Chapter IV

First Order Evolution Equations

1 Introduction

We consider first an initial-boundary value problem for the equation of heat conduction. That is, we seek a function $u : [0, \pi] \times [0, \infty] \to \mathbb{R}$ which satisfies the partial differential equation

$$u_t = u_{xx} \ , \qquad 0 < x < \pi \ , \ t > 0 \tag{1.1}$$

with the boundary conditions

$$u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \qquad t > 0 \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x) \ , \qquad 0 < x < \pi \ . \tag{1.3}$$

A standard technique for solving this problem is the method of separation of variables. One begins by looking for non-identically-zero solutions of (1.1) of the form

$$u(x,t) = v(x)T(t)$$

and is led to consider the pair of ordinary differential equations

$$v'' + \lambda v = 0 \quad , \quad T' + \lambda T = 0$$

and the boundary conditions $v(0) = v(\pi) = 0$. This is an eigenvalue problem for $v(x)$ and the solutions are given by $v_n(x) = \sin(nx)$ with corresponding eigenvalues $\lambda_n = n^2$ for integer $n \geq 1$ (cf. Section II.7.6).

The second of the pair of equations has corresponding solutions

$$T_n(t) = e^{-n^2 t}$$

and we thus obtain a countable set

$$u_n(x,t) = e^{-n^2 t} \sin(nx)$$

of functions which satisfy (1.1) and (1.2). The solution of (1.1), (1.2) and (1.3) is then obtained as the series

$$u(x,t) = \sum_{n=1}^{\infty} u_0^n e^{-n^2 t} \sin(nx) \tag{1.4}$$

where the $\{u_0^n\}$ are the Fourier coefficients

$$u_0^n = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin(nx)\, dx \ , \qquad n \geq 1 \ ,$$

of the initial function $u_0(x)$.

We can regard the representation (1.4) of the solution as a function $t \mapsto S(t)$ from the non-negative reals $\mathbb{R}_0^+$ to the bounded linear operators on $L^2[0,\pi]$. We define $S(t)$ to be the operator given by

$$S(t)u_0(x) = u(x,t) \ ,$$

so $S(t)$ assigns to each function $u_0 \in L^2[0,\pi]$ that function $u(\cdot,t) \in L^2[0,\pi]$ given by (1.4). If $t_1, t_2 \in \mathbb{R}_0^+$, then we obtain for each $u_0 \in L^2[0,\pi]$ the equalities

$$\begin{aligned}
S(t_1)u_0(x) &= \sum_{n=1}^{\infty} (u_0^n e^{-n^2 t_1}) \sin(nx) \\
S(t_2)S(t_1)u_0(x) &= \sum_{n=1}^{\infty} (u_0^n e^{-n^2 t_1}) \sin(nx) e^{-n^2 t_2} \\
&= \sum_{n=1}^{\infty} u_0^n \sin(nx) e^{-n^2 (t_1 + t_2)} \\
&= S(t_1 + t_2) u_0(x) \ .
\end{aligned}$$

Since u_0 is arbitrary, this shows that

$$S(t_1) \cdot S(t_2) = S(t_1 + t_2) \ , \qquad t_1, t_2 \geq 0 \ .$$

This is the *semigroup identity*. We can also show that $S(0) = I$, the identity operator, and that for each u_0, $S(t)u_0 \to u_0$ in $L^2[0,\pi]$ as $t \to 0^+$. Finally, we find that each $S(t)$ has norm $\leq e^{-t}$ in $\mathcal{L}(L^2[0,\pi])$. The properties of $\{S(t) : t \geq 0\}$ that we have obtained here will go into the definition of contraction semigroups. We shall find that each contraction semigroup is characterized by a representation for the solution of a corresponding Cauchy problem.

Finally we show how the semigroup $\{S(t) : t \geq 0\}$ leads to a representation of the solution of the non-homogeneous partial differential equation

$$u_t = u_{xx} + f(x,t) \ , \qquad 0 < x < \pi \ , \ t > 0 \tag{1.5}$$

with the boundary conditions (1.2) and initial condition (1.3). Suppose that for each $t > 0$, $f(\cdot,t) \in L^2[0,\pi]$ and, hence, has the eigenfunction expansion

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(nx) \quad , \quad f_n(t) \equiv \frac{2}{\pi} \int_0^{\pi} f(\xi,t) \sin(n\xi) \, d\xi \ . \tag{1.6}$$

We look for the solution in the form $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(nx)$ and find from (1.5) and (1.3) that the coefficients must satisfy

$$\begin{aligned} u_n'(t) + n^2 u_n(t) &= f_n(t) \ , \ t \geq 0 \ , \\ u_n(0) &= u_n^0 \ , \qquad\qquad n \geq 1 \ . \end{aligned}$$

Hence we have

$$u_n(t) = u_n^0 e^{-n^2 t} + \int_0^t e^{-n^2(t-\tau)} f_n(\tau) \, d\tau$$

and the solution is given by

$$u(x,t) = S(t)u_0(x) + \int_0^t \int_0^{\pi} \left\{ \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2(t-\tau)} \sin(nx) \sin(n\xi) \right\} f(\xi,\tau) \, d\xi \, d\tau \ .$$

But from (1.6) it follows that we have the representation

$$u(\cdot,t) = S(t)u_0(\cdot) + \int_0^t S(t-\tau) f(\cdot,\tau) \, d\tau \tag{1.7}$$

for the solution of (1.5), (1.2), (1.3). The preceding computations will be made precise in this chapter and (1.7) will be used to prove existence and uniqueness of a solution.

2 The Cauchy Problem

Let H be a Hilbert space, $D(A)$ a subspace of H, and $A \in L(D(A), H)$. We shall consider the evolution equation

$$u'(t) + Au(t) = 0 \ . \tag{2.1}$$

The *Cauchy problem* is to find a function $u \in C([0,\infty], H) \cap C^1((0,\infty), H)$ such that, for $t > 0$, $u(t) \in D(A)$ and (2.1) holds, and $u(0) = u_0$, where the initial value $u_0 \in H$ is prescribed.

Assume that for every $u_0 \in D(A)$ there exists a unique solution of the Cauchy problem. Define $S(t)u_0 = u(t)$ for $t \geq 0$, $u_0 \in D(A)$, where $u(\cdot)$ denotes that solution of (2.1) with $u(0) = u_0$. If $u_0, v_0 \in D(A)$ and if $a, b \in \mathbb{R}$, then the function $t \mapsto aS(t)u_0 + bS(t)v_0$ is a solution of (2.1), since A is linear, and the uniqueness of solutions then implies

$$S(t)(au_0 + bv_0) = aS(t)u_0 + bS(t)v_0 \ .$$

Thus, $S(t) \in L(D(A))$ for all $t \geq 0$. If $u_0 \in D(A)$ and $\tau \geq 0$, then the function $t \mapsto S(t+\tau)u_0$ satisfies (2.1) and takes the initial value $S(\tau)u_0$. The uniqueness of solutions implies that

$$S(t+\tau)u_0 = S(t)S(\tau)u_0 \ , \qquad u_0 \in D(A) \ .$$

Clearly, $S(0) = I$.

We define the operator A to be *accretive* if

$$\mathrm{Re}(Ax, x)_H \geq 0 \ , \qquad x \in D(A) \ .$$

If A is accretive and if u is a solution of the Cauchy problem for (2.1), then

$$\begin{aligned} D_t(\|u(t)\|^2) &= 2\,\mathrm{Re}(u'(t), u(t))H \\ &= -2\,\mathrm{Re}(Au(t), u(t))_H \leq 0 \ , \qquad t > 0 \ , \end{aligned}$$

so it follows that $\|u(t)\| \leq \|u(0)\|$, $t \geq 0$. This shows that

$$\|S(t)u_0\| \leq \|u_0\| \ , \qquad u_0 \in D(A) \ , \ t \geq 0 \ ,$$

so each $S(t)$ is a contraction in the H-norm and hence has a unique extension to the closure of $D(A)$. When $D(A)$ is dense, we thereby obtain a contraction semigroup on H.

Definition. A *contraction semigroup* on H is a set $\{S(t) : t \geq 0\}$ of linear operators on H which are contractions and satisfy

$$S(t+\tau) = S(t) \cdot S(\tau) \ , \quad S(0) = I \ , \ t, \tau \geq 0 \ , \tag{2.2}$$

$$S(\cdot)x \in C([0,\infty), H) \ , \qquad\qquad x \in H \ . \tag{2.3}$$

The *generator* of the contraction semigroup $\{S(t) : t \geq 0\}$ is the operator with domain

$$D(B) = \Big\{ x \in H : \lim_{h \to 0^+} h^{-1}(S(h) - I)x = D^+(S(0)x) \text{ exists in } H \Big\}$$

and value $Bx = \lim_{h\to 0^+} h^{-1}(S(h) - I)x = D^+(S(0)x)$. Note that Bx is the right-derivative at 0 of $S(t)x$.

The equation (2.2) is the *semigroup identity*. The definition of solution for the Cauchy problem shows that (2.3) holds for $x \in D(A)$, and an elementary argument using the uniform boundedness of the (contraction) operators $\{S(t) : t \geq 0\}$ shows that (2.3) holds for all $x \in H$. The property (2.3) is the *strong continuity* of the semigroup.

Theorem 2.1 *Let $A \in L(D(A), H)$ be accretive with $D(A)$ dense in H. Suppose that for every $u_0 \in D(A)$ there is a unique solution $u \in C^1([0,\infty), H)$ of* (2.1) *with $u(0) = u_0$. Then the family of operators $\{S(t) : t \geq 0\}$ defined as above is a contraction semigroup on H whose generator is an extension of $-A$.*

Proof: Note that uniqueness of solutions is implied by A being accretive, so the semigroup is defined as above. We need only to verify that $-A$ is a restriction of the generator. Let B denote the generator of $\{S(t) : t \geq 0\}$ and $u_0 \in D(A)$. Since the corresponding solution $u(t) = S(t)u_0$ is right-differentiable at 0, we have

$$S(h)u_0 - u_0 = \int_0^h u'(t)\,dt = -\int_0^h Au(t)\,dt \ , \qquad h > 0 \ .$$

Hence, we have $D^+(S(0)u_0) = -Au_0$, so $u_0 \in D(B)$ and $Bu_0 = -Au_0$.

We shall see later that if $-A$ is the generator of a contraction semigroup, then A is accretive, $D(A)$ is dense, and for every $u_0 \in D(A)$ there is a unique solution $u \in C^1([0,\infty), H)$ of (2.1) with $u(0) = u_0$. But first, we consider a simple example.

Theorem 2.2 *For each $B \in \mathcal{L}(H)$, the series $\sum_{n=0}^{\infty}(B^n/n!)$ converges in $\mathcal{L}(H)$; denote its sum by $\exp(B)$. The function $t \mapsto \exp(tB) : \mathbb{R} \to \mathcal{L}(H)$ is infinitely differentiable and satisfies*

$$D[\exp(tB)] = B \cdot \exp(tB) = \exp(tB) \cdot B \ , \qquad t \in \mathbb{R} \ . \tag{2.4}$$

If $B_1, B_2 \in \mathcal{L}(H)$ and if $B_1 \cdot B_2 = B_2 \cdot B_1$, then

$$\exp(B_1 + B_2) = \exp(B_1) \cdot \exp(B_2) \ . \tag{2.5}$$

Proof: The convergence of the series in $\mathcal{L}(H)$ follows from that of $\sum_{n=0}^{\infty} \|B\|^n_{\mathcal{L}(H)}/n! = \exp(\|B\|)$ in $\mathbb{R}$. To verify the differentiability of $\exp(tB)$ at $t = 0$, we note that

$$\Big[(\exp(tB) - I)/t\Big] - B = (1/t)\sum_{n=2}^{\infty}(tB)^n/n! \ , \qquad t \neq 0 \ ,$$

and this gives the estimate

$$\|\Big[(\exp(tB) - I)/t\Big] - B\| \leq (1/|t|)\Big[\exp(|t| \cdot \|B\|) - 1 - |t|\,\|B\|\Big] \ .$$

Since $t \mapsto \exp(t\|B\|)$ is (right) differentiable at 0 with (right) derivative $\|B\|$, it follows that (2.4) holds at $t = 0$. The semigroup property shows that (2.4) holds at every $t \in \mathbb{R}$. (We leave (2.5) as an exercise.)

3 Generation of Semigroups

Our objective here is to characterize those operators which generate contraction semigroups.

To first obtain necessary conditions, we assume that $B : D(B) \to H$ is the generator of a contraction semigroup $\{S(t) : t \geq 0\}$. If $t \geq 0$ and $x \in D(B)$, then the last term in the identity

$$h^{-1}(S(t+h)x - S(t)x) = h^{-1}(S(h) - I)S(t)x = h^{-1}S(t)(S(h)x - x) \ , \qquad h > 0 \ ,$$

has a limit as $h \to 0^+$, hence, so also does each term and we obtain

$$D^+S(t)x = BS(t)x = S(t)Bx \ , \qquad x \in D(B) \ , \ t \geq 0 \ .$$

Similarly, using the uniform boundedness of the semigroup we may take the limit as $h \to 0^+$ in the identity

$$h^{-1}(S(t)x - S(t-h)x) = S(t-h)h^{-1}(S(h)x - x) \ , \qquad 0 < h < t \ ,$$

to obtain

$$D^- S(t)x = S(t)Bx \ , \qquad x \in D(B) \ , \ t > 0 \ .$$

We summarize the above.

Lemma *For each $x \in D(B)$, $S(\cdot)x \in C^1(\mathbb{R}_0^+, H)$, $S(t)x \in D(B)$, and*

$$S(t)x - x = \int_0^t BS(s)x\,ds = \int_0^t S(s)Bx\,dx \ , \qquad t \geq 0 \ . \tag{3.1}$$

Corollary *B is closed.*

Proof: Let $x_n \in D(B)$ with $x_n \to x$ and $Bx_n \to y$ in H. For each $h > 0$ we have from (3.1)

$$h^{-1}(S(h)x_n - x_n) = h^{-1}\int_0^h S(s)Bx_n\,ds \ , \qquad n \geq 1 \ .$$

Letting $n \to \infty$ and then $h \to 0^+$ gives $D^+S(0)x = y$, hence, $Bx = y$.

Lemma *$D(B)$ is dense in H; for each $t \geq 0$ and $x \in H$, $\int_0^t S(s)x\,ds \in D(B)$ and*

$$S(t)x - x = B\int_0^t S(s)x\,ds \ , \qquad x \in H \ , \ t \geq 0 \ . \tag{3.2}$$

Proof: Define $x_t = \int_0^t S(s)x\,ds$. Then for $h > 0$

$$\begin{aligned} h^{-1}(S(h)x_t - x_t) &= h^{-1}\left\{\int_0^t S(h+s)x\,ds - \int_0^t S(s)x\,ds\right\} \\ &= h^{-1}\left\{\int_h^{t+h} S(s)x\,ds - \int_0^t S(s)x\,ds\right\} . \end{aligned}$$

Adding and subtracting $\int_t^h S(s)x\,ds$ gives the equation

$$h^{-1}(S(h)x_t - x_t) = h^{-1}\int_t^{t+h} S(s)x\,ds - h^{-1}\int_0^h S(s)x\,ds \ ,$$

and letting $h \to 0$ shows that $x_t \in D(B)$ and $Bx_t = S(t)x - x$. Finally, from $t^{-1}x_t \to x$ as $t \to 0^+$, it follows that $D(B)$ is dense in H.

Let $\lambda > 0$. Then it is easy to check that $\{e^{-\lambda t}S(t) : t \geq 0\}$ is a contraction semigroup whose generator is $B - \lambda$ with domain $D(B)$. From (3.1) and (3.2) applied to this semigroup we obtain

$$e^{-\lambda t}S(t)x - x = \int_0^t e^{-\lambda s}S(s)(B - \lambda)x\,ds\ , \qquad x \in D(B)\ ,\ t \geq 0\ ,$$

$$e^{-\lambda t}S(t)y - y = (B - \lambda)\int_0^t e^{-\lambda s}S(s)y\,ds\ , \qquad y \in H\ , \qquad t \geq 0\ .$$

Letting $t \to \infty$ (and using the fact that B is closed to evaluate the limit of the last term) we find that

$$x = \int_0^\infty e^{-\lambda s}S(s)(\lambda - B)x\,ds\ , \qquad x \in D(B)\ ,$$

$$y = (\lambda - B)\int_0^\infty e^{-\lambda s}S(s)y\,ds\ , \qquad y \in H\ .$$

These identities show that $\lambda - B$ is injective and surjective, respectively, with

$$\|(\lambda - B)^{-1}y\| \leq \int_0^\infty e^{-\lambda s}\,ds\|y\| = \lambda^{-1}\|y\|\ , \qquad y \in H\ .$$

This proves the necessity part of the following fundamental result.

Theorem 3.1 *Necessary and sufficient conditions that the operator $B : D(B) \to H$ be the generator of a contraction semigroup on H are that*

$D(B)$ is dense in H and $\lambda - B : D(B) \to H$ is a bijection with $\|\lambda(\lambda - B)^{-1}\|_{\mathcal{L}(H)} \leq 1$ for all $\lambda > 0$.

Proof: (Continued) It remains to show that the indicated conditions on B imply that it is the generator of a semigroup. We shall achieve this as follows: (a) approximate B by bounded operators, B_λ, (b) obtain corresponding semigroups $\{S_\lambda(t) : t \geq 0\}$ by exponentiating B_λ, then (c) show that $S(t) \equiv \lim_{\lambda\to\infty} S_\lambda(t)$ exists and is the desired semigroup.

Since $\lambda - B : D(B) \to H$ is a bijection for each $\lambda > 0$, we may define $B_\lambda = \lambda B(\lambda - B)^{-1}$, $\lambda > 0$.

Lemma *For each $\lambda > 0$, $B_\lambda \in \mathcal{L}(H)$ and satisfies*

$$B_\lambda = -\lambda + \lambda^2(\lambda - B)^{-1}\ . \tag{3.3}$$

For $x \in D(B)$, $\|B_\lambda(x)\| \le \|Bx\|$ and $\lim_{\lambda\to\infty} B_\lambda(x) = Bx$.

Proof: Equation (3.3) follows from $(B_\lambda+\lambda)(\lambda-B)x = \lambda^2 x$, $x \in D(B)$. The estimate is obtained from $B_\lambda = \lambda(\lambda - B)^{-1}B$ and the fact that $\lambda(\lambda - B)^{-1}$ is a contraction. Finally, we have from (3.3)

$$\|\lambda(\lambda - B)^{-1}x - x\| = \|\lambda^{-1}B_\lambda x\| \le \lambda^{-1}\|Bx\| \;, \qquad \lambda > 0 \;,\; x \in D(B) \;,$$

hence, $\lambda(\lambda-B)^{-1}x \mapsto x$ for all $x \in D(B)$. But $D(B)$ dense and $\{\lambda(\lambda-B)^{-1}\}$ uniformly bounded imply $\lambda(\lambda - B)^{-1}x \to x$ for all $x \in H$, and this shows $B_\lambda x = \lambda(\lambda - B)^{-1}Bx \to Bx$ for $x \in D(B)$.

Since B_λ is bounded for each $\lambda > 0$, we may define by Theorem 2.2

$$S_\lambda(t) = \exp(tB_\lambda) \;, \qquad \lambda > 0 \;,\; t \ge 0 \;.$$

Lemma *For each $\lambda > 0$, $\{S_\lambda(t) : t \ge 0\}$ is a contraction semigroup on H with generator B_λ. For each $x \in D(B)$, $\{S_\lambda(t)x\}$ converges in H as $\lambda \to \infty$, and the convergence is uniform for $t \in [0, T]$, $T > 0$.*

Proof: The first statement follows from

$$\|S_\lambda(t)\| = e^{-\lambda t}\|\exp(\lambda^2(\lambda - B)^{-1}t)\| \le e^{-\lambda t}e^{\lambda t} = 1 \;,$$

and $D(S_\lambda(t)) = B_\lambda S_\lambda(t)$. Furthermore,

$$\begin{aligned} S_\lambda(t) - S_\mu(t) &= \int_0^t D_s S_\mu(t-s)S_\lambda(s)\,ds \\ &= \int_0^t S_\mu(t-s)S_\lambda(s)(B_\lambda - B_\mu)\,ds \;, \qquad \mu, \lambda > 0 \;, \end{aligned}$$

in $\mathcal{L}(H)$, so we obtain

$$\|S_\lambda(t)x - S_\mu(t)s\| \le t\|B_\lambda x - B_\mu x\| \;, \qquad \lambda, \mu > 0 \;,\; t \ge 0 \;,\; x \in D(B) \;.$$

This shows $\{S_\lambda(t)x\}$ is uniformly Cauchy for t on bounded intervals, so the Lemma follows.

Since each $S_\lambda(t)$ is a contraction and $D(B)$ is dense, the indicated limit holds for all $x \in H$, and uniformly on bounded intervals. We define $S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x$, $x \in H$, $t \ge 0$, and it is clear that each $S(t)$ is a linear contraction. The uniform convergence on bounded intervals implies $t \mapsto$

$S(t)x$ is continuous for each $x \in H$ and the semigroup identity is easily verified. Thus $\{S(t) : t \geq 0\}$ is a contraction semigroup on H. If $x \in D(B)$ the functions $S_\lambda(\cdot)B_\lambda x$ converge uniformly to $S(\cdot)Bx$ and, hence, for $h > 0$ we may take the limit in the identity

$$S_\lambda(h)x - x = \int_0^h S_\lambda(t)B_\lambda x\,dt$$

to obtain

$$S(h)x - x = \int_0^h S(t)Bx\,dt\ , \qquad x \in D(B)\ ,\ h > 0\ .$$

This implies that $D^+(S(0)x) = Bx$ for $x \in D(B)$. If C denotes the generator of $\{S(t) : t \geq 0\}$, we have shown that $D(B) \subset D(C)$ and $Bx = Cx$ for all $x \in D(B)$. That is, C is an extension of B. But $I - B$ is surjective and $I - C$ is injective, so it follows that $D(B) = D(C)$.

Corollary 3.2 *If $-A$ is the generator of a contraction semigroup, then for each $u_0 \in D(A)$ there is a unique solution $u \in C^1([0,\infty), H)$ of* (2.1) *with $u(0) = u_0$.*

Proof: This follows immediately from (3.1).

Theorem 3.3 *If $-A$ is the generator of a contraction semigroup, then for each $u_0 \in D(A)$ and each $f \in C^1([0,\infty), H)$ there is a unique $u \in C^1([0,\infty), H)$ such that $u(0) = u_0$, $u(t) \in D(A)$ for $t \geq 0$, and*

$$u'(t) + Au(t) = f(t)\ , \qquad t \geq 0\ . \tag{3.4}$$

Proof: It suffices to show that the function

$$g(t) = \int_0^t S(t-\tau)f(\tau)\,d\tau\ , \qquad t \geq 0\ ,$$

satisfies (3.4) and to note that $g(0) = 0$. Letting $z = t - \tau$ we have

$$(g(t+h) - g(t))/h = \int_0^t S(z)(f(t+h-z) - f(t-z))h^{-1}\,dz$$
$$+h^{-1}\int_t^{t+h} S(z)f(t+h-z)\,dz$$

so it follows that $g'(t)$ exists and

$$g'(t) = \int_0^t S(z)f'(t-z)\,dz + S(t)f(0)\ .$$

Furthermore we have

$$\begin{aligned}(g(t+h)-g(t))/h &= h^{-1}\Big\{\int_0^{t+h} S(t+h-\tau)f(\tau)\,d\tau - \int_0^t S(t-\tau)f(\tau)\,d\tau\Big\} \\ &= (S(h)-I)h^{-1}\int_0^t S(t-\tau)f(\tau)\,d\tau \\ &\qquad + h^{-1}\int_t^{t+h} S(t+h-\tau)f(\tau)\,d\tau\ . \qquad (3.5)\end{aligned}$$

Since $g'(t)$ exists and since the last term in (3.5) has a limit as $h \to 0^+$, it follows from (3.5) that

$$\int_0^t S(t-\tau)f(\tau)\,d\tau \in D(A)$$

and that g satisfies (3.4).

4 Accretive Operators; two examples

We shall characterize the generators of contraction semigroups among the negatives of accretive operators. In our applications to boundary value problems, the conditions of this characterization will be more easily verified than those of Theorem 3.1. These applications will be illustrated by two examples; the first contains a first order partial differential equation and the second is the second order equation of heat conduction in one dimension. Much more general examples of the latter type will be given in Section 7.

The two following results are elementary and will be used below and later.

Lemma 4.1 *Let $B \in \mathcal{L}(H)$ with $\|B\| < 1$. Then $(I-B)^{-1} \in \mathcal{L}(H)$ and is given by the power series $\sum_{n=0}^{\infty} B^n$ in $\mathcal{L}(H)$.*

Lemma 4.2 *Let $A \in L(D(A), H)$ where $D(A) \le H$, and assume $(\mu - A)^{-1} \in \mathcal{L}(H)$, with $\mu \in \mathbb{C}$. Then $(\lambda - A)^{-1} \in \mathcal{L}(H)$ for $\lambda \in \mathbb{C}$, if and only if $[I-(\mu-\lambda)(\mu-A)^{-1}]^{-1} \in \mathcal{L}(H)$, and in that case we have*

$$(\lambda - A)^{-1} = (\mu - A)^{-1}[I-(\mu-\lambda)(\mu-A)^{-1}]^{-1}\ .$$

Proof: Let $B \equiv I - (\mu - \lambda)(\mu - A)^{-1}$ and assume $B^{-1} \in \mathcal{L}(H)$. Then we have

$$\begin{aligned}(\lambda - A)(\mu - A)^{-1}B^{-1} &= [(\lambda - \mu) + (\mu - A)](\mu - A)^{-1}B^{-1} \\ &= [(\lambda - \mu)(\mu - A)^{-1} + I]B^{-1} = I\ ,\end{aligned}$$

and

$$\begin{aligned}(\mu - A)^{-1}B^{-1}(\lambda - A) &= (\mu - A)^{-1}B^{-1}[(\lambda - \mu) + (\mu - A)] \\ &= (\mu - A)^{-1}B^{-1}[B(\mu - A)] = I\ , \quad \text{on} \quad D(A)\ .\end{aligned}$$

The converse is proved similarly.

Suppose now that $-A$ generates a contraction semigroup on H. From Theorem 3.1 it follows that

$$\|(\lambda + A)x\| \geq \lambda\|x\|\ , \qquad \lambda > 0\ ,\ x \in D(A)\ , \tag{4.1}$$

and this is equivalent to

$$2\,\mathrm{Re}(Ax, x)_H \geq -\|Ax\|^2/\lambda\ , \qquad \lambda > 0\ ,\ x \in D(A)\ .$$

But this shows A is accretive and, hence, that Theorem 3.1 implies the necessity part of the following.

Theorem 4.3 *The linear operator $-A : D(A) \to H$ is the generator of a contraction semigroup on H if and only if $D(A)$ is dense in H, A is accretive, and $\lambda + A$ is surjective* for some $\lambda > 0$.

Proof: (Continued) It remains to verify that the above conditions on the operator A imply that $-A$ satisfies the conditions of Theorem 3.1. Since A is accretive, the estimate (4.1) follows, and it remains to show that $\lambda + A$ is surjective for every $\lambda > 0$.

We are given $(\mu + A)^{-1} \in \mathcal{L}(H)$ for some $\mu > 0$ and $\|\mu(\mu + A)^{-1}\| \leq 1$. For any $\lambda \in C$ we have $\|(\lambda - \mu)(\mu + A)^{-1}\| \leq |\lambda - \mu|/\mu$, hence Lemma 4.1 shows that $I - (\lambda - \mu)(\lambda + A)^{-1}$ has an inverse which belongs to $\mathcal{L}(H)$ if $|\lambda - \mu| < \mu$. But then Lemma 4.2 implies that $(\lambda + A)^{-1} \in \mathcal{L}(H)$. Thus, $(\mu + A)^{-1} \in \mathcal{L}(H)$ with $\mu > 0$ implies that $(\lambda + A)^{-1} \in \mathcal{L}(H)$ for all $\lambda > 0$ such that $|\lambda - \mu| < \mu$, i.e., $0 < \lambda < 2\mu$. The result then follows by induction.

Example 1. Let $H = L^2(0,1)$, $c \in \mathbb{C}$, $D(A) = \{u \in H^1(0,1) : u(0) = cu(1)\}$, and $A = \partial$. Then we have for $u \in H^1(0,1)$

$$2\,\mathrm{Re}(Au,u)_H = \int_0^1 (\partial u \cdot \bar{u} + \overline{\partial u} \cdot u) = |u(1)|^2 - |u(0)|^2 \ .$$

Thus, A is accretive if (and only if) $|c| \leq 1$, and we assume this hereafter. Theorem 4.3 implies that $-A$ generates a contraction semigroup on $L^2(0,1)$ if (and only if) $I + A$ is surjective. But this follows from the solvability of the problem

$$u + \partial u = f \ , \qquad u(0) = cu(1)$$

for each $f \in L^2(0,1)$; the solution is given by

$$u(x) = \int_0^1 G(x,s) f(s)\,ds \ ,$$

$$G(x,s) = \begin{cases} [e/(e-c)]e^{-(x-s)} \ , & 0 \leq s < x \leq 1 \ , \\ [c/(e-c)]e^{-(x-s)} \ , & 0 \leq x < s \leq 1 \ . \end{cases}$$

Since $-A$ generates a contraction semigroup, the initial boundary value problem

$$\partial_t u(x,t) + \partial_x u(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t \geq 0 \tag{4.2}$$

$$u(0,t) = cu(1,t) \tag{4.3}$$

$$u(x,0) = u_0(x) \tag{4.4}$$

has a unique solution for each $u_0 \in D(A)$. This can be verified directly. Since any solution of (4.2) is locally of the form

$$u(x,t) = F(x-t)$$

for some function F; the equation (4.4) shows

$$u(x,t) = u_0(x-t) \ , \qquad 0 \leq t \leq x \leq 1 \ .$$

Then (4.3) gives $u(0,t) = cu_0(1-t)$, $0 \leq t \leq 1$, so (4.2) then implies

$$u(x,t) = cu_0(1+x-t) \ , \qquad x \leq t \leq x+1 \ .$$

An easy induction gives the representation

$$u(x,t) = c^n u_0(n+x-t) \ , \qquad n-1+x \leq t \leq n+1 \ , \ n \geq 1 \ .$$

The representation of the solution of (4.2)–(4.4) gives some additional information on the solution. First, the Cauchy problem can be solved only if $u_0 \in D(A)$, because $u(\cdot,t) \in D(A)$ implies $u(\cdot,t)$ is (absolutely) continuous and this is possible only if u_0 satisfies the boundary condition (4.3). Second, the solution satisfies $u(\cdot,t) \in H^1(0,1)$ for every $t \geq 1$ but will *not* belong to $H^2(0,1)$ unless $\partial u_0 \in D(A)$. That is, we do not in general have $u(\cdot,t) \in H^2(0,1)$, no matter how smooth the initial function u_0 may be. Finally, the representation above defines a solution of (4.2)–(4.4) on $-\infty < t < \infty$ by allowing n to be any integer. Thus, the problem can be solved backwards in time as well as forward. This is related to the fact that $-A$ generates a *group* of operators and we shall develop this notion in Section 5. Also see Section V.3 and Chapter VI.

Example 2. For our second example, we take $H = L^2(0,1)$ and let $A = -\partial^2$ on $D(A) = H_0^1(0,1) \cap H^2(0,1)$. An integration-by-parts gives

$$(Au,u)_H = \int_0^1 |\partial u|^2 \ , \qquad u \in D(A) \ ,$$

so A is accretive, and the solvability of the boundary value problem

$$u - \partial^2 u = f \ , \qquad u(0) = 0 \ , \ u(1) = 0 \ , \tag{4.5}$$

for $f \in L^2(0,1)$ shows that $I + A$ is surjective. (We may either solve (4.5) directly by the classical variation-of-parameters method, thereby obtaining the representation

$$u(x) = \int_0^1 G(x,s) f(s)\, ds \ ,$$

$$G(x,s) = \begin{cases} \dfrac{\sinh(1-x)\sinh(s)}{\sinh(1)} \ , & 0 \leq s < x \leq 1 \\[2ex] \dfrac{\sinh(1-s)\sinh(x)}{\sinh(1)} \ , & 0 \leq x < s \leq 1 \end{cases}$$

or observe that it is a special case of the boundary value problem of Chapter III.) Since $-A$ generates a contraction semigroup on $L^2(0,1)$, it follows from Corollary 3.2 that there is a unique solution of the initial-boundary value problem

$$\begin{aligned} &\partial_t u - \partial_x^2 u = 0 \ , \qquad 0 < x < 1 \ , \ t \geq 0 \\ &u(0,t) = 0 \ , \quad u(1,t) = 0 \ , \\ &u(x,0) = u_0(x) \end{aligned} \tag{4.6}$$

for each initial function $u_0 \in D(A)$.

A representation of the solution of (4.6) can be obtained by the method of separation-of-variables. This representation is the Fourier series (cf. (1.4))

$$u(x,t) = 2\int_0^1 \sum_{n=0}^{\infty} u_0(s)\sin(ns)\sin(nx)e^{-n^2 t}\,ds \tag{4.7}$$

and it gives information that is not available from Corollary 3.2. First, (4.7) defines a solution of the Cauchy problem for every $u_0 \in L^2(0,1)$, not just for those in $D(A)$. Because of the factor $e^{-n^2 t}$ in the series (4.7), every derivative of the sequence of partial sums is convergent in $L^2(0,1)$ whenever $t > 0$, and one can thereby show that the solution is infinitely differentiable in the open cylinder $(0,1)\times(0,\infty)$. Finally, the series will in general not converge if $t < 0$. This occurs because of the exponential terms, and severe conditions must be placed on the initial data u_0 in order to obtain convergence at a point where $t < 0$. Even when a solution exists on an interval $[-T, 0]$ for some $T > 0$, it will not depend continuously on the initial data (cf., Exercise 1.3). The preceding situation is typical of Cauchy problems which are resolved by *analytic semigroups*. Such Cauchy problems are (appropriately) called *parabolic* and we shall discuss these notions in Sections 6 and 7 and again in Chapters V and VI.

5 Generation of Groups; a wave equation

We are concerned here with a situation in which the evolution equation can be solved on the whole real line $\mathbb{R}$, not just on the half-line $\mathbb{R}^+$. This is the case when $-A$ generates a *group* of operators on H.

Definition. A *unitary group* on H is a set $\{G(t) : t \in \mathbb{R}\}$ of linear operators on H which satisfy

$$G(t+\tau) = G(t)\cdot G(\tau)\ , \quad G(0) = I\ , \qquad t,\tau \in \mathbb{R}\ , \tag{5.1}$$

$$G(\cdot)x \in C(\mathbb{R}, H)\ , \qquad x \in H\ , \tag{5.2}$$

$$\|G(t)\|_{\mathcal{L}(H)} = 1\ , \qquad t \in \mathbb{R}\ . \tag{5.3}$$

The *generator* of this unitary group is the operator B with domain

$$D(B) = \left\{x \in H : \lim_{h\to 0} h^{-1}(G(h) - I)x \ \text{ exists in } \ H\right\}$$

with values given by $Bx = \lim_{h\to 0} h^{-1}(G(h) - I)x = D(G(0)x)$, the (two-sided) derivative at 0 of $G(t)x$.

Equation (5.1) is the group condition, (5.2) is the condition of strong continuity of the group, and (5.3) shows that each operator $G(t)$, $t \in \mathbb{R}$, is an isometry. Note that (5.1) implies

$$G(t) \cdot G(-t) = I \ , \qquad t \in \mathbb{R} \ ,$$

so each $G(t)$ is a bijection of H onto H whose inverse is given by

$$G^{-1}(t) = G(-t) \ , \qquad t \in \mathbb{R} \ .$$

If $B \in \mathcal{L}(H)$, then (5.1) and (5.2) are satisfied by $G(t) \equiv \exp(tB)$, $t \in \mathbb{R}$ (cf., Theorem 2.2). Also, it follows from (2.4) that B is the generator of $\{G(t) : t \in \mathbb{R}\}$ and

$$D(\|G(t)x\|^2) = 2\,\mathrm{Re}(BG(t)x, G(t)x)_H \ , \qquad x \in H \ , \ t \in \mathbb{R} \ ,$$

hence, (5.3) is satisfied if and only if $\mathrm{Re}(Bx, x)_H = 0$ for all $x \in H$. These remarks lead to the following.

Theorem 5.1 *The linear operator $B : D(B) \to H$ is the generator of a unitary group on H if and only if $D(B)$ is dense in H and $\lambda - B$ is a bijection with $\|\lambda(\lambda - B)^{-1}\|_{\mathcal{L}(H)} \le 1$ for all $\lambda \in \mathbb{R}$, $\lambda \ne 0$.*

Proof: If B is the generator of the unitary group $\{G(t) : t \in \mathbb{R}\}$, then B is the generator of the contraction semigroup $\{G(t) : t \ge 0\}$ and $-B$ is the generator of the contraction semigroup $\{G(-t) : t \ge 0\}$. Thus, both B and $-B$ satisfy the necessary conditions of Theorem 3.1, and this implies the stated conditions on B. Conversely, if B generates the contraction semigroup $\{S_+(t) : t \ge 0\}$ and $-B$ generates the contraction semigroup $\{S_-(t) : t \ge 0\}$, then these operators commute. For each $x_0 \in D(B)$ we have

$$D[S_+(t)S_-(-t)x_0] = 0 \ , \qquad t \ge 0 \ ,$$

so $S_+(t)S_-(-t) = I$, $t \ge 0$. This shows that the family of operators defined by

$$G(t) = \begin{cases} S_+(t) \ , & t \ge 0 \\ S_-(-t) \ , & t < 0 \end{cases}$$

satisfies (5.1). The condition (5.2) is easy to check and (5.3) follows from

$$1 = \|G(t) \cdot G(-t)\| \le \|G(t)\| \cdot \|G(-t)\| \le \|G(t)\| \le 1 \ .$$

Finally, it suffices to check that B is the generator of $\{G(t) : t \in \mathbb{R}\}$ and then the result follows.

Corollary 5.2 *The operator A is the generator of a unitary group on H if and only if for each $u_0 \in D(A)$ there is a unique solution $u \in C^1(\mathbb{R}, H)$ of (2.1) with $u(0) = u_0$ and $\|u(t)\| = \|u_0\|$, $t \in \mathbb{R}$.*

Proof: This is immediate from the proof of Theorem 5.1 and the results of Theorem 2.1 and Corollary 3.2.

Corollary 5.3 *If A generates a unitary group on H, then for each $u_0 \in D(A)$ and each $f \in C^1(\mathbb{R}, H)$ there is a unique solution $u \in C^1(\mathbb{R}, H)$ of (3.3) and $u(0) = u_0$. This solution is given by*

$$u(t) = G(t)u_0 + \int_0^t G(t-\tau)f(\tau)\,d\tau \ , \qquad t \in \mathbb{R} \ .$$

Finally, we obtain an analogue of Theorem 4.3 by noting that both $+A$ and $-A$ are accretive exactly when A satisfies the following.

Definition. The linear operator $A \in L(D(A), H)$ is said to be *conservative* if

$$\mathrm{Re}(Ax, x)_H = 0 \ , \qquad x \in D(A) \ .$$

Corollary 5.4 *The linear operator $A : D(A) \to H$ is the generator of a unitary group on H if and only if $D(A)$ is dense in H, A is conservative, and $\lambda + A$ is surjective for some $\lambda > 0$ and for some $\lambda < 0$.*

Example. Take $H = L^2(0,1) \times L^2(0,1)$, $D(A) = H_0^1(0,1) \times H^1(0,1)$, and define

$$A[u, v] = [-i\partial v, i\partial u] \ , \qquad [u, v] \in D(A) \ .$$

Then we have

$$(A[u, v], [u, v])_H = i \int_0^1 (\partial v \cdot \bar{u} - \partial u \cdot \bar{v}) \ , \qquad [u, v] \in D(A)$$

and an integration-by-parts gives

$$2\,\mathrm{Re}(A[u,v],[u,v])_H = i(\,\bar{u}(x)v(x) - u(x)\bar{v}(x))\Big|_{x=0}^{x=1} = 0 \ , \tag{5.4}$$

since $u(0) = u(1) = 0$. Thus, A is a conservative operator. If $\lambda \neq 0$ and $[f_1, f_2] \in H$, then

$$\lambda[u,v] + A[u,v] = [f_1, f_2] \ , \qquad [u,v] \in D(A)$$

is equivalent to the system

$$-\partial^2 u + \lambda^2 u = \lambda f_1 - i\partial f_2 \ , \qquad u \in H_0^1(0,1) \ , \tag{5.5}$$

$$-i\partial u + \lambda v = f_2 \ , \qquad v \in H^1(0,1) \ . \tag{5.6}$$

But (5.5) has a unique solution $u \in H_0^1(0,1)$ by Theorem III.2.2 since $\lambda f_1 - i\partial f_2 \in (H_0^1)'$ from Theorem II.2.2. Then (5.6) has a solution $v \in L^2(0,1)$ and it follows from (5.6) that

$$(i\lambda)\partial v = \lambda f_1 - \lambda^2 u \in L^2(0,1) \ ,$$

so $v \in H^1(0,1)$. Thus $\lambda + A$ is surjective for $\lambda \neq 0$.

Corollaries 5.3 and 5.4 imply that the Cauchy problem

$$\begin{aligned} D\mathbf{u}(t) + A\mathbf{u}(t) &= [0, f(t)] \ , \qquad t \in \mathbb{R} \ , \\ \mathbf{u}(0) &= [u_0, v_0] \end{aligned} \tag{5.7}$$

is well-posed for $u_0 \in H_0^1(0,1)$, $v_0 \in H^1(0,1)$, and $f \in C^1(\mathbb{R}, H)$. Denoting by $u(t)$, $v(t)$, the components of $\mathbf{u}(t)$, i.e., $\mathbf{u}(t) \equiv [u(t), v(t)]$, it follows that $u \in C^2(\mathbb{R}, L^2(0,1))$ satisfies the wave equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = f(x,t) \ , \qquad 0 < x < 1 \ , \ t \in \mathbb{R} \ ,$$

and the initial-boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = 0 \\ u(x,0) &= u_0(x) \ , \quad \partial_t u(x,0) = -iv_0(x) \ . \end{aligned}$$

See Section VI.5 for additional examples of this type.

6 Analytic Semigroups

We shall consider the Cauchy problem for the equation (2.1) in the special case in which A is a model of an elliptic boundary value problem (cf. Corollary 3.2). Then (2.1) is a corresponding abstract parabolic equation for which Example 2 of Section IV.4 was typical. We shall first extend the definition of $(\lambda + A)^{-1}$ to a sector properly containing the right half of the complex plane $\mathbb{C}$ and then obtain an integral representation for an analytic continuation of the semigroup generated by $-A$.

Theorem 6.1 *Let V and H be Hilbert spaces for which the identity $V \hookrightarrow H$ is continuous. Let $a : V \times V \to \mathbb{C}$ be continuous, sesquilinear, and V-elliptic. In particular*

$$|a(u,v)| \le K\|u\| \, \|v\| \ , \qquad u, v \in V \ ,$$

$$\operatorname{Re} a(u,u) \ge c\|u\|^2 \ , \qquad u \in V \ ,$$

where $0 < c \le K$. Define

$$D(A) = \{u \in V : |a(u,v)| \le K_u |v|_H \ , \ v \in V\} \ ,$$

where K_u depends on u, and let $A \in L(D(A), H)$ be given by

$$a(u,v) = (Au, v)_H \ , \qquad u \in D(A) \ , \ v \in V \ .$$

Then $D(A)$ is dense in H and there is a θ_0, $0 < \theta_0 < \pi/4$, such that for each $\lambda \in S(\pi/2 + \theta_0) \equiv \{z \in \mathbb{C} : |\arg(z)| < \pi/2 + \theta_0\}$ we have $(\lambda + A)^{-1} \in \mathcal{L}(H)$. For each θ, $0 < \theta < \theta_0$, there is an M_θ such that

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(H)} \le M_\theta \ , \qquad \lambda \in S(\theta + \pi/2) \ . \tag{6.1}$$

Proof: Suppose $\lambda \in \mathbb{C}$ with $\lambda = \sigma + i\tau$, $\sigma \ge 0$. Since the form u, $v \mapsto a(u,v) + \lambda(u,v)_H$ is V-elliptic it follows that $\lambda + A : D(A) \to H$ is surjective. (This follows directly from the discussion in Section III.2.2; note that A is the restriction of $\mathcal{A}$ to $D(A) = D$.) Furthermore we have the estimate

$$\sigma |u|_H^2 \le \operatorname{Re}\{a(u,u) + \lambda(u,u)_H\} \le |(A + \lambda)u|_H |u|_H \ , \qquad u \in D(A) \ ,$$

so it follows that

$$\|\sigma(\lambda + A)^{-1}\| \le 1 \ , \qquad \operatorname{Re}(\lambda) = \sigma \ge 0 \ . \tag{6.2}$$

From the triangle inequality we obtain

$$|\tau|\,|u|_H^2 - K\|u\|_V^2 \le |\operatorname{Im}((\lambda + A)u, u)_H|\ , \qquad u \in D(A)\ , \tag{6.3}$$

where $\tau = \operatorname{Im}(\lambda)$. We show that this implies that either

$$|\operatorname{Im}((\lambda + A)u, u)_H| \ge (|\tau|/2)|u|_H^2 \tag{6.4}$$

or that

$$\operatorname{Re}((\lambda + A)u, u)_H \ge (c/2K)|\tau|\,|u|_H^2\ . \tag{6.5}$$

If (6.4) does not hold, then substitution of its negation into (6.3) gives $(|\tau|/2)\|u\|_H^2 \le K\|u\|_V^2$. But we have

$$\operatorname{Re}((\lambda + A)u, u)_H \ge c\|u\|_V^2$$

so (6.5) follows. Since one of (6.4) or (6.5) holds, it follows that

$$|((\lambda + A)u, u)_H| \ge (c/2K)|\tau|\,|u|_H^2\ , \qquad u \in D(A)\ ,$$

and this gives the estimate

$$\|\tau(\lambda + A)^{-1}\| \le 2K/c\ , \qquad \lambda = \sigma + i\tau\ ,\ \sigma \ge 0\ . \tag{6.6}$$

Now let $\lambda = \sigma + i\tau \in \mathbb{C}$ with $\tau \ne 0$ and set $\mu = i\tau$. From (6.6) we have

$$\|(\mu + A)^{-1}\| \le 2K/c|\mu|\ ,$$

so Lemma 4.1 shows that

$$\|[I - (\lambda - \mu)(\mu + A)^{-1}]^{-1}\| \le [1 - |\lambda - \mu|2K/c|\mu|]^{-1}$$

whenever $|\sigma|/|\tau| = (\lambda - \mu)/|\mu| < c/2K$.

From Lemma 4.2 we then obtain $(\lambda + A)^{-1} \in \mathcal{L}(H)$ and

$$\begin{aligned}\|\lambda(\lambda + A)^{-1}\| &\le (2K/c)(|\sigma|/|\tau| + 1)(1 - 2K|\sigma|/c|\tau|)^{-1}\ ,\\ &\lambda = \sigma + i\tau\ , \quad |\sigma|/|\tau| < c/2K\ .\end{aligned} \tag{6.7}$$

Theorem 6.1 now follows from (6.2) and (6.7) with $\theta_0 = \tan^{-1}(c/2K)$.

From (6.2) it is clear that the operator $-A$ is the generator of a contraction semigroup $\{S(t) : t \ge 0\}$ on H. We shall obtain an analytic extension of this semigroup.

Theorem 6.2 *Let $A \in L(D(A), H)$ be the operator of Theorem 6.1. Then there is a family of operators $\{T(t) : t \in S(\theta_0)\}$ satisfying*

(a) $T(t+\tau) = T(t) \cdot T(\tau)$, $\qquad t, \tau \in S(\theta_0)$,

and for $x, y \in H$, the function $t \mapsto (T(t)x, y)_H$ is analytic on $S(\theta_0)$;

(b) *for $t \in S(\theta_0)$, $T(t) \in L(H, D(A))$ and*

$$-\frac{dT(t)}{dt} = A \cdot T(t) \in \mathcal{L}(H) \ ;$$

(c) *if $0 < \varepsilon < \theta_0$, then for some constant $C(\varepsilon)$,*

$$\|T(t)\| \le C(\varepsilon) \|tAT(t)\| \le C(\varepsilon) \ , \qquad t \in S(\theta_0 - \varepsilon) \ ,$$

and for $x \in H$, $T(t) \to x$ as $t \to 0$, $t \in S(\theta_0 - \varepsilon)$.

Proof: Let θ be chosen with $\theta_0/2 < \theta < \theta_0$ and let C be the path consisting of the two rays $|\arg(z)| = \pi/2 + \theta$, $|z| \ge 1$, and the semi-circle $\{e^{it} : |t| \le \theta + \pi/2\}$ oriented so as to run from $\infty \cdot e^{-i(\pi/2+\theta)}$ to $\infty \cdot e^{i(\pi/2+\theta)}$.

If $t \in S(2\theta - \theta_0)$, then we have

$$|\arg(\lambda t)| \ge |\arg \lambda| - |\arg t| \ge \pi/2 + (\theta_0 - \theta) \ , \qquad \lambda \in C \ , \ |\lambda| \ge 1 \ ,$$

so we obtain the estimate

$$\mathrm{Re}(\lambda t) \le -\sin(\theta_0 - \theta)|\lambda t| \ , \qquad t \in S(2\theta - \theta_0) \ .$$

This shows that the (improper) integral

$$T(t) \equiv \frac{1}{2\pi i} \int_C e^{\lambda t} (\lambda + A)^{-1} \, d\lambda \ , \qquad t \in S(2\theta - \theta_0) \tag{6.8}$$

exists and is absolutely convergent in $\mathcal{L}(H)$. If $x, y \in H$ then

$$(T(t)x, y)_H = \frac{1}{2\pi i} \int_C e^{\lambda t} ((\lambda + A)^{-1} x, y)_H \, d\lambda$$

is analytic in t. If C' is a curve obtained by translating C to the right, then from Cauchy's theorem we obtain

$$(T(t)x, y)_H = \frac{1}{2\pi i} \int_{C'} e^{\lambda' t} ((\lambda' + A)^{-1} x, y)_H \, d\lambda' \ .$$

Hence, we have

$$T(t) = \frac{1}{2\pi i}\int_{C'} e^{\lambda' t}(\lambda' + A)^{-1}\, d\lambda' \ ,$$

since x, y are arbitrary and the integral is absolutely convergent in $\mathcal{L}(H)$. The semigroup identity follows from the calculation

$$\begin{aligned}
T(t)T(\tau) &= \left(\frac{1}{2\pi i}\right)^2 \int_{C'}\int_{C} e^{\lambda'\tau+\lambda t}(\lambda' + A)^{-1}(\lambda + A)^{-1}\, d\lambda\, d\lambda' \\
&= \left(\frac{1}{2\pi i}\right)^2 \Bigg[\int_{C'} e^{\lambda'\tau}(\lambda' + A)^{-1}\Big\{\int_C e^{\lambda t}(\lambda - \lambda')^{-1}\, d\lambda\Big\}\, d\lambda' \\
&\qquad - \int_C e^{\lambda t}(\lambda + A)^{-1}\Big\{\int_{C'} e^{\lambda'\tau}(\lambda - \lambda')^{-1}\, d\lambda'\Big\}\, d\lambda\Bigg] \\
&= \frac{1}{2\pi i}\int_C e^{\lambda(t+\tau)}(\lambda + A)^{-1}\, d\lambda = T(t+\tau) \ ,
\end{aligned}$$

where we have used Fubini's theorem and the identities

$$(\lambda + A)^{-1}(\lambda' + A)^{-1} = (\lambda - \lambda')^{-1}[(\lambda' + A)^{-1} - (\lambda + A)^{-1}] \ ,$$

$$\int_C e^{\lambda t}(\lambda - \lambda')^{-1}\, d\lambda = 0 \quad , \quad \int_{C'} e^{\lambda'\tau}(\lambda - \lambda')^{-1} d\lambda' = -2\pi i e^{\lambda\tau} \ .$$

Since $\theta \in (\theta_0/2, \theta_0)$ is arbitrary, (a) follows from above.

Similarly, we may differentiate (6.8) and obtain

$$\begin{aligned}
\frac{dT(t)}{dt} &= \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda(\lambda + A)^{-1}\, d\lambda \qquad (6.9)\\
&= \frac{1}{2\pi i}\int_C e^{\lambda t}[I - A(\lambda + A)^{-1}]\, d\lambda \\
&= \frac{-1}{2\pi i}\int_C e^{\lambda t}A(\lambda + A)^{-1}\, d\lambda \ .
\end{aligned}$$

Since A is closed, this implies that for $t \in S(2\theta - \theta_0)$, $\theta_0/2 < \theta < \theta_0$, we have

$$-\frac{dT(t)}{dt} = AT(t) \in \mathcal{L}(H)$$

so (b) follows.

We next consider (c). Letting $\theta = \theta_0 - \varepsilon/2$, we obtain from (6.1) and (6.8) the estimate

$$\begin{aligned}\|T(t)\| &\le \frac{1}{2\pi}\int_C |e^{\lambda t}|\cdot\|(\lambda + A)^{-1}\|\,d|\lambda| \\ &\le \frac{M_\theta}{2\pi}\int_C e^{\operatorname{Re}\lambda t}\frac{d|\lambda|}{|\lambda|}\ .\end{aligned}$$

Since $\operatorname{Re}\lambda t \le -\sin(\varepsilon/2)\cdot|\lambda t|$ in this integral, the last quantity depends only on ε. The second estimate in (c) follows similarly.

To study the behavior of $T(t)$ for $t \in S(\theta_0 - \varepsilon)$ close to 0, we first note that if $x \in D(A)$

$$\begin{aligned}T(t)x - x &= \frac{1}{2\pi i}\int_C e^{\lambda t}((\lambda + A)^{-1} - \lambda^{-1})x\,d\lambda \\ &= \frac{-1}{2\pi i}\int_C e^{\lambda t}(\lambda + A)^{-1}Ax\,d\lambda/\lambda\ ,\end{aligned}$$

and, hence, we obtain the estimate

$$\|T(t)x - x\| \le |t|\frac{M_\theta}{2\pi}\left\{\int_C e^{-\sin(\varepsilon/2)|\lambda t|}\ \frac{d|t\lambda|}{|t\lambda|^2}\right\}\|Ax\|\ .$$

Thus, $T(t)x \to x$ as $t \to 0$ with $t \in S(\theta_0 - \varepsilon)$. Since $D(A)$ is dense and $\{T(t) : t \in S(\theta_0 - \varepsilon)\}$ is uniformly bounded, this proves (c).

Definition. A family of operators $\{T(t) : t \in S(\theta_0) \cup \{0\}\}$ which satisfies the properties of Theorem 6.2 and $T(0) = I$ is called an *analytic semigroup*.

Theorem 6.3 *Let A be the operator of Theorem 6.1, $\{T(t) : t \in S(\theta_0)\}$ be given by* (6.8)*, and $T(0) = I$. Then the collection of operators $\{T(t) : t \ge 0\}$ is the contraction semigroup generated by $-A$.*

Proof: Let $u_0 \in H$ and define $u(t) = T(t)u_0$, $t \ge 0$. Theorem 6.2 shows that u is a solution of the Cauchy problem (2.1) with $u(0) = u_0$. Theorem 2.1 implies that $\{T(t) : t \ge 0\}$ is a contraction semigroup whose generator is an extension of $-A$. But $I + A$ is surjective, so the result follows.

Corollary 6.4 *If A is the operator of Theorem 6.1, then for every $u_0 \in H$ there is a unique solution $u \in C([0,\infty), H) \cap C^\infty((0,\infty), H)$ of* (2.1) *with $u(0) = u_0$. For each $t > 0$, $u(t) \in D(A^p)$ for every $p \ge 1$.*

There are some important differences between Corollary 6.4 and its counterpart, Corollary 3.2. In particular we note that Corollary 6.4 solves the Cauchy problem for all initial data in H, while Corollary 3.2 is appropriate only for initial data in $D(A)$. Also, the infinite differentiability of the solution from Corollary 6.4 and the consequential inclusion in the domain of every power of A at each $t > 0$ are properties not generally true in the situation of Corollary 3.2. These regularity properties are typical of parabolic problems (cf., Section 7).

Theorem 6.5 *If A is the operator of Theorem 6.1, then for each $u_0 \in H$ and each* Hölder continuous $f : [0,\infty) \to H$*:*

$$\|f(t) - f(\tau)\| \le K(t-\tau)^\alpha \ , \qquad 0 \le \tau \le t \ ,$$

where K and α are constant, $0 < \alpha \le 1$, there is a unique $u \in C([0,\infty), H) \cap C^1((0,\infty), H)$ such that $u(0) = u_0$, $u(t) \in D(A)$ for $t > 0$, and

$$u'(t) + Au(t) = f(t) \ , \qquad t > 0 \ .$$

Proof: It suffices to show that the function

$$g(t) = \int_0^t T(t-\tau) f(\tau)\, d\tau \ , \qquad t \ge 0 \ ,$$

is a solution of the above with $u_0 = 0$. Note first that for $t > 0$

$$g(t) = \int_0^t T(t-\tau)(f(\tau) - f(t))\, d\tau + \int_0^t T(t-\tau)\, d\tau \cdot f(t) \ .$$

from Theorem 6.2(c) and the Hölder continuity of f we have

$$\|A \cdot T(t-\tau)(f(\tau) - f(t))\| \le C(\theta_0) K |t-\tau|^{\alpha-1} \ ,$$

and since A is closed we have $g(t) \in D(A)$ and

$$Ag(t) = A \int_0^t T(t-\tau)(f(\tau) - f(t))\, d\tau + (I - T(t)) \cdot f(t) \ .$$

The result now follows from the computation (3.5) in the proof of Theorem 3.3.

7 Parabolic Equations

We were led to consider the abstract Cauchy problem in a Hilbert space H

$$u'(t) + Au(t) = f(t) \ , \qquad t > 0 \ ; \ u(0) = u_0 \tag{7.1}$$

by an initial-boundary value problem for the parabolic partial differential equation of heat conduction. Some examples of (7.1) will be given in which A is an operator constructed from an abstract boundary value problem. In these examples A will be a linear unbounded operator in the Hilbert space $L^2(G)$ of equivalence classes of functions on the domain G, so the construction of a representative $U(\cdot,t)$ of $u(t)$ is non-trivial. In particular, if such a representative is chosen arbitrarily, the functions $t \mapsto U(x,t)$ need not even be measurable for a given $x \in G$.

We begin by constructing a measurable representative $U(\cdot,\cdot)$ of a solution $u(\cdot)$ of (7.1) and then make precise the correspondence between the vector-valued derivative $u'(t)$ and the partial derivative $\partial_t U(\cdot,t)$.

Theorem 7.1 *Let $I = [a,b]$, a closed interval in $\mathbb{R}$ and G be an open (or measurable) set in $\mathbb{R}^n$.*

(a) *If $u \in C(I, L^2(G))$, then there is a measurable function $U : I \to \mathbb{R}$ such that*

$$u(t) = U(\cdot,t) \ , \qquad t \in I \ . \tag{7.2}$$

(b) *If $u \in C^1(I, L^2(G))$, U and V are measurable real-valued functions on $G \times I$ for which (7.2) holds for a.e. $t \in I$ and*

$$u'(t) = V(\cdot,t) \ , \qquad a.e. \quad t \in I \ ,$$

then $V = \partial_t U$ in $\mathcal{D}^*(G \times I)$.

Proof: (a) For each $t \in I$, let $U_0(\cdot,t)$ be a representative of $u(t)$. For each integer $n \geq 1$, let $a = t_0 < t_1 < \cdots < t_n = b$ be the uniform partition of I and define

$$U_n(x,t) = \begin{cases} U_0(x,t_k) \ , & t_k \leq t < t_{k+1} \ , \ k = 0,1,\ldots,n-1 \\ U_0(x,t) \ , & t = t_n \ . \end{cases}$$

Then $U_n : G \times I \to \mathbb{R}$ is measurable and

$$\lim_{n\to\infty} \|U_n(\cdot,t) - u(t)\|_{L^2(G)} = 0 \ ,$$

uniformly for $t \in I$. This implies

$$\lim_{m,n\to\infty} \int_I \int_G |U_m - U_n|^2 \, dx \, dt = 0$$

and the completeness of $L^2(G \times I)$ gives a $U \in L^2(G \times I)$ for which

$$\lim_{n\to\infty} \int_I \int_G |U - U_n|^2 \, dx \, dt = 0 \ .$$

It follows from the above (and the triangle inequality)

$$\int_I \|u(t) - U(\cdot, t)\|^2_{L^2(G)} \, dt = 0$$

so $u(t) = U(\cdot, t)$ for a.e. $t \in I$. The desired result follows by changing $u(t)$ to $U_0(\cdot, t)$ on a set in I of zero measure.

(b) Let $\Phi \in C_0^\infty(G \times I)$. Then $\varphi(t) \equiv \Phi(\cdot, t)$ defines $\varphi \in C_0^\infty(I, L^2(G))$. But for any $\varphi \in C_0^\infty(I, L^2(G))$ and u as given

$$-\int_I (u(t), \varphi'(t))_{L^2(G)} \, dt = \int_I (u'(t), \varphi(t))_{L^2(G)} \, dt \ ,$$

and thus we obtain

$$-\int_I \int_G U(x,t) D_t \Phi(x,t) \, dx \, dt = \int_I \int_G V(x,t) \Phi(x,t) \, dx \, dt \ .$$

This holds for all $\Phi \in C_0^\infty(G \times I)$, so the stated result holds.

We next consider the construction of the operator A appearing in (7.1) from the abstract boundary value problem of Section III.3. Assume we are given Hilbert spaces $V \subset H$, and B with a linear surjection $\gamma : V \to B$ with kernel V_0. Assume γ factors into an isomorphism of V/V_0 onto B, the injection $V \hookrightarrow H$ is continuous, and V_0 is dense in H, and H is identified with H'. (Thus, we obtain the continuous injections $V_0 \hookrightarrow H \hookrightarrow V_0'$ and $V \hookrightarrow H \hookrightarrow V'$.) (Cf. Section III.2.3 for a typical example.)

Suppose we are given a continuous sesquilinear form $a_1 : V \times V \to \mathbb{K}$ and define the formal operator $A_1 \in \mathcal{L}(V, V_0')$ by

$$A_1 u(v) = a_1(u, v) \ , \qquad u \in V \ , \ v \in V_0 \ .$$

Let $D_0 \equiv \{u \in V : A_1(u) \in H\}$ and denote by $\partial_1 \in L(D_0, B')$ the abstract Green's operator constructed in Theorem III.2.3. Thus

$$a_1(u, v) - (A_1 u, v)_H = \partial_1 u(\gamma(v)) \ , \qquad u \in D_0 \ , \ v \in V \ .$$

Suppose we are also given a continuous sesquilinear form $a_2 : B \times B \to \mathbb{K}$ and define $\mathcal{A}_2 \in \mathcal{L}(B, B')$ by

$$\mathcal{A}_2 u(v) = a_2(u, v) \ , \qquad u, v \in B \ .$$

Then we define a continuous sesquilinear form on V by

$$a(u, v) \equiv a_1(u, v) + a_2(\gamma(u), \gamma(v)) \ , \qquad u, v \in V \ .$$

Consider the triple $\{a(\cdot,\cdot), V, H\}$ above. From these we construct as in Section 6 an unbounded operator on H whose domain $D(A)$ is the set of all $u \in V$ such that there is an $F \in H$ for which

$$a(u, v) = (F, v)_H \ , \qquad v \in V \ .$$

Then define $A \in L(D(A), H)$ by $Au = F$. (Thus, A is the operator in Theorem 6.1.) From Corollary III.3.2 we can obtain the following result.

Theorem 7.2 *Let the spaces, forms and operators be as given above. Then* $D(A) \subset D_0$, $A = A_1|_{D(A)}$, *and* $u \in D(A)$ *if and only if* $u \in V$, $A_1 u \in H$, *and* $\partial_1 u + \mathcal{A}_2(\gamma(u)) = 0$ *in* B'.

(We leave a direct proof as an exercise.) We obtain the existence of a weak solution of a mixed initial-boundary value problem for a large class of parabolic boundary value problems from Theorems 6.5, 7.1 and 7.2.

Theorem 7.3 *Suppose we are given an abstract boundary value problem as above (i.e., Hilbert spaces* V, H, B, *continuous sesquilinear forms* $a_1(\cdot,\cdot)$, $a_2(\cdot,\cdot)$, *and operators* γ, ∂_1, A_1 *and* $\mathcal{A}_2$*) and that* $H = L^2(G)$ *where* G *is an open set in* $\mathbb{R}^n$. *Assume that for some* $c > 0$

$$\mathrm{Re}\Big\{a_1(v, v) + a_2(\gamma(v), \gamma(v))\Big\} \geq c\|v\|_V^2 \ , \qquad v \in V \ .$$

Let $U_0 \in L^2(G)$ *and a measurable* $F : G \times [0, T] \to \mathbb{K}$ *be given for which* $F(\cdot, t) \in L^2(G)$ *for all* $t \in [0, T]$ *and for some* $K \in L^2(G)$ *and* α, $0 < \alpha \leq 1$, *we have*

$$|F(x, t) - F(x, \tau)| \leq K(x)|t - \tau|^\alpha \ , \quad a.e. \ \ x \in G \ , \ t \in [0, T] \ .$$

Then there exists a $U \in L^2(G \times [0, T])$ *such that for all* $t > 0$

$$\left.\begin{aligned} &U(\cdot, t) \in V \ , \ \partial_t U(\cdot, t) + A_1 U(\cdot, t) = F(\cdot, t) \ \ in \ \ L^2(G) \ , \\ &and \ \ \partial_1 U(\cdot, t) + \mathcal{A}_2(\gamma U(\cdot, t)) = 0 \ \ in \ \ B' \ , \end{aligned}\right\} \tag{7.3}$$

and

$$\lim_{t\to 0}\int_G |U(x,t)-U_0(x)|^2\,dx = 0\ .$$

We shall give some examples which illustrate particular cases of Theorem 7.3. Each of the following corresponds to an elliptic boundary value problem in Section III.4, and we refer to that section for details on the computations.

7.1

Let the open set G in $\mathbb{R}^n$, coefficients a_{ij}, $a_j \in L^\infty(G)$, and sesquilinear form $a(\cdot,\cdot) = a_1(\cdot,\cdot)$, and spaces H and B be given as in Section III.4.1. Let $U_0 \in L^2(G)$ be given together with a function $F : G\times[0,T]\to\mathbb{K}$ as in Theorem 7.3. If we choose

$$V = \{v \in H^1(G) : \gamma_0 v(s) = 0\ ,\quad \text{a.e.}\quad s\in\Gamma\}$$

where Γ is a prescribed subset of ∂G, then a solution U of (7.3) satisfies

$$\left.\begin{aligned}
&\partial_t U - \sum_{i,j=1}^n \partial_j(a_{ij}\partial_i U) + \sum_{j=0}^n a_j\partial_j U = F \quad\text{in}\quad L^2(G\times[0,T])\ ,\\
&U(s,t) = 0\ ,\qquad t>0\ ,\quad \text{a.e.}\quad s\in\Gamma\ ,\quad\text{and}\\
&\frac{\partial U(s,t)}{\partial\nu_A} = 0\ ,\qquad t>0\ ,\quad\text{a.e.}\quad s\in\partial G\sim\Gamma\ ,
\end{aligned}\right\}\tag{7.4}$$

where

$$\frac{\partial U}{\partial\nu_A} \equiv \sum_{i=1}^n \partial_i U\Big(\sum_{j=1}^n a_{ij}\nu_j\Big)$$

denotes the derivative in the direction determined by $\{a_{ij}\}$ and the unit outward normal ν on ∂G. The second equation in (7.4) is called the boundary condition of *first type* and the third equation is known as the boundary condition of *second type*.

7.2

Let V be a closed subspace of $H^1(G)$ to be chosen below, $H = L^2(G)$, $V_0 = H_0^1(G)$ and define

$$a_1(u,v) = \int_G \nabla u\cdot\overline{\nabla v}\ ,\qquad u,v\in V\ .$$

Then $A_1 = -\Delta_n$ and ∂_1 is an extension of the normal derivative $\partial/\partial\nu$ on ∂G. Let $\alpha \in L^\infty(\partial G)$ and define

$$a_2(\varphi, \psi) = \int_{\partial G} \alpha(s)\varphi(s)\overline{\psi(s)}\, ds \ , \qquad \varphi, \psi \in L^2(\partial G) \ .$$

(Note that $B \subset L^2(\partial G) \subset B'$ and $\mathcal{A}_2\varphi = \alpha \cdot \varphi$.) Let $U_0 \in L^2(G)$ and F be given as in Theorem 7.3. Then (exercise) Theorem 7.3 asserts the existence of a solution of (7.3). If we choose $V = H^1(G)$, this solution satisfies

$$\left.\begin{aligned} &\partial_t U - \Delta_n U = F \quad \text{in} \quad L^2(G \times [0,T]) \ , \\ &\frac{\partial U(s,t)}{\partial \nu} + \alpha(s) U(s,t) = 0 \ , \qquad t > 0 \ , \quad \text{a.e.} \quad s \in \partial G \end{aligned}\right\} \tag{7.5}$$

If we choose $V = \{v \in H^1(G) : \gamma v = \text{constant}\}$, then U satisfies

$$\left.\begin{aligned} &\partial_t U - \Delta_n U = F \quad \text{in} \quad L^2(G \times [0,T]) \ , \\ &U(s,t) = u_0(t) \ , \qquad t > 0 \ , \quad \text{a.e.} \quad s \in \partial G \ , \\ &\int_{\partial G} \frac{\partial U(s,t)}{\partial \nu}\, ds + \int_{\partial G} \alpha(s)\, ds \cdot u_0(t) = 0 \ , \qquad t > 0 \ . \end{aligned}\right\} \tag{7.6}$$

The boundary conditions in (7.5) and (7.6) are known as the *third type* and *fourth type*, respectively. Other types of problems can be solved similarly, and we leave these as exercises. In particular, each of the examples from Section III.4 has a counterpart here.

Our final objective of this chapter is to demonstrate that the weak solutions of certain of the preceding mixed initial-boundary value problems are necessarily strong or classical solutions. Specifically, we shall show that the weak solution is smooth for problems with smooth or regular data.

Consider the problem (7.4) above with $F \equiv 0$. The solution $u(\cdot)$ of the abstract problem is given by the semigroup constructed in Theorem 6.2 as $u(t) = T(t)u_0$. (We are assuming that $a(\cdot,\cdot)$ is V-elliptic.) Since $T(t) \in L(H, D(A))$ and $AT(t) \in \mathcal{L}(H)$ for all $t > 0$, we obtain from the identity $(T(t/m))^m = T(t)$ that $T(t) \in L(H, D(A^m))$ for integer $m \geq 1$. This is an abstract regularity result; generally, for parabolic problems $D(A^m)$ consists of increasingly smooth functions as m gets large. Assume also that $a(\cdot,\cdot)$ is k-regular over V (cf. Section 6.4) for some integer $k \geq 0$. Then A^{-1} maps $H^s(G)$ into $H^{2+s}(G)$ for $0 \leq s \leq k$, so $D(A^m) \subset H^{2+k}$ whenever $2m \geq 2+k$. Thus, we have the spatial regularity result that $u(t) \in H^{2+k}(G)$ for all $t > 0$

when $a(\cdot,\cdot)$ is V-elliptic and k-regular. One can clearly use the imbedding results of Section II.4 to show $U(\cdot,t) \in C_u^p(G)$ when $2(2+k) > 2p+n$.

We consider the regularity in time of the solution of the abstract problem corresponding to (7.4). First note that $A^m : D(A^m) \to H$ defines a scalar product on $D(A^m)$ for which $D(A^m)$ is a Hilbert space. Fix $t > \tau > 0$ and consider the identity

$$(1/h)(u(t+h) - u(t)) = A^{-m}[(1/h)(T(t+h-\tau) - T(t-\tau))A^m u(t)]$$

for $0 < |h| < t - \tau$. Since $A^m u(\tau) \in H$, the term in brackets converges in H, hence $u'(t) \in D(A^m)$ for all $t > 0$ and integer m. This is an abstract temporal regularity result. Assume now that $a(\cdot,\cdot)$ is k-regular over V. The preceding remarks show that the above difference quotients converge to $u'(t) = \partial_t U(\cdot,t)$ in the space $H^{2+k}(G)$. The convergence holds in $C_u^p(G)$ if $2(2+k) > 2p+n$ as before, and the solution U is a classical solution for $p \geq 2$. Thus, (7.4) has a classical solution when the above hypotheses hold for some $k > n/2$.

Exercises

1.1. Supply all details in Section 1.

1.2. Develop analogous series representations for the solution of (1.5) and (1.3) with the boundary conditions

(a) $u_x(0,t) = u_x(\pi,t) = 0$ of Neumann type (cf. Section III.7.7),

(b) $u(0,t) = u(\pi,t)$, $u_x(0,t) = u_x(\pi,t)$ of periodic type (cf. Section III.7.8).

1.3. Find the solution of the *backward heat equation*

$$u_t + u_{xx} = 0 \ , \qquad 0 < x < \pi \ , \ t > 0$$

subject to $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \sin(nx)/n$. Discuss the dependence of the solution on the initial data $u(x,0)$.

2.1. If A is given as in Section III.7.C, obtain the eigenfunction series representation for the solution of (2.1).

2.2. Show that if $u, v \in C^1((0,T), H)$, then

$$D_t(u(t), v(t))_H = (u'(t), v(t))_H + (u(t), v'(t))_H \ , \qquad 0 < t < T \ .$$

2.3. Show (2.3) holds for all $x \in H$.

2.4. Verify (2.5).

3.1. If $\{S(t)\}$ is a contraction semigroup with generator B, show that $\{e^{-\lambda t}S(t)\}$ is a contraction semigroup for $\lambda > 0$ and that its generator is $B - \lambda$.

3.2. Verify the limits as $t \to \infty$ in the two identities leading up to Theorem 3.1.

3.3. Show $B(\lambda - B)^{-1} = (\lambda - B)^{-1}B$ for B as in Theorem 3.1.

3.4. Show that Theorem 3.3 holds if we replace the given hypothesis on f by $f : \mathbb{R}^+ \to D(A)$ and $Af(\cdot) \in C([0,\infty), H)$.

4.1. Prove Lemma 4.1.

4.2. Show that the hypothesis in Theorem 4.3 that $D(A)$ is dense in H is unnecessary. Hint: If $x \in D(A)^\perp$, then $x = (\lambda + A)z$ for some $z \in D(A)$ and $z = \theta$.

4.3. Show that (4.1) follows from A being accretive.

4.4. For the operator A in Example 4(a), find the kernel and range of $\lambda + A$ for each $\lambda \geq 0$ and c with $|c| \leq 1$.

4.5. Solve (4.5) by the methods of Chapter III.

4.6. Solve (4.5) and (4.6) when the Dirichlet conditions are replaced by Neumann conditions. Repeat for other boundary conditions.

5.1. Show that operators $\{S_+(t)\}$ and $\{S_-(t)\}$ commute in the proof of Theorem 5.1.

5.2. Verify all details in the Example of Section 5.

5.3. If A is self-adjoint on the complex Hilbert space H, show iA generates a unitary group. Discuss the Cauchy problem for the Schrodinger equation $u_t = i\Delta_n u$ on $\mathbb{R}^n \times \mathbb{R}$.

5.4. Formulate and discuss some well-posed problems for the equation $\partial_t u + \partial_x^3 u = 0$ for $0 < x < 1$ and $t > 0$.

6.1. Verify all the estimates which lead to the convergence of the integral (6.8).

6.2. Finish the proof of Theorem 6.5.

6.3. Show that $f(t) \equiv \int_0^t F(s)\,ds$ is Hölder continuous if $F(\cdot) \in L^p(0,T;H)$ for some $p > 1$.

6.4. Show that for $0 < \varepsilon < \theta_0$ and integer $n \geq 1$, there is a constant $c_{\varepsilon,n}$ for which $\|t^n A^n T(t)\| \leq c_{\varepsilon,n}$ for $t \in S(\theta_0 - \varepsilon)$ in the situation of Theorem 6.2.

7.1. In the proof of Theorem 7.1(a), verify $\lim_{n\to\infty} \|U_n(\cdot,t) - u(t)\| = 0$. For Theorem 7.1(b), show $\varphi \in C_0^\infty(I, L^2(G))$.

7.2. Give a proof of Theorem 7.2 without appealing to the results of Corollary III.3.2.

7.3. Show that the change of variable $u(t) = e^{\lambda t}v(t)$ in (7.1) gives a corresponding equation with A replaced by $A + \lambda$. Verify that (7.4) is well-posed if $a_1(\cdot,\cdot)$ is strongly elliptic.

7.4. Show that (7.3) is equivalent to (7.5) for an appropriate choice of V. Show how to solve (7.5) with a non-homogeneous boundary condition.

7.5. Show that (7.3) is equivalent to (7.6) for an appropriate choice of V. Show how to solve (7.6) with a non-homogeneous boundary condition. If G is an interval, show *periodic boundary conditions* are obtained.

7.6. Solve initial-boundary value problems corresponding to each of examples in Sections 4.3, 4.4, and 4.5 of Chapter III.

7.7. Show that $u(t) = T(t)u_0$ converges to u_0 in $D(A^m)$ if and only if $u_0 \in D(A^m)$. Discuss the corresponding limit $\lim_{t\to 0^+} U(\cdot,t)$ in (7.4).