Chapter V

Implicit Evolution Equations

1 Introduction

We shall be concerned with evolution equations in which the time-derivative of the solution is not given explicitly. This occurs, for example, in problems containing the pseudoparabolic equation

$$\partial_t u(x,t) - a\partial_x^2 \partial_t u(x,t) - \partial_x^2 u(x,t) = f(x,t) \tag{1.1}$$

where the constant a is non-zero. However, (1.1) can be reduced to the standard evolution equation (3.4) in an appropriate space because the operator $I - a\partial_x^2$ which acts on $\partial_t u(x,t)$ can be inverted. Thus, (1.1) is an example of a *regular* equation; we study such problems in Section 2. Section 3 is concerned with those regular equations of a special form suggested by (1.1).

Another example which motivates some of our discussion is the partial differential equation

$$m(x)\partial_t u(x,t) - \partial_x^2 u(x,t) = f(x,t) \tag{1.2}$$

where the coefficient is non-negative at each point. The equation (1.2) is parabolic at those points where $m(x) > 0$ and elliptic where $m(x) = 0$. For such an equation of *mixed type* some care must be taken in order to prescribe a well posed problem. If $m(x) > 0$ almost everywhere, then (1.2) is a model of a regular evolution equation. Otherwise, it is a model of a *degenerate* equation. We study the Cauchy problem for degenerate equations in Section 4 and in Section 5 give more examples of this type.

2 Regular Equations

Let V_m be a Hilbert space with scalar-product $(\cdot,\cdot)_m$ and denote the corresponding Riesz map from V_m onto the dual V_m' by $\mathcal{M}$. That is,

$$\mathcal{M}x(y) = (x,y)_m \ , \qquad x, y \in V_m \ .$$

Let D be a subspace of V_m and $L : D \to V_m'$ a linear map. If $u_0 \in V_m$ and $f \in C((0,\infty), V_m')$ are given, we consider the problem of finding $u \in C([0,\infty), V_m) \cap C^1((0,\infty), V_m)$ such that

$$\mathcal{M}u'(t) + Lu(t) = f(t) \ , \qquad t > 0 \ , \tag{2.1}$$

and $u(0) = u_0$.

Note that (2.1) is a generalization of the evolution equation IV(2.1). If we identify V_m with V_m' by the Riesz map $\mathcal{M}$ (i.e., take $\mathcal{M} = I$) then (2.1) reduces to IV(2.1). In the general situation we shall solve (2.1) by reducing it to a Cauchy problem equivalent to IV(2.1).

We first obtain our a-priori estimate for a solution $u(\cdot)$ of (2.1), with $f = 0$ for simplicity. For such a solution we have

$$D_t(u(t), u(t))_m = -2\,\mathrm{Re}\, Lu(t)(u(t))$$

and this suggests consideration of the following.

Definition. The linear operator $L : D \to V_m'$ with $D \leq V_m$ is *monotone* (or *non-negative*) if

$$\mathrm{Re}\, Lx(x) \geq 0 \ , \qquad x \in D \ .$$

We call L *strictly monotone* (or *positive*) if

$$\mathrm{Re}\, Lx(x) > 0 \ , \qquad x \in D \ , \ x \neq 0 \ .$$

Our computation above shows there is at most one solution of the Cauchy problem for (2.1) whenever L is monotone, and it suggests that V_m is the correct space in which to seek well-posedness results for (2.1).

To obtain an (explicit) evolution equation in V_m which is equivalent to (2.1), we need only operate on (2.1) with the inverse of the isomorphism $\mathcal{M}$, and this gives

$$u'(t) + \mathcal{M}^{-1} \circ Lu(t) = \mathcal{M}^{-1} f(t) \ , \qquad t > 0 \ . \tag{2.2}$$

This suggests we define $A = \mathcal{M}^{-1} \circ L$ with domain $D(A) = D$, for then (2.2) is equivalent to IV(2.1). Furthermore, since $\mathcal{M}$ is the Riesz map determined by the scalar-product $(\cdot,\cdot)_m$, we have

$$(Ax, y)_m = Lx(y) \ , \qquad x \in D \ , \ y \in V_m \ . \tag{2.3}$$

This shows that L is monotone if and only if A is accretive. Thus, it follows from Theorem IV.4.3 that $-A$ generates a contraction semigroup on V_m if and only if L is monotone and $I + A$ is surjective. Since $\mathcal{M}(I + A) = \mathcal{M} + L$, we obtain the following result from Theorem IV.3.3.

Theorem 2.1 *Let $\mathcal{M}$ be the Riesz map of the Hilbert space V_m with scalar product $(\cdot,\cdot)_m$ and let L be linear from the subspace D of V_m into V_m'. Assume that L is monotone and $\mathcal{M} + L : D \to V_m'$ is surjective. Then, for every $f \in C^1([0,\infty), V_m')$ and $u_0 \in D$ there is a unique solution $u(\cdot)$ of (2.1) with $u(0) = u_0$.*

In order to obtain an analogue of the situation in Section IV.6, we suppose L is obtained from a continuous sesquilinear form. In particular, let V be a Hilbert space for which V is a dense subset of V_m and the injection is continuous; hence, we can identify $V_m' \subset V'$. Let $\ell(\cdot,\cdot)$ be continuous and sesquilinear on V and define the corresponding linear map $\mathcal{L} : V \to V'$ by

$$\mathcal{L}x(y) = \ell(x, y) \ , \qquad x, y \in V \ .$$

Define $D \equiv \{x \in V : \mathcal{L}x \in V_m'\}$ and $L = \mathcal{L}|_D$. Then (2.3) shows that

$$\ell(x, y) = (Ax, y)_m \ , \qquad x \in D \ , \ y \in V \ ,$$

so it follows that A is the operator determined by the triple $\{\ell(\cdot,\cdot), V, V_m\}$ as in Theorem IV.6.1. Thus, from Theorems IV.6.3 and IV.6.5 we obtain the following.

Theorem 2.2 *Let $\mathcal{M}$ be the Riesz map of the Hilbert space V_m with scalar-product $(\cdot,\cdot)_m$. Let $\ell(\cdot,\cdot)$ be a continuous, sesquilinear and elliptic form on the Hilbert space V, which is assumed dense and continuously imbedded in V_m, and denote the corresponding isomorphism of V onto V' by $\mathcal{L}$. Then for every Hölder continuous $f : [0,\infty) \to V_m'$ and $u_0 \in V_m$, there is a unique $u \in C([0,\infty), V_m) \cap C^1((0,\infty), V_m)$ such that $u(0) = u_0$, $\mathcal{L}u(t) \in V_m'$ for $t > 0$, and*

$$\mathcal{M}u'(t) + \mathcal{L}u(t) = f(t) \ , \qquad t > 0 \ . \tag{2.4}$$

We give four elementary examples to suggest the types of initial-boundary value problems to which the above results can be applied. In the first three of these examples we let $V_m = H_0^1(0,1)$ with the scalar-product

$$(u,v)_m = \int_0^1 (u\bar{v} + a\partial u \partial \bar{v}\,)\ ,$$

where $a > 0$.

2.1

Let $D = \{u \in H^2(0,1) \cap H_0^1(0,1) : u'(0) = cu'(1)\}$ where $|c| \le 1$, and define $LU = -\partial^3 u$. Then we have $Lu(\varphi) = (\partial^2 u, \partial\varphi)$ for $\varphi \in H_0^1(0,1)$, and (cf., Section IV.4)

$$2\,\mathrm{Re}\, Lu(u) = |u'(1)|^2 - |u'(0)|^2 \ge 0\ , \qquad u \in D\ .$$

Thus, Theorem 2.1 shows that the initial-boundary value problem

$$\begin{aligned}
&(\partial_t - a\partial_x^2\partial_t)U(x,t) - \partial_x^3 U(x,t) = 0\ , \qquad 0 < x < 1\ ,\ t \ge 0\ ,\\
&U(0,t) = U(1,t) = 0\ , \quad \partial_x U(0,t) = c\partial_x U(1,t)\ , \qquad t \ge 0\ ,\\
&U(x,0) = U_0(x)
\end{aligned}$$

has a unique solution whenever $U_0 \in D$.

2.2

Let $V = H_0^2(0,1)$ and define

$$\ell(u,v) = \int_0^1 \partial^2 u \cdot \partial^2 \bar{v}\ , \qquad u,v \in V\ .$$

Then $D = H_0^2(0,1) \cap H^3(0,1)$ and $Lu = \partial^4 u$, $u \in D$. Theorem 2.2 then asserts the existence and uniqueness of a solution of the problem

$$\begin{aligned}
&(\partial_t - a\partial_x^2\partial_t)U(x,t) + \partial_x^4 U(x,t) = 0\ , \qquad 0 < x < 1\ ,\ t > 0\ ,\\
&U(0,t) = U(1,t) = \partial_x U(0,t) = \partial_x U(1,t) = 0\ , \qquad t > 0\ ,\\
&U(x,0) = U_0(x)\ , \qquad 0 < x < 1\ ,
\end{aligned}$$

for each $U_0 \in H_0^1(0,1)$.

2.3

Let $V = H_0^1(0,1)$ and define

$$\ell(u,v) = \int_0^1 \partial u \partial \bar{v} \ , \qquad u, v \in V \ .$$

Then $D = V = V_m$ and $Lu = -\partial^2 u$, $u \in D$. From either Theorem 2.1 or 2.2 we obtain existence and uniqueness for the problem

$$(\partial_t - a\partial_x^2\partial_t)U(x,t) - \partial_x^2 U(x,t) = 0 \ , \qquad 0 < x < 1 \ , \ t > 0 \ ,$$

$$U(0,t) = U(1,t) = 0 \ , \qquad t > 0 \ ,$$

$$U(x,0) = U_0(x) \ , \qquad 0 < x < 1 \ ,$$

whenever $U_0 \in D = V_m$.

2.4

For our last example we let V_m be the completion of $C_0^\infty(G)$ with the scalar-product

$$(u,v)_m \equiv \int_G m(x)u(x)\overline{v(x)}\, dx \ .$$

We assume G is open in $\mathbb{R}^n$ and $m \in L^\infty(G)$ is given with $m(x) > 0$ for a.e. $x \in G$. (Thus, V_m is the set of measurable functions u on G for which $m^{1/2}u \in L^2(G)$.) Let $V = H_0^1(G)$ and define

$$\ell(u,v) = \int_G \nabla u \cdot \nabla \bar{v} \ , \qquad u, v \in V \ .$$

Then Theorem 2.2 implies the existence and uniqueness of a solution of the problem

$$m(x)\partial_t U(x,t) - \Delta_n U(x,t) = 0 \ , \qquad x \in G \ , \ t > 0 \ ,$$

$$U(s,t) = 0 \ , \qquad s \in \partial G \ , \ t > 0 \ ,$$

$$U(x,0) = U_0(x) \ , \qquad x \in G \ .$$

Note that the initial condition is attained in the sense that

$$\lim_{t \to 0^+} \int_G m(x)|U(x,t) - U_0(x)|^2\, dx = 0 \ .$$

The first two of the preceding examples illustrate the use of Theorems 2.1 and 2.2 when $\mathcal{M}$ and L are both differential operators with the order of L strictly higher than the order of M. The equation in (2.2) is called *metaparabolic* and arises in special models of diffusion or fluid flow. The equation in (2.3) arises similarly and is called *pseudoparabolic*. We shall discuss this class of problems in Section 3. The last example (2.4) contains a *weakly degenerate* parabolic equation. We shall study such problems in Section 4 where we shall assume only that $m(x) \geq 0$, $x \in G$. This allows the equation to be of *mixed type*: parabolic where $m(x) > 0$ and elliptic where $m(x) = 0$. Such examples will be given in Section 5.

3 Pseudoparabolic Equations

We shall consider some evolution equations which generalize the example (2.3). Two types of solutions will be discussed, and we shall show how these two types differ essentially by the boundary conditions they satisfy.

Theorem 3.1 *Let V be a Hilbert space, suppose $m(\cdot,\cdot)$ and $\ell(\cdot,\cdot)$ are continuous sesquilinear forms on V, and denote by $\mathcal{M}$ and $\mathcal{L}$ the corresponding operators in $\mathcal{L}(V,V')$. (That is, $\mathcal{M}x(y) = m(x,y)$ and $\mathcal{L}x(y) = \ell(x,y)$ for $x,y \in V$.) Assume that $m(\cdot,\cdot)$ is V-coercive. Then for every $u_0 \in V$ and $f \in C(\mathbb{R},V')$, there is a unique $u \in C^1(\mathbb{R},V)$ for which (2.4) holds for all $t \in \mathbb{R}$ and $u(0) = u_0$.*

Proof: The coerciveness assumption shows that $\mathcal{M}$ is an isomorphism of V onto V', so the operator $A \equiv \mathcal{M}^{-1} \circ \mathcal{L}$ belongs to $\mathcal{L}(V)$. We can define $\exp(-tA) \in \mathcal{L}(V)$ as in Theorem IV.2.1 and then define

$$u(t) = \exp(-tA) \cdot u_0 + \int_0^t \exp(A(\tau - t)) \circ \mathcal{M}^{-1} f(\tau)\, d\tau\ , \qquad t > 0\ . \quad (3.1)$$

Since the integrand is continuous and appropriately bounded, it follows that (3.1) is a solution of (2.2), hence of (2.1). We leave the proof of uniqueness as an exercise.

We call the solution $u(\cdot)$ given by Theorem 3.1 a *weak solution* of (2.1). Suppose we are given a Hilbert space H in which V is a dense subset, continuously imbedded. Thus $H \subset V'$ and we can define $D(M) = \{v \in V : \mathcal{M}v \in H\}$, $D(L) = \{v \in V : \mathcal{L}v \in H\}$ and corresponding operators $M = \mathcal{M}|_{D(M)}$

and $L = \mathcal{L}|_{D(L)}$ in H. A solution $u(\cdot)$ of (2.1) for which each term in (2.1) belongs to $C(\mathbb{R}, H)$ (instead of $C(\mathbb{R}, V')$) is called a *strong solution*. Such a function satisfies

$$Mu'(t) + Lu(t) = f(t) \ , \qquad t \in \mathbb{R} \ . \tag{3.2}$$

Theorem 3.2 *Let the Hilbert space V and operators $\mathcal{M}, \mathcal{L} \in \mathcal{L}(V, V')$ be given as in Theorem 3.1. Let the Hilbert space H be given as above and define the domains $D(M)$ and $D(L)$ and operators M and L as above. Assume $D(M) \subset D(L)$. Then for every $u_0 \in D(M)$ and $f \in C(\mathbb{R}, H)$ there is a (unique) strong solution $u(\cdot)$ of (3.2) with $u(0) = u_0$.*

Proof: By making the change-of-variable $v(t) = e^{-\lambda t}u(t)$ for some $\lambda > 0$ sufficiently large, we may assume without loss of generality that $D(M) = D(L)$ and $\ell(\cdot, \cdot)$ is V-coercive. Then L is a bijection onto H so we can define a norm on $D(L)$ by $\|v\|_{D(L)} = \|Lv\|_H$, $v \in D(L)$, which makes $D(L)$ a Banach space. (Clearly, $D(L)$ is also a Hilbert space.) Since $\ell(\cdot, \cdot)$ is V-coercive, it follows that for some $c > 0$

$$c\|v\|_V^2 \leq \|Lv\|_H \|v\|_H \ , \qquad v \in D(L) \ ,$$

and the continuity of the injection $V \hookrightarrow H$ shows then that the injection $D(L) \hookrightarrow V$ is continuous. The operator $A \equiv \mathcal{M}^{-1}\mathcal{L} \in \mathcal{L}(V)$ leaves invariant the subspace $D(L)$. This implies that the restriction of A to $D(L)$ is a closed operator in the $D(L)$-norm. To see this, note that if $v_n \in D(L)$ and if $\|v_n - u_0\|_{D(L)} \to 0$, $\|Av_n - v_0\|_{D(L)} \to 0$, then

$$\begin{aligned} \|v_0 - Au_0\|_V &\leq \|v_0 - Av_n\|_V + \|A(v_n - u_0)\|_V \\ &\leq \|v_0 - Av_n\|_V + \|A\|_{\mathcal{L}(V)} \|v_n - u_0\|_V \ , \end{aligned}$$

so the continuity of $D(L) \hookrightarrow V$ implies that each of these terms converges to zero. Hence, $v_0 = Au_0$.

Since $A|_{D(L)}$ is closed and defined everywhere on $D(L)$, it follows from Theorem III.7.5 that it is continuous on $D(L)$. Therefore, the restrictions of the operators $\exp(-tA)$, $t \in \mathbb{R}$, are continuous on $D(L)$, and the formula (3.1) in $D(L)$ gives a strong solution as desired.

Corollary 3.3 *In the situation of Theorem 3.2, the weak solution $u(\cdot)$ is a strong solution if and only if $u_0 \in D(M)$.*

3.1

We consider now an abstract *pseudoparabolic* initial-boundary value problem. Suppose we are given the Hilbert spaces, forms and operators as in Theorem IV.7.2. Let $\varepsilon > 0$ and define

$$\begin{aligned} m(u,v) &= (u,v)_H + \varepsilon a(u,v) \\ \ell(u,v) &= a(u,v) \ , \qquad\qquad u,v \in V \ . \end{aligned}$$

Thus, $D(M) = D(L) = D(A)$. Let $f \in C(\mathbb{R}, H)$. If $u(\cdot)$ is a strong solution of (3.2), then we have

$$\left.\begin{aligned} & u'(t) + \varepsilon A_1 u'(t) + A_1 u(t) = f(t) \ , \\ & u(t) \in V \ , \quad \text{and} \\ & \partial_1 u(t) + \mathcal{A}_2 \gamma(u(t)) = 0 \ , \qquad\qquad t \in \mathbb{R} \ . \end{aligned}\right\} \tag{3.3}$$

Suppose instead that $F \in C(\mathbb{R}, H)$ and $g \in C(\mathbb{R}, B')$. If we define

$$f(t)(v) \equiv (F(t), v)_H + g(t)(\gamma(v)) \ , \qquad v \in V \ , \ t \in \mathbb{R} \ .$$

then a weak solution $u(\cdot)$ of (2.4) can be shown by a computation similar to the proof of Theorem III.3.1 to satisfy

$$\left.\begin{aligned} & u'(t) + \varepsilon A_1 u'(t) + A_1 u(t) = F(t) \ , \\ & u(t) \in V \ , \quad \text{and} \\ & \partial_1(\varepsilon u'(t) + u(t)) + \mathcal{A}_2(\gamma(\varepsilon u'(t) + u(t))) = g(t) \ , \qquad t \in \mathbb{R} \ . \end{aligned}\right\} \tag{3.4}$$

Note that (3.3) implies more than (3.4) with $g \equiv 0$. By taking suitable choices of the operators above, we could obtain examples of initial-boundary value problems from (3.3) and (3.4) as in Theorem IV.7.3.

3.2

For our second example we let G be open in $\mathbb{R}^n$ and choose $V = \{v \in H^1(G) : v(s) = 0.$ a.e. $s \in \Gamma\}$, where Γ is a closed subset of ∂G. We define

$$m(u,v) = \int_G \nabla u(x) \cdot \nabla \overline{v(x)}\, dx \ , \qquad u,v \in V$$

and assume $m(\cdot,\cdot)$ is V-elliptic. (Sufficient conditions for this situation are given in Corollary III.5.4.) Choose $H = L^2(G)$ and $V_0 = H_0^1(G)$; the corresponding partial differential operator $M : V \to V_0' \le \mathcal{D}^*(G)$ is given by

$Mu = -\Delta_n u$, the Laplacian (cf. Section III.2.2). Thus, from Corollary III.3.2 it follows that $D(M) = \{u \in V : \Delta_n u \in L^2(G),\ \partial u = 0\}$ where ∂ is the normal derivative ∂_ν on $\partial G \sim \Gamma$ whenever ∂G is sufficiently smooth. (Cf. Section III.2.3.) Define a second form on V by

$$\ell(u,v) = \int_G a(x)\partial_n u(x)\overline{v(x)}\,dx\ , \qquad u, v \in V\ ,$$

and note that $L = \mathcal{L} : V \to H \leq V'$ is given by $\mathcal{L}u = a(x)(\partial u/\partial x_n)$, where $a(\cdot) \in L^\infty(G)$ is given. Assume that for each $t \in \mathbb{R}$ we are given $F(\cdot,t) \in L^2(G)$ and that the map $t \mapsto F(\cdot,t) : \mathbb{R} \to L^2(G)$ is continuous. Let $g(\cdot,t) \in L^2(\partial G)$ be given similarly, and define $f \in C(\mathbb{R}, V')$ by

$$f(t)(v) = \int_G F(x,t)\overline{v(x)}\,dx + \int_{\partial G} g(s,t)\overline{v(s)}\,ds\ , \qquad t \in \mathbb{R}\ ,\ v \in V\ .$$

If $u_0 \in V$, then Theorem 3.1 gives a unique weak solution $u(\cdot)$ of (2.4) with $u(0) = u_0$. That is

$$m(u'(t), v) + \ell(u(t), v) = f(t)(v)\ , \qquad v \in V\ ,\ t \in \mathbb{R}\ ,$$

and this is equivalent to

$$\begin{aligned} &Mu'(t) + Lu(t) = F(\cdot,t)\ , && t \in \mathbb{R} \\ &u(t) \in V\ , \quad \partial_t(\partial u(t)) = g(\cdot,t)\ . && \end{aligned}$$

From Theorem IV.7.1 we thereby obtain a generalized solution $U(\cdot,\cdot)$ of the initial-boundary value problem

$$\begin{aligned} &-\Delta_n \partial_t U(x,t) + a(x)\partial_n U(x,t) = F(x,t)\ , && x \in G\ ,\ t \in \mathbb{R}\ , \\ &U(s,t) = 0\ , && s \in \Gamma\ , \\ &\partial_\nu U(s,t) = \partial_\nu U_0(s) + \int_0^t g(s,\tau)\,d\tau\ , && s \in \partial G \sim \Gamma\ , \\ &U(x,0) = U_0(x)\ , && x \in G\ . \end{aligned}$$

Finally, we note that $f \in C(\mathbb{R}, H)$ if and only if $g \equiv 0$, and then $\partial_\nu U(s,t) = \partial_\nu U_0(s)$ for $s \in \partial G \sim \Gamma$, $t \in \mathbb{R}$; thus, $U(\cdot,t) \in D(M)$ if and only if $U_0 \in D(M)$. This agrees with Corollary 3.3.

4 Degenerate Equations

We shall consider the evolution equation (2.1) in the situation where $\mathcal{M}$ is permitted to degenerate, i.e., it may vanish on non-zero vectors. Although it is not possible to rewrite it in the form (2.2), we shall essentially factor the equation (2.1) by the kernel of $\mathcal{M}$ and thereby obtain an equivalent problem which is regular.

Let V be a linear space and $m(\cdot,\cdot)$ a sesquilinear form on V that is symmetric and non-negative:

$$\begin{aligned} m(x,y) &= \overline{m(x,y)} \ , \qquad && x,y \in V \ , \\ m(x,x) &\geq 0 \ , && x \in V \ . \end{aligned}$$

Then it follows that

$$|m(x,y)|^2 \leq m(x,x) \cdot m(y,y) \ , \qquad x,y \in V \ , \tag{4.1}$$

and that $x \mapsto m(x,x)^{1/2} = \|x\|_m$ is a seminorm on V. Let V_m denote this seminorm space whose dual V_m' is a Hilbert space (cf. Theorem I.3.5). The identity

$$\mathcal{M}x(y) = m(x,y) \ , \qquad x,y \in V$$

defines $\mathcal{M} \in \mathcal{L}(V_m, V_m')$ and it is just such an operator which we shall place in the leading term in our evolution equation. Let $D \leq V$, $L \in L(D, V_m')$, $f \in C((0,\infty), V_m')$ and $g_0 \in V_m'$. We consider the problem of finding a function $u(\cdot) : [0,\infty) \to V$ such that

$$\mathcal{M}u(\cdot) \in C([0,\infty), V_m') \cap C^1((0,\infty), V_m') \ , \qquad (\mathcal{M}u)(0) = g_0 \ ,$$

and $u(t) \in D$ with

$$(\mathcal{M}u)'(t) + Lu(t) = f(t) \ , \qquad t > 0 \ . \tag{4.2}$$

(Note that when $m(\cdot,\cdot)$ is a scalar product on V_m and V_m is complete then $\mathcal{M}$ is the Riesz map and (4.2) is equivalent to (2.1).)

Let K be the kernel of the linear map $\mathcal{M}$ and denote the corresponding quotient space by V/K. If $q : V \to V/K$ is the canonical surjection, then we define by

$$m_0(q(x), q(y)) = m(x,y) \ , \qquad x,y \in V$$

a scalar product $m_0(\cdot,\cdot)$ on V/K. The completion of V/K, $m_0(\cdot,\cdot)$ is a Hilbert space W whose scalar product is also denoted by $m_0(\cdot,\cdot)$. (Cf. Theorem I.4.2.) We regard q as a map of V_m into W; thus, it is norm-preserving and has a dense range, so its dual $q' : W' \to V'_m$ is a norm-preserving isomorphism (Corollary I.5.3) defined by

$$q'(f)(x) = f(q(x)) \ , \qquad f \in W' \ , \ x \in V_m \ .$$

If $\mathcal{M}_0$ denotes the Riesz map of W with the scalar product $m_0(\cdot,\cdot)$, then we have

$$\begin{aligned} q'\mathcal{M}_0 q(x)(y) &= \mathcal{M}_0 q(x)(q(y)) = m_0(q(x), q(y)) \\ &= \mathcal{M}x(y) \ , \end{aligned}$$

hence,

$$q'\mathcal{M}_0 q = \mathcal{M} \ . \tag{4.3}$$

From the linear map $L : D \to V'_m$ we want to construct a linear map L_0 on the image $q[D]$ of $D \leq V_m$ by q so that it satisfies

$$q' L_0 q = L \ . \tag{4.4}$$

This is possible if (and, in general, only if) $K \cap D$ is a subspace of the kernel of L, $K(L)$ by Theorem I.1.1, and we shall assume this is so.

Let $f(\cdot)$ and g_0 be given as above and consider the problem of finding a function $v(\cdot) \in C([0,\infty), W) \cap C^1((0,\infty), W)$ such that $v(0) = (q'\mathcal{M}_0)^{-1} g_0$ and

$$\mathcal{M}_0 v'(t) + L_0 v(t) = (q')^{-1} f(t) \ , \qquad t > 0 \ . \tag{4.5}$$

Since the domain of L_0 is $q[D]$, if $v(\cdot)$ is a solution of (4.5) then for each $t > 0$ we can find a $u(t) \in D$ for which $v(t) = q(u(t))$. But $q'\mathcal{M}_0 : W \to V'_m$ is an isomorphism and so from (4.3), (4.4) and (4.5) it follows that $u(\cdot)$ is a solution of (4.2) with $\mathcal{M}u(0) = g_0$. This leads to the following results.

Theorem 4.1 *Let V_m be a seminorm space obtained from a symmetric and non-negative sesquilinear form $m(\cdot,\cdot)$, and let $\mathcal{M} \in \mathcal{L}(V_m, V'_m)$ be the corresponding linear operator given by $\mathcal{M}x(y) = m(x,y)$, $x, y \in V_m$. Let D be a subspace of V_m and $L : D \to V'_m$ be linear and monotone.* (a) *If $K(\mathcal{M}) \cap D \leq K(L)$ and if $\mathcal{M} + L : D \to V'_m$ is a surjection, then for every $f \in C^1([0,\infty), V'_m)$ and $u_0 \in D$ there exists a solution of* (4.2) *with $(\mathcal{M}u)(0) = \mathcal{M}u_0$.* (b) *If $K(\mathcal{M}) \cap K(L) = \{0\}$, then there is at most one solution.*

Proof: The existence of a solution will follow from Theorem 2.1 applied to (4.5) if we show $L_0 : q[D] \to W'$ is monotone and $\mathcal{M}_0 + L_0$ is onto. But (4.5) shows L_0 is monotone, and the identity

$$q'(\mathcal{M}_0 + L_0)q(x) = (\mathcal{M} + L)(x) \ , \qquad x \in D \ ,$$

implies that $\mathcal{M}_0 + L_0$ is surjective whenever $\mathcal{M} + L$ is surjective.

To establish the uniqueness result, let $u(\cdot)$ be a solution of (4.2) with $f \equiv 0$ and $\mathcal{M}u(0) = 0$; define $v(t) = qu(t)$, $t \geq 0$. Then

$$D_t m_0(v(t), v(t)) = 2\,\mathrm{Re}(\mathcal{M}_0 v'(t))(v(t)) \ , \qquad t > 0 \ ,$$

and this implies by (4.3) that

$$\begin{aligned} D_t m(u(t), u(t)) &= 2\,\mathrm{Re}(\mathcal{M}u)'(t)(u(t)) \\ &= -2\,\mathrm{Re}\, Lu(t)(u(t)) \ , \qquad t > 0 \ . \end{aligned}$$

Since L is monotone, this shows $\mathcal{M}u(t) = 0$, $t \geq 0$, and (4.2) implies $Lu(t) = 0$, $t > 0$. Thus $u(t) \in K(\mathcal{M}) \cap K(L)$, $t \geq 0$, and the desired result follows.

We leave the proof of the following analogue of Theorem 2.2 as an exercise.

Theorem 4.2 *Let V_m be a seminorm space obtained from a symmetric and non-negative sesquilinear form $m(\cdot,\cdot)$, and let $\mathcal{M} \in \mathcal{L}(V_m, V_m')$ denote the corresponding operator. Let V be a Hilbert space which is dense and continuously imbedded in V_m. Let $\ell(\cdot,\cdot)$ be a continuous, sesquilinear and elliptic form on V, and denote the corresponding isomorphism of V onto V' by $\mathcal{L}$. Let $D = \{u \in V : \mathcal{L}u \in V_m'\}$. Then, for every Hölder continuous $f : [0,\infty) \to V_m'$ and every $u_0 \in V_m$, there exists a unique solution of* (4.2) *with* $(\mathcal{M}u)(0) = \mathcal{M}u_0$.

5 Examples

We shall illustrate the applications of Theorems 4.1 and 4.2 by solving some initial-boundary value problems with partial differential equations of mixed type.

5.1

Let $V_m = L^2(0,1)$, $0 \le a < b \le 1$, and

$$m(u,v) = \int_a^b u(x)\overline{v(x)}\,dx\ , \qquad v \in V_m\ .$$

Then $V_m' = L^2(a,b)$, which we identify as that subspace of $L^2(0,1)$ whose elements are zero a.e. on $(0,a) \cup (b,1)$, and $\mathcal{M}$ becomes multiplication by the characteristic function of the interval (a,b). Let $L = \partial$ with domain $D = \{u \in H^1(0,1) : u(0) = cu(1),\ \partial u \in V_m' \subset L^2(0,1)\}$. We assume $|c| \le 1$, so L is monotone (cf. Section IV.4(a)). Note that each function in D is constant on $(0,a) \cup (b,1)$. Thus, $K(\mathcal{M}) \cap D = \{0\}$ and $K(\mathcal{M}) \cap D \le K(L)$ follows. Also, note that $K(L)$ is either $\{0\}$ or consists of the constant functions, depending on whether or not $c \ne 1$, respectively. Thus, $K(\mathcal{M}) \cap K(L) = \{0\}$. If u is the solution of (cf. Section IV.4(a))

$$u(x) + \partial u(x) = f(x)\ , \qquad a < x < b\ ,\ u(a) = cu(b)$$

and is extended to $(0,1)$ by being constant on each of the intervals, $[0,a]$ and $[b,1]$, then $(\mathcal{M} + L)u = f \in V_m'$. Hence $\mathcal{M} + L$ maps onto V_m' and Theorem 4.1 asserts the existence and uniqueness of a generalized solution of the problem

$$\left.\begin{aligned}
&\partial_t U(x,t) + \partial_x U(x,t) = F(x,t)\ , \qquad a < x < b\ ,\ t \ge 0\ ,\\
&\partial_x U(x,t) = 0\ , \qquad x \in (0,a) \cup (b,a)\ ,\\
&U(0,t) = cU(1,t)\ , \qquad U(x,0) = U_0(x)\ , \qquad a < x < b\ ,
\end{aligned}\right\} \tag{5.1}$$

for appropriate $F(\cdot,\cdot)$ and U_0. This example is trivial (i.e., equivalent to Section IV.4(a) on the interval (a,b)) but motivates the proof-techniques of Section 4.

5.2

We consider some problems with a partial differential equation of mixed elliptic-parabolic type. Let $m_0(\cdot) \in L^\infty(G)$ with $m_0(x) \ge 0$, a.e. $x \in G$, an open subset of $\mathbb{R}^n$ whose boundary ∂G is a C^1-manifold with G on one side of ∂G. Let $V_m = L^2(G)$ and

$$m(u,v) = \int_G m_0(x)u(x)\overline{v(x)}\,dx\ , \qquad u,v \in V_m\ .$$

Then $\mathcal{M}$ is multiplication by $m_0(\cdot)$ and maps $L^2(G)$ into $V'_m \equiv \{\sqrt{m_0} \cdot g : g \in L^2(G)\} \subset L^2(G)$. Let Γ be a closed subset of ∂G and define $V = \{v \in H^1(G) : \gamma_0 v = 0 \text{ on } \Gamma\}$ as in Section III.4.1. Let

$$\ell(u, v) = \int_G \nabla u \cdot \overline{\nabla v}\, dx \ , \qquad u, v \in V \tag{5.2}$$

and assume $\sum \equiv \{s \in \partial G : \nu_n(s) > 0\} \subset \Gamma$. Thus, Theorem III.5.3 implies $\ell(\cdot, \cdot)$ is V-elliptic, so $\mathcal{M} + \mathcal{L}$ maps onto V', hence, onto V'_m. Theorem 4.2 shows that if $U_0 \in L^2(G)$ and if F is given as in Theorem IV.7.3, then there is a unique generalized solution of the problem

$$\left.\begin{aligned}
&\partial_t(m_0(x)U(x,t)) - \Delta_n U(x,t) = m_0(x)F(x,t) \ , && x \in G \ , \\
&U(s,t) = 0 \ , \qquad s \in \Gamma \ , && \\
&\frac{\partial U(s,t)}{\partial \nu} = 0 \ , \quad s \in \partial G \sim \Gamma \ , && t > 0 \ , \\
&m_0(x)(U(x,0) - U_0(x)) = 0 \ . &&
\end{aligned}\right\} \tag{5.3}$$

The partial differential equation in (5.3) is parabolic at those $x \in G$ for which $m_0(x) > 0$ and elliptic where $m_0(x) = 0$. The boundary conditions are of mixed Dirichlet-Neumann type (cf. Section III.4.1) and the initial value of $U(x, 0)$ is prescribed only at those points of G at which the equation is parabolic.

Boundary conditions of the third type may be introduced by modifying $\ell(\cdot, \cdot)$ as in Section III.4.2. Similarly, by choosing

$$\ell(u, v) = \int_G \nabla u \cdot \overline{\nabla v}\, dx + (\gamma_0 u)\overline{(\gamma_0 v)}$$

on $V = \{u \in H^1(G) : \gamma_0 u \text{ is constant}\}$, we obtain a unique generalized solution of the initial-boundary value problem of *fourth type* (cf., Section III.4.2)

$$\left.\begin{aligned}
&\partial_t(m_0(x)U(x,t)) - \Delta_n U(x,t) = m_0(x)F(x,t) \ , && x \in G \ , \\
&U(s,t) = h(t) \ , && s \in \partial G \ , \\
&\left(\int_{\partial G} \frac{\partial U(s,t)}{\partial \nu}\, ds \Big/ \int_{\partial G} ds\right) + h(t) = 0 \ , && t > 0 \ , \\
&m_0(x)(U(x,0) - U_0(x)) = 0 \ . &&
\end{aligned}\right\} \tag{5.4}$$

The data $F(\cdot, \cdot)$ and U_0 are specified as before; $h(\cdot)$ is unknown and part of the problem.

5.3

Problems with a partial differential equation of mixed pseudoparabolic-parabolic type can be similarly handled. Let $m_0(\cdot)$ be given as above and define

$$m(u,v) = \int_G (u(x)\overline{v(x)} + m_0(x)\nabla u(x)\cdot \overline{\nabla v}\,(x))\,dx\ , \qquad u,v\in V_m\ ,$$

with $V_m = H^1(G)$. Then $V_m \hookrightarrow L^2(G)$ is continuous so we can identify $L^2(G) \le V_m'$. Define $\ell(\cdot,\cdot)$ by (5.2) where V is a subspace of $H^1(G)$ which contains $C_0^\infty(G)$ and is to be prescribed. Then $K(\mathcal{M}) = \{0\}$ and $m(\cdot,\cdot) + \ell(\cdot,\cdot)$ is V-coercive, so Theorem 4.2 will apply. In particular, if $U_0 \in L^2(G)$ and F as in Theorem IV.7.3 are given, then there is a unique solution of the equation

$$\partial_t(U(x,t) - \sum_{j=1}^n \partial_j(m_0(x)\partial_j U(x,t))) - \Delta_n U(x,t) = F(x,t)\ , \qquad x\in G\ ,\ t>0\ ,$$

with the initial condition

$$U(x,0) = U_0(x)\ , \qquad x\in G\ ,$$

and boundary conditions which depend on our choice of V.

5.4

We consider a problem with a time derivative and possibly a partial differential equation on a boundary. Let G be as in (5.2) and assume for simplicity that ∂G intersects the hyperplane $\mathbb{R}^{n-1}\times\{0\}$ in a set with relative interior S. Let $a_n(\cdot)$ and $b(\cdot)$ be given nonnegative, real-valued functions in $L^\infty(S)$. We define $V_m = H^1(G)$ and

$$m(u,v) = \int_G u(x)\overline{v(x)}\,dx + \int_S a(s)u(s)\overline{v(s)}\,ds\ , \qquad u,v\in V_m\ ,$$

where we suppress the notation for the trace operator, i.e., $u(s) = (\gamma_0 u)(s)$ for $s\in\partial G$. Define V to be the completion of $C^\infty(\bar{G})$ with the norm given by

$$\|v\|_V^2 \equiv \|v\|_{H^1(G)}^2 + \left(\int_S b(s)\sum_{j=1}^{n-1}|D_j v(s)|^2\,ds\right)\ .$$

Thus, V consists of these $v \in H^1(G)$ for which $b^{1/2} \cdot \partial_j(\gamma_0 v) \in L^2(S)$ for $1 \leq j \leq n-1$; it is a Hilbert space. We define

$$\ell(u,v) = \int_G \nabla u(x) \cdot \nabla \overline{v(x)}\, dx + \int_S b(s) \Big(\sum_{j=1}^{n-1} \partial_j u(s) \partial_j \overline{v(s)} \Big)\, ds \,, \qquad u, v \in V \,.$$

Then $K(\mathcal{M}) = \{0\}$ and $m(\cdot,\cdot)+\ell(\cdot,\cdot)$ is V-coercive. If $U_0 \in L^2(G)$ and $F(\cdot,\cdot)$ is given as above, then Theorem 4.2 asserts the existence and uniqueness of the solution $U(\cdot,\cdot)$ of the initial-boundary value problem

$$\begin{cases} \partial_t U(x,t) - \Delta_n U(x,t) = F(x,t) \,, & x \in G \,,\ t > 0 \,, \\ \partial_t (a(s)U(s,t)) + \dfrac{\partial U(s,t)}{\partial \nu} = \displaystyle\sum_{j=1}^{n-1} \partial_j (b(s) \partial_j U(s,t)) \,, & s \in S \,, \\ \dfrac{\partial U(s,t)}{\partial \nu} = 0 \,, & s \in \partial G \sim S \,, \\ b(s) \dfrac{\partial U(s,t)}{\partial \nu_S} = 0 \,, & s \in \partial S \,, \\ U(x,0) = U_0(x) \,, & x \in G \,, \\ a(s)(U(s,0) - U_0(s)) = 0 \,, & s \in S \,. \end{cases}$$

Similar problems with a partial differential equation of mixed type or other combinations of boundary conditions can be handled by the same technique. Also, the $(n-1)$-dimensional surface S can occur inside the region G as well as on the boundary. (Cf., Section III.4.5.)

Exercises

1.1. Use the separation-of-variables technique to obtain a series representation for the solution of (1.1) with $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = u_0(x)$, $0 < x < \pi$. Compare the rate of convergence of this series with that of Section IV.1.

2.1. Provide all details in support of the claim that Theorem 2.1 follows from Theorem IV.3.3. Show that Theorem 2.2 follows from Theorems IV.6.3 and IV.6.5.

2.2. Show that Theorem 2.2 remains true if we replace the hypothesis that $\mathcal{L}$ is V-elliptic by $\lambda\mathcal{M} + \mathcal{L}$ is V-elliptic for some $\lambda \in \mathbb{R}$.

2.3. Characterize V'_m in each of the examples (2.1)–(2.3). Construct appropriate terms for $f(t)$ in Theorems 2.1 and 2.2. Write out the corresponding initial-boundary value problems that are solved.

2.4. Show $V'_m = \{m^{1/2}v : v \in L^2(0,1)\}$ in (2.4). Describe appropriate non-homogeneous terms for the partial differential equation in (2.4).

3.1. Verify that (3.1) is a solution of (2.2) in the situation of Theorem 3.1.

3.2. Prove uniqueness holds in Theorem 3.1. [Hint: Show $\sigma(t) \equiv \|u(t)\|_V^2$ satisfies $|\sigma'(t)| \leq K\sigma(t)$, $t \in \mathbb{R}$, where u is a solution of the homogeneous equation, then show that $\sigma(t) \leq \exp(K|t|) \cdot \sigma(0)$.]

3.3. Verify that (3.4) characterizes the solution of (2.4) in the case of Section 3.1. Discuss the regularity of the solution when $a(\cdot, \cdot)$ is k-regular.

4.1. Prove (4.1).

4.2. Prove Theorem 4.2. [Hint: Compare with Theorem 2.2.]

5.1. Give sufficient conditions on the data F, u_0 in (5.1) in order to apply Theorem 4.1.

5.2. Extend the discussion in Section 5.2 to include boundary conditions of the third type.

5.3. Characterize V'_m in Section 5.3. Write out the initial-boundary value problem solved in Section 5.3 for several choices of V.

5.4. Write out the problem solved in Section 5.4 when S is an interface as in Section III.4.5.