \\ &u(s) = 0 \ , && s \in \partial G \ , \\ &u_1(s) = u_2(s) \ , \quad \frac{\partial u_1(s)}{\partial \nu_1} = \frac{\partial u_2(s)}{\partial \nu_1} \ , && s \in \Sigma \sim \Sigma_0 \ , \\ &-\Delta_{n-1} u(s) = \frac{\partial u_2(s)}{\partial \nu_1} - \frac{\partial u_1(s)}{\partial \nu_1} \ , && s \in \Sigma_0 \ , \\ &\frac{\partial u(s)}{\partial \nu_0} = 0 \ , && s \in \partial\Sigma_0 \sim \partial G_0 \ . \end{aligned}\right\} \qquad (4.8)$$

Nonhomogeneous terms could be added as in previous examples and similar problems could be solved on interfaces which are not necessarily flat.

5 Coercivity; Elliptic Forms

5.1

Let G be an open set in $\mathbb{R}^n$ and suppose we are given a collection of functions a_{ij}, $1 \le i,\ j \le n$; a_j, $0 \le j \le n$, in $L^\infty(G)$. Define the sesquilinear form

$$a(u,v) = \int_G \left\{ \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j \bar{v}(x) + \sum_{j=0}^n a_j(x)\partial_j u(x) \cdot \overline{v(x)} \right\} dx \qquad (5.1)$$

on $H^1(G)$. We saw in Section 4.1 that such forms lead to partial differential equations of second order on G.

Definition. The sesquilinear form (5.1) is called *strongly elliptic* if there is a constant $c_0 > 0$ such that

$$\operatorname{Re} \sum_{i,j=1}^n a_{ij}(x)\xi_i\bar{\xi}_j \ge c_0 \sum_{j=1}^n |\xi_j|^2 \ , \qquad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n \ , \ x \in G \ . \quad (5.2)$$

We shall show that a strongly elliptic form can be made coercive over (any subspace of) $H^1(G)$ by adding a sufficiently large multiple of the identity to it.

Theorem 5.1 *Let* (5.1) *be strongly elliptic. Then there is a* $\lambda_0 \in \mathbb{R}$ *such that for every* $\lambda > \lambda_0$*, the form*

$$a(u,v) + \lambda \int_G u(x)\bar{v}(x)\, dx$$

is $H^1(G)$*-coercive.*

Proof: Let $K_1 = \max\{\|a_j\|_{L^\infty(G)} : 1 \le j \le n\}$ and $K_0 = \operatorname{ess\,inf}\{\operatorname{Re} a_0(x) : x \in G\}$. Then, for $1 \le j \le n$ and each $\varepsilon > 0$ we have

$$\begin{aligned} |(a_j\partial_j u, u)_{L^2(G)}| &\le K_1\|\partial_j u\|_{L^2(G)} \cdot \|u\|_{L^2(G)} \\ &\le (K_1/2)\left(\varepsilon\|\partial_j u\|^2_{L^2(G)} + (1/\varepsilon)\|u\|^2_{L^2(G)}\right) \ . \end{aligned}$$

We also have $\operatorname{Re}(a_0 u, u)_{L^2(G)} \ge K_0\|u\|^2_{L^2(G)}$, so using these with (5.2) in (5.1) gives

$$\begin{aligned} \operatorname{Re} a(u,u) \ge\ & (c_0 - \varepsilon K_1/2)\|\nabla u\|^2_{L^2(G)} \\ & +(K_0 - nK_1/2\varepsilon)\|u\|^2_{L^2(G)} \ , \quad u \in H^1(G) \ . \end{aligned} \qquad (5.3)$$

We choose $\varepsilon > 0$ so that $K_1\varepsilon = c_0$. This gives us the desired result with $\lambda_0 = (nK_1^2/2c_0) - K_0$.

Corollary 5.2 *For every $\lambda > \lambda_0$, the boundary value problem* (4.2) *is well-posed, where*

$$Au = -\sum_{i,j=1}^{n} \partial_j(a_{ij}\partial_i u) + \sum_{j=1}^{n} a_j\partial_j u + (a_0 + \lambda)u \ .$$

Thus, for every $F \in L^2(G)$, there is a unique $u \in D$ such that (4.2) *holds, and we have the estimate*

$$\|(\lambda - \lambda_0)u\|_{L^2(G)} \leq \|F\|_{L^2(G)} \ . \tag{5.4}$$

Proof: The space D was defined in Section 2.2 and Corollary 3.2, so we need only to verify (5.4). For $u \in D$ and $\lambda > \lambda_0$ we have from (5.3)

$$\begin{aligned}(\lambda - \lambda_0)\|u\|^2_{L^2(G)} &\leq a(u,u) + \lambda(u,u)_{L^2(G)} = (Au,u)_{L^2(G)} \\ &\leq \|Au\|_{L^2(G)} \cdot \|u\|_{L^2(G)}\end{aligned}$$

and the estimate (5.4) now follows.

5.2

We indicate how coercivity may be obtained from the addition of boundary integrals to strongly elliptic forms.

Theorem 5.3 *Let G be open in $\mathbb{R}^n$ and suppose $0 \leq x_n \leq K$ for all $x = (x', x_n) \in G$. Let ∂G be a C^1-manifold with G on one side of ∂G. Let $\nu(s) = (\nu_1(s), \ldots, \nu_n(s))$ be the unit outward normal on ∂G and define*

$$\Sigma = \{s \in \partial G : \nu_n(s) > 0\} \ .$$

Then for all $u \in H^1(G)$ we have

$$\int_G |u|^2 \leq 2K \int_\Sigma |\gamma_0 u(s)|^2 \, ds + 4K^2 \int_G |\partial_n u|^2 \ .$$

Proof: For $u \in C^1(\bar{G})$, the Gauss Theorem gives

$$\int_{\partial G} \nu_n(s) s_n |u(s)|^2 \, ds = \int_G D_n(x_n |u(x)|^2) \, dx$$
$$= \int_G |u|^2 + \int_G x_n D_n(|u(x)|^2) \, dx \ .$$

Thus, we obtain from the inequality

$$2|a|\,|b| \le \frac{|a|^2}{2K} + 2K|b|^2 \ , \qquad a, b \in \mathbb{C} \ ,$$

the estimate

$$\int_G |u|^2 \le \int_{\partial G} \nu_n s_n |u(s)|^2 \, ds + (1/2) \int_G |u|^2 + 2K^2 \int_G |D_n u|^2 \ .$$

Since $\nu_n(s) s_n \le 0$ for $s \in \partial G \sim \Sigma$, the desired result follows.

Corollary 5.4 *If* (5.1) *is strongly elliptic,* $a_j \equiv 0$ *for* $1 \le j \le n$, $\mathrm{Re}\, a_0(x) \ge 0$, $x \in G$, *and if* $\Sigma \subset \Gamma$, *then the mixed Dirichlet-Neumann problem* (4.2) *is well-posed.*

Corollary 5.5 *If* $\alpha \in L^\infty(\partial G)$ *satisfies*

$$\mathrm{Re}\, \alpha(x) \ge 0 \ , \quad x \in \partial G \ , \quad \mathrm{Re}\, \alpha(x) \ge c > 0 \ , \quad x \in \Sigma \ ,$$

then the third boundary value problem (4.4) *is well-posed. The fourth boundary value problem* (4.5) *is well-posed if* $\mathrm{Re}(\alpha_0) > 0$.

Similar results can be obtained for the example of Section 4.3. Note that the form (4.6) satisfies

$$\mathrm{Re}\, a(u, u) = \int_G \left\{ |\partial_1 u|^2 + |\partial_2 u|^2 \right\} \ , \qquad u \in H^1(G) \ ,$$

so coercivity can be obtained over appropriate subspaces of $H^1(G)$ (as in Corollary 5.4) or by adding a positive multiple of the identity on G or boundary integrals (as in Corollary 5.5). Modification of (4.6) by restricting V, e.g., to consist of functions which vanish on a sufficiently large part of ∂G, or by adding forms, e.g., that are coercive over $L^2(G)$ or $L^2(\partial G)$, will result in a well-posed problem.

Finally, we note that the first term in the form (4.7) is coercive over $H_0^1(G)$ and, hence, over $L^2(\Sigma)$. Thus, if $\mathrm{Re}\, \alpha(x) \ge c > 0$, $x \in \Sigma_0$, then (4.7) is V-coercive and the problem (4.8) is well-posed.

5.3

In order to verify that the sesquilinear forms above were coercive over certain subspaces of $H^1(G)$, we found it convenient to verify that they satisfied the following stronger condition.

Definition. The sesquilinear form $a(\cdot,\cdot)$ on the Hilbert space V is V*-elliptic* if there is a $c > 0$ such that

$$\operatorname{Re} a(v,v) \geq c\|v\|_V^2 \ , \qquad v \in V \ . \tag{5.5}$$

Such forms will occur frequently in our following discussions.

6 Regularity

We begin this section with a consideration of the Dirichlet and Neumann problems for a simple elliptic equation. The original problems were to find solutions in $H^2(G)$ but we found that it was appropriate to seek weak solutions in $H^1(G)$. Our objective here is to show that those weak solutions are in $H^2(G)$ when the domain G and data in the equation are sufficiently smooth. In particular, this shows that the solution of the Neumann problem satisfies the boundary condition in $L^2(\partial G)$ and not just in the sense of the abstract Green's operator constructed in Theorem 2.3, i.e., in B'. (See the Example in Section 2.3.)

6.1

We begin with the Neumann problem; other cases will follow similarly.

Theorem 6.1 *Let G be bounded and open in $\mathbb{R}^n$ and suppose its boundary is a C^2-manifold of dimension $n-1$. Let $a_{ij} \in C^1(G)$, $1 \leq i,\ j \leq n$, and $a_j \in C^1(G)$, $0 \leq j \leq n$, all have bounded derivatives and assume that the sesquilinear form defined by*

$$a(\varphi,\psi) \equiv \int_G \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_i\varphi\overline{\partial_j\psi} + \sum_{j=0}^{n} a_j\partial_j\varphi\bar{\psi} \right\} dx \ , \qquad \varphi,\psi \in H^1(G) \tag{6.1}$$

is strongly elliptic. Let $F \in L^2(G)$ and suppose $u \in H^1(G)$ satisfies

$$a(u,v) = \int_G F\bar{v}\,dx \ , \qquad v \in H^1(G) \ . \tag{6.2}$$

Then $u \in H^2(G)$.

Proof: Let $\{(\varphi_k, G_k) : 1 \le k \le N\}$ be coordinate patches on ∂G and $\{\beta_k : 0 \le k \le N\}$ the partition-of-unity construction in Section II.2.3. Let B_k denote the support of β_k, $0 \le k \le N$. Since $u = \sum(\beta_k u)$ in G and each B_k is compact in $\mathbb{R}^n$, it is sufficient to show the following:

(a) $u|_{B_k \cap G} \in H^2(B_k \cap G)$, $1 \le k \le N$, and

(b) $\beta_0 u \in H^2(B_0)$.

The first case (a) will be proved below, and the second case (b) will follow from a straightforward modification of the first.

6.2

We fix k, $1 \le k \le N$, and note that the coordinate map $\varphi_k : Q \to G_k$ induces an isomorphism $\varphi_k^* : H^m(G_k \cap G) \to H^m(Q_+)$ for $m = 0, 1, 2$ by $\varphi_k^*(v) = v \circ \varphi_k$. Thus we define a continuous sesquilinear form on $H^1(Q_+)$ by

$$a^k\left(\varphi_k^*(w), \varphi_k^*(v)\right) \equiv \int_{G_k \cap G} \left\{ \sum_{i,j=1}^n a_{ij} \partial_j w \overline{\partial_j v} + \sum_{j=0}^n a_j \partial_j w \bar{v} \right\} dx \; . \tag{6.3}$$

By making the appropriate change-of-variable in (6.3) and setting $w_k = \varphi_k^*(w)$, $v_k = \varphi_k^*(v)$, we obtain

$$a^k(w_k, v_k) = \int_{Q_+} \left\{ \sum_{i,j=1}^n a_{ij}^k \partial_i(w_k) \partial_j(v_k) + \sum_{j=0}^n a_j^k \partial_j(w_k) v_k \right\} dy \; . \tag{6.4}$$

The resulting form (6.4) is strongly-elliptic on Q_+ (exercise).

Let u be the solution of (6.2) and let $v \in H^1(G \cap G_k)$ vanish in a neighborhood of ∂G_k. (That is, the support of v is contained in G_k.) Then the extension of v to all of G as zero on $G \sim G_k$ belongs to $H^1(G)$ and we obtain from (6.4) and (6.2)

$$a^k(\varphi_k^*(u), \varphi_k^*(v)) = a(u,v) = \int_{Q_+} F_k \varphi_k^*(v)\, dy \ ,$$

where $F_k \equiv \varphi_k^*(F) \cdot J(\varphi_k) \in L^2(Q_+)$. Letting $\mathcal{V}$ denote the space of those $v \in H^1(Q_+)$ which vanish in a neighborhood of ∂Q, and $u_k \equiv \varphi_k^*(u)$, we have shown that $u_k \in H^1(Q_+)$ satisfies

$$a^k(u_k, v_k) = \int_{Q_+} F_k v_k \, dy \ , \qquad v_k \in \mathcal{V} \tag{6.5}$$

where $a^k(\cdot,\cdot)$ is strongly elliptic with continuously differentiable coefficients with bounded derivatives and $F_k \in L^2(Q_+)$. We shall show that the restriction of u_k to the compact subset $K \equiv \varphi_k^{-1}(B_k)$ of Q belongs to $H^2(Q_+ \cap K)$. The first case (a) above will then follow.

6.3

Hereafter we drop the subscript "k" in (6.5). Thus, we have $u \in H^1(Q_+)$, $F \in L^2(Q_+)$ and

$$a(u,v) = \int_{Q_+} Fv \ , \qquad v \in \mathcal{V} \ . \tag{6.6}$$

Since $K \subset\subset Q$, there is by Lemma II.1.1 a $\varphi \in C_0^\infty(Q)$ such that $0 \le \varphi(x) \le 1$ for $x \in Q$ and $\varphi(x) = 1$ for $x \in K$. We shall first consider $\varphi \cdot u$.

Let w be a function defined on the half-space $\mathbb{R}^n_+$. For each $h \in \mathbb{R}$ we define a *translate* of w by

$$(\tau_h w)(x_1, x_2, \ldots, x_n) = w(x_1 + h, x_2, \ldots, x_n)$$

and a *difference* of w by

$$\nabla_h w = (\tau_h w - w)/h$$

if $h \neq 0$.

Lemma 6.2 *If $w, v \in L^2(Q_+)$ and the distance δ of the support of w to ∂Q is positive, then*

$$(\tau_h w, v)_{L^2(Q_+)} = (w, \tau_{-h} v)_{L^2(Q_+)}$$

for all $h \in \mathbb{R}$ with $|h| < \delta$.

Proof: This follows by the obvious change of variable and the observation that each of the above integrands is non-zero only on a compact subset of Q_+.

Corollary $\|\tau_h w\|_{L^2(Q_+)} = \|w\|_{L^2(Q_+)}$.

Lemma 6.3 *If* $w \in \mathcal{V}$, *then*

$$\|\nabla_h w\|_{L^2(Q_+)} \le \|\partial_1 w\|_{L^2(Q_+)} \ , \qquad 0 < |h| < \delta \ .$$

Proof: It follows from the preceding Corollary that it is sufficient to consider the case where $w \in C^1(\bar{G}) \cap \mathcal{V}$. Assuming this, and denoting the support of w by $\mathrm{supp}(w)$, we have

$$\nabla_h w(x) = h^{-1} \int_{x_1}^{x_1+h} \partial_1 w(t, x_2, \ldots, x_n)\, dt \ , \qquad w \in \mathrm{supp}(w) \ .$$

The Cauchy-Schwartz inequality gives

$$|\nabla_h w(x)| \le h^{-1/2} \Big(\int_{x_1}^{x_1+h} |\partial_1 w(t, x_2, \ldots, x_n)|^2\, dt \Big)^{1/2} \ , \qquad x \in \mathrm{supp}(w) \ ,$$

and this leads to

$$\begin{aligned}
\|\nabla_h w\|^2_{L^2(Q_+)} &\le h^{-1} \int_{\mathrm{supp}(w)} \int_{x_1}^{x_1+h} |\partial_1 w(t, x_2, \ldots, x_n)|^2\, dt\, dx \\
&= h^{-1} \int_{Q_+} \int_0^h |\partial_1 w(t + x_1, x_2, \ldots, x_n)|^2\, dt\, dx \\
&= h^{-1} \int_0^h \int_{Q_+} |\partial_1 w(t + x_1, \ldots, x_n)|^2\, dx\, dt \\
&= h^{-1} \int_0^h \int_{Q_+} |\partial_1 w(x_1, \ldots, x_n)|^2\, dx\, dt \\
&= \|\partial_1 w\|_{L^2(Q_+)} \ .
\end{aligned}$$

Corollary $\lim_{h\to 0}(\nabla_h w) = \partial_1 w$ *in* $L^2(Q_+)$.

Proof: $\{\nabla_h : 0 < |h| < \delta\}$ is a family of uniformly bounded operators on $L^2(Q_+)$, so it suffices to show the result holds on a dense subset.

We shall consider the forms

$$a_h(w,v) \equiv \int_{Q_+} \left\{ \sum_{i,j=1}^{n} (\nabla_h a_{ij}) \partial_i w \overline{\partial_j v} + \sum_{j=0}^{n} (\nabla_h a_j) \partial_j w \bar{v} \right\}$$

for $w, v \in \mathcal{V}$ and $|h| < \delta$, δ being given as in Lemma 6.2. Since the coefficients in (6.5) have bounded derivatives, the mean-value theorem shows

$$|a_h(w,v)| \leq C \|w\|_{H^1(Q_+)} \cdot \|v\|_{H^1(Q_+)} \tag{6.7}$$

where the constant is independent of w, v and h. Finally, we note that for w, v and h as above

$$a(\nabla_h w, v) + a(w, \nabla_{-h} v) = -a_{-h}(\tau_{-h} w, v) \ . \tag{6.8}$$

This follows from a computation starting with the first term above and Lemma 6.2.

After this lengthy preparation we continue with the proof of Theorem 6.1. From (6.6) we have the identity

$$\begin{aligned} a(\nabla_h(\varphi u), v) \ &= \{a(\nabla_h(\varphi u), v) + a(\varphi u, \nabla_{-h} v)\} \\ &\quad + \{a(u, \varphi \nabla_{-h} v) - a(\varphi u, \nabla_{-h} v)\} - (F, \varphi \nabla_{-h} v)_{L^2(Q_+)} \end{aligned} \tag{6.9}$$

for $v \in \mathcal{V}$ and $0 < |h| < \delta$, δ being the distance from K to ∂Q. The first term can be bounded appropriately by using (6.7) and (6.8). The third is similarly bounded and so we consider the second term in (6.9). An easy computation gives

$$\begin{aligned} &a(u, \varphi \nabla_{-h} v) - a(\varphi u, \nabla_{-h} v) \\ &\quad = \int_{Q_+} \left\{ \sum_{i,j=1}^{n} a_{ij} (\partial_i u \partial_j \varphi \nabla_{-h} v - \partial_i \varphi u \nabla_{-h}(\partial_j v)) - \sum_{j=1}^{n} a_j \partial_j \varphi u (\nabla_{-h} v) \right\} . \end{aligned}$$

Thus, we obtain the estimate

$$|a(\nabla_h(\varphi u), v)| \leq C \|v\|_{H^1(Q_+)} \ , \qquad v \in \mathcal{V} \ , \ 0 < |h| < \delta \ , \tag{6.10}$$

in which the constant C is independent of h and v. Since $a(\cdot,\cdot)$ is strongly-elliptic we may assume it is coercive (Exercise 6.2), so setting $v = \nabla_h(\varphi u)$ in (6.10) gives

$$c \|\nabla_h(\varphi u)\|^2_{H^1(Q_+)} \leq C \|\nabla_h(\varphi u)\|_{H^1(Q_+)} \ , \qquad 0 < |h| < \delta \ , \tag{6.11}$$

hence, $\{\nabla_h(\varphi u) : |h| < \delta\}$ is bounded in the Hilbert space $H^1(Q_+)$. By Theorem I.6.2 there is a sequence $h_n \to 0$ for which $\nabla_{h_n}(\varphi u)$ converges weakly to some $w \in H^1(Q_+)$. But $\nabla_{h_n}(\varphi u)$ converges weakly in $L^1(Q_+)$ to $\partial_1(\varphi u)$, so the uniqueness of weak limits implies that $\partial_1(\varphi u) = w \in H^1(Q_+)$. It follows that $\partial_1^2(\varphi u) \in L^2(Q_+)$, and the same argument shows that each of the tangential derivatives $\partial_1^2 u, \partial_2^2 u, \ldots, \partial_{n-1}^2 u$ belongs to $L^2(K)$. (Recall $\varphi = 1$ on K.) This information together with the partial differential equation resulting from (6.6) implies that $a_{nn} \cdot \partial_n^2(u) \in L^2(K)$. The strong ellipticity implies a_{nn} has a positive lower bound on K, so $\partial_n^2 u \in L^2(K)$. Since n and all of its derivatives through second order are in $L^2(K)$, it follows from Theorem II.5.5 that $u \in H^2(K)$.

The preceding proves the case (a) above. The case (b) follows by using the differencing technique directly on $\beta_0 u$. In particular, we can compute differences on $\beta_0 u$ in *any* direction. The details are an easy modification of those of this section and we leave them as an exercise.

6.4

We discuss some extensions of Theorem 6.1. First, we note that the result and proof of Theorem 6.1 also hold if we replace $H^1(G)$ by $H_0^1(G)$. This results from the observation that the subspace $H_0^1(G)$ is invariant under multiplication by smooth functions and translations and differences in tangential directions along the boundary of G. Thus we obtain a regularity result for the Dirichlet problem.

Theorem 6.4 *Let $u \in H_0^1(G)$ satisfy*

$$a(u, v) = \int_G F\bar{v} \ , \qquad v \in H_0^1(G)$$

where the set $G \subset \mathbb{R}^n$ and sesquilinear form $a(\cdot, \cdot)$ are given as in Theorem 6.1, and $F \in L^2(G)$. Then $u \in H^2(G)$.

When the data in the problem is smoother yet, one expects the same to be true of the solution. The following describes the situation which is typical of second-order elliptic boundary value problems.

Definition. Let V be a closed subspace of $H^1(G)$ with $H_0^1(G) \leq V$, and let $a(\cdot, \cdot)$ be a continuous sesquilinear form on V. Then $a(\cdot, \cdot)$ is called *k-regular on V* if for every $F \in H^s(G)$ with $0 \leq s \leq k$ and every solution $u \in V$ of

$$a(u, v) = (F, v)_{L^2(G)} \ , \qquad v \in V$$

we have $u \in H^{2+s}(G)$.

Theorems 6.1 and 6.4 give sufficient conditions for the form $a(\cdot,\cdot)$ given by (6.1) to be 0-regular over $H^1(G)$ and $H_0^1(G)$, respectively. Moreover, we have the following.

Theorem 6.5 *The form $a(\cdot,\cdot)$ given by* (6.1) *is k-regular over $H^1(G)$ and $H_0^1(G)$ if ∂G is a C^{2+k}-manifold and the coefficients $\{a_{ij}, a_j\}$ all belong to $C^{1+k}(\bar{G})$.*

7 Closed operators, adjoints and eigenfunction expansions

7.1

We were led in Section 2 to consider a linear map $A : D \to H$ whose domain D is a subspace of the Hilbert space H. We shall call such a map an (*unbounded*) *operator* on the Hilbert space H. Although an operator is frequently not continuous (with respect to the H-norm on D) it may have the property we now consider. The *graph* of A is the subspace

$$G(A) = \{[x, Ax] : x \in D\}$$

of the product $H \times H$. (This product is a Hilbert space with the scalar product

$$([x_1, x_2], [y_1, y_2])_{H \times H} = (x_1, y_1)_H + (x_2, y_2)_H \ .$$

The addition and scalar multiplication are defined componentwise.) The operator A on H is called *closed* if $G(A)$ is a closed subset of $H \times H$. That is, A is closed if for any sequence $x_n \in D$ such that $x_n \to x$ and $Ax_n \to y$ in H, we have $x \in D$ and $Ax = y$.

Lemma 7.1 *If A is closed and continuous (i.e., $\|Ax\|_H \le K\|x\|_H$, $x \in H$) then D is closed.*

Proof: If $x_n \in D$ and $x_n \to x \in H$, then $\{x_n\}$ and, hence, $\{Ax_n\}$ are Cauchy sequences. H is complete, so $Ax_n \to y \in H$ and $G(A)$ being closed implies $x \in D$.

When D is dense in H we define the *adjoint* of A as follows. The domain of the operator A^* is the subspace D^* of all $y \in H$ such that the map

$x \mapsto (Ax, y)_H : D \to \mathbb{K}$ is continuous. Since D is dense in H, Theorem I.4.5 asserts that for each such $y \in D^*$ there is a unique $A^*y \in H$ such that

$$(Ax, y) = (x, A^*y) \ , \qquad x \in D \ , \ y \in D^* \ . \tag{7.1}$$

Then the function $A^* : D^* \to H$ is clearly linear and is called the *adjoint* of A. The following is immediate from (7.1).

Lemma 7.2 *A^* is closed.*

Lemma 7.3 *If $D = H$, then A^* is continuous, hence, D^* is closed.*

Proof: If A^* is not continuous there is a sequence $x_n \in D^*$ such that $\|x_n\| = 1$ and $\|A^*x_n\| \to \infty$. From (7.1) it follows that for each $x \in H$,

$$|(x, A^*x_n)_H| = |(Ax, x_n)_H| \le \|Ax\|_H \ ,$$

so the sequence $\{A^*x_n\}$ is weakly bounded. But Theorem I.6.1 implies that it is bounded, a contradiction.

Lemma 7.4 *If A is closed, then D^* is dense in H.*

Proof: Let $y \in H$, $y \ne 0$. Then $[0, y] \notin G(A)$ and $G(A)$ closed in $H \times H$ imply there is an $f \in (H \times H)'$ such that $f[G(A)] = \{0\}$ and $f(0, y) \ne 0$. In particular, let $P : H \times H \to G(A)^\perp$ be the projection onto the orthogonal complement of $G(A)$ in $H \times H$, define $[u, v] = P[0, y]$, and set

$$f(x_1, x_2) = (u, x_1)_H + (v, x_2)_H \ , \qquad x_1, x_2 \in H \ .$$

Then we have

$$0 = f(x, Ax) = (u, x)_H + (v, Ax)_H \ , \qquad x \in D$$

so $v \in D^*$, and $0 \ne f(0, y) = (v, y)_H$. The above shows $(D^*)^\perp = \{0\}$, so D^* is dense in H.

The following result is known as the *closed-graph theorem*.

Theorem 7.5 *Let A be an operator on H with domain D. Then A is closed and $D = H$ if and only if $A \in \mathcal{L}(H)$.*

Proof: If A is closed and $D = H$, then Lemma 7.3 and Lemma 7.4 imply $A^* \in \mathcal{L}(H)$. Then Theorem I.5.2 shows $(A^*)^* \in \mathcal{L}(H)$. But (7.1) shows $A = (A^*)^*$, so $A \in \mathcal{L}(H)$. The converse in immediate.

The operators with which we are most often concerned are adjoints of another operator. The preceding discussion shows that the domain of such an operator, i.e., an adjoint, is all of H if and only if the operator is continuous. Thus, we shall most often encounter unbounded operators which are closed and densely defined.

We give some examples in $H = L^2(G)$, $G = (0, 1)$.

7.2

Let $D = H_0^1(G)$ and $A = i\partial$. If $[u_n, Au_n] \in G(A)$ converges to $[u, v]$ in $H \times H$, then in the identity

$$\int_0^1 Au_n\varphi\,dx = -i\int_0^1 u_n\,D\varphi\,dx\ , \qquad \varphi \in C_0^\infty(G)\ ,$$

we let $n \to \infty$ and thereby obtain

$$\int_0^1 v\varphi\,dx = -i\int_0^1 u\,D\varphi\,dx\ , \qquad \varphi \in C_0^\infty(G)\ .$$

This means $v = i\partial u = Au$ and $u_n \to u$ in $H^1(G)$. Hence $u \in H_0^1(G)$, and we have shown A is closed.

To compute the adjoint, we note that

$$\int_0^1 Au\bar{v}\,dx = \int_0^1 u\bar{f}\,dx\ , \qquad u \in H_0^1(G)$$

for some pair $v, f \in L^2(G)$ if and only if $v \in H^1(G)$ and $f = i\partial v$. Thus $D^* = H^1(G)$ and $A^* = i\partial$ is a proper extension of A.

7.3

We consider the operator A^* above: on its domain $D^* = H^1(G)$ it is given by $A^* = i\partial$. Since A^* is an adjoint it is closed. We shall compute $A^{**} = (A^*)^*$, the second adjoint of A. We first note that the pair $[u, f] \in H \times H$ is in the graph of A^{**} if and only if

$$\int_0^1 A^*v\bar{u}\,dx = \int_0^1 v\bar{f}\,dx\ , \qquad v \in H^1(G)\ .$$

This holds for all $v \in C_0^\infty(G)$, so we obtain $i\partial u = f$. Substituting this into the above and using Theorem II.1.6, we obtain

$$i\int_0^1 \partial(v\bar{u})\,dx = \int_0^1 \Big[(i\partial v)\bar{u} - v\overline{(i\partial u)}\Big]\,dx = 0\ ,$$

hence, $v(1)\bar{u}(1) - v(0)\bar{u}(0) = 0$ for all $v \in H^1(G)$. But this implies $u(0) = u(1) = 0$, hence, $u \in H_0^1(G)$. From the above it follows that $A^{**} = A$.

7.4

Consider the operator $B = i\partial$ on $L^2(G)$ with domain $D(B) = \{u \in H^1(G) : u(0) = cu(1)\}$ where $c \in \mathbb{C}$ is given. If $v, f \in L^2(G)$, then $B^*v = f$ if and only if

$$\int_0^1 i\partial u \cdot \bar{v}\,dx = \int_0^1 u\bar{f}\,dx\ , \qquad u \in D\ .$$

But $C_0^\infty(G) \le D$ implies $v \in H^1(G)$ and $i\partial v = f$. We substitute this identity in the above and obtain

$$0 = i\int_0^1 \partial(u\bar{v}\,)\,dx = iu(1)[\bar{v}(1) - c\bar{v}(0)]\ , \qquad u \in D\ .$$

The preceding shows that $v \in D(B^*)$ only if $v \in H^1(0,1)$ and $v(1) = \bar{c}v(0)$. It is easy to show that every such v belongs to $D(B^*)$, so we have shown that $D(B^*) = \{v \in H^1(G) : v(1) = \bar{c}v(0)\}$ and $B^* = i\partial$.

7.5

We return to the situation of Section 2.2. Let $a(\cdot,\cdot)$ be a continuous sesquilinear form on the Hilbert space V which is dense and continuously imbedded in the Hilbert space H. We let D be the set of all $u \in V$ such that the map $v \mapsto a(u,v)$ is continuous on V with the norm of H. For such a $u \in D$, there is a unique $Au \in H$ such that

$$a(u,v) = (Au,v)_H\ , \qquad u \in D\ ,\ v \in V\ . \tag{7.2}$$

This defines a linear operator A on H with domain D.

Consider the (adjoint) sesquilinear form on V defined by $b(u,v) = \overline{a(v,u)}$, $u, v \in V$. This gives another operator B on H with domain $D(B)$ determined as before by

$$b(u,v) = (Bu,v)_H\ , \qquad u \in D(B)\ ,\ v \in V\ .$$

Theorem 7.6 *Assume there is a $\lambda > 0$ and $c > 0$ such that*

$$\text{Re } a(u,u) + \lambda|u|_H^2 \geq c\|u\|_V^2 , \qquad u \in V . \tag{7.3}$$

Then D is dense in H, A is closed, and $A^* = B$, hence, $D^* = D(B)$.

Proof: Theorem 2.2 shows D is dense in H. If we prove $A^* = B$, then by symmetry we obtain $B^* = A$, hence A is closed by Lemma 7.2.

Suppose $v \in D(B)$. Then for all $u \in D(A)$ we have $(Au, v)_H = a(u,v) = \overline{b(v,u)} = \overline{(Bv,u)_H}$, hence, $(Au,v)_H = (u, Bv)_H$. This shows $D(B) \leq D^*$ and $A^*|_{D(B)} = B$. We need only to verify that $D(B) = D^*$. Let $u \in D^*$. Since $B + \lambda$ is surjective, there is a $u_0 \in D(B)$ such that $(B+\lambda)u_0 = (A^* + \lambda)u$. Then for all $v \in D$ we have

$$\begin{aligned} ((A+\lambda)v, u)_H &= (v, (B+\lambda)u_0)_H = a(v, u_0) + \lambda(v, u_0)_H \\ &= ((A+\lambda)v, u_0)_H . \end{aligned}$$

But $A+\lambda$ is a surjection, so this implies $u = u_0 \in D(B)$. Hence, $D^* = D(B)$.

For those operators as above which arise from a symmetric sesquilinear form on a space V which is compactly imbedded in H, we can apply the eigenfunction expansion theory for self-adjoint compact operators.

Theorem 7.7 *Let V and H be Hilbert spaces with V dense in H and assume the injection $V \hookrightarrow H$ is compact. Let $A : D \to H$ be the linear operator determined as above by a continuous sesquilinear form $a(\cdot,\cdot)$ on V which we assume is V-elliptic and symmetric:*

$$a(u,v) = \overline{a(v,u)} , \qquad u, v \in V .$$

Then there is a sequence $\{v_j\}$ of eigenfunctions of A with

$$\left.\begin{aligned} Av_j = \lambda_j v_j , \qquad |v_j|_H = 1 , \\ (v_i, v_j)_H = 0 , \qquad i \neq j , \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty \quad \text{as } n \to +\infty , \end{aligned}\right\} \tag{7.4}$$

and $\{v_j\}$ is a basis for H.

Proof: From Theorem 7.6 it follows that $A = A^*$ and, hence, $A^{-1} \in \mathcal{L}(H)$ is self-adjoint. The V-elliptic condition (5.5) shows that $A^{-1} \in \mathcal{L}(H,V)$.

Since the injection $V \hookrightarrow H$ is compact, it follows that $A^{-1} : H \to V \to H$ is compact. We apply Theorem I.7.5 to obtain a sequence $\{v_j\}$ of eigenfunctions of A^{-1} which are orthonormal in H and form a basis for $D = \mathrm{Rg}(A^{-1})$. If their corresponding eigenvalues are denoted by $\{\mu_j\}$, then the symmetry of $a(\cdot,\cdot)$ and (5.5) shows that each μ_j is positive. We obtain (7.4) by setting $\lambda_j = 1/\mu_j$ for $j \geq 1$ and noting that $\lim_{j\to\infty} \mu_j = 0$.

It remains to show $\{v_j\}$ is a basis for H. (We only know that it is a basis for D.) Let $f \in H$ and $u \in D$ with $Au = f$. Let $\sum b_j v_j$ be the Fourier series for f, $\sum c_j v_j$ the Fourier series for u, and denote their respective partial sums by

$$u_n = \sum_{j=1}^{n} c_j v_j \quad , \quad f_n = \sum_{j=1}^{n} b_j v_j \ .$$

We know $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} f_n = f_\infty$ exists in H (cf. Exercise I.7.2). For each $j \geq 1$ we have

$$b_j = (Au, v_j)_H = (u, Av_j)_H = \lambda_j c_j \ ,$$

so $Au_n = f_n$ for all $n \geq 1$. Since A is closed, it follows $Au = f_\infty$, hence, $f = \lim_{n\to\infty} f_n$ as was desired.

If we replace A by $A + \lambda$ in the proof of Theorem 7.7, we observe that ellipticity of $a(\cdot,\cdot)$ is not necessary but only that $a(\cdot,\cdot) + \lambda(\cdot,\cdot)_H$ be V-elliptic for some $\lambda \in \mathbb{R}$.

Corollary 7.8 *Let V and H be given as in Theorem 7.7, let $a(\cdot,\cdot)$ be continuous, sesquilinear, and symmetric. Assume also that*

$$a(v,v) + \lambda |v|_H^2 \geq c\|v\|_V^2 \ , \qquad v \in V$$

for some $\lambda \in \mathbb{R}$ and $c > 0$. Then there is an orthonormal sequence of eigenfunctions of A which is a basis for H and the corresponding eigenvalues satisfy $-\lambda < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$ as $n \to +\infty$.

We give some examples in $H = L^2(G)$, $G = (0,1)$. These eigenvalue problems are known as *Sturm-Liouville problems*. Additional examples are described in the exercises.

7.6

Let $V = H_0^1(G)$ and define $a(u,v) = \int_0^1 \partial u \overline{\partial v}\, dx$. The compactness of $V \to H$ follows from Theorem II.5.7 and Theorem 5.3 shows $a(\cdot,\cdot)$ is $H_0^1(G)$-elliptic. Thus Theorem 7.7 holds; it is a straightforward exercise to compute the eigenfunctions and corresponding eigenvalues for the operator $A = -\partial^2$ with domain $D(A) = H_0^1(G) \cap H^2(G)$:

$$v_j(x) = 2\sin(j\pi x)\ , \quad \lambda = (j\pi)^2\ , \qquad j = 1, 2, 3, \ldots\ .$$

Since $\{v_j\}$ is a basis for $L^2(G)$, each $F \in L^2(G)$ has a Fourier sine-series expansion. Similar results hold in higher dimension for, e.g., the eigenvalue problem

$$\begin{cases} -\Delta_n v(x) = \lambda v(x)\ , & x \in G\ , \\ v(s) = 0\ , & s \in \partial G\ , \end{cases}$$

but the actual computation of the eigenfunctions and eigenvalues is difficult except for very special regions $G \subset \mathbb{R}^n$.

7.7

Let $V = H^1(G)$ and choose $a(\cdot,\cdot)$ as above. The compactness follows from Theorem II.5.8 so Corollary 7.8 applies for any $\lambda > 0$ to give a basis of eigenfunctions for $A = -\partial^2$ with domain $D(A) = \{v \in H^2(G) : v'(0) = v'(1) = 0\}$:

$$\begin{aligned} v_0(x) &= 1\ , \quad v_j(x) = 2\cos(j\pi x)\ ,\ j \geq 1\ , \\ \lambda_j &= (j\pi)^2\ , \qquad\qquad\qquad\quad j \geq 0\ . \end{aligned}$$

As before, similar results hold for the Laplacean with boundary conditions of second type in higher dimensions.

7.8

Let $a(\cdot,\cdot)$ be given as above but set $V = \{v \in H^1(G) : v(0) = v(1)\}$. Then we can apply Corollary 7.8 to the *periodic eigenvalue problem* (cf. (4.5))

$$\begin{aligned} -\partial^2 v(x) &= \lambda v(x)\ , \qquad 0 < x < 1\ , \\ v(0) &= v(1)\ , \quad v'(0) = v'(1)\ . \end{aligned}$$

The eigenfunction expansion is just the standard Fourier series.

Exercises

1.1. Use Theorem 1.1 to show the problem $-\Delta_n u = F$ in G, $u = 0$ on ∂G is well-posed. Hint: Use Theorem II.2.4 to obtain an appropriate norm on $H_0^1(G)$.

1.2. Use Theorem 1.1 to solve (1.1) with the boundary condition $\partial u/\partial\nu + u = 0$ on ∂G. Hint: Use $(u,v)_V \equiv (u,v)_{H^1(G)} + (\gamma u, \gamma v)_{L^2(\partial G)}$ on $H^1(G)$.

2.1. Give the details of the construction of α, β in (2.2).

2.2. Verify the remark on $H = L^2(G)$ following (2.5) (cf. Section I.5.3).

2.3. Use Theorem I.1.1 to construct the F which appears after (2.6). Check that it is continuous.

2.4. Show that $a(u,v) = \int_0^1 \partial u(x) \partial \bar{v}(x)\, dx$, $V = \{u \in H^1(0,1) : u(0) = 0\}$, and $f(v) \equiv v(1/2)$ are admissible data in Theorem 2.1. Find a formula for the unique solution of the problem.

2.5. In Theorem 2.1 the continuous dependence of the solution u on the data f follows from the estimate made in the theorem. Consider the two abstract boundary value problems $\mathcal{A}_1 u_1 = f$ and $\mathcal{A}_2 u_2 = f$ where $f \in V'$, and $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{L}(V, V')$ are coercive with constants c_1, c_2, respectively. Show that the following estimates holds:

$$\|u_1 - u_2\| \leq (1/c_1)\|(\mathcal{A}_2 - \mathcal{A}_1)u_2\| \ ,$$
$$\|u_1 - u_2\| \leq (1/c_1 c_2)\|\mathcal{A}_2 - \mathcal{A}_1\| \, \|f\| \ .$$

Explain how these estimates show that the solution of (2.1) depends continuously on the form $a(\cdot,\cdot)$ or operator $\mathcal{A}$.

3.1. Show (3.3) implies (3.1) in Theorem 3.1.

3.2. (Non-homogeneous Boundary Conditions.) In the situation of Theorem 3.1, assume we have a closed subspace V_1 with $V_0 \subset V_1 \subset V$ and $u_0 \in V$. Consider the problem to find

$$u \in V \ , \quad u - u_0 \in V_1 \ , \quad a(u,v) = f(v) \ , \quad v \in V_1 \ .$$

(a) Show this problem is well-posed if $a(\cdot,\cdot)$ is V_1-coercive.

(b) Characterize the solution by $u - u_0 \in V_1$, $u \in D_0$, $Au = F$, and $\partial u(v) + a_2(\gamma u, \gamma v) = g(\gamma v)$, $v \in V_1$.

(c) Construct an example of the above with $V_0 = H_0^1(G)$, $V = H^1(G)$, $V_1 = \{v \in V : v|_\Gamma = 0\}$, where $\Gamma \subset \partial G$ is given.

4.1. Verify that the formal operator and Green's theorem are as indicated in Section 4.1.

4.2. Characterize the boundary value problem resulting from the choice of $V = \{v \in H^1(G) : v = \text{const. on } G_0\}$ in Section 4.2, where $G_0 \subset G$ is given.

4.3. When G is a cube in $\mathbb{R}^n$, show (4.5) is related to a problem on $\mathbb{R}^n$ with *periodic* solutions.

4.4. Choose V in Section 4.2 so that the solution $u : \mathbb{R}^n \to \mathbb{K}$ is periodic in each coordinate direction.

5.1. Formulate and solve the problem (4.8) with non-homogeneous data prescribed on ∂G and Σ.

5.2. Find choices for V in Section 4.3 which lead to well-posed problems. Characterize the solution by a boundary value problem.

5.3. Prove Corollary 5.4.

5.4. Discuss coercivity of the form (4.6). Hint: $\mathrm{Re}(\int_{\partial G} \frac{\partial u}{\partial \tau} \bar{u}\, ds) = 0$.

6.1. Show (6.4) is strongly-elliptic on Q_+.

6.2. Show that the result of Theorem 6.1 holds for $a(\cdot,\cdot)$ if and only if it holds for $a(\cdot,\cdot) + \lambda(\cdot,\cdot)_{L^2(G)}$. Hence, one may infer coercivity from strong ellipticity without loss of generality.

6.3. If $u \in H^1(G)$, show $\nabla_h(u)$ converges weakly in $L^2(G)$ to $\partial_1(u)$.

6.4. Prove the case (b) in Theorem 6.1.

6.5. Prove Theorem 6.5.

6.6. Give sufficient conditions for the solution of (6.2) to be a classical solution in $C^2_u(G)$.

7.1. Prove Lemma 7.2 of Section 7.1.

7.2. Compute the adjoint of $\partial : \{v \in H^1(G) : v(0) = 0\} \to L^2(G)$, $G = (0,1)$.

7.3. Let $D \leq H^2(G)$, $G = (0,1)$, $a_1(\cdot), a_2(\cdot) \in C^1(\bar{G})$, and define $L : D \to L^2(G)$ by $Lu = \partial^2 u + a_1 \partial u + a_2 u$. The *formal adjoint* of L is defined by

$$L^* v(\varphi) = \int_0^1 v(x)\overline{L\varphi(x)}\, dx \ , \qquad v \in L^2(G) \ , \ \varphi \in C_0^\infty(G) \ .$$

(a) Show $L^* v = \partial^2 v - \partial(\bar{a}_1 v) + \bar{a}_2 v$ in $\mathcal{D}^*(G)$.

(b) If $u, v \in H^2(G)$, then $\int_0^1 (Lu\bar{v} - uL^*\bar{v}\,)\, dx = J(u,v)|_{x=0}^{x=1}$, where $J(u,v) = \bar{v}\partial u - u\partial\bar{v} + a_1 u\bar{v}$.

(c) $D(L^*) = \{v \in H^2(G) : J(u,v)|_{x=0}^{x=1} = 0$, all $u \in D\}$ determines the domain of the $L^2(G)$-adjoint.

(d) Compute $D(L^*)$ when $L = \partial^2 + 1$ and each of the following:

(i) $D = \{u : u(0) = u'(0) = 0\}$,

(ii) $D = \{u : u(0) = u(1) = 0\}$,

(iii) $D = \{u : u(0) = u(1),\ u'(0) = u'(1)\}$.

7.4. Let A be determined by $\{a(\cdot,\cdot), V, H\}$ as in (7.2) and A_λ by $\{a(\cdot,\cdot) + \lambda(\cdot,\cdot)_H, V, H\}$. Show $D(A_\lambda) = D(A)$ and $A_\lambda = A + \lambda I$.

7.5. Let H_j, V_j be Hilbert spaces with V_j continuously embedded in H_j for $j = 1, 2$. Show that if $T \in \mathcal{L}(H_1, H_2)$ and if $T_1 \equiv T|_{V_1} \in L(V_1, V_2)$, then $T_1 \in \mathcal{L}(V_1, V_2)$.

7.6. In the situation of Section 6.4, let $a(\cdot,\cdot)$ be 0-regular on V and assume $a(\cdot,\cdot)$ is also V-elliptic. Let A be determined by $\{a(\cdot,\cdot), V, L^2(G)\}$ as in (7.2).

(a) Show $A^{-1} \in \mathcal{L}(L^2(G), V)$.

(b) Show $A^{-1} \in \mathcal{L}(L^2(G), H^2(G))$.

(c) If $a(\cdot,\cdot)$ is k-regular, show $A^{-p} \in \mathcal{L}(L^2(G), H^{2+k}(G))$ if p is sufficiently large.

7.7. Let A be *self-adjoint* on the complex Hilbert space H. That is, $A = A^*$.

(a) Show that if $\mathrm{Im}(\lambda) \neq 0$, then $\lambda - A$ is invertible and $|\,\mathrm{Im}(\lambda)|\, \|x\|_H \leq \|(\lambda - A)x\|_H$ for all $x \in D(A)$.

(b) $\mathrm{Rg}(\lambda - A)$ is dense in H.

(c) Show $(\lambda - A)^{-1} \in \mathcal{L}(H)$ and $\|(\lambda - A)^{-1}\| \le |\,\mathrm{Im}(\lambda)|^{-1}$.

7.8. Show Theorem 7.7 applies to the mixed Dirichlet-Neumann eigenvalue problem

$$-\partial^2 v = \lambda v(x)\ , \qquad 0 < x < 1\ ,\ v(0) = v'(1) = 0\ .$$

Compute the eigenfunctions.

7.9. Show Corollary 7.8 applies to the eigenvalue problem with boundary conditions of third type

$$-\partial^2 v(x) = \lambda v(x)\ , \qquad 0 < x < 1\ ,$$
$$\partial v(0) - hv(0) = 0 \quad , \quad \partial v(1) + hv(1) = 0\ ,$$

where $h > 0$. Compute the eigenfunctions.

7.10. Take $c\bar{c} = 1$ in Section 7.4 and discuss the eigenvalue problem $Bv = \lambda v$.

7.11. In the proof of Theorem 7.7, deduce that $\{v_j\}$ is a basis for H directly from the fact that $\bar{D} = H$.