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PALAIS-SMALE APPROACHES TO SEMILINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

HWAI-CHIUAN WANG

ABSTRACT. Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$, and $2^* = \infty$ if $N = 1, 2$, $2^* = \frac{2N}{N-2}$ if $N > 2$, $2 < p < 2^*$. Consider the semilinear elliptic problem

$$-\Delta u + u = |u|^{p-2}u \quad \text{in } \Omega;$$
$$u \in H_0^1(\Omega).$$

Let $H_0^1(\Omega)$ be the Sobolev space in Ω. The existence, the nonexistence, and the multiplicity of positive solutions are affected by the geometry and the topology of the domain Ω. The existence, the nonexistence, and the multiplicity of positive solutions have been the focus of a great deal of research in recent years.

That the above equation in a bounded domain admits a positive solution is a classical result. Therefore the only interesting domains in which this equation admits a positive solution are proper unbounded domains. Such elliptic problems are difficult because of the lack of compactness in unbounded domains. Remarkable progress in the study of this kind of problem has been made by P. L. Lions. He developed the concentration-compactness principles for solving a large class of minimization problems with constraints in unbounded domains. The characterization of domains in which this equation admits a positive solution is an important open question. In this monograph, we present various analyses and use them to characterize several categories of domains in which this equation admits a positive solution or multiple solutions.

CONTENTS

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1. Introduction

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$, and $2^* = \infty$ if $N = 1, 2$, $2^* = \frac{2N}{N-2}$ if $N > 2$, $2 < p < 2^*$. Consider the semilinear elliptic problem

$$\begin{gathered} -\Delta u + u = |u|^{p-2}u \quad \text{in } \Omega; \\ u \in H_0^1(\Omega). \end{gathered} \tag{1.1}$$

Let $H_0^1(\Omega)$ be the Sobolev space in Ω. For the general theory of Sobolev spaces $H_0^1(\Omega)$, see Adams [2]. Associated with (1.1), we consider the energy functionals a, b and J for $u \in H_0^1(\Omega)$

$$\begin{gathered} a(u) = \int_\Omega (|\nabla u|^2 + u^2); \\ b(u) = \int_\Omega |u|^p; \\ J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u). \end{gathered}$$

As in Rabinowitz [64, Proposition B.10], a, b and J are of C^2. It is well known that the solutions of (1.1) and the critical points of the energy functional J are the same.

The existence, the nonexistence, and the multiplicity of positive solutions of (1.1) are affected by the geometry and the topology of the domain Ω. The existence, the nonexistence, and the multiplicity of positive solutions of (1.1) have been the focus of a great deal of research in recent years. That Equation (1.1) in a bounded domain admits a positive solution is a classical result. Gidas-Ni-Nirenberg [35] and Kwong [46] asserted that (1.1) in the whole space $\mathbb{R}^N$ admits a “unique” positive spherically symmetric solution. Therefore the only interesting domains in which (1.1) admits a positive solution are proper unbounded domains. Such elliptic problems are difficult because of the lack of compactness in unbounded domains. Remarkable progress in the study of this kind of problem has been made by P. L. Lions [49] and [50].

He developed the concentration-compactness principles for solving a large class of minimization problems with constraints in unbounded domains. The cornerstone is the paper of Esteban-Lions [33], in which they asserted : no any nonzeroal solutions $H_0^1(\Omega)$ for the (1.1) exist in an Esteban-Lions domain (see Definition 2.6.) The characterization of domains in which (1.1) admits a positive solution is an important open question. In this monograph, we present various analyses and use them to characterize several categories of domains in which (1.1) admits a positive solution or multiple solutions.

In Section 2 we define the Palais-Smale (denoted by (PS))-sequences, (PS)-values, and (PS)-conditions. We study the properties of (PS)-values. We recall the classical compactness theorems such as the Lebesgue dominated convergence theorem and the Vitali convergence theorem. We then come to study (PS)-conditions: the modern concepts for compactness.

In Section 3 we recall the (PS) decomposition theorems in $\mathbb{R}^N$ of Lions [49] and the (PS) decomposition theorems in the infinite strip $\mathbf{A}^r$ of Lien-Tzeng-Wang [47].

In Section 4 we assert the four classical (PS)-values in Ω: the constrained maximizing value, the Nehari minimizing value, the mountain pass minimax value, and the minimal positive (PS)-value are the same. We call any one of them the index of the functional J in the domain Ω. We also study in detail various indexes of the functional J in domain Ω.

In Section 5 we use the indexes of the functional J in domains Ω to characterize the (PS)-conditions: we obtain a theorem in which eight conditions are equivalent to the (PS)-conditions.

In Section 6 we establish y-symmetric (PS)-conditions. The development is interesting in its own right and will also be used to prove the multiplicity of nonzero solutions in Section 13.

In Section 7 we present the y-symmetric (PS) decomposition theorems in the infinite strip $\mathbf{A}^r$.

In Section 8 we study the fundamental properties, regularity, and asymptotic behavior of solutions of (1.1).

In Section 9 we use the asymptotic behavior of solutions developed in Section 8 and apply the "moving plane" method to prove the symmetry of positive solutions to (1.2) in the infinite strip $\mathbf{A}^r$. Our approach is similar to those in Gidas-Ni-Nirenberg [34, Theorem 1] and [35, Theorem 2] but is more complicated. Finally we propose an open question—are positive solutions of (1.1) in the generalized infinite strip $\mathbf{S}^r$ unique up to a translation—?

In Section 10 we characterize Esteban-Lions domains. We prove that proper large domains, Esteban-Lions domains, and some interior flask domains are nonachieved.

Nonachieved domains may admit higher energy solutions. Berestycki conjectured that there is a positive solution of (1.1) in an Esteban-Lions domain with a hole. In Section 11 we answer the Berestycki conjecture affirmatively. We also study the dynamic system of those solutions.

In Section 12 we assert that a bounded domain, a quasibounded domain, a periodic domain, some interior flask domains, some flat interior flask domains, canal domains, and manger domains are achieved. Finally we propose an open question: in Theorem 12.7, is $s_0 = r$?

In Section 12 we prove that there is a ground state solution in an achieved domain. In Section 13 we prove that if we perturb (1.1), then we obtain three nontrivial solutions of (1.2) or if we perturb the achieved domain by adding or removing a domain, then we obtain three positive solutions of (1.1).

For the simplicity and the convenience of the reader, we present results for (1.1). As a matter of fact, our results also hold for more general semilinear elliptic equations as follows:

$$\begin{gathered} -\Delta u + u = |u|^{p-2}u + h(z) \quad \text{in } \Omega; \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.2}$$

$$\begin{gathered} -\Delta u + u = g(u) \quad \text{in } \Omega; \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.3}$$

$$\begin{gathered} -\Delta u = f(z, u) \quad \text{in } \Omega; \\ u \in H_0^1(\Omega). \end{gathered} \tag{1.4}$$

Readers interested in other aspects of critical point theory may consult the following books: Aubin-Ekeland [6], Brézis [14], Chabrowski [18], [19], Ghoussoub [37], Mawhin-Willem [56], Ni [58], Rabinowitz [64], Struwe [66], Willem [78], and Zeidler [79]. For the study of semilinear elliptic equations in unbounded domains, we recommend the following articles: Ambrosetti-Rabinowitz [4], Benci-Cerami [11], Berestycki-Lions [13], Esteban-Lions [33], Lions [49], [50], Palais [59], and Palais-Smale [60].

I am grateful to Roger Temam for inviting me to visit Université de Paris-Sud in 1983, to Haïm Brézis for introducing me to critical point theory in 1983, to Wei-Ming Ni for introducing me to the semilinear elliptic problems in 1987, to Henri Berestycki, Maria J. Esteban, and H. Attouch, and to Pierre-Louis Lions for giving me his preprints and for enlightening discussions.

2. Preliminaries

Throughout this monograph, let $X(\Omega)$ be a closed linear subspace of $H_0^1(\Omega)$ with dual $X^{-1}(\Omega)$ with the space $X(\Omega)$ satisfying the following three properties:

(p1) If $u \in X(\Omega)$, then $|u| \in X(\Omega)$

(p2) If $u \in X(\Omega)$, then $\xi_n u \in X(\Omega)$ for each $n = 1, 2, \dots$, where $\xi \in C^\infty([0,\infty))$ satisfies $0 \le \xi \le 1$,

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1]; \\ 1 & \text{for } t \in [2,\infty), \end{cases}$$

and

$$\xi_n(z) = \xi(\frac{2|z|}{n}). \tag{2.1}$$

(p3) If $u \in X(\Omega)$, then $\eta_n u \in X(\Omega)$ for each $n = 1, 2, \dots$, where $\eta \in C_c^\infty([0,\infty))$ satisfies $0 \le \eta \le 1$ and

$$\eta(t) = \begin{cases} 1 & \text{for } t \in [0,1]; \\ 0 & \text{for } t \in [2,\infty), \end{cases}$$

and

$$\eta_n(z) = \eta(\frac{2|z|}{n}). \tag{2.2}$$

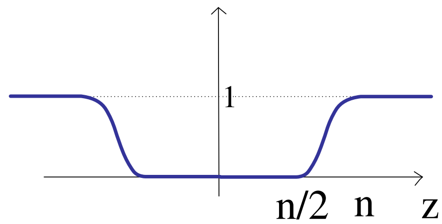

FIGURE 1. $\xi_n(z)$.

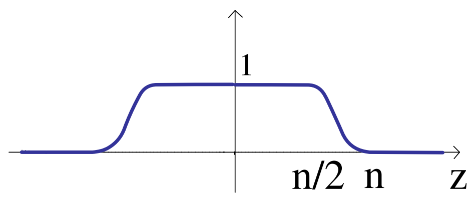

FIGURE 2. $\eta_n(z)$.

Typical examples of $X(\Omega)$ are the whole space $H_0^1(\Omega)$ and the y-symmetric space $H_s^1(\Omega)$.

Let $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$. In this monograph, we refer to three universal domains: the whole space $\mathbb{R}^N$, the infinite strip $\mathbf{A}^r$, the infinite hole strip $\mathbf{A}^{r_1,r_2}$ (in this case, $N \geq 3$), and their subdomains: the ball $B^N(z_0; s)$, the upper semi-strip $\mathbf{A}_s^r$, the interior flask domain $\mathbf{F}_s^r$, the infinite cone $\mathbf{C}$, and the epigraph Π as follows.

$$\mathbf{A}^r = \{(x, y) \in \mathbb{R}^N : |x| < r\};$$
$$\mathbf{A}_{s,t}^r = \{(x, y) \in \mathbf{A}^r : s < y < t\};$$
$$\mathbf{A}_s^r = \{(x, y) \in \mathbf{A}^r : s < y\};$$
$$\mathbf{A}^r \backslash \omega, \quad \text{where } \omega \subset \mathbf{A}^r \quad \text{is a bounded domain};$$
$$\widetilde{\mathbf{A}}_s^r = \mathbf{A}^r \backslash \overline{\mathbf{A}_s^r};$$
$$B^N(z_0; s) = \{z \in \mathbb{R}^N : |z - z_0| < s\};$$
$$\mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0; s);$$
$$\mathbf{A}^{r_1,r_2} = \{(x, y) \in \mathbb{R}^N : r_1 < |x| < r_2\};$$
$$\mathbf{A}_{s,t}^{r_1,r_2} = \{(x, y) \in \mathbf{A}^{r_1,r_2} |\ s < y < t\};$$
$$\mathbf{A}_s^{r_1,r_2} = \{(x, y) \in \mathbf{A}^{r_1,r_2} |\ s < y\};$$
$$\widetilde{\mathbf{A}}_s^{r_1,r_2} = \mathbf{A}^{r_1,r_2} \backslash \overline{\mathbf{A}_s^{r_1,r_2}};$$
$$\mathbb{R}_+^N = \{(x, y) \in \mathbb{R}^N : 0 < y\};$$
$$\mathbb{R}_{-\rho,\rho}^N = \{(x, y) \in \mathbb{R}^N : -\rho < y < \rho\};$$
$$\mathbf{P}^+ = \{(x, y) \in \mathbb{R}^N : y > |x|^2\};$$
$$\mathbf{P}^- = \{(x, -y) : (x, y) \in \mathbf{P}^+\};$$

$$\mathbf{C} = \{(x,y) \in \mathbb{R}^N : |x| < y\};$$
$$\Pi = \{(x,y) \in \mathbb{R}^N : f(x) < y\}, \text{ where } f : \mathbb{R}^{N-1} \to \mathbb{R} \quad \text{is a function.}$$

Definition 2.1. (i) We say that Ω is a large domain in $\mathbb{R}^N$ if for any $r > 0$, $z \in \Omega$ exists such that $B(z;r) \subset \Omega$;
(i') We say that Ω is a strictly large domain in $\mathbb{R}^N$ if Ω contains an infinite cone of $\mathbb{R}^N$;
(ii) We call Ω a large domain in $\mathbf{A}^r$ if for any positive number m, a, b exist such that $b - a = m$ and $\mathbf{A}^r_{a,b} \subset \Omega$;
(ii') We call Ω a strictly large domain in $\mathbf{A}^r$ if Ω contains a semi-strip of $\mathbf{A}^r$;
(iii) We call Ω a large domain in $\mathbf{A}^{r_1,r_2}$ if for any positive number m, a, b exist with $a < b$ such that $b - a = m$ and $\mathbf{A}^{r_1,r_2}_{a,b} \subset \Omega$;
(iii') We call Ω a strictly large domain in $\mathbf{A}^{r_1,r_2}$ if Ω contains a semi-strip of $\mathbf{A}^{r_1,r_2}$.

Let Ω be any one of $\mathbb{R}^N$, $\mathbf{A}^r$, or $\mathbf{A}^{r_1,r_2}$. Then a strictly large domain in Ω is a large domain in Ω.

Example 2.2. The infinite cone $\mathbf{C}$, the upper semi-space $\mathbb{R}^N_+$, the paraboloid $\mathbf{P}^+$, and the epigraph Π are strictly large domains in $\mathbb{R}^N$.

Example 2.3. $\mathbf{A}^r_s$ and $\mathbf{A}^r_s \setminus D$ are strictly large domains in $\mathbf{A}^r$, where $s \in \mathbb{R}$ and $D \subset \mathbf{A}^r_s$ is a bounded domain.

Example 2.4. $\mathbf{A}^{r_1,r_2}_s$ and $\mathbf{A}^{r_1,r_2}_s \setminus D$ are strictly large domains in $\mathbf{A}^{r_1,r_2}$, where $s \in \mathbb{R}$ and $D \subset \mathbf{A}^{r_1,r_2}_s$ is a bounded domain.

There is a large domains in $\mathbf{A}^r$ which is not a strictly large domain in $\mathbf{A}^r$.

Example 2.5. Let $\Omega = A^r_0 \setminus \bigcup_{n=1}^{\infty} B(z_n, \frac{r}{4})$ where $z_n = (0, 0, \dots, 2^n)$. Then Ω is a large domain in A^r which is not a strictly large domain in $\mathbf{A}^r$.

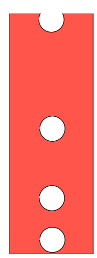

FIGURE 3. Large domains 1.

Definition 2.6. A proper smooth unbounded domain Ω in $\mathbb{R}^N$ is an Esteban-Lions domain if $\chi \in \mathbb{R}^N$ exists with $\|\chi\| = 1$ such that $n(z) \cdot \chi \geq 0$, and $n(z) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(z)$ is the unit outward normal vector to $\partial\Omega$ at the point z.

Example 2.7. An upper half strip $\mathbf{A}^r_s$, a lower half strip $\widetilde{\mathbf{A}^r_t}$, the epigraph Π, the infinite cone $\mathbf{C}$, the upper half space $\mathbb{R}^N_+$, and the paraboloid P^+ are Esteban-Lions domains.

We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $X(\Omega)$ for J as follows.

Definition 2.8. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$;
(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $X(\Omega)$ for J if there is a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J;
(iii) J satisfies the $(\mathrm{PS})_\beta$-condition in $X(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J contains a convergent subsequence;
(iv) J satisfies the (PS)-condition in $X(\Omega)$ if for every $\beta \in \mathbb{R}$, J satisfies the $(\mathrm{PS})_\beta$-condition in $X(\Omega)$.

A $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J.

Lemma 2.9. (i) *For $\mu \in X^{-1}(\Omega)$, we can extend it to be $\mu \in H^{-1}(\Omega)$ such that $\|\mu\|_{X^{-1}} = \|\mu\|_{H^{-1}}$;*
(ii) Let $\{u_n\}$ be in $X(\Omega)$ and satisfy $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$, then $J'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;
(iii) If $J'(u) = 0$ in $X^{-1}(\Omega)$, then $J'(u) = 0$ in $H^{-1}(\Omega)$.

Proof. (i) Since $X(\Omega)$ is a closed linear subspace of the Hilbert space $H_0^1(\Omega)$, we have

$$H_0^1(\Omega) = X(\Omega) \oplus X(\Omega)^\perp.$$

Since μ is a bounded linear functional in $X(\Omega)$, by the Riesz representation theorem, there is a $w \in X(\Omega)$ such that

$$\mu(\varphi) = \langle w, \varphi\rangle_{H^1} \quad \text{for each } \varphi \in X(\Omega)$$

and $\|\mu\|_{X^{-1}} = \|w\|_{H^1}$. Define

$$\mu(\varphi) = \langle w, \varphi\rangle_{H^1} \quad \text{for each } \varphi \in H_0^1(\Omega).$$

Note that $\langle w, \phi\rangle_{H^1} = 0$ for each $\phi \in X(\Omega)^\perp$. For any $v \in H_0^1(\Omega)$ with $\|v\|_{H^1} \le 1$, $v_s \in X(\Omega)$ and $v_s^\perp \in X(\Omega)^\perp$ exist such that $v = v_s + v_s^\perp$. Then

$$|\mu(v)| = |\langle w, v\rangle_{H^1}| = |\langle w, v_s + v_s^\perp\rangle_{H^1}| = |\langle w, v_s\rangle_{H^1}| \le \|w\|_{H^1} = \|\mu\|_{X^{-1}}.$$

Thus, $\|\mu\|_{H^{-1}} \le \|\mu\|_{X^{-1}}$. Moreover,

$$\begin{aligned}\|\mu\|_{X^{-1}} &= \sup\{|\mu(\varphi)| \mid \varphi \in X(\Omega), \|\varphi\|_{H^1} \le 1\}\\ &\le \sup\{|\mu(\varphi)| \mid \varphi \in H_0^1(\Omega), \|\varphi\|_{H^1} \le 1\}\\ &\le \|\mu\|_{H^{-1}}.\end{aligned}$$

Therefore, $\|\mu\|_{H^{-1}} = \|\mu\|_{X^{-1}}$. Part ($ii$) follows from part ($i$). Part ($iii$) follows form ($i$). □

Bound and weakly convergence are the same.

Lemma 2.10. *Let Y be a normed linear space and $u_n \rightharpoonup u$ weakly in Y, then $\{u_n\}$ is bounded in Y and*

$$\|u\| \le \liminf_{n\to\infty} \|u_n\|.$$

Lemma 2.11. *Let $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then there exists a subsequence $\{u_n\}$ such that:*
(i) $\{u_n\}$ is bounded in $X(\Omega)$ and $\|u\|_{H^1} \le \liminf_{n\to\infty} \|u_n\|_{H^1}$;
(ii) $u_n \rightharpoonup u$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$, and $u_n \to u$ a.e. in Ω;
(iii) $\|u_n - u\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o(1)$.

Proof. Part (i) follows from Lemma 2.10. (ii) For $v \in (L^2(\Omega))^N$, define

$$f(u) = \int_\Omega v\nabla u \quad \text{for } u \in X(\Omega),$$

then $|f(u)| \le \|v\|_{L^2}\|\nabla u\|_{L^2} \le \|v\|_{L^2}\|u\|_{H^1}$. Thus, f is a bounded linear functional in $X(\Omega)$. By the Riesz representation theorem, $w \in X^{-1}(\Omega)$ exists such that

$$f(u) = \langle u, w\rangle_{H^1} \quad \text{for } u \in X(\Omega).$$

Hence, if $u_n \rightharpoonup u$ weakly in $X(\Omega)$, then $f(u_n) \to f(u)$, or

$$\int_\Omega (\nabla u_n)v \to \int_\Omega (\nabla u)v \quad \text{for } v \in (L^2(\Omega))^N.$$

Thus, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$. Similarly, for $v \in L^2(\Omega)$, define

$$g(u) = \int_\Omega vu \quad \text{for } u \in X(\Omega),$$

then we have $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$. Recall that the embedding $X(\Omega) \hookrightarrow L^p_{\rm loc}(\Omega)$ is compact. There is a subsequence $\{u_n^i\}$ of $\{u_n^{i-1}\}$ and u in $X(\Omega)$ such that $u_n^i \to u$ in $L^p(\Omega \cap B^N(0;i))$ and a.e. in $\Omega \cap B^N(0;i)$. Then we have $u_n^n \to u$ a.e. in Ω.
(iii) By the definition of weak convergence in $X(\Omega)$, we have

$$\int_\Omega (\nabla u_n \nabla u + u_n u) = \int_\Omega (|\nabla u|^2 + |u|^2) + o(1).$$

Therefore,

$$\begin{aligned}
\|u_n - u\|^2_{H^1} &= \int_\Omega |\nabla u_n - \nabla u|^2 + \int_\Omega |u_n - u|^2 \\
&= \int_\Omega (|\nabla u_n|^2 + |u_n|^2) + \int_\Omega (|\nabla u|^2 + |u|^2) - 2\int_\Omega (\nabla u_n \nabla u + u_n u) \\
&= \|u_n\|^2_{H^1} - \|u\|^2_{H^1} + o(1).
\end{aligned}$$

□

There is a sequence which converges weakly to zero.

Lemma 2.12. *For $u \in H^1(\mathbb{R}^N)$ and $\{z_n\}$ in $\mathbb{R}^N$ satisfying $|z_n| \to \infty$ as $n \to \infty$, then $u(z + z_n) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. For $\varepsilon > 0$, $\varphi \in H^1(\mathbb{R}^N)$, and $\phi \in C_c^1(\mathbb{R}^N)$ exist such that

$$\|\varphi - \phi\|_{H^1} < \varepsilon/2(\|u\|_{H^1} + 1).$$

Let $K = \operatorname{supp}\phi$, then K is compact. We have

$$\begin{aligned}
\langle u(z+z_n), \phi(z)\rangle_{H^1} &= \int_{\mathbb{R}^N} \nabla u(z+z_n)\nabla\phi(z)dz + \int_{\mathbb{R}^N} u(z+z_n)\phi(z)dz \\
&= \int_K \nabla u(z+z_n)\nabla\phi(z)dz + \int_K u(z+z_n)\phi(z)dz \\
&\le \|\nabla u(z+z_n)\|_{L^2(K)}\|\nabla\phi\|_{L^2(K)} + \|u(z+z_n)\|_{L^2(K)}\|\phi\|_{L^2(K)} \\
&= o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, for some $N > 0$ such that $|\langle u(z+z_n), \phi(z)\rangle_{H^1}| < \frac{\varepsilon}{2}$ for $n \ge N$. In addition,

$$\begin{aligned}
\langle u(z+z_n), \varphi(z)\rangle_{H^1} &= \langle u(z+z_n), \varphi(z) - \phi(z)\rangle_{H^1} + \langle u(z+z_n), \phi(z)\rangle_{H^1} \\
&\le \|u(z+z_n)\|_{H^1(\mathbb{R}^N)}\|\varphi(z) - \phi(z)\|_{H^1(\mathbb{R}^N)}
\end{aligned}$$

$$
\begin{aligned}
&+ \langle u(z+z_n), \phi(z)\rangle_{H^1} \\
&\le \|u(z)\|_{H^1(\mathbb{R}^N)}\|\varphi(z)-\phi(z)\|_{H^1(\mathbb{R}^N)} + \frac{\varepsilon}{2} \\
&< \varepsilon \text{ for } n \ge N.
\end{aligned}
$$

Therefore, $u(z+z_n) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. □

Lemma 2.13. *For $u \in H_0^1(\mathbf{A}^r)$ and $\{z_n\}$ in $\mathbf{A}^r$ satisfying $|z_n| \to \infty$ as $n \to \infty$, then $u(z+z_n) \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$ as $n \to \infty$.*

The proof of this lemma is the same as the proof of Lemma 2.12. Therefore, we omit it. Bounded $L^p(\Omega)$ sequence admits interesting convergent properties.

Lemma 2.14 (Brézis-Lieb Lemma)**.** *Suppose $u_n \to u$ a.e. in Ω and there is a $c > 0$ such that $\|u_n\|_{L^p(\Omega)} \le c$ for $n = 1, 2, \dots$. Then*
(i)$\|u_n - u\|_{L^p}^p = \|u_n\|_{L^p}^p - \|u\|_{L^p}^p + o(1)$;
(ii) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ in $L^{\frac{p}{p-1}}(\Omega)$.

Proof. (*i*) Let $\varphi(t) = t^p$ for $t > 0$, then $\varphi'(t) = pt^{p-1}$ and

$$|u_n - u|^p - |u_n|^p = \varphi(|u_n - u|) - \varphi(|u_n|) = \varphi'(t)(|u_n - u| - |u_n|),$$

where $t = (1-\theta)|u_n| + \theta|u_n - u| \le |u_n| + |u|$ for some $\theta \in [0,1]$. Thus, by the Young inequality, for $\varepsilon > 0$

$$||u_n - u|^p - |u_n|^p| \le p(|u_n| + |u|)^{p-1}|u| \le d(|u_n|^{p-1}|u|) + d|u|^p \le \varepsilon|u_n|^p + c_\varepsilon|u|^p.$$

Thus,

$$||u_n - u|^p - |u_n|^p + |u|^p| \le \varepsilon|u_n|^p + (c_\varepsilon + 1)|u|^p.$$

We have

$$\int_\Omega ||u_n - u|^p - |u_n|^p + |u|^p| \le \varepsilon c^p + (c_\varepsilon + 1)\int_\Omega |u|^p.$$

Since $\|u\|_{L^p} \le \liminf_{n\to\infty} \|u_n\|_{L^p} \le c$. For some $\delta > 0$ $|E| < \delta$ implies $\int_E |u|^p < \varepsilon$. In addition, K in $\mathbb{R}^N$ exists such that $|K| < \infty$ and $\int_{K^c} |u|^p < \varepsilon$. Thus,

$$
\begin{aligned}
&\int_E ||u_n - u|^p - |u_n|^p + |u|^p| \le (c^p + c_\varepsilon + 1)\varepsilon, \\
&\int_{K^c} ||u_n - u|^p - |u_n|^p + |u|^p| \le (c^p + c_\varepsilon + 1)\varepsilon.
\end{aligned}
$$

Clearly, $||u_n - u|^p - |u_n|^p + |u|^p| = o(1)$ a.e. in Ω. By Theorem 2.23 below, $\int_\Omega ||u_n - u|^p - |u_n|^p + |u|^p| = o(1)$, or

$$\|u_n - u\|_{L^p}^p = \|u_n\|_{L^p}^p - \|u\|_{L^p}^p + o(1).$$

(*ii*) Let $\varphi(t) = |t|^{p-2}t$, then $\varphi'(t) = (p-1)|t|^{p-2}$. The proof is similar to part (*i*) □

New (PS)-sequences can be produced as follows.

Lemma 2.15. *Let $u_n \rightharpoonup u$ weakly in $X(\Omega)$ and*

$$J'(u_n) = -\Delta u_n + u_n - |u_n|^{p-2}u_n = o(1) \quad \text{in } X^{-1}(\Omega).$$

Then
(i) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ in $X^{-1}(\Omega)$;
(ii) $J'(\varphi_n) = -\Delta\varphi_n + \varphi_n - |\varphi_n|^{p-2}\varphi_n = o(1)$ in $X^{-1}(\Omega)$ where $\varphi_n = u_n - u$;
(iii) if $\{u_n\}$ is a (PS)$_\beta$-sequence, then $\{\varphi_n\}$ is a (PS)$_{(\beta - J(u))}$-sequence.

Proof. (i) By Lemma 2.14,

$$\int_\Omega ||u_n-u|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u|^{\frac{p}{p-1}}=o(1).$$

Now for $\varphi\in H^1(\Omega)$,

$$\begin{aligned}&|\langle|u_n-u|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u,\varphi\rangle|\\&=|\int_\Omega\varepsilon_n\varphi|\le(\int_\Omega|\varepsilon_n|^{\frac{p}{p-1}})^{\frac{p-1}{p}}(\int_\Omega|\varphi|^p)^{1/p}\\&\le c\|\varepsilon_n\|_{L^{\frac{p}{p-1}}}\|\varphi\|_{H^1},\end{aligned}$$

where $\varepsilon_n=|u_n-u|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u$. Therefore,

$$\||u_n-u|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u\|_{X^{-1}}\le c\|\varepsilon_n\|_{L^{\frac{p}{p-1}}}=o(1).$$

(ii) Since

$$J'(u_n)=-\Delta u_n+u_n-|u_n|^{p-2}u_n=o(1)\quad\text{in } X(\Omega)\tag{2.3}$$

and $u_n\rightharpoonup u$, then by Lemma 2.11, we have $J'(u)=0$, or

$$-\Delta u+u-|u|^{p-2}u=0.\tag{2.4}$$

Now by part (i), (2.3), and (2.4),

$$\begin{aligned}J'(\varphi_n)&=-\Delta\varphi_n+\varphi_n-|\varphi_n|^{p-2}\varphi_n\\&=-\Delta(u_n-u)+(u_n-u)-|u_n-u|^{p-2}(u_n-u)\\&=(-\Delta u_n+u_n-|u_n|^{p-2}u_n)-(-\Delta u+u-|u|^{p-2}u)\\&\quad-(|u_n-u|^{p-2}(u_n-u)-|u_n|^{p-2}u_n+|u|^{p-2}u)\\&=o(1).\end{aligned}$$

(iii) Since $u_n\rightharpoonup u$ weakly in $X(\Omega)$ and $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence, by Lemma 2.11, 2.14 and Theorem 2.28 below, a subsequence $\{u_n\}$ exists such that $a(\varphi_n)=a(u_n)-a(u)+o(1)$ and $b(\varphi_n)=b(u_n)-b(u)+o(1)$. Thus, $J(\varphi_n)=J(u_n)-J(u)+o(1)=\beta-J(u)+o(1)$. Therefore, by part (ii), $\{\varphi_n\}$ is a $(\mathrm{PS})_{(\beta-J(u))}$-sequence. □

Define the concentration function of $|u_n|^2$ in $\mathbb{R}^N$ by

$$Q_n(t)=\sup_{z\in\mathbb{R}^N}\int_{z+B^N(0;t)}|u_n|^2.$$

Then we have the following concentration lemma.

Lemma 2.16. *Let $\{u_n\}$ be bounded in $H^1(\mathbb{R}^N)$ and for some $t_0>0$, let $Q_n(t_0)=o(1)$. Then*
(i) $u_n=o(1)$ strongly in $L^q(\mathbb{R}^N)$ for $2<q<2^*$;
(ii) in addition, if u_n satisfies

$$-\Delta u_n+u_n-|u_n|^{p-2}u_n=o(1)\quad\text{in } H^{-1}(\mathbb{R}^N),$$

then $u_n=o(1)$ strongly in $H^1(\mathbb{R}^N)$.

Proof. (i) Decompose $\mathbb{R}^N$ into the family $\mathcal{F}_0=\{P_i^0\}_{i=1}^\infty$ of unit cubes P_i^0 of edge 1. Continue to bisect the cubes to obtain the family $\mathcal{F}_m=\{P_i^m\}_{i=1}^\infty$ of unit cubes P_i^m of edge $\frac{1}{2^m}$. Let m_0 satisfy $\sqrt{N}\frac{1}{2^{m_0}}<t_0$. For each i, let $B_i^{m_0}$ be a ball in $\mathbb{R}^N$ with radius t_0 such that the centers of $B_i^{m_0}$ and $P_i^{m_0}$ are the same. Then

$P_i^{m_0} \subset B_i^{m_0}$, $\mathbb{R}^N = \cup_{i=1}^{\infty} P_i^{m_0}$ and $\{P_i^{m_0}\}_{i=1}^{\infty}$ are nonoverlapping. Write $P_i = P_i^{m_0}$, $2 < q < r < 2^*$, and

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^q &= \sum_{i=1}^{\infty} \int_{P_i} |u_n|^q = \sum_{i=1}^{\infty} \int_{P_i} |u_n|^{2(1-t)} |u_n|^{rt} \\ &\le \sum_{i=1}^{\infty} \Big(\int_{P_i} |u_n|^2\Big)^{1-t} \Big(\int_{P_i} |u_n|^r\Big)^t \\ &\le (Q_n(t_0))^{(1-t)} \sum_{i=1}^{\infty} \Big(\int_{P_i} |u_n|^r\Big)^t \\ &\le c(Q_n(t_0))^{(1-t)} \sum_{i=1}^{\infty} \Big(\int_{P_i} |\nabla u_n|^2 + u_n^2\Big)^{rt/2},\end{aligned}$$

where $0 < t < 1$. Since $\frac{rt}{2} \to \frac{q}{2} > 1$ as $r \to q$, we may choose r satisfying $2 < q < r < 2^*$ and $s = \frac{rt}{2} > 1$. Recall that

$$\|\{a_n\}\|_{\ell^s} = (\sum_{n=1}^{\infty} |a_n|^s)^{1/s} \le \sum_{n=1}^{\infty} |a_n| = \|\{a_n\}\|_{\ell^1}, \quad \ell^1 \subset \ell^2 \subset \cdots \subset \ell^{\infty}.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{\infty} \Big(\int_{P_i} |\nabla u_n|^2 + |u_n|^2\Big)^{rt/2} &\le \Big(\sum_{i=1}^{\infty} \int_{P_i} (|\nabla u_n|^2 + |u_n|^2)\Big)^s \\ &= \Big(\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2)\Big)^s \\ &= \|u_n\|_{H^1(\mathbb{R}^N)}^{2s} \le c \quad \text{for } n = 1, 2, \dots .\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} |u_n|^q \le c(Q_n(t_0))^{(1-t)}, \quad \text{or} \quad \int_{\mathbb{R}^N} |u_n|^q = o(1) \quad \text{as } n \to \infty.$$

(ii) In addition, if u_n satisfies

$$-\Delta u_n + u_n - |u_n|^{p-2} u_n = o(1) \quad \text{in } H^{-1}(\mathbb{R}^N), \tag{2.5}$$

then $\{u_n\}$ is bounded. Multiply Equation (2.5) by u_n and integrate it to obtain

$$a(u_n) = b(u_n) + o(1).$$

By part (i), $b(u_n) = o(1)$. Thus, $a(u_n) = o(1)$, or

$$\|u_n\|_{H^1} = o(1) \quad \text{strongly in } H^1(\mathbb{R}^N).$$

□

Lemma 2.17. *Let $\{u_n\}$ be bounded in $H_0^1(\mathbf{A}^r)$ and for some $t_0 > 0$,*

$$Q_n^r(t_0) = \sup_{y \in \mathbb{R}} \int_{(0,y)+\mathbf{A}^r_{-t_0,t_0}} |u_n|^2 = o(1).$$

Then
(i) $u_n = o(1)$ strongly in $L^q(\mathbf{A}^r)$ for $2 < q < 2^*$;
(ii) In addition, if u_n satisfies

$$-\Delta u_n + u_n - |u_n|^{p-2} u_n = o(1) \quad \text{in } H^{-1}(\mathbf{A}^r),$$

then $u_n = o(1)$ *strongly in* $H_0^1(\mathbf{A}^r)$.

The proof of the above lemma is the same as the proof of Lemma 2.16. We have a sufficient condition for a solution of (1.1) to be zero.

Lemma 2.18. *Let* $N \geq 2$. *For* $c > 0$, *there is a* $\delta > 0$ *such that if* $v \in H_0^1(\Omega)$ *solves* (1.1) *in* Ω *satisfying* $\|v\|_{H^1} \leq c$ *and* $\|v\|_{L^2} \leq \delta$, *then* $v \equiv 0$.

Proof. For $0 < t_0 < 1$ and $p < q < \infty$, let

$$\gamma = \begin{cases} 2t_0 & \text{for } n \geq 3; \\ qt_0 & \text{for } n = 2, \end{cases}$$

and $p = 2(1-t_0) + \gamma$. Since $\|v\|_{H^1} \leq c$ and $\|v\|_{L^2} \leq \delta$, multiply $-\Delta v + v = |v|^{p-2}v$ by v and integrate it to obtain

$$\|v\|_{H^1}^2 = \int_\Omega |v|^p = \int_\Omega |v|^{2(1-t_0)}|v|^\gamma \leq \|v\|_{L^2}^{2(1-t_0)} \|v\|_{L^{\gamma/t_0}}^\gamma \leq d\delta^{2(1-t_0)}\|v\|_{H^1}^\gamma.$$

Thus, we have

$$\|v\|_{H^1}^2 \leq d\delta^{2(1-t_0)}\|v\|_{H^1}^\gamma. \tag{2.6}$$

Suppose that $\|v\|_{H^1} > 0$.

(i) Let $\gamma - 2 \geq 0$. Note that $2(1-t_0) > 0$. By (2.6), we have

$$1 \leq d\delta^{2(1-t_0)}\|v\|_{H^1}^{\gamma-2} \leq dc^{\gamma-2}\delta^{2(1-t_0)}.$$

Let $\delta_1 > 0$ satisfy $dc^{\gamma-2}\delta_1^{2(1-t_0)} < 1$. If $\delta \leq \delta_1$, then

$$1 \leq dc^{\gamma-2}\delta^{2(1-t_0)} \leq dc^{\gamma-2}\delta_1^{2(1-t_0)} < 1,$$

which is a contradiction.

(ii) Let $\gamma - 2 < 0$. By (2.6), we have

$$\|v\|_{H^1} \leq \delta^{\frac{2(1-t_0)}{2-\gamma}} d^{\frac{1}{2-\gamma}},$$

since

$$\|v\|_{H^1}^2 = \int_\Omega |v|^p \leq c_1\|v\|_{H^1}^p, \quad \text{or} \quad 1 \leq c_1\|v\|_{H^1}^{p-2}.$$

Thus, we have

$$1 \leq c_1\|v\|_{H^1}^{p-2} \leq c_2\delta^{\frac{2(1-t_0)(p-2)}{2-\gamma}},$$

where $c_2 = c_1 d^{\frac{p-2}{2-\gamma}} > 0$. Note that $\frac{2(1-t_0)(p-2)}{2-\gamma} > 0$. Let $\delta_2 > 0$ such that

$$c_2\delta_2^{\frac{2(1-t_0)(p-2)}{2-\gamma}} < 1.$$

If $\delta \leq \delta_2$, then $1 \leq c_2\delta^{\frac{2(1-t_0)(p-2)}{2-\gamma}} < 1$, which is a contradiction.

Take $\delta_0 = \min\{\delta_1, \delta_2\}$, if $\delta \leq \delta_0$, from parts (i) and (ii), and we obtain $\|v\|_{H^1} = 0$ or $v = 0$. □

Let

$$\tilde{u}(z) = \begin{cases} u(z) & \text{for } z \in \Omega; \\ 0 & \text{for } z \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

Then we have the following characterization of a function in $W_0^{1,p}(\Omega)$.

Lemma 2.19. *Let Ω be a $C^{0,1}$ domain in $\mathbb{R}^N$ and $u \in L^p(\Omega)$ with $1 < p < \infty$. Then the following are equivalent:*
(i) $u \in W_0^{1,p}(\Omega)$*;*
(ii) there is a constant $c > 0$ such that

$$|\int_\Omega u \frac{\partial \varphi}{\partial x_i}| \le c \|\varphi\|_{L^p}, \quad \text{for each } \varphi \in C_c^1(\mathbb{R}^N),\ i = 1, 2, \dots, N;$$

(iii) $\tilde{u} \in W_0^{1,p}(\mathbb{R}^N)$ *and* $\frac{\partial \tilde{u}}{\partial z_i} = \widetilde{\frac{\partial u}{\partial z_i}}$.

For the proof of this lemma, see Brézis [14, Proposition IX.18], Gilbarg-Trudinger [36, Theorem 7.25], and Grisvard [38, p26].

We recall the classical compactness theorems. The Lebesgue dominated convergence theorem is a well-known compactness theorem.

Theorem 2.20 (Lebesgue Dominated Convergence Theorem)**.** *Suppose Ω is a domain in $\mathbb{R}^N$, $\{u_n\}_{n=1}^\infty$ and u are measurable functions in Ω such that $u_n \to u$ a.e. in Ω. If $\varphi \in L^1(\Omega)$ exists such that for each n*

$$|u_n| \le \varphi \quad a.e.\ in\ \Omega,$$

then $u_n \to u$ in $L^1(\Omega)$.

The converse of the Lebesgue dominated convergence theorem fails.

Example 2.21. For $n = 1, 2, \dots$, let $u_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$u_n(z) = \begin{cases} 0 & \text{for } z \le n; \\ 2 & \text{for } z = n + 1/2n; \\ 0 & \text{for } z \ge n + 1/n; \\ & \text{linear otherwise.} \end{cases}$$

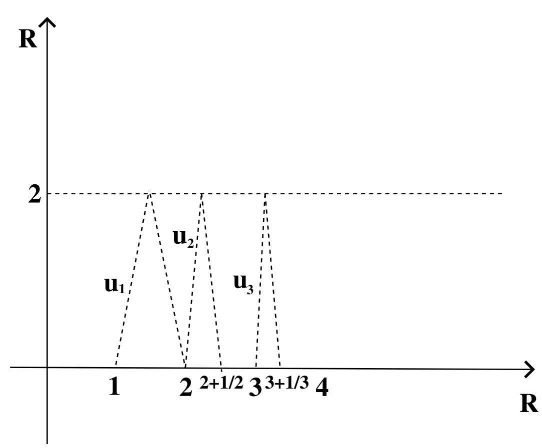

FIGURE 4. Counter example 1.

We have

$$\int_{\mathbb{R}} u_n(z) dz = \frac{1}{n} < \infty \qquad \text{for each } n \in \mathbb{N}.$$

Hence, $u_n \to 0$ a.e. in $\mathbb{R}$ and strongly in $L^1(\mathbb{R})$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy $|u_n| \le \varphi$ a.e. in $\mathbb{R}$ for each $n \in \mathbb{N}$. Then $\infty = \sum_{n=1}^{\infty} \frac{1}{n} = \int_{\mathbb{R}} \sum_{n=1}^{\infty} u_n \le \int_{\mathbb{R}} \varphi$. Consequently, $\varphi \notin L^1(\mathbb{R})$.

However, the generalized Lebesgue dominated convergence theorem is a necessary and sufficient result for L^1 convergence.

Theorem 2.22 (Generalized Lebesgue Dominated Convergence Theorem:)**.** *Suppose Ω is a domain in $\mathbb{R}^N$, $\{u_n\}_{n=1}^{\infty}$ and u are measurable functions in Ω such that $u_n \to u$ a.e. in Ω. Then $u_n \to u$ in $L^1(\Omega)$ if and only if $\{\varphi_n\}_{n=1}^{\infty}, \varphi \in L^1(\Omega)$ exist such that $\varphi_n \to \varphi$ a.e. in Ω, $|u_n| \le \varphi_n$ a.e. in Ω for each n, and $\varphi_n \to \varphi$ in $L^1(\Omega)$.*

Proof. ($\Longrightarrow$) Suppose that $u_n \to u$ in $L^1(\Omega)$, take $\varphi_n = |u_n|$ and $\varphi = |u|$, then $\varphi_n \to \varphi$ in $L^1(\Omega)$.
($\Longleftarrow$) Suppose that a sequence of measurable functions $\{\varphi_n\}_{n=1}^{\infty}$ and φ in Ω exist such that $\varphi_n \in L^1(\Omega)$, $\varphi_n \to \varphi$ a.e. in Ω, $|u_n| \le \varphi_n$ a.e. in Ω for each n, and $\varphi_n \to \varphi$ in $L^1(\Omega)$. Applying the Fatou lemma, we have

$$\int_{\Omega} \liminf_{n\to\infty} (\varphi_n - u_n) \le \liminf_{n\to\infty} \int_{\Omega} (\varphi_n - u_n),$$

or

$$\int_{\Omega} u \ge \limsup_{n\to\infty} \int_{\Omega} u_n.$$

Applying the Fatou lemma again, we have

$$\int_{\Omega} \liminf_{n\to\infty} (\varphi_n + u_n) \le \liminf_{n\to\infty} \int_{\Omega} (\varphi_n + u_n),$$

or

$$\int_{\Omega} u \le \liminf_{n\to\infty} \int_{\Omega} u_n.$$

Thus,

$$\int_{\Omega} u = \lim_{n\to\infty} \int_{\Omega} u_n.$$

□

Another necessary and sufficient result for L^1 convergence is the Vitali convergence theorem.

Theorem 2.23 (Vitali Convergence Theorem for $L^1(\Omega)$)**.** *Suppose Ω is a domain in $\mathbb{R}^N$, $\{u_n\}_{n=1}^{\infty}$ in $L^1(\Omega)$, and $u \in L^1(\Omega)$. Then $\|u_n - u\|_{L^1} \to 0$ if the following three conditions hold:*
(i) $u_n \to u$ a.e in Ω;
(ii) (Uniformly integrable) For each $\varepsilon > 0$, a measurable set $E \subset \Omega$ exists such that $|E| < \infty$ and

$$\int_{E^c} |u_n| d\mu < \varepsilon$$

for each $n \in \mathbb{N}$, where $E^c = \Omega \backslash E$;
(iii) (Uniformly continuous) For each $\varepsilon > 0$, $\delta > 0$ exists such that $|E| < \delta$ implies

$$\int_{E} |u_n| d\mu < \varepsilon \quad \text{for each } n \in \mathbb{N}.$$

Conversely, if $\|u_n - u\|_{L^1} \to 0$, then conditions (ii) and (iii) hold and there is a subsequence $\{u_n\}$ such that (i) holds. Furthermore, if $|\Omega| < \infty$, then we can drop condition (ii).

Proof. Assume the three conditions hold. Choose $\varepsilon > 0$ and let $\delta > 0$ be the corresponding number given by condition (iii). Condition (ii) provides a measurable set $E \subset \Omega$ with $|E| < \infty$ such that

$$\int_{E^c} |u_n| d\mu < \varepsilon$$

for all positive integers n. Since $|E| < \infty$, we can apply the Egorov theorem to obtain a measurable set $B \subset E$ with $|E \backslash B| < \delta$ such that u_n converges uniformly to u on B. Now write

$$\int_\Omega |u_n - u| d\mu = \int_B |u_n - u| d\mu + \int_{E\backslash B} |u_n - u| d\mu + \int_{E^c} |u_n - u| d\mu.$$

Since $u_n \to u$ uniformly in B, the first integral on the right can be made arbitrarily small for large n. The second and third integrals will be estimated with the help of the inequality

$$|u_n - u| \leq |u_n| + |u|.$$

From condition (iii), we have $\int_{E\backslash B} |u_n| d\mu < \varepsilon$ for all $n \in \mathbb{N}$ and the Fatou Lemma shows that $\int_{E\backslash B} |u| d\mu \leq \varepsilon$ as well. The third integral can be handled in a similar way using condition (ii). Thus, it follows that $\|u_n - u\|_{L^1} \to 0$.

Now suppose $\|u_n - u\|_{L^1} \to 0$. Then for each $\varepsilon > 0$, a positive integer n_0 exists such that $\|u_n - u\|_{L^1} < \varepsilon/2$ for $n > n_0$, and measurable sets A and B of finite measure exist such that

$$\int_{A^c} |u| d\mu < \varepsilon/2 \quad \text{and} \quad \int_{B^c} |u_n| d\mu < \varepsilon \quad \text{for } n = 1, 2, \dots, n_0.$$

Minkowski's inequality implies that

$$\|u_n\|_{L^1(A^c)} \leq \|u_n - u\|_{L^1(A^c)} + \|u\|_{L^1(A^c)} < \varepsilon \quad \text{for } n > n_0.$$

Then let $E = A \cup B$ to obtain the necessity of condition (ii). Similar reasoning establishes the necessity of condition (iii).

Convergence in L^1 implies convergence in measure. Hence, condition (i) holds for a subsequence. □

There is a bounded sequence $\{u_n\}$ in $L^1(\mathbb{R})$ that violates Theorem 2.23 condition (ii).

Example 2.24. For $n = 1, 2, \dots$, let $u_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$u_n(z) = \begin{cases} 0 & \text{for } z \leq n; \\ 2 & \text{for } z = n + 1/2; \\ 0 & \text{for } z \geq n + 1; \\ & \text{linear otherwise,} \end{cases}$$

then $\int_{\mathbb{R}} u_n(z) dz = 1$ for each $n \in \mathbb{N}$. Clearly, $\{u_n\}$ violates Theorem 2.23 (ii).

There is a bounded sequence $\{u_n\}$ in $L^1(\mathbb{R})$ that violates Theorem 2.23 condition (iii).

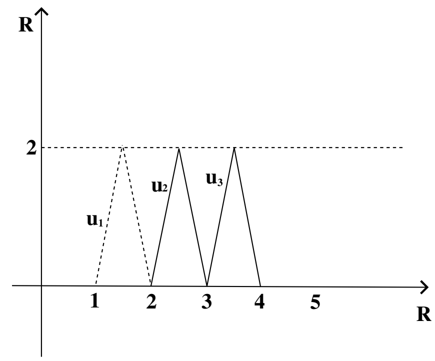

FIGURE 5. counter example violating Theorem 2.23 condition (ii).

Example 2.25. For $n = 1, 2, \dots$, let $u_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$u_n(z) = \begin{cases} 0 & \text{for } z \leq n; \\ 2n & \text{for } z = n + 1/2n; \\ 0 & \text{for } z \geq n + 1/n; \\ & \text{linear therwise.} \end{cases}$$

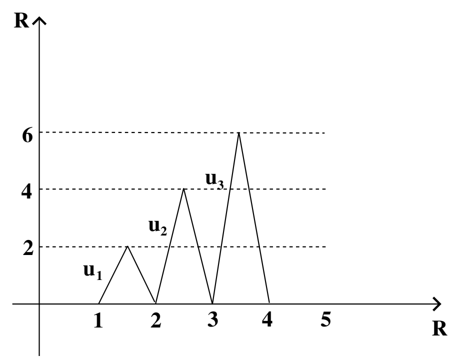

FIGURE 6. counter example violating Theorem 2.23 condition (iii).

Then

$$\int_{\mathbb{R}} u_n(z)dz = 1 \quad \text{for each } n \in \mathbb{N}.$$

Clearly, $\{u_n\}$ violates Theorem 2.23 condition (iii).

Lemma 2.26. *In the Vitali convergence theorem 2.23 condition* (ii)*, the set* E *with* $|E| < \infty$ *can be replaced by the condition that* E *is bounded.*

Proof. Let $E_n = E \cap B^N(0; n)$ for $n = 1, 2, \dots$. Then $E_1 \subset E_2 \subset \cdots \nearrow E$. Thus $|E_1| \leq |E_2| \leq \cdots \nearrow |E|$. For $\delta > 0$ as in Theorem 2.23 condition (iii), there is an E_N such that $|E \backslash E_N| < \delta$. Now

$$\int_{E_N^c} |u_n| dz = \int_{E^c} |u_n| dz + \int_{E \backslash E_N} |u_n| dz < 2\varepsilon$$

for each $n \in \mathbb{N}$. □

Lemma 2.27. *Let Ω be a domain in $\mathbb{R}^N$, $1 \le r < q < s$, and $\{u_n\}$ in $L^r(\Omega) \cap L^s(\Omega)$. Suppose that either $\|u_n\|_{L^r} = o(1)$ and $\|u_n\|_{L^s} = O(1)$, or $\|u_n\|_{L^r} = O(1)$ and $\|u_n\|_{L^s} = o(1)$, then $\|u_n\|_{L^q} = o(1)$.*

Proof. Note that $q = (1-t)r + ts$, $0 < t < 1$, so by the Hölder inequality,

$$\int_\Omega |u_n|^q dz \le \Big(\int_\Omega |u_n|^r dz \Big)^{1-t} \Big(\int_\Omega |u_n|^s dz \Big)^t.$$

Then the conclusion follows. □

We recall the Sobolev embedding theorem as follows.

Theorem 2.28 (Sobolev Embedding Theorem in $W_0^{m,p}(\Omega)$)). *Let $m \in \mathbb{N}$ and $1 \le p < \infty$. Then we have the following continuous injections.*
(i) If $\frac{1}{p} - \frac{m}{N} > 0$, then $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [p, p^*]$, $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{N}$;
(ii) If $\frac{1}{p} - \frac{m}{N} = 0$, then $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [p, \infty)$;
(iii) If $\frac{1}{p} - \frac{m}{N} < 0$, then $W_0^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.
Moreover, if $m - \frac{N}{p} > 0$ is not an integer, let $k = \Big[m - \frac{N}{p}\Big]$ and $\theta = m - \frac{N}{p} - k$ $(0 < \theta < 1)$, then we have for $u \in W_0^{m,p}(\Omega)$

$$\|D^\beta u\|_{L^\infty} \le c\|u\|_{W^{m,p}} \quad \text{for } |\beta| \le k$$
$$|u(x) - u(y)| \le c\|u\|_{W^{m,p}} |x-y|^\theta \qquad \text{a.e. for } x, y \in \Omega.$$

In particular, $W_0^{m,p}(\Omega) \hookrightarrow C^{k,\theta}(\overline{\Omega})$.

For the proof ot the theorem above, see Gilbarg-Trudinger [36, p.164].

Definition 2.29. Ω satisfies a uniform interior cone condition if a fixed cone K_Ω exists such that each $x \in \partial\Omega$ is the vertex of a cone $K_\Omega(x) \subset \overline{\Omega}$ and congruent to K_Ω.

Theorem 2.30 (Sobolev Embedding Theorem in $W^{m,p}(\Omega)$)). *Let Ω satisfy a uniform interior cone condition, $m \in \mathbb{N}$ and $1 \le p < \infty$. Then we have the following continuous injections.*
(i) If $\frac{1}{p} - \frac{m}{N} > 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [p, p^*]$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{N}$;
(ii) If $\frac{1}{p} - \frac{m}{N} = 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [p, \infty)$;
(iii) If $\frac{1}{p} - \frac{m}{N} < 0$, then $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.
Moreover, if $m - \frac{N}{p} > 0$ is not an integer, let

$$k = \big[m - \frac{N}{p}\big] \quad \text{and} \quad \theta = m - \frac{N}{p} - k \quad (0 < \theta < 1),$$

then we have for $u \in W^{m,p}(\Omega)$,

$$\|D^\beta u\|_{L^\infty} \le c\|u\|_{W^{m,p}} \quad \text{for } \beta \quad \text{with } |\beta| \le k$$
$$|D^\beta u(x) - D^\beta u(y)| \le c\|u\|_{W^{m,p}} |x-y|^\theta \quad \text{a.e. for } x, y \in \Omega \quad \text{and } |\beta| = k.$$

In particular, $W^{m,p}(\Omega) \hookrightarrow C^{k,\theta}(\overline{\Omega})$.

For the proof of the theorem above, see Brézis [14, Cor. IX.13] and Gilbarg-Trudinger [36, Theorem 7.26].

Theorem 2.31 (Rellich-Kondrakov Theorem in $W_0^{m,p}(\Omega)$)**.** *Let Ω be a bounded domain, $m \in \mathbb{N}$ and $1 \le p < \infty$. Then we have the following compact injections.*
(i) If $\frac{1}{p} - \frac{m}{N} > 0$, then $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [1, p^*)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{N}$;
(ii) If $\frac{1}{p} - \frac{m}{N} = 0$, then $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [1, \infty)$;
(iii) If $\frac{1}{p} - \frac{m}{N} < 0$, then $W_0^{m,p}(\Omega) \hookrightarrow C^k(\overline{\Omega})$, where $m - \frac{N}{p} > 0$ is not an integer and $k = \left[m - \frac{N}{p}\right]$.

For the proof of the aboved theroem, see Gilbarg-Trudinger [36, Theorem 7.22].

Theorem 2.32 (Rellich-Kondrakov Theorem in $W^{m,p}(\Omega)$)**.** *Let Ω be a bounded $C^{0,1}$ domain in $\mathbb{R}^N$, $m \in \mathbb{N}$ and $1 \le p < \infty$. Then we have the following compact injections.*
(i) If $\frac{1}{p} - \frac{m}{N} > 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [1, p^*)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{N}$;
(ii) If $\frac{1}{p} - \frac{m}{N} = 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $q \in [1, \infty)$;
(iii) If $\frac{1}{p} - \frac{m}{N} < 0$, then $W^{m,p}(\Omega) \hookrightarrow C^{k,\beta}(\overline{\Omega})$, where $m - \frac{N}{p} > 0$ is not an integer, $0 < \beta < \theta$, $k = \left[m - \frac{N}{p}\right]$, and $\theta = m - \frac{N}{p} - k$ $(0 < \theta < 1)$.

For the proof of the above theorem, see Brézis [14, p. 169] and Gilbarg-Trudinger [36, Theorem 7.26].

For the Sobolev space $X(\Omega)$, we can drop condition *(iii)* of the Vitali convergence theorem 2.23 through the interpolation results.

Theorem 2.33 (Rellich-Kondrakov Theorem)**.** *Let Ω be a domain in $\mathbb{R}^N$ of finite measure. Then the embedding $X(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

Proof. Let $\{u_n\}$ be a bounded sequence in $X(\Omega)$, then by Lemma 2.11, a subsequence $\{u_n\}$ and $u \in X(\Omega)$ exist such that $u_n \to u$ a.e. in Ω. By the Egorov theorem, for $\varepsilon > 0$, a closed subset F in $\mathbb{R}^N$ exists such that $F \subset \Omega$, $|\Omega \backslash F| < \varepsilon$, and $u_n \to u$ uniformly in F. Thus,

$$\int_F |u_n - u|^p = o(1) \quad \text{as } n \to \infty.$$

For $N > 2$, we have

$$\begin{aligned} \int_{\Omega\backslash F} |u_n - u|^p &\le \Big(\int_{\Omega\backslash F} 1\Big)^{1/r} \Big(\int_{\Omega\backslash F} |u_n - u|^{ps}\Big)^{1/s} \\ &\le |\Omega\backslash F|^{1/r} \Big(\int_{\Omega} |u_n - u|^{ps}\Big)^{1/s} \\ &\le c\|u_n - u\|_{H^1}^p |\Omega\backslash F|^{1/r} < c\varepsilon^{1/r}, \end{aligned}$$

where $ps = 2^*$ and $\frac{1}{r} + \frac{1}{s} = 1$. For $N = 2$, take any $s > 1$ to obtain the above inequality. Hence, $u_n \to u$ strongly in $L^p(\Omega)$. □

Theorem 2.34 (Vitali Convergence Theorem for $X(\Omega)$)**.** (*i*) *Let Ω be a domain in $\mathbb{R}^N$ of finite measure. Then the embedding $X(\Omega) \hookrightarrow L^p(\Omega)$ is compact;*
(ii) Let Ω be a domain in $\mathbb{R}^N$ and let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $X(\Omega)$. Suppose that a constant $c > 0$ exists such that $\|u_n\|_{H^1} \le c$ for each n and $u_n \to u$ a.e. in Ω. Then for each $\varepsilon > 0$, a measurable set $E \subset \Omega$ exists such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dz < \varepsilon$ for each $n \in \mathbb{N}$ if and only if $\|u_n - u\|_{L^p(\Omega)} = o(1)$.

Proof. Part (i) follows from Willem [78]. (ii) By the Fatou lemma, $\int_{E^c} |u|^p dz \le \varepsilon$. Since $|E| < \infty$ and $\|u_n\|_{H^1} \le c$, by (i), there is a subsequence $\{u_n\}_{n=1}^{\infty}$ satisfying

$$\int_E |u_n - u|^p dz = o(1).$$

Therefore,

$$\int_\Omega |u_n - u|^p dz = \int_{E\cap\Omega} |u_n - u|^p dz + \int_{E^c\cap\Omega} |u_n - u|^p dz = o(1).$$

Now suppose $\|u_n - u\|_{L^p(\Omega)} = o(1)$. Then for each $\varepsilon > 0$, a positive integer n_0 exists such that $\|u_n - u\|_{L^p(\Omega)} < \frac{\varepsilon^{1/p}}{2}$ for $n > n_0$, and measurable sets A and B of finite measure exist such that

$$\int_{A^c} |u|^p dz < \frac{\varepsilon}{2^p} \quad \text{and} \quad \int_{B^c} |u_n|^p dz < \varepsilon \quad \text{for } n = 1, 2, \dots, n_0.$$

The Minkowski inequality implies

$$\|u_n\|_{L^p(A^c)} \le \|u_n - u\|_{L^p(A^c)} + \|u\|_{L^p(A^c)} < \varepsilon^{1/p} \quad \text{for } n > n_0.$$

Then let $E = A \cup B$ to obtain the conclusion. □

Let $L^p_w(\mathbb{R}^N) = \{u \in L^p_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(z)|^p w(z) dz < \infty\}$ be a weighted Lebesgue space, where the weight w is nonnegative with

$$\|u\|^p_{L^p_w(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u(z)|^p w(z) dz.$$

We denote by $Q(x, l)$ the cube of the form

$$Q(x, l) = \{y \in \mathbb{R}^N : |y_j - x_j| < l/2,\ j = 1, \dots, N\}.$$

Theorem 2.35 (Vitali Convergence Theorem for $H^1(\mathbb{R}^N)$)**.** *The embedding of $H^1(\mathbb{R}^N)$ into $L^p_w(\mathbb{R}^N)$ is compact:*

(i) Let $N > 2$. Suppose that $w \in L_w^{\frac{p+\delta}{\delta}}(\mathbb{R}^N)$, with $2 \le p < p + \delta < 2^*$ for some $\delta > 0$, and

$$\lim_{|x|\to\infty} \int_{Q(x,l)} w(z)^{\frac{p+\delta}{\delta}} dz = 0 \tag{2.7}$$

for some $l > 0$. Then $H^1(\mathbb{R}^N)$ is compactly embedded in $L^p_w(\mathbb{R}^N)$;
(ii) Let $N = 2$ and suppose that $w \in L^s_w(\mathbb{R}^N)$ for some $s > 1$ and

$$\lim_{|x|\to\infty} \int_{Q(x,l)} w(z)^s dz = 0 \tag{2.8}$$

for some $l > 0$. Then $H^1(\mathbb{R}^N)$ is compactly embedded in $L^p_w(\mathbb{R}^N)$ for every $p \ge 2$;
(iii) Let $N = 1$ and suppose that $w \in L^1_{\text{loc}}(\mathbb{R}^N)$ and

$$\lim_{|x|\to\infty} \int_{Q(x,l)} w(z) dz = 0 \tag{2.9}$$

for some $l > 0$. Then $H^1(\mathbb{R}^N)$ is compact embedded in $L^p_w(\mathbb{R}^N)$ for every $p \ge 2$.

Proof. (i) It suffices to show that for every $\varepsilon > 0$, a $R > 0$ exists such that

$$\|u - u\chi_{Q(0,R)}\|_{L^p_w(\mathbb{R}^N)} < \varepsilon \tag{2.10}$$

for each $u \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1(\mathbb{R}^N)} \le 1$, where χ_Q is the characteristic function of the cube. Indeed, let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. We assume that $\|u_n\|_{H^1(\mathbb{R}^N)} \leq 1$ for all $n \in \mathbb{N}$. Consequently, a subsequence $\{u_n\}$ and a $u \in H^1(\mathbb{R}^N)$ exist such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n \to u$ in $L^p(Q(0,R))$. On the other hand, by (2.10), we have

$$\|u_n - u\|_{L^p_w(\mathbb{R}^N\setminus Q(0,R))} \leq \|u_n\|_{L^p_w(\mathbb{R}^N\setminus Q(0,R))} + \|u\|_{L^p_w(\mathbb{R}^N\setminus Q(0,R))} \leq 2\varepsilon.$$

Combining this with the previous observation, it is easy to conclude that $u_n \to u$ in $L^p_w(\mathbb{R}^N)$.

To show (2.10), we cover $\mathbb{R}^N$ with cubes $Q(\hat{z},1)$, $\hat{z} \in \mathbb{Z}^N$. We may assume that (i) holds with $l = 1$. For $\eta > 0$, we use (2.7) to find a positive constant n_0 such that $\int_Q w(z)^{\frac{p+\delta}{\delta}} dz < \eta$ for each $Q = Q(\hat{z},1)$ outside $Q(0,n_0)$. By the Sobolev embedding theorem, for any $u \in H^1(\mathbb{R}^N)$, a constant $c > 0$ exists such that

$$\|u\|_{L^p(Q)} \leq c\|u\|_{H^1(Q)} \quad \text{for all } 2 \leq p < 2^*.$$

Thus, by the Hölder inequality, we have

$$\int_Q |u|^p w dz \leq \Big(\int_Q w^{\frac{p+\delta}{\delta}} dz\Big)^{\frac{\delta}{p+\delta}} \Big(\int_Q |u|^{p+\delta} dz\Big)^{\frac{p}{p+\delta}} \leq c'\eta^{1/s}\|u\|^p_{H^1(Q)}$$

where $c' = c^{p/(p+\delta)}$. Now, choose $c'\eta^{1/s} < \varepsilon$ and add these inequalities over all $Q(\hat{z},1)$ outside $Q(0,n_0)$ to obtain $R = n_0$.
(ii) and (iii) are similar to (i). □

We define $H^1_r(\Omega) = \{u \in H^1_0(\Omega) : u \quad \text{is radially symmetric}\}$.

Lemma 2.36. *For $N \geq 2$, every $u \in H^1_r(\mathbb{R}^N)$ is equal to a continuous function U a.e. in $\mathbb{R}^N\setminus\{0\}$ such that for $z \neq 0$*

$$|U(z)| \leq (\frac{2}{\omega_N})^{1/2}|z|^{\frac{1-N}{2}}\Big(\int_{|t|\geq|z|}|u(t)|^2 dt\Big)^{1/4}\Big(\int_{|y|\geq|z|}|\nabla u(t)|^2 dt\Big)^{1/4},$$

where ω_N is the area of the unit ball in $\mathbb{R}^N$.

Proof. Let $\varphi \in C^\infty_c(\mathbb{R}^N)$ be a radially symmetric function. Then for $0 \leq r < \infty$,

$$\begin{aligned} r^{N-1}\varphi(r)^2 &= \int_0^r (s^{N-1}\varphi(s)^2)' ds \\ &= (N-1)\int_0^r s^{N-2}\varphi(s)^2 ds + 2\int_0^r s^{N-1}\varphi(s)\varphi'(s) ds. \end{aligned}$$

Thus,

$$0 = (N-1)\int_0^\infty s^{N-2}\varphi(s)^2 ds + 2\int_0^\infty s^{N-1}\varphi(s)\varphi'(s) ds.$$

Consequently,

$$\begin{aligned} r^{N-1}\varphi(r)^2 &\leq (N-1)\int_0^\infty s^{N-2}\varphi(s)^2 ds + 2\int_0^r s^{N-1}\varphi(s)\varphi'(s) ds \\ &= -2\int_r^\infty s^{N-1}\varphi(s)\varphi'(s) ds \\ &= (\frac{-2}{\omega_N})\int_{|t|\geq r}\varphi(t)\varphi'(t) dt \\ &\leq (\frac{2}{\omega_N})\Big(\int_{|t|\geq r}|\varphi(t)|^2 dt\Big)^{1/2}\Big(\int_{|t|\geq r}|\nabla\varphi(t)|^2 dt\Big)^{1/2}. \end{aligned}$$

For $u \in H_r^1(\mathbb{R}^N)$, take a sequence $\{\varphi_n\}$ radially symmetric in $C_c^\infty(\mathbb{R}^N)$, such that

$$\varphi_n \to u \quad \text{in } H^1(\mathbb{R}^N),$$

then there is a subsequence $\{\varphi_n(r)\}$ such that

$$\begin{aligned} r^{N-1}u(r)^2 &= \lim_{n\to\infty} r^{N-1}\varphi_n(r)^2 \\ &\le \lim_{n\to\infty}(\frac{2}{\omega_N})\Big(\int_{|t|\ge r}|\varphi_n(t)|^2dt\Big)^{1/2}\Big(\int_{|t|\ge r}|\nabla\varphi_n(t)|^2dt\Big)^{1/2} \\ &\le (\frac{2}{\omega_N})\Big(\int_{|t|\ge r}|u(t)|^2dt\Big)^{1/2}\Big(\int_{|t|\ge r}|\nabla u(t)|^2dt\Big)^{1/2}. \end{aligned}$$

Since $u \in H_r^1(\mathbb{R}^N)$, it is a function in $H^1(\mathbb{R})$, and there is a continuous function U in $\mathbb{R}$ such that $u = U$ a.e. and

$$|U(z)| \le (\frac{2}{\omega_N})^{1/2}|z|^{\frac{1-N}{2}}\Big(\int_{|t|\ge|z|}|u(t)|^2dt\Big)^{1/4}\Big(\int_{|t|\ge|z|}|\nabla u(t)|^2dt\Big)^{1/4}.$$

□

Let Θ be an annulus, say $\Theta = \{z \in \mathbb{R}^N : 1 < |z|\}$ with $N \ge 3$.

Theorem 2.37 (Rellich-Kondrakov Theorem for $H_r^1(\Theta)$)**.** *The embedding $H_r^1(\Theta) \hookrightarrow L^p(\Theta)$ is compact.*

Proof. Let $\{u_n\}$ be a bounded sequence in $H_r^1(\Theta)$. Then a subsequence $\{u_n\}$ exists such that $u_n \to u$ a.e. in Θ and $u_n \rightharpoonup u$ weakly in $H_0^1(\Theta)$. By Lemma 2.36, $\lim_{|z|\to\infty} u_n(z) = 0$ uniformly in n and $\lim_{s\to 0}\frac{|s|^p}{|s|^2+|s|^{2^*}} = 0$. Thus, for $\varepsilon > 0$, there is a $K > 0$ such that if $|z| \ge K$, for each n, we have

$$|u_n(z)|^p \le \varepsilon(|u_n(z)|^2 + |u_n(z)|^{2^*}),$$

or

$$\int_{\Theta_K^c}|u_n|^p \le c\varepsilon,$$

where $\Theta_K = \{z \in \Theta : \ z| < K\}$. By the Fatou lemma,

$$\int_{\Theta_K^c}|u|^p \le c\varepsilon.$$

By the Rellich-Kondrakov compactness theorem, a subsequence $\{u_n\}$ exists such that

$$\lim_{n\to\infty}\int_{\Theta_K}|u_n - u|^p = 0.$$

Thus,

$$\lim_{n\to\infty}\int_{\Theta}|u_n - u|^p = 0.$$

□

For any $\beta \in \mathbb{R}$, a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J is bounded. Moreover, a (PS)-value β should be nonnegative.

Lemma 2.38. *Let $\beta \in \mathbb{R}$ and let $\{u_n\}$ be a (PS)$_\beta$-sequence in $X(\Omega)$ for J, then a positive sequence $\{c_n(\beta)\}$ exists such that $\|u_n\|_{H^1} \le c_n(\beta) \le c$ for each n and $c_n(\beta) = o(1)$ as $n \to \infty$ and $\beta \to 0$. Furthermore,*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

and $\beta \ge 0$.

Proof. Since $\{u_n\}$ is a (PS)$_\beta$-sequence in $X(\Omega)$ for J, we have

$$|\beta| + \delta_n + \frac{\varepsilon_n \|u_n\|_{H^1}}{p} \ge J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle = (\frac{1}{2} - \frac{1}{p})\|u_n\|^2_{H^1},$$

where $\delta_n = o(1)$ and $\varepsilon_n = o(1)$. Take

$$c_n(\beta) = \frac{1}{p-2}(\varepsilon_n + \sqrt{\varepsilon_n^2 + 2p(p-2)(|\beta| + \delta_n)}),$$

then $c_n(\beta) = o(1)$ as $n \to \infty$ and $\beta \to 0$ and $\|u_n\|_{H^1} \le c(\beta) \le c$ for each n. Since $\{u_n\}$ is bounded, we have

$$o(1) = \langle J'(u_n), u_n\rangle = a(u_n) - b(u_n),$$

or

$$\beta + o(1) = J(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) = \frac{p-2}{2p}a(u_n) + o(1).$$

Therefore,

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1).$$

This implies $\beta \ge 0$. □

Lemma 2.39. *Let $\{u_n\}$ be in $X(\Omega)\backslash\{0\}$ satisfying $a(u_n) = b(u_n) + o(1)$ and let $J(u_n) = \beta + o(1)$ with $\beta > 0$, then $c > 0$ exists such that $\|u_n\|_{H^1} \ge c$ for each n.*

Proof. Suppose that a subsequence $\{u_n\}$ satisfies $\lim_{n\to\infty} \|u_n\|_{H^1} = 0$. Then $J(u_n) = o(1)$, but this contradicts $\beta > 0$. Thus, $c > 0$ exists such that $\|u_n\|_{H^1} \ge c$ for each n. □

Let Ω be an unbounded domain and ξ_n as in (2.1), then we have the following lemma.

Lemma 2.40. *Let $\{u_n\}$ be a (PS)$_\beta$-sequence in $X(\Omega)$ for J such that*

$$\int_{\Omega_n} |u_n|^p = o(1),$$

where $\Omega_n = \Omega \cap B^N(0;n)$. Then for any $r \ge 1$, we have
(i) $\int_\Omega \xi_n^r |u_n|^p = \int_\Omega |u_n|^p + o(1) = \frac{2p}{p-2}\beta + o(1)$;
(ii) $\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p + o(1) = \frac{2p}{p-2}\beta + o(1)$;
(iii) $\int_\Omega (\xi_n^r - 1) u_n \varphi = o(1)\|\varphi\|_{H^1}$ *for every* $\varphi \in X(\Omega)$;
(iv) $|\int_\Omega (\xi_n^r - 1)|u_n|^{p-2} u_n \varphi| = o(1)\|\varphi\|_{H^1}$ *for every* $\varphi \in X(\Omega)$;
(v) $|\int_\Omega (\xi_n^r - 1)\nabla u_n \nabla\varphi| = o(1)\|\varphi\|_{H^1}$ *for every* $\varphi \in X(\Omega)$.

Proof. (*i*) Clearly, we have

$$\int_\Omega \xi_n^r |u_n|^p = \int_\Omega |u_n|^p + o(1) = \frac{2p}{p-2}\beta + o(1).$$

(*ii*) Let $w_n = \xi_n^r u_n$. Since $\{w_n\}$ is bounded in $X(\Omega)$, we have

$$\begin{aligned} o(1) &= \langle J'(u_n), w_n\rangle \\ &= \int_\Omega (\xi_n^r |\nabla u_n|^2 + r\xi_n^{r-1} u_n \nabla\xi_n \cdot \nabla u_n + \xi_n^r u_n^2) - \int_\Omega \xi_n^r |u_n|^p. \end{aligned}$$

Note that $|\nabla \xi_n(z)| \le \frac{c}{n}$ and $\{u_n\}$ is bounded in $X(\Omega)$, so

$$\int_\Omega \xi_n^{r-1} u_n \nabla\xi_n \cdot \nabla u_n = o(1).$$

We conclude that

$$\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p + o(1) = \frac{2p}{p-2}\beta + o(1).$$

Therefore, the results follow.
(*iii*) By the Hölder and Sobolev inequalities, we have

$$|\int_\Omega (\xi_n^r - 1) u_n \varphi| \le \Big(\int_{\Omega_n} |u_n|^2\Big)^{1/2} \Big(\int_\Omega |\varphi|^2\Big)^{1/2} \le o(1)\|\varphi\|_{H^1}.$$

(*iv*) By the Hölder and Sobolev inequalities, we have

$$\Big|\int_\Omega (\xi_n^r - 1)|u_n|^{p-2} u_n \varphi\Big| \le (\int_{\Omega_n} |u_n|^p)^{\frac{p-1}{p}} (\int_\Omega |\varphi|^p)^{1/p} \le o(1)\|\varphi\|_{H^1}.$$

(*v*) By the hypothesis and part (*i*), we have

$$\begin{aligned} o(1) &= \langle J'(u_n), w_n\rangle \\ &= \langle J'(u_n), w_n\rangle - \langle J'(u_n), u_n\rangle + \langle J'(u_n), u_n\rangle \\ &= \int_\Omega (\xi_n^r - 1)|\nabla u_n|^2 + \int_\Omega (\xi_n^r - 1) u_n^2 - \int_\Omega (\xi_n^r - 1)|u_n|^p + o(1) \\ &= \int_\Omega (\xi_n^r - 1)|\nabla u_n|^2 + o(1). \end{aligned}$$

Thus,

$$\Big|\int_\Omega (\xi_n^r - 1)|\nabla u_n|^2\Big| = \int_\Omega (1 - \xi_n^r)|\nabla u_n|^2 = o(1).$$

Therefore, by the Hölder inequality,

$$\begin{aligned} |\int_\Omega (\xi_n^r - 1)\nabla u_n \nabla\varphi| &\le \Big(\int_\Omega (\xi_n^r - 1)^2 |\nabla u_n|^2\Big)^{1/2} \|\varphi\|_{H^1} \\ &\le \Big(\int_\Omega (1 - \xi_n^r)|\nabla u_n|^2\Big)^{1/2} \|\varphi\|_{H^1} \\ &\le o(1)\|\varphi\|_{H^1}. \end{aligned}$$

□

Lemma 2.41. (*i*) *Suppose that* $\{u_n\}$ *is a sequence in* $X(\Omega)$ *satisfying* $u_n \rightharpoonup 0$ *weakly in* $X(\Omega)$, *then there is a subsequence* $\{u_n\}$ *in* $X(\Omega)$ *such that* $\int_{\Omega_n} |u_n|^p = o(1)$ *as* $n \to \infty$;

(ii) For any $\beta > 0$, suppose that $\{u_n\}$ is a $(PS)_\beta$-sequence in $X(\Omega)$ for J satisfying $\int_{\Omega_n} |u_n|^p = o(1)$ as $n \to \infty$, then $\{\xi_n u_n\}$ is also a $(PS)_\beta$-sequence in $X(\Omega)$ for J.

Proof. (*i*) Since $u_n \rightharpoonup 0$ weakly in $X(\Omega)$, there is a subsequence $\{u_n\}$ such that $u_n \to u$ strongly in $L^p_{\rm loc}(\Omega)$, or there is a subsequence $\{u_n\}$ such that

$$\int_{\Omega_n} |u_n|^p = o(1),$$

where $\Omega_n = \Omega \cap B^N(0; n)$.
(*ii*) Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $X(\Omega)$ for J satisfying $\int_{\Omega_n} |u_n|^p = o(1)$ as $n \to \infty$. By Lemma 2.40, we have

$$\begin{aligned}
J(\xi_n u_n) &= \frac{1}{2}\int_\Omega \left[|\nabla(\xi_n u_n)|^2 + (\xi_n u_n)^2\right] - \frac{1}{p}\int_\Omega |\xi_n u_n|^p \\
&= \frac{1}{2}\int_\Omega \left[|\nabla \xi_n|^2 u_n^2 + \xi_n^2(|\nabla u_n|^2 + u_n^2) + 2\xi_n u_n \nabla\xi_n \nabla u_n\right] - \frac{1}{p}\int_\Omega \xi_n^p |u_n|^p \\
&= \frac{1}{2} a(u_n) - \frac{1}{p} b(u_n) + o(1) = \beta + o(1).
\end{aligned}$$

Then for $\varphi \in X(\Omega)$, we have

$$\begin{aligned}
&|\langle J'(\xi_n u_n), \varphi\rangle| \\
&= \big|\langle J'(\xi_n u_n), \varphi\rangle - \langle J'(u_n), \varphi\rangle + \langle J'(u_n), \varphi\rangle\big| \\
&= \Big| \int_\Omega (\xi_n \nabla u_n \nabla\varphi + u_n \nabla\xi_n \nabla\varphi + \xi_n u_n \varphi - \xi_n^{p-1}|u_n|^{p-2} u_n \varphi) \\
&\quad - \langle J'(u_n), \varphi\rangle + \langle J'(u_n), \varphi\rangle\Big| \\
&= \Big| \int_\Omega \left[(\xi_n - 1)\nabla u_n \nabla\varphi + (\xi_n - 1) u_n \varphi - (\xi_n^{p-1} - 1)|u_n|^{p-2} u_n \varphi\right] + \langle J'(u_n), \varphi\rangle\Big| \\
&\le o(1)\|\varphi\|_{H^1}
\end{aligned}$$

Thus, $J'(\xi_n u_n) = o(1)$. □

Moreover, we have the following lemma.

Lemma 2.42. *Let $\{u_n\}$ be a (PS)-sequence in $H_0^1(\Omega)$ for J satisfying $u_n \rightharpoonup 0$ weakly in $X(\Omega)$ and let $v_n = \xi_n u_n$. Then $\|u_n - v_n\|_{H^1} = o(1)$ as $n \to \infty$.*

Proof. Note that

$$\begin{aligned}
a(u_n - v_n) &= \langle u_n - v_n, u_n - v_n\rangle_{H^1} \\
&= a(u_n) + a(v_n) - 2\langle u_n, v_n\rangle_{H^1} \\
&= 2a(u_n) - 2\langle u_n, v_n\rangle_{H^1} + o(1).
\end{aligned}$$

Thus, it suffices to show that

$$a(u_n) = \langle u_n, v_n\rangle_{H^1} + o(1).$$

We have

$$\begin{aligned}
\langle u_n, v_n\rangle_{H^1} &= \int_\Omega \nabla u_n \nabla v_n + u_n v_n \\
&= \int_\Omega \xi_n \left[|\nabla u_n|^2 + (u_n)^2\right] + \int_\Omega u_n \nabla u_n \nabla \xi_n.
\end{aligned}$$

Note that $|\nabla \xi_n| \le \frac{c}{n}$ and $\{u_n\}$ is a (PS)-sequence in $H_0^1(\Omega)$ for J, so

$$\int_\Omega u_n \nabla u_n \nabla \xi_n = o(1).$$

Hence,

$$\langle u_n, v_n\rangle_{H^1} = \int_\Omega \xi_n \left[|\nabla u_n|^2 + (u_n)^2\right] + o(1).$$

By Lemma 2.40 (i), (ii) and Lemma 2.41 (i), we have

$$\langle u_n, v_n\rangle_{H^1} = \int_\Omega \xi_n \left[|\nabla u_n|^2 + (u_n)^2\right] + o(1) = a(u_n) + o(1).$$

□

Bibliographical notes: The (PS)-sequences were originally introduced by Palais-Smale [60]. Lemma 2.10 is from Brézis [14, p. 35]. Lemma 2.11 is from Zeidler [79, II/A, p. 303]. Lemma 2.16 is from Bahri-Lions [10]. Lemma 2.19 is from Grisvard [38, p. 24].

3. Palais-Smale Decomposition Theorems

In this section, we present the Palais-Smale decomposition theorem in $H_0^1(\Omega)$ for J. This is the concentration-compactness method of P. L. Lions.

Theorem 3.1 (Palais-Smale Decomposition Theorem in $\mathbb{R}^N$))**.** *Let Ω be strictly large domain (see Definition 2.1) in $\mathbb{R}^N$ and let $\{u_n\}$ be a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J. Then there are a subsequence $\{u_n\}$, a positive integer m, sequences $\{z_n^i\}_{n=1}^\infty$ in $\mathbb{R}^N$, a function $\bar u \in H_0^1(\Omega)$, and $0 \ne w^i \in H^1(\mathbb{R}^N)$ for $1 \le i \le m$ such that*

$$|z_n^i| \to \infty, \quad \text{for } i = 1,2,\dots,m,$$
$$-\Delta \bar u + \bar u = |\bar u|^{p-2}\bar u \quad \text{in } \Omega,$$
$$-\Delta w^i + w^i = | w^i|^{p-2} w^i \quad \text{in } \mathbb{R}^N,$$

and

$$u_n = \bar u + \sum_{i=1}^m w^i(\cdot - z_n^i) + o(1) \text{ strongly} \quad \text{in} \quad H^1(\mathbb{R}^N),$$
$$a(u_n) = a(\bar u) + \sum_{i=1}^m a(w^i) + o(1),$$
$$b(u_n) = b(\bar u) + \sum_{i=1}^m b(w^i) + o(1),$$
$$J(u_n) = J(\bar u) + \sum_{i=1}^m J(w^i) + o(1).$$

In addition, if $u_n \ge 0$, then $\bar u \ge 0$ and $w^i \ge 0$ for each $1 \le i \le m$.

Proof. **Step 0**. Since $\{u_n\}$ is a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J, by Lemma 2.38 there is a $c > 0$ such that $\|u_n\|_{H^1} \le c$. In the following proof of this theorem, we fix such a c. There is a subsequence $\{u_n\}$ and a $\bar u$ in $H_0^1(\Omega)$ such that $u_n \rightharpoonup \bar u$ weakly in $H_0^1(\Omega)$ and $\bar u$ solves

$$-\Delta \bar u + \bar u = |\bar u|^{p-2}\bar u \quad \text{in } \Omega.$$

Suppose that $u_n \to \bar{u}$ strongly in $H_0^1(\Omega)$, then we have $u_n = \overline{u} + o(1)$ strongly in $H_0^1(\Omega)$, $a(u_n) = a(\overline{u}) + o(1)$, $b(u_n) = b(\overline{u}) + o(1)$, $J(u_n) = J(\overline{u}) + o(1)$.

Step 1. Suppose that $u_n \nrightarrow \bar{u}$ strongly in $H_0^1(\Omega)$. Let

$$u_n^1 = u_n - \bar{u} \quad \text{for } n = 1, 2, \dots.$$

By Lemma 2.15, $\{u_n^1\}$ is a $(\mathrm{PS})_{(\beta - J(\bar{u}))}$-sequence in $H_0^1(\Omega)$ for J.

(1-0) $\int_{B^N(0;1)} |w_n^1(z)|^2 dz \geq \frac{d_1}{2}$ for some constant $d_1 > 0$ and $n = 1, 2, \dots$, where $w_n^1(z) = u_n^1(z + y_n^1)$ for some $\{y_n^1\} \subset \mathbb{R}^N$: since $\{u_n^1\}$ is bounded, $J'(u_n^1) = o(1)$, and $u_n^1 \nrightarrow 0$ strongly in $H_0^1(\Omega)$. By Lemma 2.16 there is a subsequence $\{u_n^1\}$, a constant $d_1 > 0$ such that

$$Q_n^1 = \sup_{z \in \mathbb{R}^N} \int_{z + B^N(0;1)} |u_n^1|^2 \geq d_1 \quad \text{for } n = 1, 2, \dots.$$

Take $\{y_n^1\}$ in $\mathbb{R}^N$ such that

$$\int_{y_n^1 + B^N(0;1)} |u_n^1(z)|^2 dz \geq \frac{d_1}{2}.$$

Let $w_n^1(z) = u_n^1(z + y_n^1)$, then

$$\int_{B^N(0;1)} |w_n^1(z)|^2 dz \geq \frac{d_1}{2} \quad \text{for } n = 1, 2, \dots.$$

(1-1) $u_n(z) = \bar{u}(z) + w_n^1(z - y_n^1)$ in $H^1(\mathbb{R}^N)$.

(1-2) $\|w_n^1\|_{H^1(\mathbb{R}^N)} \leq c$ for $n = 1, 2, \dots$ and $\|w^1\|_{H^1} \leq c$, where $w_n^1 \rightharpoonup w^1$ weakly in $H^1(\mathbb{R}^N)$: by Lemma 2.11 (iii),

$$\|w_n^1\|_{H^1}^2 = \|u_n^1\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|\bar{u}\|_{H^1}^2 + o(1) \leq c^2 + o(1),$$

we have $\|w_n^1\|_{H^1(\mathbb{R}^N)} \leq c$ for $n = 1, 2, \dots$. Then there is a subsequence $\{w_n^1\}$ and a w^1 in $H^1(\mathbb{R}^N)$ such that $w_n^1 \rightharpoonup w^1$ weakly in $H^1(\mathbb{R}^N)$. By Lemma 2.11 (i), we have

$$\|w^1\|_{H^1} \leq \liminf_{n \to \infty} \|w_n^1\|_{H^1} \leq c.$$

(1-3) $\{w_n^1\}$ is a $(\mathrm{PS})_{(\beta - J(\bar{u}))}$-sequence in $H^1(\mathbb{R}^N)$ for J: note that $J'(u_n^1) = o(1)$ in $H^{-1}(\Omega)$. Because Ω is a strictly large domain, (1-7) below and Theorem 2.35, we have for every $\varphi \in H_0^1(\mathbb{R}^N)$,

$$\langle J'(w_n^1), \varphi \rangle = \int_{\mathbb{R}^N} \nabla w_n^1 \nabla \varphi + w_n^1 \varphi - \int_{\mathbb{R}^N} |w_n^1|^{p-2} w_n^1 \varphi = o(1)$$

Therefore, $J'(w_n^1) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N)$. Moreover, we have

$$J(w_n^1) = J(u_n^1(z + y_n^1)) = J(u_n^1) = (\beta - J(\bar{u})) + o(1).$$

(1-4) $-\Delta w^1 + w^1 - |w^1|^{p-2} w^1 = 0$ in $\mathbb{R}^N$: by Theorem 5.6 (i) below.

(1-5) $w^1 \not\equiv 0$: by the Rellich-Kondrakov theorem 2.31 and $(1 - 0)$, we have

$$\int_{B^N(0;1)} |w^1|^2 = \lim_{n \to \infty} \int_{B^N(0;1)} |w_n^1|^2 \geq \frac{d_1}{2},$$

thus $w^1 \not\equiv 0$.

(1-6) By $(1-2)$, $(1-4)$, $(1-5)$, and Lemma 2.18, $\delta > 0$ exists such that
$$\|w^1\|_{H^1(\mathbb{R}^N)} \geq \|w^1\|_{L^2(\mathbb{R}^N)} > \delta.$$
Therefore,
$$J(w^1) = (\frac{1}{2} - \frac{1}{p})a(w^1) > (\frac{1}{2} - \frac{1}{p})\delta^2 = \delta'.$$

(1-7) $|y_n^1| \to \infty$: otherwise, $R > 0$ exists such that $y_n^1 + B^N(0;1) \subset B^N(0;R)$ for $n = 1, 2, \dots$. Then by $(1-0)$, we have
$$0 = \lim_{n\to\infty} \int_{B^N(0;R)} |u_n^1|^2 \geq \overline{\lim_{n\to\infty}} \int_{y_n^1 + B^N(0;1)} |u_n^1|^2 \geq \frac{d_1}{2},$$
which is a contradiction.

(1-8) $a(u_n) = a(\bar{u}) + a(w_n^1) + o(1)$: since $u_n \rightharpoonup \bar{u}$ weakly in $H^1(\mathbb{R}^N)$, by Lemma 2.11 (iii), we have
$$a(u_n) - a(\bar{u}) = a(u_n - \bar{u}) + o(1) = a(u_n^1) + o(1) = a(w_n^1) + o(1).$$
Thus, $a(u_n) = a(\bar{u}) + a(w_n^1) + o(1)$.

(1-9) $b(u_n) = b(\bar{u}) + b(w_n^1) + o(1)$: since $u_n \to \bar{u}$ a.e. in Ω and $\{u_n\}$ is bounded in $L^p(\Omega)$, by Lemma 2.14 (i), we have
$$b(u_n) - b(\bar{u}) = b(u_n - \bar{u}) + o(1) = b(u_n^1) + o(1) = b(w_n^1) + o(1).$$
Thus, $b(u_n) = b(\bar{u}) + b(w_n^1) + o(1)$.

(1-10) $J(u_n) = J(\bar{u}) + J(w_n^1) + o(1)$: by $(1-8)$ and $(1-9)$, we have
$$J(u_n) = J(\bar{u}) + J(w_n^1) + o(1).$$

Step 2. Suppose that $w_n^1(z) \nrightarrow w^1(z)$ strongly in $H^1(\mathbb{R}^N)$. Let
$$u_n^2(z) = w_n^1(z) - w^1(z).$$
We have $u_n^2 \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ but $u_n^2 \nrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$.

(2-0) $\int_{B^N(0;1)} |w_n^2(z)|^2 dz \geq \frac{d_2}{2}$ for some constant $d_2 > 0$ and $n = 1, 2, \dots$, where $w_n^2(z) = u_n^2(z + y_n^2)$ for some $\{y_n^2\} \subset \mathbb{R}^N$: since $\{u_n^2\}$ is bounded, $J'(u_n^2) = o(1)$, and $u_n^2 \nrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$, by Lemma 2.16 there are a subsequence $\{u_n^2\}$, and a constant $d_2 > 0$ such that
$$Q_n^2 = \sup_{z\in\mathbb{R}^N} \int_{z+B^N(0;1)} |u_n^2(z)|^2 dz \geq d_2 \quad \text{for } n = 1, 2, \dots.$$

For $n = 1, 2, \dots$, take $\{y_n^2\}$ in $\mathbb{R}^N$ such that
$$\int_{y_n^2 + B^N(0;1)} |u_n^2(z)|^2 dz \geq \frac{d_2}{2} \quad \text{for } n = 1, 2, \dots.$$
Let $w_n^2(z) = u_n^2(z + y_n^2)$, then
$$\int_{B^N(0;1)} |w_n^2(z)|^2 dz \geq \frac{d_2}{2} \quad \text{for } n = 1, 2, \dots.$$
As in Step 1, we have the following results.

(2-1) $u_n(z) = \bar{u}(z) + w^1(z - y_n^1) + w_n^2(z - y_n^1 - y_n^2)$ in $H^1(\mathbb{R}^N)$;

(2-2) $\|w_n^2\|_{H^1} \leq c$ for $n = 1, 2, \dots$ and $\|w^2\|_{H^1} \leq c$, where $w_n^2 \rightharpoonup w^2$ weakly in $H^1(\mathbb{R}^N)$;

(2-3) $\{w_n^2\}$ is a (PS)-sequence in $H^1(\mathbb{R}^N)$ for J;

(2-4) $-\Delta w^2 + w^2 - |w^2|^{p-2} w^2 = 0$ in $\mathbb{R}^N$;

(2-5) $w^2 \not\equiv 0$;
(2-6) $\|w^2\|_{L^2(\mathbb{R}^N)} > \delta$ and $J(w^2) > \delta'$;
(2-7) $|y_n^2| \to \infty$;
(2-8) $a(u_n) = a(\bar{u}) + a(w^1) + a(w_n^2) + o(1)$: since

$$u_n^2(z) = w_n^1(z) - w^1(z) \rightharpoonup 0,$$

we have

$$a(w_n^2) = a(u_n^2) = a(w_n^1) - a(w^1) + o(1).$$

Further by $(1-8)$, we have

$$a(u_n) - a(\bar{u}) = a(w_n^1) + o(1) = a(w^1) + a(w_n^2) + o(1).$$

(2-9) $b(u_n) = b(\bar{u}) + b(w^1) + b(w_n^2) + o(1)$;
(2-10) $J(u_n) = J(\bar{u}) + J(w^1) + J(w_n^2) + o(1)$.

Continuing this process, we arrive at the m-th step.

(m-0) $\int_{B^N(0;1)} |w_n^m(z)|^2 dz \geq \frac{d_m}{2}$ for some constant $d_m > 0$ and $n = 1, 2, \dots$, where $w_n^m(z) = u_n^m(z + y_n^m)$ for some $\{y_n^m\} \subset \mathbb{R}^N$;
(m-1) $u_n(z) = \bar{u}(z) + \sum_{i=1}^{m-1} w^i(z - z_n^i) + w_n^m(z - z_n^m)$ in $H^1(\mathbb{R}^N)$, where $z_n^i = y_n^1 + \dots + y_n^i$ for $i = 1, 2, \dots, m$: since

$$w_n^m(z) = u_n^m(z + y_n^m) = w_n^{m-1}(z + y_n^m) - w^{m-1}(z + y_n^m),$$

thus

$$w_n^m(z) + w^{m-1}(z + y_n^m) = w_n^{m-1}(z + y_n^m).$$

Continuing this way, we obtain

$$\begin{aligned} &w_n^m(z) + w^{m-1}(z + y_n^m) + \dots + w^1(z + y_n^2 + \dots + y_n^m) \\ &= w_n^1(z + y_n^2 + \dots + y_n^m) \\ &= u_n^1(z + y_n^1 + y_n^2 + \dots + y_n^m) \end{aligned}$$

(m-2) $\|w_n^m\|_{H^1} \leq c$ for $n = 1, 2, \dots$ and $\|w^m\|_{H^1} \leq c$, where $w_n^m \rightharpoonup w^m$ weakly in $H^1(\mathbb{R}^N)$;
(m-3) $\{w_n^m\}$ is a (PS)-sequence in $H^1(\mathbb{R}^N)$ for J;
(m-4) $-\Delta w^m + w^m - |w^m|^{p-2} w^m = 0$ in $\mathbb{R}^N$;
(m-5) $w^m \not\equiv 0$;
(m-6) $\|w^m\|_{L^2(\mathbb{R}^N)} > \delta$ and $J(w^m) > \delta'$;
(m-7) $|y_n^i| = |z_n^i - z_n^{i-1}| \to \infty$ and $|z_n^i| \to \infty$, for each $i = 1, 2, \dots, m$: we show it by induction on i. For $i = 1$, $|z_n^1| = |y_n^1| \to \infty$. Assume that $|z_n^i| \to \infty$, for $i = 1, 2, \dots, k$, for some $k < m$. By Lemma 2.12, we have $w^i(z - z_n^i) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ for $i = 1, 2, \dots, k$. We claim that $|z_n^{k+1}| \to \infty$. Otherwise, suppose that $\{z_n^{k+1}\}$ is bounded. Since $\|w^{k+1}\|_{L^2(\mathbb{R}^N)} > \delta$, $R > 0$ exists such that

$$z_n^{k+1} + B^N(0; R) \subset B^N(0; 2R)$$

and

$$\int_{B^N(0;R)} |w^{k+1}(z)|^2 \geq (\frac{\delta}{2})^2.$$

We have

$$
\begin{aligned}
(\frac{\delta}{2})^2 &\le \int_{B^N(0;R)} \int_{B^N(0;R)} |w^{k+1}(z)|^2 \\
&= \lim_{n\to\infty} \int_{B^N(0;R)} |u_n^1(z+z_n^{k+1})|^2 dz \\
&\le \lim_{n\to\infty} \int_{B^N(0;2R)} |u_n^1(z)|^2 = 0,
\end{aligned}
$$

which is a contradiction. By the induction hypothesis, we have

$$|z_n^i| \to \infty \quad \text{for } i = 1, 2, \dots, m.$$

(m-8) $a(u_n) = a(\bar{u}) + \sum_{i=1}^{m-1} a(w^i) + a(w_n^m) + o(1)$;
(m-9) $b(u_n) = b(\bar{u}) + \sum_{i=1}^{m-1} b(w^i) + b(w_n^m) + o(1)$;
(m-10) $J(u_n) = J(\bar{u}) + \sum_{i=1}^{m-1} J(w^i) + J(w_n^m) + o(1)$.

By the Archimedean principle, $l \in \mathbb{N}$ exists such that $l\delta^2 > \beta$. Then after step $(l+1)$, we obtain

$$a(u_n) = a(\bar{u}) + a(w^1) + a(w^2) + \dots + a(w^l) + a(w_n^{l+1}) + o(1).$$

Since $a(w_n^{l+1}) \ge 0$, $a(\bar{u}) > 0$, and $a(w^i) > \delta^2$ for $i = 1, 2, \dots, l$, we have $\beta + o(1) \ge l\delta^2 > \beta$, which is a contradiction. Therefore, there is an $m \in \mathbb{N}$, such that $w_n^m(z) = w^m(z) + o(1)$ strongly in $H^1(\mathbb{R}^N)$, $w_n^i(z) = w^i(z) + o(1)$ weakly, and $w_n^i(z) \ne w^i(z) + o(1)$ strongly in $H^1(\mathbb{R}^N)$ for $i = 1, 2, \dots m-1$. Then we have

(sm-0) $\int_{B^N(0;1)} |w_n^m(z)|^2 dz \ge \frac{d_m}{2}$ for some constant $d_m > 0$ and $n = 1, 2, \dots$, where $w_n^m(z) = u_n^m(z + y_n^m)$ for some $\{y_n^m\} \subset \mathbb{R}^N$;
(sm-1) $u_n(z) = \bar{u}(z) + \sum_{i=1}^{m} w^i(z - z_n^i) + o(1)$ strongly in $H^1(\mathbb{R}^N)$, where $z_n^i = y_n^1 + \dots + y_n^i$ for $i = 1, 2, \dots, m$;
(sm-2) $\|w_n^m\|_{H^1} \le c$ for $n = 1, 2, \dots$ and $\|w^m\|_{H^1} \le c$, where $w_n^m \rightharpoonup w^m$ weakly in $H^1(\mathbb{R}^N)$;
(sm-3) $\{w_n^m\}$ is a (PS)-sequence in $H^1(\mathbb{R}^N)$ for J;
(sm-4) $-\Delta w^m + w^m - |w^m|^{p-2} w^m = 0$ in $\mathbb{R}^N$;
(sm-5) $w^m \not\equiv 0$;
(sm-6) $\|w^m\|_{L^2(\mathbb{R}^N)} > \delta$ and $J(w^m) > \delta'$;
(sm-7) $|y_n^i| = |z_n^i - z_n^{i-1}| \to \infty$ and $|z_n^i| \to \infty$, for each $i = 1, 2, \dots, m$;
(sm-8) $a(u_n) = a(\bar{u}) + \sum_{i=1}^{m} a(w^i) + o(1)$;
(sm-9) $b(u_n) = b(\bar{u}) + \sum_{i=1}^{m} b(w^i) + o(1)$;
(sm-10) $J(u_n) = J(\bar{u}) + \sum_{i=1}^{m} J(w^i) + o(1)$.

Finally, suppose $u_n \ge 0$ for $n = 1, 2, \dots$. Then
(i) Since $u_n \rightharpoonup \bar{u}$ weakly in $H_0^1(\Omega)$. By Lemma 2.11 (ii), there is a subsequence $\{u_n\}$ such that $u_n \to \bar{u}$ a.e. in Ω. Thus, $\bar{u} \ge 0$.
(ii) Since $w_n^1(z) = u_n(z+y_n^1) - \bar{u}(z+y_n^1) \rightharpoonup w^1(z)$ weakly in $H^1(\mathbb{R}^N)$ and $\bar{u}(z+y_n^1) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Thus, $u_n(z+y_n^1) \to w^1(z)$ a.e. in Ω, or $w^1 \ge 0$.
(iii) Continuing this process, we obtain $w^i \ge 0$ for each $i = 1, 2, \dots, m$. □

We have the following useful corollary.

Corollary 3.2. *Let Ω be a strictly large domain in $\mathbb{R}^N$. If $\{u_n\}$ is a positive $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J.*
(i) If $\beta \neq j\alpha(\mathbb{R}^N)$ for each $j \in \mathbb{N}$, then there is a positive solution $\overline{u}$ of (1.1) in Ω;
(ii) If $\alpha(\mathbb{R}^N) < \beta < 2\alpha(\mathbb{R}^N)$, then $\{u_n\}$ contains a strongly convergent subsequence.

Proof. By Theorem 3.1, we have

$$J(u_n) = J(\overline{u}) + \sum i = 1^m J(w^i) + o(1).$$

By Corollary 3.6 below, the positive solutions of (1.1) in $\mathbb{R}^N$ are unique, and we obtain $J(w^i) = \alpha(\mathbb{R}^N)$ for each i. Thus, we have

$$\beta = J(\overline{u}) + mJ(w^i) + o(1).$$

(i) If $\beta \neq j\alpha(\mathbb{R}^N)$ for each $j \in \mathbb{N}$, then $J(\overline{u}) \neq 0$, or $\overline{u} \neq 0$. By Theorem 5.6 (i) below, there is a positive solution $\overline{u}$ of (1.1) in Ω.
(ii) Recall that we always have $\beta \geq \alpha(\Omega) \geq \alpha(\mathbb{R}^N)$. Suppose that $m \geq 1$ and $\alpha(\mathbb{R}^N) < \beta < 2\alpha(\mathbb{R}^N)$, then $J(\overline{u}) \neq 0$ or $J(\overline{u}) \geq \alpha(\Omega)$. Thus,

$$2\alpha(\mathbb{R}^N) > \beta + o(1) = J(\overline{u}) + m\alpha(\mathbb{R}^N) \geq (m+1)\alpha(\mathbb{R}^N).$$

This is a contradiction. Hence, $m = 0$. By the proof of Theorem 3.1, we have

$$u_n = \overline{u} + o(1) \quad \text{strongly in } H_0^1(\Omega).$$

□

Remark 3.3. Note that if we replace a strictly large domain by a domain in Theorem 3.1, then the theorem may fail. Let $\mathbf{A}_0^r$ be an upper semi-strip with sufficiently large r, then $\alpha(\mathbb{R}^N) < \alpha(\mathbf{A}_0^r) < 2\alpha(\mathbb{R}^N)$. By the Esteban-Lions theorem 10.7, (1.1) in $\mathbf{A}_0^r$ admits only trivial solution, but if Theorem 3.1 holds, by Corollary 3.2, (1.1) in $\mathbf{A}_0^r$ admits a positive solution, a contradiction.

Definition 3.4. A domain Θ in $\mathbb{R}^N$ is a periodic domain if a partition $\{Q_m\}_{m=0}^\infty$ of Θ and points $\{z_m\}_{m=1}^\infty$ in $\mathbb{R}^N$ exist, satisfying the following conditions:
(i) $\{z_m\}_{m=1}^\infty$ forms a subgroup of $\mathbb{R}^N$;
(ii) Q_0 is bounded;
(iii) $Q_m = z_m + Q_0$ for each m.

Typical examples of periodic domains are the infinite strip $\mathbf{A}^r$, the infinite hollow strip $\mathbf{A}^{r_1,r_2}$, and the whole space $\mathbb{R}^N$.

Similarly, we have the Palais-Smale decomposition theorem in $H_0^1(\Omega)$ for J in a periodic domain in $\Theta \subset \mathbb{R}^N$.

Theorem 3.5 (Palais-Smale Decomposition Theorem in a Periodic Domain)**.** *Let Ω be a strictly large domain in Θ and let $\{u_n\}$ be a positive $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J. Then there are a subsequence $\{u_n\}$, a positive integer m, a subsequence $\{z_n^i\}_{n=1}^\infty$ of $\{z_m\}_{m=1}^\infty$ in Θ, and a function $\bar{u} \in H_0^1(\Omega)$, and $0 \neq w^i \in H^1(\Theta)$, for $1 \leq i \leq m$ such that*
(i) $|z_n^i| \to, \infty$ for $i = 1, 2, \dots, m$;
(ii) $-\Delta\bar{u} + \bar{u} = \mid \bar{u} \mid^{p-2} \bar{u}$ in Ω;
(iii) $-\Delta w^i + w^i = |w^i|^{p-2} w^i$ in Θ;
(iv) $u_n = \bar{u} + \sum_{i=1}^m w^i(\cdot - z_n^i) + o(1)$ strongly in $H^1(\Theta)$;
(v) $a(u_n) = a(\bar{u}) + \sum_{i=1}^m a(w^i) + o(1)$;
(vi) $b(u_n) = b(\bar{u}) + \sum_{i=1}^m b(w^i) + o(1)$;

(vii) $J(u_n) = J(\bar{u}) + \sum_{i=1}^{m} J(w^i) + o(1)$.
In addition, if $u_n \geq 0$*, then* $\bar{u} \geq 0$ *and* $w^i \geq 0$ *for each* $1 \leq i \leq m$.

Proof. The proof is similar to those of Theorem 3.1: see Lien-Tzeng-Wang [47]. Note that instead of

$$Q_n = \sup_{z \in \mathbb{R}^N} \int_{z + B^N(0;1)} |u_n(z)|^2 dz$$

we use

$$Q_n^r = \sup_{y \in \mathbb{R}} \int_{(0,y) + \mathbf{A}^r_{-1,1}} |u_n(z)|^2 dz,$$

where $\mathbf{A}^r_{-1,1} = \{(x, y) \in \mathbf{A}^r \mid -1 < y < 1\}$. □

Corollary 3.6. *Let* $\mathbf{\Theta}$ *be a periodic domain in* $\mathbb{R}^N$ *and let* Ω *be a strictly large domain in* $\mathbf{\Theta}$*, and let* $\{u_n\}$ *be a positive* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J*. Suppose that the only positive solutions in* $\mathbf{\Theta}$ *are ground state solutions.*
(i) If $\beta \neq j\alpha(\mathbf{\Theta})$ *for each* $j \in \mathbb{N}$*, then there is a positive solution* $\overline{u}$ *of* (1.1) *in* Ω*;*
(ii) If $\alpha(\mathbf{\Theta}) < \beta < 2\alpha(\mathbf{\Theta})$*, then* $\{u_n\}$ *contains a strongly convergent subsequence.*

The proof of the above corollary is same as the proof of Corollary 3.2.
Bibliographical notes: Theorem 3.1 is from Lions [49, Lemma 19] and Struwe [66]. Theorem 3.5 is from Lien-Tzeng-Wang [47, Theorem 4.1].

4. Palais-Smale Values and Indexes of Domains

In this section, we prove that four classical important (PS)-values in $X(\Omega)$ for J are the same. Then any one of them is called the index of a domain. The index of a domain Ω is important in studying the existence of solutions of (1.1) in Ω.
(A) Consider the constrained maximization problem

$$\alpha_\gamma(\Omega) = (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{\frac{2p}{2-p}},$$

where $\gamma(\Omega) = \sup\{b(u) \mid u \in X(\Omega), \quad a(u) = 1\}$. By the Sobolev embedding theorem, we have $\alpha_\gamma(\Omega) > 0$. Moreover, $\alpha_\gamma(\Omega)$ is a (PS)-value in $X(\Omega)$ for J.

Theorem 4.1. $\alpha_\gamma(\Omega)$ *is a (PS)-value in* $X(\Omega)$ *for* J*.*

Proof. Let $\{u_n\}$ in $X(\Omega)$ be a maximizing sequence of $\gamma(\Omega)$. Then $a(u_n) = 1$ for $n = 1, 2, \dots$, and

$$\int_\Omega |u_n|^p = \gamma(\Omega)^p + o(1) \quad \text{as } n \to \infty.$$

Let $v_n = \gamma(\Omega)^{\frac{p}{2-p}} u_n$ for each $n = 1, 2, \dots$. Then we have

$$a(v_n) = \int_\Omega (|\nabla v_n|^2 + v_n^2) = \gamma(\Omega)^{\frac{2p}{2-p}} \quad \text{for each } n = 1, 2, \dots,$$

$$b(v_n) = \int_\Omega |v_n|^p = \gamma(\Omega)^{\frac{2p}{2-p}} + o(1) \quad \text{as } n \to \infty,$$

and

$$\begin{aligned} J(v_n) &= \frac{1}{2}a(v_n) - \frac{1}{p}b(v_n) \\ &= (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{\frac{2p}{2-p}} + o(1) \quad \text{as } n \to \infty \\ &= \alpha_\gamma(\Omega) + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

For each $n = 1, 2, \dots$ and $\varphi \in X(\Omega)$, denote

$$l_n(\varphi) = \int_\Omega |v_n|^{p-2} v_n \varphi.$$

Let $\phi \in X(\Omega)$ satisfy $\|\phi\|_{H^1} = 1$. Then $\gamma(\Omega) \geq \|\phi\|_{L^p}$ and

$$\begin{aligned} |l_n(\phi)| = \Big|\int_\Omega |v_n|^{p-2} v_n \phi\Big| &\leq \Big(\int_\Omega |v_n|^p\Big)^{(p-1)/p} \Big(\int_\Omega |\phi|^p\Big)^{1/p} \\ &\leq \gamma(\Omega)^{\frac{2p-2}{2-p}} \gamma(\Omega) + o(1) = \gamma(\Omega)^{\frac{p}{2-p}} + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

Thus,

$$\|l_n\|_{X^{-1}} \leq \gamma(\Omega)^{\frac{p}{2-p}} + o(1) \quad \text{as } n \to \infty.$$

Furthermore,

$$l_n\Big(\frac{v_n}{\|v_n\|_{H^1}}\Big) = \frac{\int_\Omega |v_n|^p}{\|v_n\|_{H^1}} = \frac{\gamma(\Omega)^{2p/(2-p)}}{\gamma(\Omega)^{p/(2-p)}} + o(1) = \gamma(\Omega)^{\frac{p}{2-p}} + o(1)$$

as $n \to \infty$. We conclude that

$$\|l_n\|_{X^{-1}} = \gamma(\Omega)^{\frac{p}{2-p}} + o(1) \quad \text{as } n \to \infty.$$

Since l_n is a continuous linear functional in $X(\Omega)$, by the Riesz representation theorem, for each n, $w_n \in X(\Omega)$ exists such that

$$l_n(\varphi) = \langle w_n, \varphi\rangle_{H^1} = \int_\Omega (\nabla w_n \cdot \nabla \varphi + w_n \varphi) \quad \text{for each } \varphi \in X(\Omega),$$

and $\|w_n\|_{H^1} = \|l_n\|_{X^{-1}}$. Since

$$\langle w_n, v_n\rangle_{H^1} = l_n(v_n) = \int_\Omega |v_n|^p = \gamma(\Omega)^{\frac{2p}{2-p}} + o(1) \quad \text{as } n \to \infty,$$

we obtain

$$\begin{aligned} \|v_n - w_n\|_{H^1}^2 &= \langle v_n, v_n\rangle_{H^1} - 2\langle v_n, w_n\rangle_{H^1} + \langle w_n, w_n\rangle_{H^1} \\ &= \|v_n\|_{H^1}^2 - 2\langle v_n, w_n\rangle_{H^1} + \|w_n\|_{H^1}^2 \\ &= \gamma(\Omega)^{\frac{2p}{2-p}} - 2\gamma(\Omega)^{\frac{2p}{2-p}} + \gamma(\Omega)^{\frac{2p}{2-p}} + o(1) \\ &= o(1) \quad \text{as } n \to \infty. \end{aligned}$$

For $\varphi \in X(\Omega)$ satisfying $\|\varphi\|_{H^1} = 1$, we have

$$\begin{aligned} \langle J'(v_n), \varphi\rangle &= \int_\Omega (\nabla v_n \cdot \nabla \varphi + v_n \varphi) - \int_\Omega |v_n|^{p-2} v_n \varphi \\ &= \langle v_n, \varphi\rangle_{H^1} - \langle w_n, \varphi\rangle_{H^1} = \langle v_n - w_n, \varphi\rangle_{H^1}, \end{aligned}$$

so

$$|\langle J'(v_n), \varphi\rangle| \leq \|v_n - w_n\|_{H^1}.$$

We conclude that

$$J'(v_n) = o(1) \quad \text{strongly in } X^{-1}(\Omega) \quad \text{as } n \to \infty.$$

□

(B) Consider the Nehari minimizing problem

$$\alpha_{\mathbf{M}}(\Omega) = \inf_{v\in\mathbf{M}(\Omega)} J(v),$$

where $\mathbf{M}(\Omega) = \{u \in X(\Omega)\backslash\{0\} : a(u) = b(u)\}$. Note that $\mathbf{M}(\Omega)$ contains every nonzero solution of (1.1). Consider the unit sphere $\mathbf{U}(\Omega)$ and the zero energy manifold $\mathbf{Z}(\Omega)$, where

$$\mathbf{U}(\Omega) = \{u \in X(\Omega) : \|u\|_{H^1} = 1\},$$
$$\mathbf{Z}(\Omega) = \{u \in X(\Omega)\backslash\{0\} : \frac{1}{2}a(u) = \frac{1}{p}b(u)\}.$$

$\alpha_{\mathbf{M}}(\Omega) > 0$ is a consequence of part (i) of the following lemma. Part (ii) of the following lemma will be used later in Lemma 4.6 and Theorem 4.12.

Lemma 4.2. (i) *There is a bijective $C^{1,1}$ map m from $\mathbf{U}(\Omega)$ to $\mathbf{M}(\Omega)$. Moreover, $\mathbf{M}(\Omega)$ is path-connected and a constant $c > 0$ exists such that for $u \in \mathbf{M}(\Omega)$, $\|u\|_{H^1} \geq c$ and $J(u) \geq c$;*
(ii) *There is a bijective $C^{1,1}$ map z from $\mathbf{U}(\Omega)$ to $\mathbf{Z}(\Omega)$. Moreover, $\mathbf{Z}(\Omega)$ is path-connected and a constant $c' > 0$ exists such that for $u \in \mathbf{Z}(\Omega)$, $\|u\|_{H^1} \geq c'$.*

Proof. (i) For $t \geq 0$, $u \in \mathbf{U}(\Omega)$, let

$$h_u(t) = J(tu) = \frac{1}{2}t^2 - \frac{1}{p}t^p b(u).$$

Then $h_u'(t) = t - t^{p-1}b(u)$. We take uniquely $s_u \in \mathbb{R}^+$ such that $s_u > 0$, $s_u u \in \mathbf{M}(\Omega)$, and $0 = h_u'(s_u)$. For $v \in \mathbf{U}(\Omega)$, a $s_v \in \mathbb{R}^+$ exists such that $s_v v \in \mathbf{M}(\Omega)$: that is

$$\langle J'(s_v v), s_v v\rangle = s_v^2 - s_v^p b(v) = 0.$$

Consider the function $g(t,u) : \mathbb{R}^+ \times \mathbf{U}(\Omega) \to \mathbb{R}$ defined by

$$g(t,u) = \langle J'(tu), tu\rangle = t^2 a(u) - t^p b(u).$$

Note that $g(s_v, v) = \langle J'(s_v v), s_v v\rangle = 0$. Thus,

$$\frac{\partial g}{\partial t}(t,u)\big|_{(s_v,v)} = 2s_v - ps_v^{p-1}b(v) = s_v(2-p) < 0.$$

By the implicit function theorem, a neighborhood $\mathbf{W}$ of v in $\mathbf{U}(\Omega)$ and a unique function $t \in C^{1,1}$ exist such that

$$t : \mathbf{W} \to \mathbb{R}^+,\ t(v) = s_v,$$
$$g(t(u), u) = 0 \text{ for all } u \in \mathbf{W}.$$

Therefore, for each $v \in \mathbf{U}(\Omega)$, $t : \mathbf{U}(\Omega) \to \mathbb{R}^+$ and $m : \mathbf{U}(\Omega) \to \mathbf{M}(\Omega)$, t, $m \in C^{1,1}$ exist such that $t(v) = s_v$, $m(v) = s_v v$. Clearly, t and m are injective. For each $u \in \mathbf{M}(\Omega)$, write $u = s_v v$, where $s_v = \|u\|_{H^1}$ and $v = \frac{u}{\|u\|_{H^1}} \in \mathbf{U}(\Omega)$. Since $m(v) = u$, m is surjective. Since $\mathbf{U}(\Omega)$ is path-connected, $\mathbf{M}(\Omega)$ is path-connected. Note that $u \in \mathbf{M}(\Omega)$, so $J'(u) = 0$, or $s_v^2 = \int_\Omega s_v^p |v|^p$. By the Sobolev embedding theorem, we have $s_v^2 = \int_\Omega s_v^p |v|^p \leq d s_v^p$, or $c \leq s_v$, where d and c are two positive constants. Therefore, $\|u\|_{H^1} = \|s_v v\|_{H^1} = s_v \geq c$ for $u \in \mathbf{M}(\Omega)$.
(ii) The proof is similar to part (i). $\square$

Theorem 4.3. *Let $\beta > 0$ and let $\{u_n\}$ in $X(\Omega)\backslash\{0\}$ be a sequence for J such that $J(u_n) = \beta + o(1)$ and $a(u_n) = b(u_n) + o(1)$. Then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + o(1)$, $\{s_n u_n\}$ is in $\mathbf{M}(\Omega)$ and $J(s_n u_n) = \beta + o(1)$. In particular, if $\{u_n\}$ is a (PS)$_\beta$-sequence for J, then there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\}$ is in $\mathbf{M}(\Omega)$ and there is also a (PS)$_\beta$-sequence in $X(\Omega)$ for J.*

Proof. By Lemma 4.2, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\}$ is in $\mathbf{M}(\Omega)$: $s_n^2 a(u_n) = s_n^p b(u_n)$ for each n, because $a(u_n) = b(u_n) + o(1)$ and $J(u_n) = \beta + o(1)$ imply $s_n = 1 + o(1)$. Therefore, $J(s_n u_n) = \beta + o(1)$. The last part follows from Lemma 2.38. □

A minimizing sequence $\{u_n\}$ in $\mathbf{M}(\Omega)$ of $\alpha_{\mathbf{M}}(\Omega)$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}$-sequence in $X(\Omega)$ for J .

Theorem 4.4. *Let $\{u_n\}$ be in $X(\Omega)$. Then $\{u_n\}$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}$-sequence for J if and only if $J(u_n) = \alpha_{\mathbf{M}}(\Omega) + o(1)$ and $a(u_n) = b(u_n) + o(1)$. In particular, every minimizing sequence $\{u_n\}$ in $\mathbf{M}(\Omega)$ of $\alpha_{\mathbf{M}}(\Omega)$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}$-sequence in $X(\Omega)$ for J . In particular, $\alpha_{\mathbf{M}}(\Omega)$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}-$value in $X(\Omega)$ for J .*

Proof. Suppose $\{u_n\}$ is a (PS)$_{\alpha_{\mathbf{M}}(\Omega)}$-sequence in $X(\Omega)$ for J. By Lemma 2.38, we have $a(u_n) = b(u_n) + o(1)$.

Conversely, let $\{u_n\}$ satisfy $J(u_n) = \alpha_{\mathbf{M}}(\Omega) + o(1)$ and $a(u_n) = b(u_n) + o(1)$. Then we have

$$a(u_n) = \frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega) + o(1) \quad \text{as } n \to \infty. \tag{4.1}$$

For $n = 1, 2, \dots$, denote

$$l_n(\varphi) = \int_\Omega |u_n|^{p-2} u_n \varphi \quad \text{for } \varphi \in X(\Omega). \tag{4.2}$$

Let $\phi \in \mathbf{U}(\Omega)$. By Lemma 4.2, $t > 0$ exists such that $t\phi \in \mathbf{M}(\Omega) : \|t\phi\|_{H^1}^2 = \|t\phi\|_{L^p}^p$; we conclude that $t = \|\phi\|_{L^p}^{\frac{-p}{p-2}}$ and

$$\alpha_{\mathbf{M}(\Omega)} \leq (\frac{1}{2} - \frac{1}{p})\|t\phi\|_{H^1}^2 = \frac{p-2}{2p} t^2 = \frac{p-2}{2p}\|\phi\|_{L^p}^{\frac{-2p}{p-2}}.$$

Therefore, $\|\phi\|_{L^p} \leq (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{\frac{2-p}{2p}}$. For each n,

$$\begin{aligned}
|l_n(\phi)| &= |\int_\Omega |u_n|^{p-2} u_n \phi| \\
&\leq \Big(\int_\Omega |u_n|^p\Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\phi|^p\Big)^{1/p} \\
&\leq (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{\frac{p-1}{p}} (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{\frac{2-p}{2p}} + o(1) \\
&= \big(\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega)\big)^{1/2} + o(1) \quad \text{as } n \to \infty,
\end{aligned}$$

we have

$$\|l_n\|_{X^{-1}} \leq (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{1/2} + o(1) \quad \text{as } n \to \infty. \tag{4.3}$$

Furthermore, by (4.2), we have

$$\begin{aligned} l_n(\frac{u_n}{\|u_n\|_{H^1}}) &= \frac{\int_\Omega |u_n|^p}{\|u_n\|_{H^1}} \\ &= (b(u_n))^{1/2} + o(1) \qquad (4.4) \\ &= (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{1/2} + o(1) \quad \text{as } n \to \infty \end{aligned}$$

By (4.3) and (4.4), we conclude that

$$\|l_n\|_{X^{-1}} = (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{1/2} + o(1) \quad \text{as } n \to \infty.$$

By the Riesz representation theorem, for each n, $w_n \in X(\Omega)$ exists such that, for each $\varphi \in X(\Omega)$,

$$l_n(\varphi) = \langle w_n, \varphi\rangle_{H^1} = \int_\Omega (\nabla w_n \cdot \nabla\varphi + w_n\varphi),$$

and

$$\|w_n\|_{H^1} = \|l_n\|_{X^{-1}} = (\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega))^{1/2} + o(1). \qquad (4.5)$$

Consequently,

$$\langle w_n, u_n\rangle_{H^1} = l_n(u_n) = \int_\Omega |u_n|^p = \frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega) + o(1). \qquad (4.6)$$

By (4.1), (4.5), and (4.6), we obtain

$$\begin{aligned} \|u_n - w_n\|_{H^1}^2 &= \langle u_n, u_n\rangle_{H^1} - 2\langle u_n, w_n\rangle_{H^1} + \langle w_n, w_n\rangle_{H^1} \\ &= \|u_n\|_{H^1}^2 - 2\langle u_n, w_n\rangle_{H^1} + \|w_n\|_{H^1}^2 \\ &= \frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega) - 2\frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega) + \frac{2p}{p-2}\alpha_{\mathbf{M}}(\Omega) + o(1) \\ &= o(1) \quad \text{as } n \to \infty. \end{aligned}$$

For $\varphi \in \mathbf{U}(\Omega)$, we have

$$\begin{aligned} \langle J'(u_n), \varphi\rangle &= \int_\Omega (\nabla u_n \cdot \nabla\varphi + u_n\varphi) - \int_\Omega |u_n|^{p-2}u_n\varphi \\ &= \langle u_n, \varphi\rangle_{H^1} - \langle w_n, \varphi\rangle_{H^1} = \langle u_n - w_n, \varphi\rangle_{H^1}, \end{aligned}$$

so

$$\|J'(u_n)\|_{X^{-1}} \le \|u_n - w_n\|_{H^1} = o(1).$$

We conclude that $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$. □

If u achieves $\alpha_{\mathbf{M}}(\Omega)$, then u is a nonzero solution of (1.1).

Theorem 4.5. *Let $u \in \mathbf{M}(\Omega)$ be such that $J(u) = \min_{v\in\mathbf{M}(\Omega)} J(v)$. Then u is a nonzero solution of* (1.1).

Proof. Set $g(v) = a(v) - b(v)$ for $v \in X(\Omega)$. Note that $\langle g'(u), u\rangle = (2-p)a(u) \neq 0$. Since the minimum of J is achieved at u and is constrained in $\mathbf{M}(\Omega)$, by the Lagrange multiplier theorem, $\lambda \in \mathbb{R}$ exists such that $J'(u) = \lambda g'(u)$ in $X(\Omega)$. Thus,

$$0 = \langle J'(u), u\rangle = \lambda\langle g'(u), u\rangle,$$

or $\lambda = 0$. Thus, $J'(u) = 0$. Hence, u is a weak solution of (1.1) such that $J(u) = \alpha_{\mathbf{M}}(\Omega)$. □

(C) Consider the mountain pass minimax problem

$$\alpha_\Gamma(\Omega) = \inf_{g\in\Gamma(\Omega)} \max_{t\in[0,1]} J(g(t)),$$

where $e \neq 0$, $J(e) = 0$, and

$$\Gamma(\Omega) = \{g \in C([0,1], X(\Omega)) : g(0) = 0, g(1) = e\}.$$

Then $\alpha_\Gamma(\Omega) > 0$ is a consequence of the following lemma.

Lemma 4.6. *A ball $B(0;r)$ in $X(\Omega)$, $c > 0$, and $e \notin \overline{B(0;r)}$ exist such that $J(e) = 0$ and $\min_{v\in\partial B(0;r)} J(v) \geq c$.*

Proof. By Lemma 4.2 (*ii*), for each $u \in \mathbf{U}(\Omega)$, there is a $t > 0$ such that $J(tu) = 0$. Let $e = tu$, then $J(e) = 0$. Since for each $v \in X(\Omega)\backslash\{0\}$

$$J(v) = \frac{1}{2}a(v) - \frac{1}{p}b(v),$$

by the Sobolev inequality, there is a constant $c_1 > 0$ such that $b(v) \leq c_1 a(v)^{p/2}$, and we have

$$J(v) \geq a(v)\{\frac{1}{2} - \frac{c_1}{p}a(v)^{\frac{p-2}{2}}\}.$$

Take $r > 0$ such that $e \notin \overline{B(0;r)}$ and $\frac{1}{2} - \frac{c_1}{p}r^{p-2} \geq \frac{1}{4}$, then for $\|v\|_{H^1} = r$, we have

$$J(v) \geq c,$$

where $c = \frac{1}{4}r^2$. □

We require the following lemma.

Theorem 4.7 (Ekeland variational principle)**.** *Let M be a complete metric space with metric d and let $F : M \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, bounded from below, and $\not\equiv \infty$. Then for any $\varepsilon > 0$ and $\lambda > 0$, and any $u \in M$ with*

$$F(u) \leq \inf_M F + \varepsilon,$$

there is an element $v \in M$ such that

$$F(v) \leq F(u),$$
$$d(u,v) \leq \frac{1}{\lambda},$$
$$F(w) + \varepsilon\lambda d(v,w) > F(v) \quad \text{for } w \neq v.$$

Proof. It is sufficient to prove our assertion for $\lambda = 1$. The general case is obtained by replacing d by an equivalent metric λd. We define the relation on M:

$$w \leq v \Longleftrightarrow F(w) + \varepsilon d(v,w) \leq F(v).$$

It is easy to see that this relation define a partial ordering on M. We now construct inductively a sequence $\{u_m\}$ as follows: $u_0 = u$; also assuming that u_m has been defined, we set

$$S_n = \{w \in M \mid w \leq u_n\}$$

and choose $u_{n+1} \in S_n$ so that

$$F(u_{n+1}) \leq \inf_{S_n} F + \frac{1}{n+1}.$$

Since $u_{n+1} \le u_n$, $S_{n+1} \subset S_n$, and by the lower semicontinuity of F, S_n is closed. We now show that diam $S_n \to 0$. Indeed, if $w \in S_{n+1}$, then $w \le u_{n+1} \le u_n$ and consequently,

$$\varepsilon d(w, u_{n+1}) \le F(u_{n+1}) - F(w) \le \inf_{S_n} F + \frac{1}{n+1} - \inf_{S_n} F = \frac{1}{n+1}.$$

This estimate implies

$$\operatorname{diam} S_{n+1} \le \frac{2}{\varepsilon(n+1)}$$

and our claim follows. The fact that M is complete implies that

$$\cap_{n\ge 0} S_n = \{v\}$$

for some $v \in M$. In particular, $v \in S_0$, that is, $v \le u_0 = u$. Hence,

$$F(v) \le F(u) - \varepsilon d(u, v) \le F(u).$$

Moreover,

$$d(u, v) \le \varepsilon^{-1}(F(u) - F(v)) \le \varepsilon^{-1}\big(\inf_M F + \varepsilon - \inf_M F\big) = 1.$$

To complete the proof we must show $w \le v$ implies $w = v$. If $w \le v$, then $w \le u_n$ for each integer $n \ge 0$, that is $w \in \cap_{n\ge 0} S_n = \{v\}$. □

Lemma 4.8. *Let $\Gamma(\Omega)$ be the complete metric space with the usual L^∞ distance d and $J \in C^1(X(\Omega), \mathbb{R})$. Then for each $\varepsilon > 0$ and each $f \in \Gamma(\Omega)$ such that*

$$\max_{s\in[0,1]} J(f(s)) \le \alpha_\Gamma(\Omega) + \varepsilon, \tag{4.7}$$

$v \in X(\Omega)$ exists such that

$$\alpha_\Gamma(\Omega) - \varepsilon \le J(v) \le \max_{s\in[0,1]} J(f(s)),$$
$$\operatorname{dist}(v, f([0,1])) \le \varepsilon^{1/2},$$
$$|J'(v)| \le \varepsilon^{1/2}.$$

Proof. Without loss of generality, we can assume that

$$0 < \varepsilon < \alpha_\Gamma(\Omega). \tag{4.8}$$

Let $f \in \Gamma(\Omega)$ satisfy the condition (4.7). We define the function $\Phi : \Gamma(\Omega) \to \mathbb{R}$ by

$$\Phi(g) = \max_{s\in[0,1]} J(g(s)).$$

Then (i) Φ is bounded below: $\Phi(g) \ge \alpha_\Gamma(\Omega) > 0$.
(ii) Φ is continuous at each $g \in \Gamma(\Omega)$: since J is continuous on the compact set $K = g([0,1])$, for each $\varepsilon > 0$, $u \in K$, there is a $\delta_u > 0$ such that if $w \in B(u; \delta_u)$ is an open ball in $X(\Omega)$, then $|J(w) - J(u)| < \frac{1}{2}\varepsilon$. Since K is compact, finite values $B(u_i; \delta_{u_i})$, $i = 1, \dots, n$, exist such that

$$K \subset B(u_1; \frac{\delta_{u_1}}{2}) \cup \dots \cup B(u_n; \frac{\delta_{u_n}}{2}).$$

Take $\delta = \min\{\frac{\delta_{u_1}}{2}, \dots, \frac{\delta_{u_n}}{2}\}$. Let $k \in \Gamma(\Omega)$ satisfy $\|k - g\|_{L^\infty} < \delta$. For each $s \in [0,1]$, we have

$$|k(s) - g(s)| < \delta,$$

or $g(s) \in B(u_i; \frac{\delta_{u_i}}{2})$, $k(s) \in B(u_i; \delta_{u_i})$. Thus

$$|J(k(s)) - J(g(s))| < \varepsilon, \quad \text{or} \quad |\Phi(k) - \Phi(g)| \leq \varepsilon.$$

The Ekeland variational principle (Theorem 4.7) implies the existence of $h \in \Gamma(\Omega)$ such that

$$\Phi(h) \leq \Phi(f) \leq \alpha_\Gamma(\Omega) + \varepsilon,$$
$$\max_{s \in [0,1]} |h(s) - f(s)| \leq \varepsilon^{1/2},$$

and

$$\Phi(g) > \Phi(h) - \varepsilon^{\frac{1}{2}} d(h, g) \quad \text{whenever } g \in \Gamma(\Omega) \quad \text{and } g \neq h. \tag{4.9}$$

Let $A = \{s \in [0,1] : \alpha_\Gamma(\Omega) - \varepsilon \leq J(h(s))\}$, then A is nonempty since

$$\alpha_\Gamma(\Omega) - \varepsilon < \alpha_\Gamma(\Omega) = \inf_{g \in \Gamma(\Omega)} \max_{s \in [0,1]} J(g(s)) \leq \max_{s \in [0,1]} J(h(s)).$$

Note that for $s \in A$,

$$|J'(h(s))| \leq \varepsilon^{1/2},$$

if and only if

$$|\langle J'(h(s)), v\rangle| \leq \varepsilon^{1/2} \quad \text{for } v \in \mathbf{U}(\Omega),$$

if and only if

$$\langle J'(h(s)), v\rangle \geq -\varepsilon^{1/2} \quad \text{for } v \in \mathbf{U}(\Omega).$$

We claim that there is some $s \in A$ satisfying $|J'(h(s))| \leq \varepsilon^{1/2}$. If this is not the case, then for each $s \in A$, $v_s \in \mathbf{U}(\Omega)$ exists such that $\langle J'(h(s)), v_s\rangle < -\varepsilon^{1/2}$. By the continuity of J', $\delta_s > 0$ and an open ball B_s in $[0,1]$ containing s exist such that for $t \in B_s$ and $u \in X(\Omega)$ with $|u| \leq \delta_s$, we have

$$\langle J'(h(t) + u), v_s\rangle < -\varepsilon^{1/2}. \tag{4.10}$$

Since A is compact, a finite subcovering B_{s_1}, $B_{s_2} \dots B_{s_k}$ of A exists. We define the Lipschitz continuous functions, for each $j = 1, 2, \dots, k$, $\psi_j : [0,1] \to [0,1]$ by

$$\psi_j(t) = \begin{cases} \operatorname{dist}(t, B_{s_j}^c) / \sum_{i=1}^k \operatorname{dist}(t, B_{s_i}^c) & \text{for } t \in A; \\ 0 & \text{for } t \notin \cup_{i=1}^k B_{s_i}. \end{cases}$$

Then

$$\sum_{j=1}^k \psi_j(t) = 1 \text{for } t \in A;$$
$$\|\sum_{j=1}^k \psi_j(t) v_{s_j}\|_{H^1} \leq 1 \quad \text{for } t \in A.$$

Let $\delta = \min\{\delta_{s_1}, \dots \delta_{s_k}\}$ and let $\psi : [0,1] \to [0,1]$ be a continuous function such that

$$\psi(t) = \begin{cases} 1 & \text{if } J(h(t)) \geq \alpha_\Gamma(\Omega); \\ 0 & \text{if } J(h(t)) \leq \alpha_\Gamma(\Omega) - \varepsilon, \end{cases}$$

and let $g \in C([0,1], X(\Omega))$ be defined by

$$g(t) = h(t) + \delta\psi(t) \sum_{j=1}^k \psi_j(t) v_{s_j}.$$

It follows from (4.8) that, for $t \in \{0,1\}$, we have $J(h(t)) = 0 < \alpha_\Gamma(\Omega) - \varepsilon$, or $\psi(t) = 0$. Consequently, $g(0) = h(0) = 0$ and $g(1) = h(1) = e$, that is, $g \in \Gamma(\Omega)$. The mean value theorem and (4.10) imply that, for each $t \in A$, there is some $0 < \tau < 1$ for which

$$\begin{aligned} &J(g(t)) - J(h(t)) \\ &= \langle J'(h(t) + \tau\delta\psi(t)\sum_{j=1}^{k}\psi_j(t)v_{s_j}), \delta\psi(t)\sum_{j=1}^{k}\psi_j(t)v_{s_j}\rangle \\ &= \delta\psi(t)\sum_{j=1}^{k}\psi_j(t)\langle J'\big(h(t) + \tau\delta\psi(t)\sum_{j=1}^{k}\psi_j(t)v_{s_j}\big), v_{s_j}\rangle \\ &\le -\varepsilon^{1/2}\delta\psi(t). \end{aligned} \tag{4.11}$$

Thus

$$J(g(t)) \le J(h(t)) - \varepsilon^{1/2}\delta\psi(t) \le J(h(t)).$$

If $t \notin A$, then $\psi(t) = 0$ and hence $J(g(t)) = J(h(t))$. Let $\bar{t} \in [0,1]$ satisfy $J(g(\bar{t})) = \Phi(g)$, then we obtain

$$J(h(\bar{t})) \ge J(g(\bar{t})) \ge \alpha_\Gamma(\Omega),$$

so that $\bar{t} \in A$ and $\psi(\bar{t}) = 1$. By (4.11), we obtain

$$J(g(\bar{t})) - J(h(\bar{t})) \le -\varepsilon^{1/2}\delta$$

and in particular

$$\Phi(g) + \varepsilon^{1/2}\delta \le J(h(\bar{t})) \le \Phi(h),$$

so that $g \ne h$. However, by the definition of g, we have $d(g,h) \le \delta$ and

$$\Phi(g) + \varepsilon^{1/2}d(g,h) \le \Phi(h)$$

which contradicts (4.9). The proof is complete . □

$\alpha_\Gamma(\Omega)$ is a (PS)-value in $X(\Omega)$ for J.

Theorem 4.9. *Under the conditions of Lemma 4.8, for each minimizing sequence $\{f_k\} \subset \Gamma(\Omega)$ such that*

$$\Phi(f_k) = \max_{s\in[0,1]} J(f_k(s)) = \alpha_\Gamma(\Omega) + o(1),$$

there is a (PS)-sequence $\{v_k\}$ in $X(\Omega)$ for J satisfying

$$\begin{gathered} J(v_k) = \alpha_\Gamma(\Omega) + o(1), \\ \operatorname{dist}(v_k,\ f_k([0,1])) = o(1), \\ J'(v_k) = o(1) \quad \textit{strongly in } X^{-1}(\Omega) \end{gathered}$$

as $k \to \infty$. In particular, $\alpha_\Gamma(\Omega)$ is a (PS)-value in $X(\Omega)$ for J.

Proof. We define $\varepsilon_k = \max_{s\in[0,1]} J(f_k(s)) - \alpha_\Gamma(\Omega)$ if $\max_{s\in[0,1]} J(f_k(s)) - \alpha_\Gamma(\Omega) > 0$ and $\varepsilon_k = \frac{1}{k}$ in the other case. Then we apply Lemma 4.8 to ε_k and f_k:

$$\begin{gathered} \alpha_\Gamma(\Omega) - \varepsilon_k \le J(v_k) \le \max_{s\in[0,1]} J(f_k(s)) \le \alpha_\Gamma(\Omega) + \varepsilon_k, \\ \operatorname{dist}(v_k, f_k([0,1])) \le \varepsilon_k^{1/2}, \\ |J'(v_k)| \le \varepsilon_k^{\frac{1}{2}} \qquad \text{for each } k > 0. \end{gathered}$$

This completes the proof. □

(D) Consider the infimum of positive (PS)-values in $X(\Omega)$ for J :

$$\alpha_{\mathbf{P}}(\Omega) = \inf_{\beta \in \mathbf{P}(\Omega)} \beta,$$

where $\mathbf{P}(\Omega)$ is the set of all positive (PS)-values in $X(\Omega)$ for J. That $\alpha_{\mathbf{P}}(\Omega) > 0$ is a consequence of the following theorem.

Theorem 4.10. *There is a $\beta_0 > 0$ such that $\beta \geq \beta_0$ for every positive (PS)-value β in $X(\Omega)$ for J.*

Proof. Let $\{u_n\}$ be a $(\text{PS})_\beta$-sequence in $X(\Omega)$ for J with $\beta > 0$. By Lemma 2.38, a positive sequence $\{c_n(\beta)\}$ exists such that $c_n(\beta) = o(1)$ as $n \to \infty$, $\beta \to 0$, and

$$a(u_n) \leq c_n(\beta)^2. \tag{4.12}$$

By the Sobolev embedding theorem, there is a constant $d > 0$ such that

$$b(u_n) \leq d a(u_n)^{p/2}. \tag{4.13}$$

By Lemma 2.38, (4.12), and (4.13), we have

$$o(1) = a(u_n) - b(u_n) \geq a(u_n)\left[1 - dc_n(\beta)^{p-2}\right].$$

Take $\beta_0 > 0$ and $n_0 > 0$ such that if $\beta < \beta_0$ and $n \geq n_0$, then $1 - dc_n(\beta)^{p-2} > \frac{1}{2}$. Consequently, $a(u_n) = b(u_n) = o(1)$, or $J(u_n) = o(1)$. Thus, $\beta \geq \beta_0$. □

$\alpha_{\mathbf{P}}(\Omega)$ is a (PS)-value in $X(\Omega)$ for J.

Theorem 4.11. $\alpha_{\mathbf{P}}(\Omega) \in \mathbf{P}(\Omega)$.

Proof. For each $n \in \mathbb{N}$, take $u_n \in X(\Omega)$ and $c_n \in \mathbf{P}(\Omega)$ such that

$$\begin{gathered}
|c_n - \alpha_{\mathbf{P}}(\Omega)| < \frac{1}{n}, \\
|J(u_n) - c_n| < \frac{1}{n}, \\
\|J'(u_n)\|_{X^{-1}} < \frac{1}{n}.
\end{gathered}$$

Then $J(u_n) = \alpha_{\mathbf{P}}(\Omega) + o(1)$ and $J'(u_n) = o(1)$. Thus, $\alpha_{\mathbf{P}}(\Omega) \in \mathbf{P}(\Omega)$. □

The following theorem is very useful.

Theorem 4.12. *Let β be a positive (PS)-value in $X(\Omega)$ for J. Then*
(i) $\beta \geq \alpha_\gamma(\Omega)$; (ii) $\beta \geq \alpha_{\mathbf{M}}(\Omega)$; (iii) $\beta \geq \alpha_\Gamma(\Omega)$; (iv) $\beta \geq \alpha_{\mathbf{P}}(\Omega)$.

Proof. Let $\{u_n\}$ be a nonzero $(\text{PS})_\beta$-sequence in $X(\Omega)$ for J with $\beta > 0$. By Lemma 2.38, we have

$$\begin{gathered}
J(u_n) = \beta + o(1), \\
a(u_n) - b(u_n) = o(1).
\end{gathered}$$

(*i*) Let $w_n = u_n(a(u_n))^{-\frac{1}{2}}$, then $a(w_n) = 1$ and $b(w_n) = a(u_n)^{-p/2} b(u_n) \leq \gamma(\Omega)^p$. Thus, $a(u_n) \geq \gamma(\Omega)^{2p/(2-p)} + o(1)$, or $\beta \geq (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{2p/(2-p)} = \alpha_\gamma(\Omega)$.
(*ii*) By Theorem 4.3, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\} \subset \mathbf{M}(\Omega)$ and $J(s_n u_n) = \beta + o(1)$. Therefore, $\beta \geq \alpha_{\mathbf{M}}(\Omega)$.
(*iii*) By Theorem 4.3 and Lemma 4.2 (*ii*), there are sequences $\{s_n\}$ and $\{t_n\}$ in $\mathbb{R}^+$ such that $\{s_n u_n\} \subset \mathbf{M}(\Omega)$, $\{t_n u_n\} \subset \mathbf{Z}(\Omega)$, and $J(s_n u_n) = \beta + o(1)$. Since the

manifold $\mathbf{Z}(\Omega)$ is path-connected, there is a path ζ_n in $\mathbf{Z}(\Omega)$ that connects $t_n u_n$ to e. Let γ_n' be the line segment connecting 0 and $t_n u_n$ and the path $\gamma_n = \gamma_n' \cup \zeta_n$, then

$$\alpha_\Gamma(\Omega) \le \max_{0\le t\le 1} J(\gamma_n(t)) = J(s_n u_n) = \beta + o(1).$$

Thus, $\beta \ge \alpha_\Gamma(\Omega)$.
(iv) Clearly, $\beta \ge \alpha_{\mathbf{P}}(\Omega)$. □

By Theorems 4.1, 4.4, 4.9, 4.11, and 4.12, we have the following theorem.

Theorem 4.13. $\alpha_\gamma(\Omega) = \alpha_{\mathbf{M}}(\Omega) = \alpha_\Gamma(\Omega) = \alpha_{\mathbf{P}}(\Omega)$.

Definition 4.14. By Theorem 4.13, we conclude that the positive (PS)-values $\alpha_\gamma(\Omega)$, $\alpha_\Gamma(\Omega)$, $\alpha_{\mathbf{M}}(\Omega)$, and $\alpha_{\mathbf{P}}(\Omega)$ in $X(\Omega)$ for J are the same. Any one of them is called the index of J in $X(\Omega)$ and denoted by $\alpha_X(\Omega)$. By the definition of $\alpha_{\mathbf{M}}(\Omega)$, if u is a nonzero solution of Equation (1.1), then $u \in \mathbf{M}(\Omega)$. Thus, $J(u) \ge \alpha_{\mathbf{M}}(\Omega) = \alpha_X(\Omega)$. We say that a nonzero solution u of Equation (1.1) in $X(\Omega)$ is a ground state solution if $J(u) = \alpha_X(\Omega)$, and is a higher energy solution if $J(u) > \alpha_X(\Omega)$.

Remark 4.15. We denote $\alpha_X(\Omega)$ by $\alpha(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$ and by $\alpha_s(\Omega)$ for $X(\Omega) = H_s^1(\Omega)$ (see Definition 6.1).

Remark 4.16. By Theorem 8.2, a ground state solution in $X(\Omega)$ is of constant sign. Note that if u is a solution of (1.1), then $-u$ is also a solution of (1.1). By the maximum principle, if u is a nonzero and nonnegative solution of (1.1), then u is positive. From now on, by a ground state solution in $X(\Omega)$, we mean a positive solution of (1.1).

Definition 4.17. We say that a domain Ω in $\mathbb{R}^N$ is an achieved domain if there is a solution u in $H_0^1(\Omega)$ of (1.1) such that $J(u) = \alpha(\Omega)$, by Remark 4.16, we may assume that u be positive. Otherwise, we say that Ω is a nonachieved domain.

Theorem 4.18. *(i) If Ω is a large domain in $\mathbb{R}^N$, then $\alpha(\Omega) = \alpha(\mathbb{R}^N)$;*
(ii) If Ω is a large domain in $\mathbf{A}^r$, then $\alpha(\Omega) = \alpha(\mathbf{A}^r)$;
(iii) If Ω is a large domain in $\mathbf{A}^{r_1,r_2}$, then $\alpha(\Omega) = \alpha(\mathbf{A}^{r_1,r_2})$.

Proof. It suffices to prove part (i). Let $w \in H^1(\mathbb{R}^N)$ be a ground state solution of Equation (1.1) satisfying

$$a(w) = \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) = b(w) = \int_{\mathbb{R}^N} |w|^p = (\frac{2p}{p-2})\alpha(\mathbb{R}^N).$$

For $r_n \to \infty$, take $\{z_n\} \subset \Omega$ such that $B^N(z_n; r_n) \subset \Omega$. Consider the cut-off function $\eta \in C_c^\infty([0,\infty))$ as in (2.2), and for each n, let

$$w_n(z) = \eta(\frac{2|z - z_n|}{r_n}) w(z - z_n).$$

Then $w_n \in H_0^1(\Omega)$ and

$$a(w_n) = \int_\Omega (|\nabla w_n|^2 + w_n^2) = (\frac{2p}{p-2})\alpha(\mathbb{R}^N) + o(1),$$
$$b(w_n) = \int_\Omega |w_n|^p = (\frac{2p}{p-2})\alpha(\mathbb{R}^N) + o(1) \quad \text{as } n \to \infty.$$

Thus,

$$J(w_n) = \alpha(\mathbb{R}^N) + o(1),$$

$$a(w_n) = b(w_n) + o(1) \quad \text{as } n \to \infty.$$

By Theorem 4.4, $\{w_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbb{R}^N)}$-sequence in $H_0^1(\Omega)$ for J. Therefore, $\alpha(\Omega) \leq \alpha(\mathbb{R}^N)$. Clearly, $\alpha(\mathbb{R}^N) \leq \alpha(\Omega)$, thus we have $\alpha(\Omega) = \alpha(\mathbb{R}^N)$. □

Theorem 4.19. *Let Ω be a large domain in $\mathbb{R}^N$. If β is a positive (PS)-value in $H_0^1(\Omega)$ for J, then $m\beta$ is also a positive (PS)-value in $H_0^1(\Omega)$ for J, where $m = 2, 3, \dots$.*

Proof. It suffices to prove the case $m = 2$. First embed $H_0^1(\Omega)$ into $H^1(\mathbb{R}^N)$. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$. Then by Lemma 2.38, there is a constant $c > 0$ such that, for each n, $a(u_n) \leq c$ and $b(u_n) \leq c$. For $r_n \to \infty$, since $\Omega \backslash B^N(0; 5r_n)$ is also a large domain in $\mathbb{R}^N$, $z_n \in \Omega$ exists such that $B^N(z_n; 2r_n) \subset \Omega$ and

$$\int_{B^N(0;r_n)^c} |\nabla u_n|^2 + u_n^2 < \frac{1}{n} \quad \text{and} \quad \int_{B^N(0;r_n)^c} |u_n|^p < \frac{1}{n}.$$

Note that $|z_n| \geq 5r_n$. Let $\eta_n(z) = \eta(\frac{|z|}{r_n})$, where η is as in (2.2), $v_n(z) = \eta_n(z)u_n(z)$ and $w_n(z) = v_n(z - z_n)$. Then we have $|\nabla \eta_n| \leq \frac{2}{r_n}$ and $\operatorname{supp} w_n \subset B^N(z_n; 2r_n)$.
(*i*) $J(v_n) = \beta + o(1)$: note that

$$|\nabla v_n|^2 = |\eta_n|^2 |\nabla u_n|^2 + |\nabla \eta_n|^2 |u_n|^2 + 2\eta_n u_n \nabla \eta_n \nabla u_n.$$

Thus, for $z \in B^N(0; r_n)$, we have $|\nabla v_n| = |\nabla u_n|$ and

$$\begin{aligned}
\int_\Omega |\nabla v_n|^2 &= \int_{B^N(0;r_n)} |\nabla v_n|^2 + \int_{B^N(0;2r_n) \backslash B^N(0;r_n)} |\nabla v_n|^2 \\
&= \int_{B^N(0;r_n)} |\nabla u_n|^2 + o(1) \\
&= \int_\Omega |\nabla u_n|^2 + o(1).
\end{aligned}$$

Similarly, we have

$$\int_\Omega |v_n|^2 = \int_\Omega |u_n|^2 + o(1), \quad \int_\Omega |v_n|^p = \int_\Omega |u_n|^p + o(1).$$

Thus, $J(v_n) = J(u_n) + o(1) = \beta + o(1)$. Clearly, for each n, $J(w_n) = J(v_n)$, and hence $J(w_n) = \beta + o(1)$.
(*ii*) $J(v_n + w_n) = 2\beta + o(1)$: since the supports of v_n and w_n are disjoint, we have

$$\begin{aligned}
a(v_n + w_n) &= \int_\Omega |\nabla(v_n + w_n)|^2 + (v_n + w_n)^2 \\
&= \int_\Omega |\nabla v_n|^2 + v_n^2 + \int_\Omega |\nabla w_n|^2 + w_n^2 + 2\int_\Omega \nabla v_n \nabla w_n + 2\int_\Omega v_n w_n \\
&= a(v_n) + a(w_n).
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_\Omega |v_n + w_n|^p - |v_n|^p - |w_n|^p \\
&= \int_{B^N(0;2r_n)} |v_n + w_n|^p - |v_n|^p - |w_n|^p + \int_{B^N(0;2r_n)^c \cap \Omega} |v_n + w_n|^p - |v_n|^p - |w_n|^p \\
&= 0.
\end{aligned}$$

Thus,

$$b(v_n + w_n) = \int_\Omega |v_n + w_n|^p = \int_\Omega |v_n|^p + \int_\Omega |w_n|^p = b(v_n) + b(w_n).$$

Hence,

$$J(v_n + w_n) = \frac{1}{2}a(v_n + w_n) - \frac{1}{p}b(v_n + w_n) = J(v_n) + J(w_n) = 2\beta + o(1).$$

(iii) $\|J'(v_n + w_n)\| = o(1)$: for $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned}
\langle J'(v_n), \varphi\rangle &= \int_{B^N(0;r_n)} u_n(\nabla\eta_n)\cdot\nabla\varphi + \int_{B^N(0;r_n)} \eta_n(\nabla u_n)\cdot\nabla\varphi + \eta_n u_n\varphi \\
&\quad - \int_{B^N(0;r_n)} |\eta_n u_n|^{p-2}\eta_n u_n\varphi + o(1) \\
&= \int_{B^N(0;r_n)} \nabla u_n(z)\cdot\nabla\varphi(z) + u_n(z)\varphi(z) \\
&\quad - \int_{B^N(0;r_n)} |u_n|^{p-2}u_n\varphi(z) + o(1) \\
&= \langle J'(u_n), \varphi\rangle + o(1).
\end{aligned}$$

Thus, $\|J'(v_n)\|_{H^{-1}} = o(1)$. Similarly, $\|J'(w_n)\|_{H^{-1}} = o(1)$.
We have

$$\begin{aligned}
&\int_\Omega |v_n + w_n|^{p-2}(v_n + w_n)\varphi - |v_n|^{p-2}v_n\varphi - |w_n|^{p-2}w_n\varphi \\
&= \int_{B^N(0;2r_n)} |v_n + w_n|^{p-2}(v_n + w_n)\varphi - |v_n|^{p-2}v_n\varphi - |w_n|^{p-2}w_n\varphi \\
&\quad + \int_{B^N(0;2r_n)^c\cap\Omega} |v_n + w_n|^{p-2}(v_n + w_n)\varphi - |v_n|^{p-2}v_n\varphi - |w_n|^{p-2}w_n\varphi \\
&= 0.
\end{aligned}$$

Now for $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned}
\langle J'(v_n + w_n), \varphi\rangle &= \int_\Omega \nabla(v_n + w_n)\nabla\varphi + (v_n + w_n)\varphi \\
&\quad - \int_\Omega |v_n + w_n|^{p-2}(v_n + w_n)\varphi \\
&= \int_\Omega \nabla v_n\nabla\varphi + v_n\varphi + \int_\Omega \nabla w_n\nabla\varphi + w_n\varphi \\
&\quad - \int_\Omega |v_n|^{p-2}v_n\varphi - \int_\Omega |w_n|^{p-2}w_n\varphi \\
&= \langle J'(v_n), \varphi\rangle + \langle J'(w_n), \varphi\rangle.
\end{aligned}$$

Therefore, $\|J'(v_n + w_n)\|_{H^{-1}} = o(1)$. We conclude that

$$\begin{gathered}
J(v_n + w_n) = 2\beta + o(1), \\
J'(v_n + w_n) = o(1) \qquad \text{strongly in } H^{-1}(\Omega).
\end{gathered}$$

□

The following theorem has a proof similar to that of Theorem 4.19.

Theorem 4.20. *Let Ω be a large domain in $\mathbf{A}^r$. If β is a positive (PS)-value in $H_0^1(\Omega)$ for J, then $m\beta$ is also a positive (PS)-value in $H_0^1(\Omega)$ for J, where $m = 2, 3, \dots$.*

Lemma 4.21. *The set $\mathbf{P}(\Omega)$ is closed.*

The proof of this lemma is similar to the proof of Theorem 4.11; so we omit it.

By Lemma 4.2, $J(\mathbf{M}(\Omega))$ is bounded below away from zero. Actually for any domain Ω in $\mathbb{R}^N$, $J(\mathbf{M}(\Omega))$ is unbounded above.

Theorem 4.22. *If Ω is a domain in $\mathbb{R}^N$, then $J(\mathbf{M}(\Omega)) = (\alpha(\Omega), \infty)$ for a nonachieved domain Ω and $J(\mathbf{M}(\Omega)) = [\alpha(\Omega), \infty)$ for an achieved domain Ω.*

Proof. (i) Suppose that Ω is bounded. By Struwe [66, p.116 Theorem 6.6], an unbounded sequence $\{u_n\}$ exists in $\mathbf{M}(\Omega)$ for J. Since $J(u_n) = (\frac{1}{2} - \frac{1}{p})a(u_n)$ and $\mathbf{M}(\Omega)$ is path connected, then we have $J(\mathbf{M}(\Omega)) = [\alpha(\Omega), \infty)$.
(ii) Let Ω be an unbounded domain and Ω_1 be a bounded domain in Ω. Then $\mathbf{M}(\Omega_1) \subset \mathbf{M}(\Omega)$ and $\alpha(\Omega) \leq \alpha(\Omega_1)$. By part (i), we have

$$[\alpha(\Omega_1), \infty) = J(\mathbf{M}(\Omega_1)) \subset J(\mathbf{M}(\Omega)).$$

Since $\mathbf{M}(\Omega)$ is path connected, the result follows. □

Theorem 4.23. *Let Ω be an Esteban-Lions domain as well as a large domain in $\mathbb{R}^N$. Then we have $\mathbf{P}(\Omega) = \{\alpha(\Omega),\ 2\alpha(\Omega),\ 3\alpha(\Omega), \dots\}$.*

Proof. By Theorem 4.19, $\mathbf{P}(\Omega) \supset \{\alpha(\Omega),\ 2\alpha(\Omega),\ 3\alpha(\Omega), \dots\}$. Suppose that a $(\mathrm{PS})_\beta$-sequence $\{u_n\}$ exists for J, where $k\alpha(\Omega) < \beta < (k+1)\alpha(\Omega)$ for some $k \in \mathbb{N}$. By the Palais-Smale decomposition theorem 3.1, Equation (1.1) has a nonzero solution. This contradicts Theorem 10.7. □

By Lemma 2.38, if $\{u_n\}$ is a $(\mathrm{PS})_\beta-$sequence in $H_0^1(\Omega)$ for J, then $a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$. By Theorems 4.22 and 4.23, we have:

Lemma 4.24. *Let Ω be an Esteban-Lions domain as well as a large domain in $\mathbb{R}^N$. For each β and $m = 0, 1, \dots,$ satisfying $m\alpha(\Omega) < \beta < (m+1)\alpha(\Omega)$, then there is a sequence $\{u_n\}$ in $H_0^1(\Omega)$ for J satisfying*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

but

$$J'(u_n) \nrightarrow 0 \quad \textit{strongly in } H^{-1}(\Omega).$$

Let Ω be an unbounded domain in $\mathbb{R}^N$ and $\Omega_n = \Omega \cap \mathbf{B}^N(0; r_n)$, then we have the following theorem.

Theorem 4.25. *$\alpha_X(\Omega_n) = \alpha_X(\Omega) + o(1)$ as $n \to \infty$.*

Proof. Suppose that $\{u_n\}$ in $X(\Omega)$ is a minimizing sequence in $\mathbf{M}(\Omega)$ of $\alpha_X(\Omega)$, then by Lemma 2.38, $\{u_n\}$ is bounded in $X(\Omega)$. Let $\{r_n\}$ be a sequence of strictly increasing positive integers such that $r_n \geq n$,

$$\int_{\Omega \cap \{|z| \geq \frac{r_n}{2}\}} |\nabla u_n|^2 + |u_n|^2 < \frac{1}{n} \tag{4.14}$$

and

$$\int_{\Omega \cap \{|z| \geq \frac{r_n}{2}\}} |u_n|^p < \frac{1}{n}. \tag{4.15}$$

Define $\eta_n(z) = \eta(\frac{2|z|}{r_n})$, where η is as in (2.2). Then $\eta_n u_n \in X(\Omega_n) \subset X(\Omega)$. From the inequalities (4.14) and (4.15), we obtain

$$a(\eta_n u_n) = a(u_n) + o(1) \quad \text{and } b(\eta_n u_n) = b(u_n) + o(1).$$

Therefore, we have

$$J(\eta_n u_n) = J(u_n) + o(1) = \alpha_X(\Omega) + o(1).$$

and

$$a(\eta_n u_n) = b(\eta_n u_n) + o(1).$$

By Theorem 4.3, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + o(1)$, $\{s_n \eta_n u_n\}$ is in $\mathbf{M}(\Omega)$ and $J(s_n \eta_n u_n) = \alpha_X(\Omega) + o(1)$. Note that $J(s_n \eta_n u_n) \geq \alpha_X(\Omega_n) > \alpha_X(\Omega)$. Hence, $\alpha_X(\Omega_n) = \alpha_X(\Omega) + o(1)$. □

Let Ω be a domain containing zero in $\mathbb{R}^N$. For $\delta > 0$, we define

$$\delta\Omega = \{\delta z \mid z \in \Omega\}.$$

Then we have the following theorem.

Theorem 4.26. *(i)* $\lim_{\delta \to \infty} \alpha(\delta\Omega) = \alpha(\mathbb{R}^N)$*;*
(ii) Let $\delta\Omega$ *be achieved for each* $\delta > 0$*, then* $lim_{\delta \to 0^+} \alpha(\delta\Omega) = \infty$*.*

Proof. *(i)* By Theorem 12.5 below, there is a ground state solution u in $H^1(\mathbb{R}^N)$ such that $a(u) = b(u)$ and $J(u) = \alpha(\mathbb{R}^N)$. A sequence $\{\delta_n\}$ exists such that $\delta_n \to \infty$ and $B^N(0; n) \subset \delta_n\Omega$. Consider the cut-off function η and $\eta_n(z) = \eta(\frac{2|z|}{n})$ for $n = 1, 2, \dots$. Let $u_n(z) = \eta_n(z)u(z)$, then $u_n(z) \in H_0^1(B^N(0; n))$, $J(u_n) = \alpha(\mathbb{R}^N) + o(1)$, and $a(u_n) = b(u_n) + o(1)$ as $n \to \infty$. By Theorem 4.3, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + o(1)$, $\{s_n u_n\}$ is in $\mathbf{M}(B^N(0; n))$ and $J(s_n u_n) = \alpha(\mathbb{R}^N) + o(1)$. Then we have

$$\lim_{n\to\infty} \alpha(\delta_n\Omega) \leq \lim_{n\to\infty} \alpha(B^N(0; n)) \leq \lim_{n\to\infty} J(s_n u_n) = \alpha(\mathbb{R}^N) + o(1).$$

However, $\alpha(\mathbb{R}^N) \leq \alpha(\delta\Omega)$ for each $\delta > 0$. Thus, $\lim_{\delta\to\infty} \alpha(\delta\Omega) = \alpha(\mathbb{R}^N)$.
(ii) Let u be a ground state solution of Equation (1.1) in $H_0^1(\delta\Omega)$, then $a(u) = b(u)$ and $J(u) = \alpha(\delta\Omega)$. Set $v(z) = u(\delta z)$, then $v \in H_0^1(\Omega)$. Note that

$$a(u) = \int_{\delta\Omega} (|\nabla u(z)|^2 + u(z)^2) dz = \delta^{N-2} \int_\Omega |\nabla v|^2 + \delta^N \int_\Omega v^2,$$

and

$$(b(u))^{2/p} = (\int_{\delta\Omega} |u(z)|^p)^{2/p} = \delta^{\frac{2N}{p}} (\int_\Omega |v|^p)^{\frac{2}{p}}.$$

Therefore, by the Sobolev continuous embedding theorem,

$$a(u) \geq \delta^{N-2} \int_\Omega |\nabla v|^2 \geq c\delta^{N-2} (\int_\Omega |v|^p)^{\frac{2}{p}} \geq c\delta^{N-2-\frac{2N}{p}} (b(u))^{2/p}.$$

That is, $(a(u))^{\frac{p-2}{p}} \geq c\delta^{N-2-\frac{2N}{p}}$. Thus,

$$J(u) = (\frac{1}{2} - \frac{1}{p}) a(u) \geq c(\frac{1}{2} - \frac{1}{p})(\delta^{N-2-\frac{2N}{p}})^{p/(p-2)}.$$

Therefore, $\alpha(\delta\Omega) \geq c(\frac{1}{2} - \frac{1}{p})(\delta^{N-2-\frac{2N}{p}})^{p/(p-2)}$. Since $p < \frac{2N}{N-2}$, we have $N - 2 - \frac{2N}{p} < 0$. We conclude that $\lim_{\delta\to 0+} \alpha(\delta\Omega) = \infty$. □

As a corollary of Theorem 4.26, we have

Theorem 4.27. *(i)* $\lim_{r\to\infty}\alpha(B^N(0;r))=\alpha(\mathbb{R}^N)$*; (ii)* $\lim_{r\to 0+}\alpha(B^N(0;r))=\infty$.

Using the same argument as for the proof of Theorem 4.26, we obtain the following theorem.

Theorem 4.28. *(i)* $\lim_{t\to\infty}\alpha(\mathbf{A}^r_{-t,t})=\alpha(\mathbf{A}^r)$*; (ii)* $\lim_{t\to 0^+}\alpha(\mathbf{A}^r_{-t,t})=\infty$.

Let $\Omega=O\times\mathbb{R}^l$, where O is a bounded domain in $\mathbb{R}^m$, $m\ge 1$, $n\ge 1$. With the same argument of the proof in Theorem 4.26, we get

Theorem 4.29. *(i)* $\lim_{\delta\to\infty}\alpha(\delta\Omega)=\alpha(\mathbb{R}^N)$*; (ii)* $\lim_{\delta\to 0+}\alpha(\delta\Omega)=\infty$.

We have the following continuity property.

Theorem 4.30. *We have*
$$\lim_{\delta\to 1}\alpha(\delta\mathbf{A}^r)=\alpha(\mathbf{A}^r).$$

Proof. *(i)*. $\lim\limits_{\delta\to 1^+}\alpha(\delta\mathbf{A}^r)=\alpha(\mathbf{A}^r)$: let $1<\delta<2$. By Theorem 5.7 *(ii)* below, $\alpha(2\mathbf{A}^r)<\alpha(\delta\mathbf{A}^r)<\alpha(\mathbf{A}^r)$. In addition, $\alpha(\delta\mathbf{A}^r)$ is increasing as δ is decreasing. Let $c\equiv\lim_{\delta\to 1^+}\alpha(\delta\mathbf{A}^r)$, then $c\le\alpha(\mathbf{A}^r)$. We claim that $c\ge\alpha(\mathbf{A}^r)$. By Theorem 12.5 below, a ground state solution u_n in $\mathbf{M}((1+\frac{1}{n})\mathbf{A}^r)$ exists such that
$$a(u_n)=b(u_n)\quad\text{and}\quad J(u_n)=\alpha\big((1+\frac{1}{n})\mathbf{A}^r\big)\quad\text{for each } n\in\mathbb{N}.$$
Moreover, we have $a(u_n)=b(u_n)=(\frac{2p}{p-2})c+o(1)$ and $J(u_n)=\alpha((1+\frac{1}{n})\mathbf{A}^r)=c+o(1)$. Define $v_n(z)=u_n((1+\frac{1}{n})z)\in H^1_0(\mathbf{A}^r)$. Since
$$a(u_n)=(1+\frac{1}{n})^{N-2}\int_{\mathbf{A}^r}|\nabla v_n|^2dz+(1+\frac{1}{n})^N\int_{\mathbf{A}^r}v_n^2dz$$
and
$$b(u_n)=(1+\frac{1}{n})^N\int_{\mathbf{A}^r}|v_n|^pdz,$$
then $\{v_n\}$ is bounded in $H^1_0(\mathbf{A}^r)$. Thus, $a(v_n)=b(v_n)+o(1)$ and $J(v_n)=c+o(1)$. By Theorem 4.3, $s_n>0$ exists such that $s_nv_n\in\mathbf{M}(\mathbf{A}^r)$, $s_n=1+o(1)$, and $J(s_nv_n)=c+o(1)$. Hence, $c\ge\alpha(\mathbf{A}^r)$. We conclude the proof.
(ii) $\lim\limits_{\delta\to 1^-}\alpha(\delta\mathbf{A}^r)=\alpha(\mathbf{A}^r)$: by Theorem 12.5, $u\in\mathbf{M}(\mathbf{A}^r)$ satisfies $a(u)=b(u)$ and $J(u)=\alpha(\mathbf{A}^r)$. Let $v_n(x,y)=u((1+\frac{1}{n})x,y)$ for $n\in\mathbb{N}$. Then $v_n(x,y)\in H^1_0(\frac{n}{n+1}\mathbf{A}^r)$, $a(v_n)=a(u)+o(1)$, and $b(v_n)=b(u)+o(1)$. Thus $a(v_n)=b(v_n)+o(1)$ as $n\to\infty$ and
$$J(v_n)=\frac{1}{2}a(v_n)-\frac{1}{p}b(v_n)=J(u)+o(1).$$
By Theorem 4.3, for each n, there is an $s_n>0$ such that $s_nv_n\in\mathbf{M}(\frac{n}{n+1}\mathbf{A}^r)$, $s_n=1+o(1)$ and $J(s_nv_n)=J(u)+o(1)$ as $n\to\infty$. Moreover, $J(s_nv_n)\ge\alpha(\frac{n}{n+1}\mathbf{A}^r)>\alpha(\mathbf{A}^r)$ for each $n\in\mathbb{N}$. By the squeeze theorem,
$$\lim_{\delta\to 1^-}\alpha(\delta\mathbf{A}^r)=\alpha(\mathbf{A}^r).$$
□

Bibliographical notes: The constrained maximization problem $\alpha_\gamma(\Omega)$ is a classical problem. Theorem 4.1 is from Lien-Tzeng-Wang [47]. The Nehari minimizing problem $\alpha_{\mathbf{M}}(\Omega)$ was first studied by Nehari [57]. Theorem 4.3 is from Chen-Wang [26, p.158]. For Theorem 4.4, Stuart [67] proved that there is a $(\mathrm{PS})_{\alpha_{\mathbf{M}(\Omega)}}$-sequence.

However, Chen-Wang [26, Lemma 2.1] proved that every minimizing sequence in $\mathbf{M}(\Omega)$ for $\alpha_{\mathbf{M}}(\Omega)$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}(\Omega)}$-sequence. The mountain pass minimax problem $\alpha_{\Gamma}(\Omega)$ is originally from the mountain pass lemmas 4.8 and 4.9 by Ambrosetti-Rabinowitz [4] and the new version is from Brézis-Nirenberg [15]. Theorem 4.13 is due to Willem [78] and Wang [71, p.4241]. Theorem 2.7 is from Lien-Tzeng-Wang [47, Lemma 2.5].

5. Palais-Smale Conditions

The Palais-Smale conditions are conditions for compactness. They are useful in ascertaining the existence of solutions of (1.1). In this section, we assert that eight related (PS)-conditions in $X(\Omega)$ for J are actually equivalent.

Theorem 5.1. *The $(PS)_{\alpha_X(\Omega)}$-condition for J holds in a bounded domain Ω. In particular, there is a ground state solution of* (1.1) *in a bounded domain Ω.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J, by Lemma 2.38, $\{u_n\}$ is bounded and

$$J(u_n) = \alpha_X(\Omega) + o(1),\ a(u_n) = b(u_n) + o(1).$$

Take a subsequence $\{u_n\}$ and $u \in X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. By the compactness theorem, $u_n \to u$ strongly in $L^p(\Omega)$. Suppose $u = 0$, then $b(u_n) = o(1)$. Thus, $a(u_n) = o(1)$ and $J(u_n) = o(1)$, contradicting that $\alpha(\Omega) > 0$. By Theorem 5.6, u is a ground state solution in $X(\Omega)$ for J and $u_n \to u$ strongly in $X(\Omega)$. □

The $(\mathrm{PS})_{\alpha_X(\Omega)}$-condition holds in unbounded domains and quasibounded domains.

Definition 5.2. A domain Ω is quasibounded if

$$\lim_{z\in\Omega, |z|\to\infty} \operatorname{dist}(z, \partial\Omega) = 0.$$

Example 5.3. (i) Let $f, g : \mathbb{R}^{N-1} \to \mathbb{R}$ be two functions of C^1, $f \le g$,

$$\lim_{|x|\to\infty} (g(x) - f(x)) = 0,$$

and

$$\Omega = \{z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : f(x) < y < g(x)\}.$$

Then Ω is a quasibounded domain;
(ii) Let A be the domain in $\mathbb{R}^{N-1} \times \mathbb{R}$ with a hypersurface boundary. For each $t \in \mathbb{R}$, let $A_t = \{(x, y) \in A : y = t\}$ be the section of A at t. If $\lim_{t\to\infty} diam A_t = 0$, then A is a quasibounded domain.

Theorem 5.4. (i) *Let Ω be a C^1 quasibounded domain, then the embedding $H_0^1(\Omega) \to L^q(\Omega)$ is compact, where $2 < q < 2^*$;*
(ii) *The $(PS)_{\alpha_X(\Omega)}$-condition holds for J in a C^1 quasibounded domain Ω, and there is a ground state solution of* (1.1) *in a C^1 quasibounded domain.*

Proof. (i) By Adams [2]. (ii) Similar to the proof of Theorem 5.1. □

Theorem 5.5. *Let $m \ge 1$, $k \ge 2$, ω be a smooth bounded open set in $\mathbb{R}^m$, and let $E = \omega \times \mathbb{R}^k$. Denote by (x, y) a generic point in $\mathbb{R}^m \times \mathbb{R}^k$ and consider the space $H_s^1(E)$ consisting of functions in $H_0^1(E)$ that are spherically symmetric in y. Then the Sobolev embedding from $H_s^1(E)$ into $L^q(E)$ is compact for every $q \in (2, \frac{2N}{N-2})$ with $N = m + k$.*

The proof of this theorem can be found in Wang [72] and Lions [50].

As a consequence of Lemma 2.38, for each $(\mathrm{PS})_\beta$-sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$ and a u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution of Equation (1.1).

Theorem 5.6. *(i) Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J and u in $X(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution of Equation (1.1);*
(ii) Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J and u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$ and u is nonzero. Then u is a ground state solution of Equation (1.1) and $u_n \to u$ strongly in $X(\Omega)$;
(iii) The $(PS)_{\alpha_X(\Omega)}$-condition holds for J if and only if for each $(PS)_{\alpha_X(\Omega)}-$ sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$and a nonzero u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$.

Proof. (i) By Lemmas 2.38, 2.11, and Theorem 2.33, there is a subsequence $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$, a.e. in Ω, and strongly in $L^q_{\ell oc}(\Omega)$ where $1 \le q < 2^*$. By Lemma 2.11, we obtain, for $\phi \in X(\Omega) \cap C_c^\infty(\Omega)$, $\operatorname{supp}\phi = K$,

$$\int_\Omega \nabla u_n \cdot \nabla\phi \to \int_\Omega \nabla u \cdot \nabla\phi,$$
$$\int_\Omega u_n\phi \to \int_\Omega u\phi.$$

Let $g_n = |u_n|^{p-1}\phi$ and $g = |u|^{p-1}\phi$. Then $||u_n|^{p-2}u_n\phi| \le g_n$ for each n, $g_n \to g$ a.e. By the Rellich-Kondrakov theorem 2.33, $g_n \to g$ in $L^1(K)$. By Theorem 2.22,

$$\int_\Omega |u_n|^{p-2}u_n\phi \to \int_\Omega |u|^{p-2}u\phi.$$

Thus, $\langle J'(u_n), \phi\rangle \to \langle J'(u), \phi\rangle$ for each $\phi \in X(\Omega)\cap C_c^\infty(\Omega)$. Since $\langle J'(u_n), \phi\rangle = o(1)$, for each $\phi \in X(\Omega)\cap C_c^\infty(\Omega)$, we have $J'(u) = 0$ in $X^{-1}(\Omega)$. Therefore, u is a solution of Equation (1.1).
(ii) By part (i), u is a nonzero solution of Equation (1.1), hence $u \in \mathbf{M}(\Omega)$ and

$$J(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) = \alpha_X(\Omega) + o(1),$$
$$\langle J'(u_n), u_n\rangle = a(u_n) - b(u_n) = o(1).$$

Thus,

$$a(u_n) = \frac{2p}{p-2}\alpha_X(\Omega) + o(1). \tag{5.1}$$

Since a is weakly lower semicontinuous, we have

$$\alpha_X(\Omega) \le J(u) = (\frac{1}{2} - \frac{1}{p})a(u) \le (\frac{1}{2} - \frac{1}{p})\liminf_{n\to\infty} a(u_n) = \alpha_X(\Omega),$$

or $J(u) = \alpha_X(\Omega)$. By Lemma 8.2 below, u is of constant sign. Recall that if w is a solution of Equation (1.1), then $-w$ is also a solution of (1.1). By the maximum principle, we may assume that u is positive. Let $p_n = u_n - u$. By Lemma 2.11 and 2.14, we have

$$J(p_n) = J(u_n) - J(u) + o(1) = o(1).$$

By Lemma 2.15, $\{p_n\}$ is a Palais-Smale sequence for J, thus $\langle J'(p_n), p_n\rangle = \mathrm{o}\ (1)$. Similar to (5.1), we have

$$a(p_n) = \frac{2p}{p-2}J(p_n) + o(1) = o(1).$$

Thus, $u_n \to u$ strongly in $X(\Omega)$. (iii) follows by part (ii). $\square$

Let $\Omega^1 \subsetneqq \Omega^2$ and $\alpha_X^i = \alpha_X(\Omega^i)$ for $i = 1, 2$, then clearly $\alpha_X^2 \le \alpha_X^1$. If $\alpha_X^2 = \alpha_X^1$, then we have the following useful results.

Theorem 5.7. *Let $\Omega^1 \subsetneqq \Omega^2$ and $J : X(\Omega^2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_X^2 = \alpha_X^1$. Then*
(i) α_X^1 does not admit any ground state solution;
(ii) J does not satisfy the $(PS)_{\alpha_X^1}$-condition;
(iii) J does not satisfy the $(PS)_{\alpha_X^2}$-condition.

Proof. (i) Suppose that α_X^1 admits a ground state solution $u \in \mathbf{M}(\Omega^1) \subset \mathbf{M}(\Omega^2)$ such that $J(u) = \alpha_X^1$. Then we have $J(u) = \alpha_X^1 = \alpha_X^2 = \min_{v \in \mathbf{M}(\Omega^2)} J(v)$. By Lemma 4.5 and Theorem 5.6 (ii), (iii), u is a ground state solution of (1.1) in Ω^2. Thus, $u > 0$ in Ω^2, which contradicts the fact that $u \in X(\Omega^1)$.
(ii) By part (i) and Theorem 5.6 (ii), (iii).
(iii) Let $\{u_n\}$ in $X(\Omega^1)$ satisfy $J(u_n) \to \alpha_X^1$ and $J'(u_n) \to 0$ strongly in $X^{-1}(\Omega^1)$. By Theorem 4.3, $\{s_n\}$ in $\mathbb{R}^+$ exists such that $s_n = 1 + o(1)$, $w_n = s_n u_n \in \mathbf{M}(\Omega^1)$ and $J(w_n) = \alpha_X^1 + o(1)$ and $J'(w_n) = o(1)$ strongly in $X^{-1}(\Omega^1)$. Since $\mathbf{M}(\Omega^1) \subset \mathbf{M}(\Omega^2)$, $\{w_n\} \subset \mathbf{M}(\Omega^2)$ and $J(w_n) = \alpha_X^2 + o(1)$. By Theorem 4.4, we have

$$
\begin{aligned}
J(w_n) &= \alpha_X^2 + o(1), \\
J'(w_n) &= o(1) \quad \text{strongly in } X^{-1}(\Omega^2).
\end{aligned}
$$

Suppose that J satisfies the $(\mathrm{PS})_{\alpha_X^2}$-condition. Then there is a subsequence $\{w_n\}$ and a $w \in X(\Omega^2)$ satisfying $w_n \to w$ strongly in $X(\Omega^2)$ and $J(w) = \alpha_X^2$. Hence, $w \neq 0$. By Lemma 8.2 below and the maximum principle, w is a ground state solution of (1.1) in Ω^2. Since $\{w_n\} \subset \mathbf{M}(\Omega^1)$ and $w_n \to w$ strongly in $X(\Omega^2)$, we have $w = 0$ in $(\Omega^1)^c$. This contradicts the fact that w is a positive solution of Equation (1.1) in Ω^2. Thus, J does not satisfy the $(\mathrm{PS})_{\alpha_X^2}$-condition. $\square$

Compare Theorem 5.7 for $X(\Omega) = H_0^1(\Omega)$ and the following results.

Corollary 5.8. *Let E be either $\mathbb{R}^N$, or $\mathbf{A}^r$, or $\mathbf{A}^{r_1,r_2}$ and Ω a proper large domain of E. Then there is no any ground state solution of* (1.1) *in Ω.*

The proof of this corollary follows by Theorem 5.7 (ii) and Theorem 4.18.

Theorem 5.9. *Let $X(\Omega) = H_0^1(\Omega)$. We have: (i) $\alpha(\mathbf{A}^r) > \alpha(\Theta)$ for each domain $\Theta \supsetneqq \mathbf{A}^r$; (ii) J does not satisfy $(PS)_{\alpha(\mathbf{A}^r)}$-condition.*

Proof. (i) Since the infinite strip $\mathbf{A}^r$ is a periodic domain, by Theorem 12.5 below, there is a ground state solution $u_0 \in \mathbf{M}(\mathbf{A}^r)$ such that

$$J(u_0) = \alpha(\mathbf{A}^r).$$

The result follows from Theorem 5.7.
(ii) Let $u_n = u_0(x, y + n)$ for each n. Since $\mathbf{A}^r$ is a periodic domain, we have $u_n \in H_0^1(\mathbf{A}^r)$ for each $n \in \mathbb{N}$,

$$J(u_n) = \alpha(\mathbf{A}^r) \quad \text{and} \quad a(u_n) = b(u_n).$$

By Theorem 4.4, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-sequence for J. Moreover, for $\varphi \in C_c^\infty(\mathbf{A}^r)$ and $K = \mathrm{supp}\varphi$, we have

$$\langle u_n, \varphi(z) \rangle_{H^1} = \langle u_0(x, y + n), \varphi(z) \rangle_{H^1}$$

$$
\begin{aligned}
&= \int_{\mathbf{A}^r} \nabla u_0(x, y+n)\nabla\varphi(z)dz + \int_{\mathbf{A}^r} u_0(x, y+n)\varphi(z)dz \\
&= \int_K \nabla u_0(x, y+n)\nabla\varphi(z)dz + \int_K u_0(x, y+n)\varphi(z)dz \\
&= o(1) \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

For $\varepsilon > 0$, $\phi \in H_0^1(\mathbf{A}^r)$, there is $\varphi \in C_c^\infty(\mathbf{A}^r)$ such that

$$
\|\phi - \varphi\|_{H^1} < \varepsilon/(\|u\|_{H^1} + 1)
$$

and

$$
\begin{aligned}
\langle u_n, \phi(z)\rangle_{H^1} &= \langle u_n, \phi(z) - \varphi(z)\rangle_{H^1} + \langle u_n, \varphi(z)\rangle_{H^1} \\
&\le \|u_n\|_{H^1}\|\phi(z) - \varphi(z)\|_{H^1} + \langle u_n, \varphi(z)\rangle_{H^1} \\
&< \varepsilon \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus, $u_n \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$. Suppose that J satisfies the $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-condition, then there is a subsequence $\{u_n\}$ such that $u_n \to 0$ strongly in $H_0^1(\mathbf{A}^r)$. This contradicts $\alpha(\mathbf{A}^r) > 0$. Therefore, J does not satisfy the $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-condition. □

From now on $\alpha_X(\Omega)$ is simply denoted by α_X. Let $\{u_n\}$ in $X(\Omega)$ be a $(\mathrm{PS})_{\alpha_X}$-sequence for J in Ω. Clearly, $\{u_n\}$ is bounded in $X(\Omega)$. Then a subsequence $\{u_n\}$ and $u \in X(\Omega)$ exist such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$ a.e. in Ω, and strongly in $L^p_{\mathrm{loc}}(\Omega)$. Define

$$
a_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\Omega\cap\{R<|z|\}} (|\nabla u_n|^2 + u_n^2),
$$

$$
b_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\Omega\cap\{R<|z|\}} |u_n|^p.
$$

We have the following results.

Lemma 5.10. *Let $\{u_n\}$ be a (PS)$_{\alpha_X}$-sequence in $X(\Omega)$ for J, then a subsequence $\{u_n\}$ exists such that*
(i) $\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^2 + u_n^2 = (\int_\Omega |\nabla u|^2 + u^2) + a_\infty$;
(ii) $\limsup_{n\to\infty} \int_\Omega |u_n|^p = \int_\Omega |u|^p + b_\infty$;
(iii) $a_\infty = b_\infty$ *and* $\alpha_X = J_\infty + J(u)$, *where* $J_\infty = (\frac{p-2}{2p})b_\infty$.

Proof. Since $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_X}$-sequence in $X(\Omega)$ for J, by Lemma 2.38, $\{u_n\}$ is bounded in $X(\Omega)$. A subsequence $\{u_n\}$ and $u \in X(\Omega)$ exist such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$, a.e. in Ω, and strongly in $L^p_{\mathrm{loc}}(\Omega)$.
(i) Let $\eta_R(z) = \eta(\frac{2|z|}{R})$, then

$$
\begin{aligned}
\int_\Omega |\nabla u_n|^2 + u_n^2 &= \int_{\Omega\cap\{|z|<R\}} (\eta_R + 1 - \eta_R)(|\nabla u_n|^2 + u_n^2) \\
&\quad + \int_{\Omega\cap\{R<|z|\}} (|\nabla u_n|^2 + u_n^2) \\
&= \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) + \int_{\Omega\cap\{\frac{R}{2}<|z|<R\}} (1 - \eta_R)(|\nabla u_n|^2 + u_n^2) \\
&\quad + \int_{\Omega\cap\{R<|z|\}} (|\nabla u_n|^2 + u_n^2).
\end{aligned} \tag{5.2}
$$

Step 1: Since $\{\eta_R u_n\}$ is bounded in $X(\Omega)$, we have for each $R > 0$

$$o(1) = \langle J'(u_n), \eta_R u_n\rangle = \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) + u_n \nabla\eta_R \nabla u_n - \int_\Omega \eta_R |u_n|^p.$$

Thus,

$$\int_\Omega \eta_R\big(|\nabla u_n|^2 + u_n^2\big) + u_n\nabla\eta_R\nabla u_n - \int_\Omega \eta_R|u_n|^p = o(1).$$

Then, for each $R > 0$,

$$\begin{aligned} &-|\int_\Omega u_n\nabla\eta_R\nabla u_n| + \int_{\Omega\cap\{|z|<R\}} \eta_R|u_n|^p + o(1)\\ &\le \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) \\ &\le |\int_\Omega u_n\nabla\eta_R\nabla u_n| + \int_{\Omega\cap\{|z|<R\}} \eta_R|u_n|^p + o(1). \end{aligned} \tag{5.3}$$

Since $\{u_n\}$ is bounded in $X(\Omega)$ and $|\nabla\eta_R| \le \frac{c}{R}$ for each R, then by the Hölder inequality,

$$|\int_\Omega u_n\nabla\eta_R\nabla u_n| \le \frac{c}{R}.$$

Recall that for bounded sequences $\{s_n\}$ and $\{t_n\}$, we have

$$\begin{gathered} \limsup_{n\to\infty}(s_n + t_n) \le \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n,\\ \liminf_{n\to\infty}(s_n + t_n) \ge \liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n,\\ -\limsup_{n\to\infty} s_n = \liminf_{n\to\infty}(-s_n). \end{gathered}$$

Thus, by (5.3), we have for each $R > 0$

$$\limsup_{n\to\infty} \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) \le \frac{c}{R} + \int_{\Omega\cap\{|z|<R\}} \eta_R|u|^p, \tag{5.4}$$

$$\liminf_{n\to\infty} \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) \ge \liminf_{n\to\infty}(-|\int_\Omega u_n\nabla\eta_R\nabla u_n|) + \int_{\Omega\cap\{|z|<R\}} \eta_R|u|^p, \tag{5.5}$$

and

$$\lim_{R\to\infty} \lim \liminf_{n\to\infty}(-|\int_\Omega u_n\nabla\eta_R\nabla u_n|) = -\lim_{R\to\infty} \limsup_{n\to\infty}(|\int_\Omega u_n\nabla\eta_R\nabla u_n|) = 0 \tag{5.6}$$

By (5.4), (5.5), and (5.6), letting $R \to \infty$ we obtain

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) \le \int_\Omega |u|^p = \int_\Omega |\nabla u|^2 + u^2 \tag{5.7}$$

and

$$\lim_{R\to\infty} \liminf_{n\to\infty} \int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2) \ge \int_\Omega |u|^p = \int_\Omega |\nabla u|^2 + u^2. \tag{5.8}$$

Step 2: Let $\varphi \in C_c^\infty([0,\infty))$ satisfy

$$\varphi(t) = \begin{cases} 1 - \eta(t) & \text{for } t \in [1,2];\\ \eta(t-1) & \text{for } t \in [2,3];\\ 0 & \text{otherwise,} \end{cases}$$

then $0 \le \varphi \le 1$. Let $\varphi_R(z) = \varphi(\frac{2|z|}{R})$. Since $\{\varphi_R u_n\}$ is bounded in $X(\Omega)$, we have

$$\begin{aligned} o(1) &= \langle J'(u_n), \varphi_R u_n\rangle \\ &= \int_\Omega \varphi_R(|\nabla u_n|^2 + u_n^2) + u_n \nabla\varphi_R \nabla u_n - \int_\Omega \varphi_R |u_n|^p. \end{aligned}$$

Then

$$\int_\Omega \varphi_R(|\nabla u_n|^2 + u_n^2) \le \big|\int_\Omega u_n \nabla\varphi_R\nabla u_n\big| + \int_{\Omega\cap\{\frac{R}{2}<|z|<\frac{3R}{2}\}} \varphi_R|u_n|^p + o(1).$$

Note that $|\int_\Omega u_n\nabla\varphi_R\nabla u_n| \le \frac{c}{R}$. Similarly to (5.4), we obtain

$$\limsup_{n\to\infty}\int_\Omega \varphi_R(|\nabla u_n|^2 + u_n^2) \le \frac{c}{R} + \limsup_{n\to\infty}\int_{\Omega\cap\{\frac{R}{2}<|z|<\frac{3R}{2}\}} \varphi_R|u|^p.$$

Thus,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_\Omega \varphi_R(|\nabla u_n|^2 + u_n^2) = 0. \tag{5.9}$$

Note that $(1-\eta_R)(z) = \varphi_R(z)$ for each $\frac{R}{2} < |z| < R$. Let $R\to\infty$, and by (5.9) we have,

$$\begin{aligned} &\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\Omega\cap\{\frac{R}{2}<|z|<R\}} (1-\eta_R)(|\nabla u_n|^2 + u_n^2) \\ &\le \lim_{R\to\infty}\limsup_{n\to\infty}\int_\Omega \varphi_R(|\nabla u_n|^2 + u_n^2) = 0. \end{aligned}\tag{5.10}$$

Step 3: by (5.2), (5.7), and (5.10), we obtain

$$\begin{aligned} \limsup_{n\to\infty}\int_\Omega |\nabla u_n|^2 + u_n^2 &\le [\lim_{R\to\infty}\limsup_{n\to\infty}\int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2)] \\ &\quad + \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\Omega\cap\{\frac{R}{2}<|z|<R\}} (1-\eta_R)(|\nabla u_n|^2 + u_n^2) + a_\infty \\ &\le \Big(\int_\Omega |\nabla u|^2 + u^2\Big) + a_\infty. \end{aligned}$$

On the other hand, by (5.2), (5.8), and (5.10), we have

$$\begin{aligned} a_\infty &\le \limsup_{n\to\infty}\int_\Omega (|\nabla u_n|^2 + u_n^2) + \lim_{R\to\infty}\limsup_{n\to\infty}\Big[-\int_\Omega \eta_R(|\nabla u_n|^2 + u_n^2)\Big] \\ &\quad + \lim_{R\to\infty}\limsup_{n\to\infty}\Big[-\int_{\Omega\cap\{\frac{R}{2}<|z|<R\}} (1-\eta_R)(|\nabla u_n|^2 + u_n^2)\Big] \\ &\le (\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^2 + u_n^2) - \int_\Omega |\nabla u|^2 + u^2. \end{aligned}$$

Hence, we have $\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^2 + u_n^2 = (\int_\Omega |\nabla u|^2 + u^2) + a_\infty$.

(*ii*) Recall that for bounded sequences $\{s_n\}$ and $\{t_n\}$ such that $\lim_{n\to\infty} s_n = s$, then

$$\limsup_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

For each $R > 0$

$$\limsup_{n\to\infty}\int_\Omega |u_n|^p = \int_{\Omega\cap\{|z|<R\}} |u|^p + \limsup_{n\to\infty}\int_{\Omega\cap\{R<|z|\}} |u_n|^p.$$

Let $R \to \infty$, we have

$$\limsup_{n\to\infty} \int_\Omega |u_n|^p = \int_\Omega |u|^p + b_\infty.$$

(*iii*) There is a subsequence $\{u_n\}$ such that

$$a(u_n) = b(u_n) + o(1) = (\frac{2p}{p-2})\alpha_X + o(1).$$

Note that

$$b(u) + b_\infty = b(u_n) + o(1) = a(u_n) + o(1) = a(u) + a_\infty,$$

thus, $a_\infty = b_\infty$. Moreover,

$$\begin{aligned}\alpha_X &= \lim_{n\to\infty}(\frac{1}{2}a(u_n) - \frac{1}{p}b(u_n)) \\ &= \frac{1}{2}a(u) + \frac{1}{2}a_\infty - \frac{1}{p}b(u) - \frac{1}{p}b_\infty \\ &= J(u) + J_\infty.\end{aligned}$$

□

We require the following results to assert our main result.

Theorem 5.11. *Let $\Omega_n = \Omega \cap B^N(0;n)$, then the following properties are equivalent:*
(i) J does not satisfy the (PS)$_{\alpha_X(\Omega)}$ -condition in $X(\Omega)$ for J;
(ii) There is a (PS)$_{\alpha_X(\Omega)}$ -sequence $\{u_n\}$ in $X(\Omega)$ for J such that

$$\int_{\Omega_n} |u_n|^p = o(1);$$

(iii) There is a (PS)$_{\alpha_X(\Omega)}$ -sequence $\{u_n\}$ in $X(\Omega)$ for J such that $\{\xi_n u_n\}$ is also a (PS)$_{\alpha_X(\Omega)}$ -sequence $\{u_n\}$ in $X(\Omega)$ for J.

Proof. (*i*) $\Longrightarrow$ (*ii*) Suppose J does not satisfy the (PS)$_{\alpha_X(\Omega)}$-condition in $X(\Omega)$ for J. By Lemma 5.8, there is a (PS)$_{\alpha_X(\Omega)}$ -sequence $\{u_n\}$ in $X(\Omega)$ for J such that $u_n \rightharpoonup 0$ weakly in $X(\Omega)$. By Theorem 5.7 and the Rellich compactness lemma, $u_n \rightharpoonup 0$ a.e. in Ω and strongly in $L^p_{\rm loc}(\Omega)$. Thus, $\lim_{m\to\infty} \int_{\Omega_n} |u_m|^p = 0$. We can take a subsequence $\{u_{m_n}\}$ such that $\int_{\Omega_n} |u_{m_n}|^p < \frac{1}{n}$, or $\int_{\Omega_n} |v_n|^p = {\rm o}(1)$.
(*ii*) $\Longrightarrow$ (*iii*) Suppose there is a (PS)$_{\alpha_X(\Omega)}$-sequence $\{u_n\}$ in $X(\Omega)$ for J such that

$$\int_{\Omega_n} |u_n|^p = o(1). \tag{5.11}$$

By (5.11) and Lemma 2.11, we have

$$\int_\Omega \xi_n^q |u_n|^p = \int_\Omega |u_n|^p + o(1) = \frac{2p}{p-2}\alpha_X(\Omega) + o(1) \quad \text{for } q > 0. \tag{5.12}$$

Let $v_n = \xi_n u_n$. Then $v_n \in X(\Omega)$. By (5.12), we have

$$b(v_n) = \int_\Omega \xi_n^p |u_n|^p = \frac{2p}{p-2}\alpha_X(\Omega) + o(1). \tag{5.13}$$

Since $\{\xi_n^2 u_n\}$ is bounded in $X(\Omega)$, we have

$$o(1) = \langle J'(u_n), \xi_n^2 u_n\rangle = \int_\Omega (\xi_n^2|\nabla u_n|^2 + 2\xi_n u_n \nabla\xi_n\cdot\nabla u_n + \xi_n^2 u_n^2) - \int_\Omega \xi_n^2|u_n|^p. \quad (5.14)$$

By $|\nabla\xi_n(z)| \le \frac{c}{n}$ and (5.11), we have

$$\int_\Omega \xi_n u_n \nabla\xi_n \cdot \nabla u_n = o(1).$$

By (5.13) and (5.14) we have

$$a(v_n) = \int_\Omega \xi_n^2(|\nabla u_n|^2 + u_n^2) = b(v_n) + o(1) = \frac{2p}{p-2}\alpha_X(\Omega) + o(1).$$

Thus

$$\begin{aligned} J(v_n) &= \frac{1}{2}a(v_n) - \frac{1}{p}b(v_n) \\ &= \frac{1}{2}\frac{2p}{p-2}\alpha_X(\Omega) - \frac{1}{p}\frac{2p}{p-2}\alpha_X(\Omega) + o(1) \\ &= \alpha_X(\Omega) + o(1). \end{aligned}$$

Since $J(v_n) = \alpha_X(\Omega) + o(1)$ and $a(v_n) = b(v_n) + o(1)$, by Theorem 4.4, $\{v_n\}$ is a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J.
(iii) $\Longrightarrow$ *(i)* Let $\{u_n\}$ be a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J such that $\{\xi_n u_n\}$ is also a $(\mathrm{PS})_{\alpha_X(\Omega)}$-sequence in $X(\Omega)$ for J. Let $v_n = \xi_n u_n$. Claim: $v_n \rightharpoonup 0$ as $n \to \infty$. For $\phi \in C_c^1(\Omega)$ and $K = \mathrm{supp}\ \phi$, $K \subset \Omega$ is compact and there is an n_0 such that $v_n(z) = 0$ in K for all $n \ge n_0$. We have

$$\langle v_n(z), \phi(z)\rangle_{H^1} = \int_\Omega \nabla v_n(z)\nabla\phi(z)dz + \int_\Omega v_n(z)\phi(z)dz = 0 \qquad \text{for all } n \ge n_0.$$

By Lemma 2.11, there is a $C > 0$ such that $\|v_n(z)\|_{H^1} \le C$. For $\varepsilon > 0$, $\varphi \in H_0^1(\Omega)$, $\phi \in C_c^1(\Omega)$ exists such that

$$\|\varphi - \phi\|_{H^1} < \varepsilon/2(C+1)$$

Moreover,

$$\begin{aligned} \langle v_n(z), \varphi(z)\rangle_{H^1} &= \langle v_n(z), \varphi(z) - \phi(z)\rangle_{H^1} + \langle v_n(z), \phi(z)\rangle_{H^1} \\ &\le \|v_n(z)\|_{H^1}\|\varphi(z) - \phi(z)\|_{H^1} + \langle v_n(z), \phi(z)\rangle_{H^1} \\ &\le C\|\varphi(z) - \phi(z)\|_{H^1} \\ &< \varepsilon \text{ for } n \ge n_0. \end{aligned}$$

This implies $v_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. Therefore, by Lemma 5.6 J does not satisfy the $(\mathrm{PS})_{\alpha_X(\Omega)}$-condition in $X(\Omega)$. □

For $k \ge 1$, $i = 1, 2, \dots, k$, let Ω be an unbounded domain and let Ω_i be a proper domain in Ω such that $\Omega = \cup_{i=1}^k \Omega_i$, $\Omega_i \cap \Omega_j$ is bounded, and at least one of Ω_i is unbounded. Let $\alpha_X = \alpha_X(\Omega)$ and $\alpha_X^i = \alpha_X(\Omega_i)$, then

$$\mathbf{M} = \{u \in X(\Omega)\backslash\{0\} \mid a(u) = b(u)\},$$
$$\mathbf{M}_i = \{u \in H_0^1(\Omega_i)\backslash\{0\} \mid a(u) = b(u)\} \quad \text{for } i = 1, 2, \dots, k.$$

Since $X(\Omega_i) \subset X(\Omega)$ and $\mathbf{M}_i \subset \mathbf{M}$, for $i = 1, 2, \dots, k$, $\alpha_X \le \min\{\alpha_X^1, \alpha_X^2, \dots, \alpha_X^k\}$. Let Ω be an unbounded domain in $\mathbb{R}^N$ and $\Omega^0 \subsetneqq \Omega$ with the indexes $\alpha_X = \alpha_X(\Omega)$ and $\alpha_X^0 = \alpha_X(\Omega^0)$. Then we have $\alpha_X \le \alpha_X^0$. Let

$$\widetilde{\Omega}_n = \Omega \setminus \overline{B^N(0;n)};$$
$$\widetilde{\mathbf{M}}_n = \{u \in H_0^1(\widetilde{\Omega}_n)\backslash\{0\} \mid a(u) = b(u)\};$$
$$\widetilde{\alpha}_X^n = \alpha(\widetilde{\Omega}_n) = \inf_{u \in \widetilde{M}_n} J(u).$$

Theorem 5.12. *The following properties are equivalent:*
(i) J satisfies the (PS)$_{\alpha_X}$-condition;
(ii) For every (PS)$_{\alpha_X}$-sequence $\{u_n\}$ in $X(\Omega)$ for J, there are a subsequence $\{u_n\}$ and $u \ne 0$ in $X(\Omega)$ such that $u_n \to u$ strongly in $X(\Omega)$;
(iii) For every (PS)$_{\alpha_X}$-sequence $\{u_n\}$ in $X(\Omega)$ for J, there are $c > 0$, a subsequence $\{u_n\}$, and positive integers K and n_0 such that for each $n \ge n_0$, we have

$$\int_{\Omega \cap \{|z| < K\}} |u_n|^p \ge c;$$

(iv) For every (PS)$_{\alpha_X}$-sequence $\{u_n\} \subset X(\Omega)$ for J, there is a subsequence $\{u_n\}$ such that for any $\varepsilon > 0$, there is a measurable set E such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dz < \varepsilon$ for each $n \in \mathbb{N}$;
(v) $\alpha_X < \tilde{\alpha}_X^n$ for each $n \in \mathbb{N}$;
(vi) $\alpha_X < \min\{\alpha_X^1, \alpha_X^2, \dots, \alpha_X^k\}$;
(vii) $J_\infty < \alpha_X$;
(viii) $\alpha_X < \alpha_X^0$ for each proper subdomain Ω^0 of Ω.

Proof. *(i)* $\Longrightarrow$ *(ii)* Suppose that J satisfies the (PS)$_{\alpha_X}$−condition. Let $\{u_n\}$ be a (PS)$_{\alpha_X}$-sequence in $X(\Omega)$ for J. Then there are a subsequence $\{u_n\}$ and a u in $X(\Omega)$ such that $u_n \to u$ strongly in $X(\Omega)$. We conclude that $J(u) = \alpha_X > 0$. Thus, $u \ne 0$.
(ii) $\Longrightarrow$ *(iii)* Suppose that $\{u_n\}$ is a (PS)$_{\alpha_X}$-sequence in $X(\Omega)$ for J that has a subsequence $\{u_n\}$ and $u \ne 0$ in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. By Theorem 5.6, $\lim_{n\to\infty} \int_\Omega |u_n|^p = \int_\Omega |u|^p$. Take $K > 0$ and $c > 0$ with $\int_{\Omega \cap \{|z|<K\}} |u|^p \ge 2c$. $n_0 > 0$ exists such that

$$\int_{\Omega \cap \{|z| < K\}} |u_n|^p \ge c \quad \text{for } n \ge n_0.$$

Then *(iii)* follows.
(iii) $\Longrightarrow$ *(iv)* Given a (PS)$_{\alpha_X}$-sequence $\{u_n\} \subset X(\Omega)$ for J, there are a subsequence $\{u_n\}$ and a u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. By *(iii)*, there are $c > 0$, a subsequence $\{u_n\}$, positive integers K and n_0 such that for each $n \ge n_0$, we have

$$\int_{\Omega \cap \{|z| < K\}} |u_n|^p \ge c.$$

Since $\lim_{n\to\infty} \int_{\Omega \cap \{|z|<K\}} |u_n|^p = \int_{\Omega \cap \{|z|<K\}} |u|^p$, we have $u \ne 0$. By Theorem 5.6, $u_n \to u$ in $L^p(\Omega)$. Thus, by Theorem 2.23, for $\varepsilon > 0$, there is a set E such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dz < \varepsilon$ for each $n \in \mathbb{N}$.
(iv) $\Longrightarrow$ *(v)* For every (PS)$_{\alpha_X}$-sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$ such that for $\varepsilon > 0$, there is a set E such that $|E| < \infty$ and $\int_{E^c} |u_n|^p dz < \varepsilon$

for each $n \in \mathbb{N}$. Then $\{u_n\}$ is bounded and there is a subsequence $\{u_n\}$ and a u in $X(\Omega)$ such that $u_n \to u$ a.e. in Ω. By Theorem 2.34, $u_n \to u$ in $L^p(\Omega)$. Note that

$$\alpha_X + o(1) = J(u_n) = (\frac{1}{2} - \frac{1}{p})b(u_n) + o(1) = (\frac{1}{2} - \frac{1}{p})b(u) + \text{o}(1).$$

Thus, $u \neq 0$. By Theorem 5.6, J satisfies the $(\text{PS})_{\alpha_X}$-condition in Ω. Suppose that $\widetilde{\alpha}_X^{n_0} = \alpha_X$ for some $n_0 \in \mathbb{N}$, by Theorem 5.7 J does not satisfy the $(\text{PS})_{\alpha_X}$-condition in Ω, which is a contradiction. Hence, we have $\alpha_X < \tilde{\alpha}_X^n$ for each n.

$(v) \Longrightarrow (vi)$ On the contrary, suppose that $\alpha_X = \min\{\alpha_X^1, \alpha_X^2, \dots, \alpha_X^k\}$, say $\alpha_X = \alpha_X^1$. Since $\Omega_1 \subsetneqq \Omega$, by Theorem 5.7, J does not satisfy the $(\text{PS})_{\alpha_X}$-condition in Ω. By Theorem 5.6, there is a $(\text{PS})_{\alpha_X}$-sequence $\{u_n\}$ such that $u_n \rightharpoonup 0$ weakly in $X(\Omega)$. There is a subsequence $\{u_n\}$ and a sequence $\{\Omega_n\}$ such that

$$\int_{\Omega_n} |u_n|^p = o(1).$$

Let ξ_n be as in (2.1) and $v_n = \xi_n u_n$. By Lemma 2.41, we have

$$J(v_n) = \alpha_X + o(1).$$
$$J'(v_n) = o(1) \quad \text{strongly in } X^{-1}(\Omega).$$

Then by Theorem 4.3, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $w_n = s_n v_n$, $\{w_n\} \in \widetilde{\mathbf{M}}_n$, and $J(w_n) = J(v_n) + o(1) = \alpha_X + o(1)$. Note that $\widetilde{\alpha}_X^n \le J(w_n)$ for each $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \widetilde{\alpha}_X^n \le \alpha_X$. Since $\Omega \supset \widetilde{\Omega}_n \supset \widetilde{\Omega}_{n+1}$, we have $\alpha_X \le \widetilde{\alpha}_X^n \le \widetilde{\alpha}_X^{n+1}$ for each $n \in \mathbb{N}$. Then we can conclude that $\alpha_X = \widetilde{\alpha}_X^n$ for each $n \in \mathbb{N}$, which is a contradiction.

$(vi) \Longrightarrow (vii)$ Let $\{u_n\}$ be a $(\text{PS})_{\alpha_X}$-sequence in $X(\Omega)$. Then a subsequence $\{u_n\}$ and a u in $X(\Omega)$ exist such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. By Lemma 5.10, $\alpha_X = J_\infty + J(u)$. On the contrary, suppose that $J_\infty = \alpha_X$, and we have $u = 0$. Thus, $u_n \rightharpoonup 0$ weakly in $X(\Omega)$. There are a subsequence $\{u_n\}$ and a sequence $\{\Omega_n\}$ such that

$$\int_{\Omega_n} |u_n|^p = o(1).$$

Let $v_n = \xi_n u_n$. By Lemma 2.41, $\{v_n\}$ is a $(\text{PS})_{\alpha_X}$-sequence in $X(\Omega)$. Since $\Omega_i \cap \Omega_j$ is bounded for $i \neq j$, $n_0 > 0$, $v_n = 0$ in $B^N(0; n_0)$ exists for $n \geq n_0$, where $B^N(0; n_0) \supset \Omega_i \cap \Omega_j$ for $i \neq j$. Set $v_n = v_n^1 + v_n^2 + \dots + v_n^k$, where $v_n^i \in H_0^1(\Omega_i)$, and for $i = 1, 2, \dots, k$,

$$v_n^i(z) = \begin{cases} v_n(z) & \text{for } z \in \Omega_i; \\ 0 & \text{otherwise.} \end{cases}$$

As in the proof of Lemma 2.41, we obtain

$$J'(v_n^i) = o(1) \text{ strongly in } X^{-1}(\Omega) \quad \text{for } i = 1, 2, \dots, k.$$

Assume

$$J(v_n^i) = c_i + o(1) \quad \text{for } i = 1, 2, \dots, k.$$

Since $J(u_n) = \alpha_X + o(1)$, we have $c_1 + c_2 + \dots + c_k = \alpha_X$. Since c_i are (PS)-values in $X(\Omega)$ for J, by Lemma 2.38, they are nonnegative. There is at least one of the c_i that is positive, say $c_1 > 0$. By Theorem 4.10, $c_1 \geq \alpha_X^1$, thus $\alpha_X \geq c_1 \geq \alpha_X^1$. This proves $\alpha_X \geq \min\{\alpha_X^1, \alpha_X^2, \dots, \alpha_X^k\}$. We conclude that $\alpha_X = \min\{\alpha_X^1, \alpha_X^2, \dots, \alpha_X^k\}$.

$(vii) \Longrightarrow (viii)$ Let $\{u_n\}$ be a $(\text{PS})_\alpha$-sequence in Ω. Then a subsequence $\{u_n\}$ and

a u in $X(\Omega)$ exist such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. By Lemma 5.10, $J(u) = \alpha_X - J_\infty$. Suppose that $J_\infty < \alpha_X$, then $u \neq 0$. By Theorem 5.6, J satisfies $(\mathrm{PS})_{\alpha_X}$-condition in $X(\Omega)$. By Theorem 5.7, $\alpha_X < \alpha_X^0$ for each proper subdomain Ω^0 of Ω.
$(viii) \implies (i)$ Suppose that J does not satisfy the $(\mathrm{PS})_{\alpha_X}$-condition in Ω. By Lemma 2.41 and Theorem5.7, $\{u_n\}$ exists in $X(\Omega)$ and is a $(\mathrm{PS})_{\alpha_X}$-sequence for J such that $\{\xi_n u_n\}$ also is a $(\mathrm{PS})_{\alpha_X}$-sequence in $X(\Omega)$for J. Let $v_n = \xi_n u_n$, then

$$J(v_n) = \alpha + o(1),$$
$$J'(v_n) = o(1) \quad \text{strongly in } X^{-1}(\Omega).$$

By Theorem 4.3, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $w_n = s_n v_n$, $\{w_n\} \in \mathbf{M}(\Omega\backslash\overline{B^N(0;\frac{n}{2})})$ and $J(w_n) = J(v_n) + o(1) = \alpha_X + o(1)$. $n_0 > 0$ exists such that $\Omega \setminus \overline{B^N(0;n_0)} \subsetneqq \Omega$. Let $\Omega^0 = \Omega\backslash\overline{B^N(0;n_0)}$. Then $w_n \in \mathbf{M}(\Omega^0)$ for $n \geq n_0$. Since $\mathbf{M}(\Omega^0) \subset \mathbf{M}(\Omega)$ and $J(w_n) = \alpha_X + o(1)$, thus, $\alpha_X^0 = \alpha_X$, which is a contradiction. □

Bibliographical notes: Theorem 5.6 is from Chen-Lee-Wang [24, Lemma 19]. Theorem 5.12 is from Chen-Lin-Wang [25, Theorem 23].

6. Symmetric Palais-Smale Conditions

In this section, we focus on the symmetric Palais-Smale conditions which will be used in Section 13.

Definition 6.1. (i) Suppose that $(x,y) \in \Omega$ if and only if $(x,-y) \in \Omega$, then we call Ω a y-symmetric domain;
(ii) Let Ω be a y-symmetric domain and Θ be a y-symmetric bounded domain in $\mathbb{R}^N$. If two disjoint subdomains Ω_1 and Ω_2 of Ω exist such that

$$(x,y) \in \Omega_2 \quad \text{if and only if} \quad (x,-y) \in \Omega_1,$$
$$\Omega\backslash\bar{\Theta} = \Omega_1 \cup \Omega_2,$$

then we say that Ω is separated by Θ;
(iii) Let Ω be a y-symmetric domain in $\mathbb{R}^N$. If a function $u : \Omega \to \mathbb{R}$ satisfies $u(x,y) = u(x,-y)$ for $(x,y) \in \Omega$, then we call u a y-symmetric (axially symmetric) function;
(iv) Let Ω be a y-symmetric domain in $\mathbb{R}^N$ and denote the space $H_s^1(\Omega)$ by the H^1-closure of the space $\{u \in C_0^\infty(\Omega) : u \text{ is } y\text{-symmetric}\}$.

Remark 6.2. (i) Note that $H_s^1(\Omega)$ is a closed linear subspace of $H_0^1(\Omega)$. Let $H_s^{-1}(\Omega)$ be the dual space of $H_s^1(\Omega)$;
(ii) Let Ω be a y-symmetric domain in $\mathbf{A}^r$ and let $B^N(0;r+1)$ be a $N-$ball. Then clearly Ω is separated by $B^N(0;r+1)$.

Example 6.3. (i) For each $\rho > 0$, let $\Omega = (\mathbb{R}^N\backslash\overline{K_\rho}) \cup \mathbf{A}^r$. Then Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by a bounded domain $\mathbf{A}^r_\rho$;
(ii) Let $\Omega = [\mathbf{P}^+ + (0,\frac{R}{2})] \cup B^N(0;R) \cup [\mathbf{P}^- - (0,\frac{R}{2})]$, then Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by the bounded domain $B^N(0;R)$.

Theorem 6.4. (i) $\alpha_s(B^N(0;R)) = \alpha(B^N(0;R))$;
(ii) $\alpha_s(\mathbb{R}^N) = \alpha(\mathbb{R}^N)$;

(iii) $\alpha_s(\mathbf{A}^r_{-t,t}) = \alpha(\mathbf{A}^r_{-t,t})$;
(iv) $\alpha_s(\mathbf{A}^r) = \alpha(\mathbf{A}^r)$.

Proof. By Lien-Tzeng-Wang [47] and Theorem 12.3 below, there is a ground state solution of (1.1) in $B^N(0;R)$, $\mathbb{R}^N$, $\mathbf{A}^r_{-t,t}$, and $\mathbf{A}^r$. By Gidas-Ni-Nirenberg [34] and [35] and Chen-Chen-Wang [23], every positive solution of (1.1) in $B^N(0;R)$, $\mathbb{R}^N$, $\mathbf{A}^r_{-t,t}$, and $\mathbf{A}^r$ is y-symmetric. □

The following symmetric results are required to assert our main result.

Theorem 6.5. *Suppose that Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by a y-symmetric bounded domain, then $\alpha_s(\Omega) \le 2\alpha(\Omega)$.*

Proof. First, by Lien-Tzeng-Wang [47] and Gidas-Ni-Nirenberg [35], there is a positive solution u_0 of Equation (1.1) with radial symmetry such that $J(u_0) = \alpha(\mathbb{R}^N)$. Since Ω is a y-symmetric proper large domain in $\mathbb{R}^N$, for $n = 1, 2, \dots$, sequences $\{z_n\}$ and $\{r_n\}$ exist such that $B^N(z_n; r_n) \subset \Omega$ and $r_n \to \infty$ as $n \to \infty$. Let $\eta_n(z) = \eta(\frac{2|z-z_n|}{r_n})$ as in (2.2), and $u_n(z) = \eta_n(z)u_0(z - z_n)$. Then $u_n(z) \in H^1_0(\Omega)$, and

$$J(u_n) = J(u_0) + o(1) = \alpha(\mathbb{R}^N) + o(1)$$
$$a(u_n) = b(u_n) + o(1).$$

By Theorem 4.4 and Theorem 4.18, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence in $H^1_0(\Omega)$ for J. Moreover, if we let $w_n = u_n(x, -y)$, then w_n is also a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence in $H^1_0(\Omega)$ for J such that $\mathrm{supp} w_n \cap \mathrm{supp} u_n = \emptyset$ and $\{u_n + w_n\} \subset H^1_s(\Omega)$. We have

$$\begin{aligned} a(u_n + w_n) &= \int_\Omega |\nabla(u_n + w_n)|^2 + (u_n + w_n)^2 \\ &= \int_\Omega |\nabla u_n|^2 + u_n^2 + \int_\Omega |\nabla w_n|^2 + w_n^2 \\ &\quad + 2\int_\Omega \nabla u_n \nabla w_n + 2\int_\Omega u_n w_n \\ &= a(u_n) + a(w_n), \end{aligned}$$

and $b(u_n + w_n) = \int_\Omega |u_n + w_n|^p = \int_\Omega |u_n|^p + \int_\Omega |w_n|^p = b(u_n) + b(w_n)$. Hence,

$$\begin{aligned} J(u_n + w_n) &= \frac{1}{2} a(u_n + w_n) - \frac{1}{p} b(u_n + w_n) \\ &= J(u_n) + J(w_n) \\ &= 2\alpha(\Omega) + o(1). \end{aligned}$$

Moreover, for $\varphi \in C^\infty_c(\Omega)$ with y-symmetry, we have

$$\begin{aligned} &|\langle J'(u_n + w_n), \varphi\rangle| \\ &= big|\int_\Omega \nabla(u_n + w_n)\nabla\varphi + (u_n + w_n)\varphi - \int_\Omega |u_n + w_n|^{p-2}(u_n + w_n)\varphi\big| \\ &= \big|\int_\Omega \nabla u_n \nabla\varphi + u_n\varphi + \int_\Omega \nabla w_n \nabla\varphi + w_n\varphi - \int_\Omega |u_n|^{p-2}u_n\varphi - \int_\Omega |w_n|^{p-2}w_n\varphi\big| \\ &= |\langle J'(u_n), \varphi\rangle| + |\langle J'(w_n), \varphi\rangle| \\ &\le \|J'(u_n)\|_{H^{-1}} + \|J'(w_n)\|_{H^{-1}}. \end{aligned}$$

Therefore, $\|J'(u_n + w_n)\|_{H_s^{-1}} = o(1)$. We conclude that $\{u_n + w_n\}$ is a $(\mathrm{PS})_{2\alpha(\Omega)}$-sequence in $H_s^1(\Omega)$ for J. By Theorems 4.12 and 4.13, $\alpha_s(\Omega) \le 2\alpha(\Omega)$. □

We have the following symmetric Palais-Smale condition.

Theorem 6.6. *Suppose that Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by a y-symmetric bounded domain. Then $\alpha_s(\Omega) < 2\alpha(\Omega)$ if and only if J satisfies the $(PS)_{\alpha_s(\Omega)}$-condition in $H_s^1(\Omega)$.*

Proof. Let $\alpha_s(\Omega) < 2\alpha(\Omega)$. Suppose J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}$-condition. By Theorem 5.11, a $(\mathrm{PS})_{\alpha_s(\Omega)}$-sequence $\{u_n\}$ in $H_s^1(\Omega)$ for J exists such that $\{\xi_n u_n\}$ is also a $(\mathrm{PS})_{\alpha_s(\Omega)}$-sequence in $H_s^1(\Omega)$ for J, where ξ_n is as at (2.1). Let $w_n = \xi_n u_n$, then by Lemma 2.9, we obtain

$$\begin{aligned} J(w_n) &= \alpha_s(\Omega) + o(1), \\ J'(w_n) &= o(1) \quad \text{in } H^{-1}(\Omega). \end{aligned} \tag{6.1}$$

Since Ω is a y-symmetric domain in $\mathbb{R}^N$ separated by a bounded domain Q, $n_0 > 0$, exists such that $w_n = 0$ in $\overline{Q}$ for $n \ge n_0$ and two disjoint subdomains Ω_1 and Ω_2 of Ω exist such that

$$\begin{gathered} (x, y) \in \Omega_2 \quad \text{if and only if } (x, -y) \in \Omega_1, \\ \Omega \backslash \overline{Q} = \Omega_1 \cup \Omega_2. \end{gathered}$$

Note that, for $n \ge n_0$, $w_n = w_n^1 + w_n^2$ and $w_n^1(x, y) = w_n^2(x, -y)$, where for $i = 1, 2$,

$$w_n^i(x) = \begin{cases} w_n(x) & \text{for } x \in \Omega_i, \\ 0 & \text{for } x \notin \Omega_i. \end{cases}$$

Then $w_n^i \in H_0^1(\Omega_i)$. We obtain $J(w_n^1) = J(w_n^2)$ and

$$\alpha_s(\Omega) + o(1) = J(w_n) = J(w_n^1) + J(w_n^2) = 2J(w_n^i) \quad \text{for } i = 1, 2,$$

or

$$J(w_n^i) = \frac{1}{2}\alpha_s(\Omega) + o(1) \quad \text{for } i = 1, 2.$$

By (6.1), we have

$$J'(w_n^i) = o(1) \quad \text{in } H_0^1(\Omega_i) \quad \text{for } i = 1, 2.$$

Therefore $\frac{1}{2}\alpha_s(\Omega)$ is a (PS)-value in $H_0^1(\Omega)$ for J. By Theorems 4.12 and 4.13,

$$\frac{1}{2}\alpha_s(\Omega) \ge \alpha(\Omega_i).$$

Since Ω and Ω_i are large domains of $\mathbb{R}^N$, by Theorem 4.18, we have

$$\alpha(\Omega_i) = \alpha(\mathbb{R}^N) = \alpha(\Omega).$$

Thus $\alpha_s(\Omega) \ge 2\alpha(\Omega)$, which is a contradiction.
Conversely, suppose that J satisfies the $(\mathrm{PS})_{\alpha_s(\Omega)}$-condition in $H_s^1(\Omega)$. By Theorem 6.5, we have $\alpha_s(\Omega) \le 2\alpha(\Omega)$. Suppose that $\alpha_s(\Omega) = 2\alpha(\Omega)$. By the definition of the large domain in $\mathbb{R}^N$, we may take a domain $\tilde{\Omega} = \Omega \setminus \overline{B^N(0; \tilde{r})}$ for some $\tilde{r} > 0$ such that $\tilde{\Omega} \subsetneqq \Omega$, and $\tilde{\Omega}$ is a proper y-symmetric large domain in $\mathbb{R}^N$ separated by a y-symmetric bounded domain. By Theorem 5.7, we have $2\alpha(\mathbb{R}^N) = 2\alpha(\Omega) = \alpha_s(\Omega) < \alpha_s(\tilde{\Omega})$. By Theorem 6.5, $\alpha_s(\tilde{\Omega}) \le 2\alpha(\tilde{\Omega}) = 2\alpha(\mathbb{R}^N)$. Thus, $2\alpha(\mathbb{R}^N) < 2\alpha(\mathbb{R}^N)$, which is a contradiction. □

As a consequence of Theorem 6.6, we have the following result.

Theorem 6.7. *If Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by a y-symmetric bounded domain, then $\alpha(\Omega) < \alpha_s(\Omega)$.*

Proof. Since $\Omega \subsetneqq \mathbb{R}^N$, we have $\alpha_s(\mathbb{R}^N) \le \alpha_s(\Omega)$. Assume that $\alpha_s(\mathbb{R}^N) = \alpha_s(\Omega)$. Then by Theorem 5.7, J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}$-condition in $H^1_s(\Omega)$ for J. By Theorem 4.18, $\alpha(\mathbb{R}^N) = \alpha(\Omega)$, by Theorem 6.4, $\alpha_s(\mathbb{R}^N) = \alpha(\mathbb{R}^N)$, and by Theorem 6.6, $2\alpha(\Omega) \le \alpha_s(\Omega)$. We conclude that

$$2\alpha(\mathbb{R}^N) = 2\alpha(\Omega) \le \alpha_s(\Omega) = \alpha_s(\mathbb{R}^N) = \alpha(\mathbb{R}^N),$$

which is a contradiction. □

Consider the y-symmetric large domain Ω_R in $\mathbb{R}^N$ separated by a y-symmetric bounded domain, where $\Omega_R = \big[\mathbf{P}^+ + (0, \frac{R}{2})\big] \cup B^N(0;R) \cup \big[\mathbf{P}^- - (0, \frac{R}{2})\big]$. Then we have the following existence result.

Theorem 6.8. *An $R_0 > 0$ exists such that for $R \ge R_0$, there is a positive y-symmetric solution of* (1.1) *in Ω_R.*

Proof. By Lien-Tzeng-Wang [47], $\alpha(B^N(0,R))$ is strictly decreasing as R is strictly increasing and

$$\alpha(B^N(0,R)) \searrow \alpha(\mathbb{R}^N) \quad \text{as } R \to \infty.$$

By Theorem 6.4, $\alpha(B^N(0,R)) = \alpha_s(B^N(0,R))$ for each R. Thus, there is a $R_0 > 0$ such that $\alpha_s(\Omega_R) \le \alpha_s(B^N(0,R)) < 2\alpha(\mathbb{R}^N) = 2\alpha(\Omega_R)$ for each $R \ge R_0$. By Theorem 6.5 and Theorem 6.6, there is a y-symmetric positive solution of Equation (1.1) in Ω_R for each $R \ge R_0$. □

Bibliographical notes: The results of this section are from Wang-Wu [74].

7. Symmetric Palais-Smale Decomposition Theorems

In this section, we present the symmetric Palais-Smale decomposition theorem in $\mathbf{A}^r$.

Lemma 7.1. *Let $\Theta_1 \subset \Theta_2 \subset \Theta_3 \subset \dots$, where $\overset{\infty}{\underset{n=1}{\cup}} \Theta_n = \mathbf{A}^r$. If*

$$f_n(z) = \begin{cases} g_n(z) - h_n(z), & \text{for } z \in \Theta_n, \\ 0, & \text{otherwise,} \end{cases}$$

$f_n \to f$ a.e., $g_n \ge 0$, and $h_n \to 0$ a.e., then $f \ge 0$.

Proof. For $z \in \mathbf{A}^r$, we have $z \in \Theta_m$ for some $m \in \mathbb{N}$, then $z \in \Theta_{m+i}$ for $i = 0, 1, 2, \dots$. Since $g_{m+i}(z) = f_{m+i}(z) + h_{m+i}(z)$, $f_{m+i} \to f$ a.e., $h_{m+i} \to 0$ a.e. for $i \to \infty$, hence $g_{m+i} \to f$ a.e., and since $g_{m+i} \ge 0$, we have $f \ge 0$. □

Theorem 7.2 (Symmetric Palais-Smale Decomposition Theorem in $\mathbf{A}^r$)**.** *Let $\{u_n\}$ be a $(PS)_\beta$-sequence in $H^1_s(\mathbf{A}^r)$ for J. Then there are a subsequence $\{u_n\}$, a positive integer m, sequences $\{\tilde{z}^{i,j}_n\}_{n=1}^\infty$ in $\mathbf{A}^r$, a function $\bar{u} \in H^1_s(\mathbf{A}^r)$, and $0 \ne w^{i,j} \in H^1_0(\mathbf{A}^r)$ for $1 \le i \le m$, $j = 1, 2$ such that*

$$w^{i,1}(x,y) = w^{i,2}(x,-y),$$

$$|\tilde{z}^{i,j}_n| \to \infty \quad \text{for } i = 1, 2, \dots, m,$$

$$-\Delta \bar{u} + \bar{u} = |\, \bar{u}\, |^{p-2}\, \bar{u} \quad \text{in } \mathbf{A}^r,$$

$$-\Delta w^{i,j} + w^{i,j} =| w^{i,j}|^{p-2} w^{i,j} \quad \textit{in } \mathbf{A}^r,$$

and

$$u_n = \bar{u} + \sum_{j=1}^{2}\sum_{i=1}^{m} w^{i,j}(\cdot - \tilde{z}_n^{i,j}) + o(1) \textit{ strongly} \quad \textit{in} \quad H_0^1(\mathbf{A}^r),$$

$$a(u_n) = a(\bar{u}) + 2\sum_{i=1}^{m} a(w^{i,j}) + o(1) \quad \textit{for some } j = 1, 2,$$

$$b(u_n) = b(\bar{u}) + 2\sum_{i=1}^{m} b(w^{i,j}) + o(1) \quad \textit{for some } j = 1, 2,$$

$$J(u_n) = J(\bar{u}) + 2\sum_{i=1}^{m} J(w^{i,j}) + o(1) \quad \textit{for some } j = 1, 2.$$

In addition, if $u_n \geq 0$*, then* $\bar{u} \geq 0$ *and* $w^{i,j} \geq 0$ *for each* $1 \leq i \leq m$*,* $j = 1, 2$*.*

Proof. **Step 0**. Since $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_s^1(\mathbf{A}^r)$ for J, by Lemma 2.38 there is a $c > 0$ such that $\|u_n\|_{H^1} \leq c$. In the following proof of this theorem, we fix the value of c. There is a subsequence $\{u_n\}$ and a $\bar{u}$ in $H_s^1(\mathbf{A}^r)$ such that $u_n \rightharpoonup \bar{u}$ weakly in $H_s^1(\mathbf{A}^r)$ and $\bar{u}$ solves

$$-\Delta \bar{u} + \bar{u} = |\bar{u}|^{p-2}\bar{u} \quad \text{in } \mathbf{A}^r.$$

Step 1. Suppose that $u_n \nrightarrow \bar{u}$ strongly in $H_s^1(\mathbf{A}^r)$. Let

$$u_n^1 = u_n - \bar{u} \quad \text{for } n = 1, 2, \dots.$$

By Lemma 2.15, $\{u_n^1\}$ is a $(\mathrm{PS})_{(\beta - J(\bar{u}))}$-sequence in $H_s^1(\mathbf{A}^r)$ for J. Let $v_n^1 = \xi_n u_n^1$, where ξ_n are as in 2.1. Note that $u_n^1 \rightharpoonup 0$ weakly in $H_s^1(\mathbf{A}^r)$ and $u_n^1 \nrightarrow 0$ strongly in $H_s^1(\mathbf{A}^r)$. By Lemma 2.41, $\{v_n^1\}$ is also a $(\mathrm{PS})_{(\beta - J(\bar{u}))}$-sequence in $H_s^1(\mathbf{A}^r)$ for J. Moreover, $J(u_n^1) = J(v_n^1) + o(1)$, $v_n^1 \rightharpoonup 0$ weakly in $H_s^1(\mathbf{A}^r)$ and $v_n^1 \nrightarrow 0$ strongly in $H_s^1(\mathbf{A}^r)$. Let $K = 2([r] + 1)$ and $Q_K = \mathbf{A}^r \cap B^N(0; K)$. Then $v_n^1 = 0$ in $\overline{Q_K}$ for $n \geq 2K$. Two disjoint strictly large domains $\Omega^1 = \mathbf{A}_K^r$ and $\Omega^2 = \widetilde{\mathbf{A}}_{-K}^r$ in $\mathbf{A}^r$ exist such that

$$(x, y) \in \Omega^2 \quad \text{if and only if} \quad (x, -y) \in \Omega^1,$$
$$\Omega \backslash \overline{Q_K} = \Omega^1 \cup \Omega^2.$$

For $j = 1, 2$, let

$$v_n^{1,j}(z) = \begin{cases} v_n^1(z) & \text{for } z \in \Omega^j, \\ 0 & \text{for } z \notin \Omega^j. \end{cases}$$

Then $v_n^{1,j} \in H_0^1(\Omega^j)$, $v_n^{1,1}(x, y) = v_n^{1,2}(x, -y)$, $v_n^1 = v_n^{1,1} + v_n^{1,2}$, and $J(v_n^{1,1}) = J(v_n^{1,2})$. We claim that $\{v_n^{1,j}\}$ is a $(\mathrm{PS})_{\frac{1}{2}(\beta - J(\bar{u}))}$-sequence in $H_0^1(\Omega^j)$ for J. In fact,

$$J(v_n^{1,j}) = \frac{1}{2} J(v_n^1) = \frac{1}{2} J(u_n^1) + o(1) = \frac{1}{2}\left[\beta - J(\bar{u})\right] + o(1),$$

and for $\varphi \in C_c^\infty(\Omega^j)$, $\|\varphi\|_{H^1} = 1$, we have

$$|\langle J'(v_n^{1,j}), \varphi\rangle| = |\langle J'(v_n^1), \varphi\rangle| \leq \|J'(v_n^1)\|_{H^{-1}} \|\varphi\|_{H^1}.$$

Thus,

$$\|J'(v_n^{1,j})\|_{H^{-1}} \leq \|J'(v_n^1)\|_{H^{-1}} = o(1).$$

Note that $v_n^1 \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $v_n^1 \nrightarrow 0$ strongly in $H_0^1(\Omega)$, so we have $v_n^{1,j} \rightharpoonup 0$ weakly in $H_0^1(\Omega^j)$ and $v_n^{1,j} \nrightarrow 0$ strongly in $H_0^1(\Omega^j)$.

(1-0) $\int_{A^r_{-1,1}} |w_n^{1,j}(z)|^2 dz \geq \frac{d_1}{2}$ for some constant $d_1 > 0$, $n = 1, 2, \dots$, and $j = 1, 2$, where $w_n^{1,j}(z) = v_n^{1,j}(z + z_n^{1,j})$ for some $\{z_n^{1,j}\} \subset \mathbf{A}^r$: for $j = 1, 2$, since $\{v_n^{1,j}\}$ is bounded, $J'(v_n^{1,j}) = o(1)$, and $v_n^{1,j} \nrightarrow 0$ strongly in $H_0^1(\Omega^j)$. By Lemma 2.17, there is a subsequence $\{v_n^{1,j}\}$, and a constant $d_1 > 0$ such that

$$Q_n^{r,1} = \sup_{y \in \mathbb{R}} \int_{(0,y)+A^r_{-1,1}} |v_n^{1,j}(z)|^2 dz \geq d_1 \quad \text{for } n = 1, 2, \dots.$$

For $n = 1, 2, \dots$, take $z_n^{1,1} = (0, y_n^1)$ and $z_n^{1,2} = (0, -y_n^1)$ in $\mathbf{A}^r$ such that

$$\int_{z_n^{1,j}+A^r_{-1,1}} |v_n^{1,j}(z)|^2 dz \geq \frac{d_1}{2} \text{ for } \quad n = 1, 2, \dots.$$

Let

$$w_n^{1,j}(z) = v_n^{1,j}(z + z_n^{1,j})$$

then

$$\int_{A^r_{-1,1}} |w_n^{1,j}(z)|^2 dz \geq \frac{d_1}{2} \text{ for } \quad n = 1, 2, \dots.$$

(1-1) $u_n(z) = \bar{u}(z) + \sum_{j=1}^2 w_n^{1,j}(z - z_n^{1,j}) + o(1)$ strongly in $H_0^1(\mathbf{A}^r)$. By Lemma 2.42, we have the following equalities in the strong sense in $H_0^1(\mathbf{A}^r)$

$$\sum_{j=1}^2 w_n^{1,j}(z - z_n^{1,j}) = \sum_{j=1}^2 v_n^{1,j}(z) = v_n^1(z) = u_n^1(z) + o(1) = u_n(z) - \overline{u}(z) + o(1),$$

or

$$u_n(z) = \overline{u}(z) + \sum_{j=1}^2 w_n^{1,j}(z - z_n^{1,j}) + o(1) \quad \text{strongly in } H_0^1(\mathbf{A}^r).$$

(1-2) $\|w_n^{1,j}\|_{H^1} \leq c$ for $n = 1, 2, \dots$, and $\|w^{1,j}\|_{H^1} \leq c$, where $w_n^{1,j} \rightharpoonup w^{1,j}$ weakly in $H_0^1(\mathbf{A}^r)$ for $j = 1, 2$. By Lemma 2.11(*iii*),

$$\begin{aligned}
\|w_n^{1,j}\|_{H^1}^2 = \|v_n^{1,j}\|_{H^1}^2 &= \frac{1}{2}\|v_n^1\|_{H^1}^2 = \frac{1}{2}\|u_n^1\|_{H^1}^2 + o(1) \\
&= \frac{1}{2}(\|u_n\|_{H^1}^2 - \|\bar{u}\|_{H^1}^2) + o(1) \\
&\leq \frac{1}{2}c^2 + o(1),
\end{aligned}$$

we have $\|w_n^{1,j}\|_{H_0^1(\mathbf{A}^r)} \leq c$ for $n = 1, 2, \dots$. Then there is a subsequence $\{w_n^{1,j}\}$ and $w^{1,j}$ in $H_0^1(\mathbf{A}^r)$ such that $w_n^{1,j} \rightharpoonup w^{1,j}$ weakly in $H_0^1(\mathbf{A}^r)$. In addition, $w^{1,1}(x, y) = w^{1,2}(x, -y)$. By Lemma 2.11 (*i*), we have

$$\|w^{1,j}\|_{H^1} \leq \liminf_{n\to\infty} \|w_n^{1,j}\|_{H^1} \leq c \text{ for } j = 1, 2.$$

(1-3) $\{w_n^{1,j}\}$ is a $(\mathrm{PS})_{\frac{1}{2}(\beta - J(\bar{u}))}$-sequence in $H_0^1(\mathbf{A}^r)$ for J: note that $J'(v_n^{1,j}) = o(1)$ in $H^{-1}(\Omega^j)$. Because Ω^j is a half infinite strip, $(1-7)$ below and Theorem 2.35, we have for every $\varphi \in H_0^1(\mathbf{A}^r)$,

$$\langle J'(w_n^{1,j}), \varphi \rangle = \int_{\mathbf{A}^r} \nabla w_n^{1,j} \nabla \varphi + w_n^{1,j} \varphi - \int_{\mathbf{A}^r} |w_n^{1,j}|^{p-2} w_n^{1,j} \varphi = o(1).$$

Therefore, $J'(w_n^{1,j}) = o(1)$ in $H^{-1}(\mathbf{A}^r)$. Moreover, we have

$$J(w_n^{1,j}) = J(v_n^{1,j}(z + z_n^{1,j})) = \frac{1}{2}J(v_n^1) = \frac{1}{2}(\beta - J(\bar{u})) + o(1).$$

(1-4) $-\Delta w^{1,j} + w^{1,j} - |w^{1,j}|^{p-2}w^{1,j} = 0$ in $\mathbf{A}^r$: by Theorem 5.6 (i).

(1-5) $w^{1,j} \not\equiv 0$: by the Rellich-Kondrakov theorem 2.31 and $(1-0)$, we have

$$\int_{A^r_{-1,1}} |w^{1,j}|^2 = \lim_{n\to\infty} \int_{A^r_{-1,1}} |w_n^{1,j}|^2 \geq \frac{d_1}{2},$$

thus $w^{1,j} \not\equiv 0$.

(1-6) By (1-2), (1-4), (1-5), and Lemma 2.18, there is a $\delta > 0$ such that

$$\|w^{1,j}\|_{H_0^1(\mathbf{A}^r)} \geq \|w^{1,j}\|_{L^2(\mathbf{A}^r)} > \delta.$$

Therefore,

$$J(w^{1,j}) = (\frac{1}{2} - \frac{1}{p})a(w^{1,j}) > (\frac{1}{2} - \frac{1}{p})\delta^2 = \delta'.$$

(1-7) $|z_n^{1,j}| \to \infty$: otherwise, there is a $R > 0$ such that $z_n^{1,j} + A^r_{-1,1} \subset A^r_{-R,R}$ for $n = 1, 2, \dots$. Then by (1-0), we have

$$0 = \lim_{n\to\infty} \int_{A^r_{-R,R}} |v_n^{1,j}|^2 \geq \overline{\lim_{n\to\infty}} \int_{z_n^{1,j} + A^r_{-1,1}} |v_n^{1,j}|^2 \geq \frac{d_1}{2},$$

which is a contradiction.

(1-8) $a(u_n) = a(\bar{u}) + 2a(w_n^{1,j}) + o(1)$ for $j = 1, 2$: since $u_n \rightharpoonup \bar{u}$ weakly in $H_0^1(\mathbf{A}^r)$, by Lemma 2.11(iii),

$$\begin{aligned}
a(u_n) - a(\bar{u}) &= a(u_n - \bar{u}) + o(1) \\
&= a(u_n^1) + o(1) \\
&= a(v_n^1) + o(1) \\
&= a(v_n^{1,1}) + a(v_n^{1,2}) + o(1) \\
&= a(w_n^{1,1}) + a(w_n^{1,2}) + o(1) \\
&= 2a(w_n^{1,j}) + o(1) \quad \text{for } j = 1, 2,
\end{aligned}$$

thus, $a(u_n) = a(\bar{u}) + 2a(w_n^{1,j}) + o(1)$ for $j = 1, 2$.

(1-9) $b(u_n) = b(\bar{u}) + 2b(w_n^{1,j}) + o(1)$ for $j = 1, 2$: since $u_n \to \bar{u}$ a.e. in Ω and $\{u_n\}$ is bounded in $L^p(\Omega)$, by Lemma 2.14(i), we have

$$\begin{aligned}
b(u_n) - b(\bar{u}) &= b(u_n - \bar{u}) + o(1) \\
&= b(u_n^1) + o(1) \\
&= b(v_n^1) + o(1) \\
&= b(v_n^{1,1}) + b(v_n^{1,2}) + o(1) \\
&= b(w_n^{1,1}) + b(w_n^{1,2}) + o(1) \\
&= 2b(w_n^{1,j}) + o(1) \quad \text{for } j = 1, 2
\end{aligned}$$

thus $b(u_n) = b(\bar{u}) + 2b(w_n^{1,j}) + o(1)$ for $j = 1, 2$.

(1-10) $J(u_n) = J(\bar{u}) + 2J(w_n^{1,j}) + o(1)$ for $j = 1, 2$: by $(1-8)$, $(1-9)$ and $J(w^{1,1}) = J(w^{1,2})$, we have

$$J(u_n) = J(\bar{u}) + J(w_n^{1,1}) + J(w_n^{1,2}) + o(1)$$
$$= J(\bar{u}) + 2J(w_n^{1,j}) + o(1) \text{ for } j = 1, 2.$$

Step 2. Suppose that $w_n^{1,j}(z) \nrightarrow w^{1,j}(z)$ strongly in $H_0^1(\mathbf{A}^r)$. Let

$$v_n^{2,j}(z) = w_n^{1,j}(z) - w^{1,j}(z).$$

We have $v_n^{2,j} \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$ but $v_n^{2,j} \nrightarrow 0$ strongly in $H_0^1(\mathbf{A}^r)$.

(2-0) $\int_{A^r_{-1,1}} |w_n^{2,j}(z)|^2 dz \geq \frac{d_2}{2}$ for some constant $d_2 > 0$, $n = 1, 2, \dots$, and $j = 1, 2$, where $w_n^{2,j}(z) = v_n^{2,j}(z + z_n^{2,j})$ for some $\{z_n^{2,j}\} \subset \mathbf{A}^r$: for $j = 1, 2$, since $\{v_n^{2,j}\}$ is bounded, $J'(v_n^{2,j}) = o(1)$, and $v_n^{2,j} \nrightarrow 0$ strongly in $H_0^1(\mathbf{A}^r)$, by Lemma 2.17, there is a subsequence $\{v_n^{2,j}\}$, a constant $d_2 > 0$ such that

$$Q_n^{r,2} = \sup_{y \in \mathbb{R}} \int_{(0,y) + A^r_{-1,1}} |v_n^{2,j}(z)|^2 dz \geq d_2 \quad \text{for } n = 1, 2, \dots.$$

For $n = 1, 2, \dots$, take $z_n^{2,1} = (0, y_n^2)$ and $z_n^{2,2} = (0, -y_n^2)$ in $\mathbf{A}^r$ such that

$$\int_{z_n^{2,j} + A^r_{-1,1}} |v_n^{2,j}(z)|^2 dz \geq \frac{d_2}{2} \quad \text{for } n = 1, 2, \dots.$$

Let $w_n^{2,j}(z) = v_n^{2,j}(z + z_n^{2,j})$, then

$$\int_{A^r_{-1,1}} |w_n^{2,j}(z)|^2 dz \geq \frac{d_2}{2} \quad \text{for} \quad n = 1, 2, \dots.$$

As in Step 1, we have the following results.

(2-1) $u_n(z) = \bar{u}(z) + \sum_{j=1}^2 w^{1,j}(z - z_n^{1,j}) + \sum_{j=1}^2 w_n^{2,j}(z - z_n^{1,j} - z_n^{2,j}) + o(1)$ strongly in $H_0^1(\mathbf{A}^r)$;

(2-2) $\|w_n^{2,j}\|_{H^1} \leq c$ for $n = 1, 2, \dots$ and $\|w^{2,j}\|_{H^1} \leq c$, where $w_n^{2,j} \rightharpoonup w^{2,j}$ weakly in $H_0^1(\mathbf{A}^r)$ for $j = 1, 2$;

(2-3) $\{w_n^{2,j}\}$ is a (PS)-sequence in $H_0^1(\mathbf{A}^r)$ for J;

(2-4) $-\Delta w^{2,j} + w^{2,j} - |w^{2,j}|^{p-2} w^{2,j} = 0$ in $\mathbf{A}^r$;

(2-5) $w^{2,j} \not\equiv 0$;

(2-6) $\|w^{2,j}\|_{L^2(\mathbf{A}^r)} > \delta$ and $J(w^{2,j}) > \delta'$;

(2-7) $|z_n^{2,j}| \to \infty$;

(2-8) $a(u_n) = a(\bar{u}) + 2a(w^{1,j}) + 2a(w_n^{2,j}) + o(1)$: since

$$v_n^{2,j}(z) = w_n^{1,j}(z) - w^{1,j}(z) \rightharpoonup 0,$$

we have

$$a(w_n^{2,j}) = a(v_n^{2,j}) = a(w_n^{1,j}) - a(w^{1,j}) + o(1).$$

Further, by $(1-8)$, we have

$$a(u_n) - a(\bar{u}) = a(w_n^{1,1}) + a(w_n^{1,2}) + o(1)$$
$$= 2a(w^{1,j}) + 2a(w_n^{2,j}) + o(1).$$

(2-9) $b(u_n) = b(\bar{u}) + 2b(w^{1,j}) + 2b(w_n^{2,j}) + o(1)$;

(2-10) $J(u_n) = J(\bar{u}) + 2J(w^{1,j}) + 2J(w_n^j) + o(1)$.

Continuing this process, we arrive at the m-th step

(m-0) $\int_{A^r_{-1,1}} |w_n^{m,j}(z)|^2 dz \geq \frac{d_m}{2}$ for some constant $d_m > 0$, $n = 1, 2, \dots$ and $j = 1, 2$, where $w_n^{m,j}(z) = v_n^{m,j}(z + z_n^{m,j})$ for some $\{z_n^{m,j}\} \subset \mathbf{A}^r$;

(m-1) $u_n(z) = \bar{u}(z) + \sum_{j=1}^2 \sum_{i=1}^{m-1} w^{i,j}(z - \tilde{z}_n^{i,j}) + \sum_{j=1}^2 w_n^{m,j}(z - \tilde{z}_n^{m,j}) + o(1)$ strongly in $H_0^1(\mathbf{A}^r)$, where $\tilde{z}_n^{i,j} = z_n^{1,j} + \dots + z_n^{i,j}$ for $i = 1, 2, \dots, m$ and $j = 1, 2$;

(m-2) $\|w_n^{m,j}\|_{H^1} \leq c$ for $n = 1, 2, \dots$ and $\|w^{m,j}\|_{H^1} \leq c$, where $w_n^{m,j} \rightharpoonup w^{m,j}$ weakly in $H_0^1(\mathbf{A}^r)$;

(m-3) $\{w_n^{m,j}\}$ is a (PS)-sequence in $H_0^1(\mathbf{A}^r)$ for J;

(m-4) $-\Delta w^{m,j} + w^{m,j} - |w^{m,j}|^{p-2} w^{m,j} = 0$ in $\mathbf{A}^r$;

(m-5) $w^{m,j} \not\equiv 0$;

(m-6) $\|w^{m,j}\|_{L^2(\mathbf{A}^r)} > \delta$ and $J(w^{m,j}) > \delta'$;

(m-7) $|z_n^{i,j}| = |\tilde{z}_n^{i,j} - \tilde{z}_n^{i-1,j}| \to \infty$ and $|\tilde{z}_n^{i,j}| \to \infty$, for each $i = 1, 2, \dots, m$: we show this by induction on i. For $i = 1$, $|\tilde{z}_n^{i,j}| = |z_n^{1,j}| \to \infty$. Assume that $|\tilde{z}_n^{i,j}| \to \infty$, for $i = 1, 2, \dots, k$, for some $k < m$. By Lemma 2.12, we have $w^{i,j}(z - \tilde{z}_n^{i,j}) \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$ for $i = 1, 2, \dots, k$. We claim that $|\tilde{z}_n^{k+1,j}| \to \infty$. Otherwise, $\{\tilde{z}_n^{k+1,j}\}$ is bounded. Since $\|w^{k+1,j}\|_{L^2(\mathbf{A}^r)} > \delta$, $R > 0$ exists such that

$$\tilde{z}_n^{k+1,j} + A^r_{-R,R} \subset A^r_{-2R,2R}$$

and

$$\int_{A^r_{-R,R}} |w^{k+1,j}|^2 \geq (\frac{\delta}{2})^2.$$

We have

$$\begin{aligned}(\frac{\delta}{2})^2 &\leq \int_{A^r_{-R,R}} |w^{k+1,j}|^2 \\ &= \lim_{n\to\infty} \int_{A^r_{-R,R}} |v_n^{1,j}(z + \tilde{z}_n^{k+1,j})|^2 \\ &\leq \lim_{n\to\infty} \int_{A^r_{-2R,2R}} |v_n^{1,j}(z)|^2 = 0,\end{aligned}$$

which is a contradiction. By the induction hypothesis, we have $|\tilde{z}_n^{i,j}| \to \infty$ for $i = 1, 2, \dots, m$.

(m-8) $a(u_n) = a(\bar{u}) + 2\sum_{i=1}^{m-1} a(w^{i,j}) + 2a(w_n^{i,j}) + o(1)$;

(m-9) $b(u_n) = b(\bar{u}) + 2\sum_{i=1}^{m-1} b(w^{i,j}) + 2b(w_n^{i,j}) + o(1)$;

(m-10) $J(u_n) = J(\bar{u}) + 2\sum_{i=1}^{m-1} J(w^{i,j}) + 2J(w_n^{i,j}) + o(1)$.

By the Archimedean principle, $k \in \mathbb{N}$ exists such that $k\delta^2 > \beta$. Take $l = [\frac{k}{2}] + 1$, then after step $(l + 1)$, we obtain

$$a(u_n) = a(\bar{u}) + 2a(w^{1,j}) + 2a(w^{2,j}) + \dots + 2a(w^{l,j}) + 2a(w_n^{l+1,j}) + o(1).$$

Since $a(w_n^{l+1,j}) \geq 0$, $a(\bar{u}) > 0$, and $a(w^{i,j}) > \delta^2$ for $i = 1, 2, \dots, l$, we have $\beta + o(1) \geq 2l\delta^2 > k\delta^2 > \beta$, which is a contradiction. Therefore, there is an $m \in \mathbb{N}$, such that $w_n^{m,j}(z) = w^{m,j}(z) + o(1)$ strongly in $H_0^1(\mathbf{A}^r)$, $w_n^{i,j}(z) = w^{i,j}(z) + o(1)$ weakly in $H_0^1(\mathbf{A}^r)$, and $w_n^{i,j}(z) \neq w^{i,j}(z) + o(1)$ strongly in $H_0^1(\mathbf{A}^r)$ for $i = 1, 2, \dots, m - 1$. Then we have

(sm-0) $\int_{A^r_{-1,1}} |w_n^{m,j}(z)|^2 dz \geq \frac{d_m}{2}$ for some constant $d_m > 0$, $n = 1, 2, \dots,$ and $j = 1, 2$, where $w_n^{m,j}(z) = v_n^{m,j}(z + z_n^{m,j})$ for some $\{z_n^{m,j}\} \subset \mathbf{A}^r$;

(*sm*-1) $u_n(z) = \bar{u}(z) + \sum_{j=1}^{2}\sum_{i=1}^{m} w^{i,j}(z - \tilde{z}_n^{i,j}) + o(1)$ strongly in $H_0^1(\mathbf{A}^r)$, where $\tilde{z}_n^{i,j} = z_n^{1,j} + \cdots + z_n^{i,j}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2$;
(*sm*-2) $\|w_n^{m,j}\|_{H^1} \le c$ for $n = 1, 2, \ldots$ and $\|w^{m,j}\|_{H^1} \le c$, where $w_n^{m,j} \rightharpoonup w^{m,j}$ weakly in $H_0^1(\mathbf{A}^r)$;
(*sm*-3) $\{w_n^{m,j}\}$ is a (PS)-sequence in $H_0^1(\mathbf{A}^r)$ for J;
(*sm*-4) $-\Delta w^{m,j} + w^{m,j} - |w^{m,j}|^{p-2} w^{m,j} = 0$ in $\mathbf{A}^r$;
(*sm*-5) $w^{m,j} \not\equiv 0$;
(*sm*-6) $\|w^{m,j}\|_{L^2(\mathbf{A}^r)} > \delta$ and $J(w^{m,j}) > \delta'$;
(*sm*-7) $|z_n^{i,j}| = |\tilde{z}_n^{i,j} - \tilde{z}_n^{i-1,j}| \to \infty$ and $|\tilde{z}_n^{i,j}| \to \infty$, for each $i = 1, 2, \ldots, m$;
(*sm*-8) $a(u_n) = a(\bar{u}) + 2\sum_{i=1}^{m} a(w^{i,j}) + o(1)$;
(*sm*-9) $b(u_n) = b(\bar{u}) + 2\sum_{i=1}^{m} b(w^{i,j}) + o(1)$;
(*sm*-10) $J(u_n) = J(\bar{u}) + 2\sum_{i=1}^{m} J(w^{i,j}) + o(1)$.

Finally, suppose $u_n \ge 0$ for $n = 1, 2, \ldots$. Then
(*i*) Since $u_n \rightharpoonup \bar{u}$ weakly in $H_0^1(\Omega)$. By Lemma 2.11(*ii*), there is a subsequence $\{u_n\}$ such that $u_n \to \bar{u}$ a.e. in Ω. Thus, $\bar{u} \ge 0$.
(*ii*) For $j = 1, 2$, let $z \in \Omega^j - z_n^{1,j}$, then

$$w_n^{1,j}(z) = v_n^{1,j}(z + z_n^{1,j}) = v_n^1(z + z_n^{1,j}) = \xi_n(z + z_n^{1,j}) u_n^1(z + z_n^{1,j}).$$

Thus,

$$w_n^{1,j}(z) = \begin{cases} \xi_n(z + z_n^{1,j}) u_n^1(z + z_n^{1,j}) & \text{if } z \in \Omega^j - z_n^{1,j} \\ 0 & \text{otherwise.} \end{cases}$$

Let $h_n(z) = \xi_n(z + z_n^{1,j})\bar{u}(z + z_n^{1,j})$ and $g_n(z) = \xi_n(z + z_n^{1,j}) u_n(z + z_n^{1,j}) \ge 0$. Since $w_n^{1,j}(z) \rightharpoonup w^{1,j}(z)$ weakly in $H_0^1(\mathbf{A}^r)$ and $h_n(z) \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$, we have $w_n^{1,j}(z) \to w^{1,j}(z)$ a.e. in $\mathbf{A}^r$ and $h_n(z) \to 0$ a.e. in $\mathbf{A}^r$. By Lemma 7.1, we have $w^{1,j} \ge 0$.
(*iii*) In fact, we have $w_n^{2,j}(z) = v_n^{2,j}(z + z_n^{2,j}) = w_n^{1,j}(z + z_n^{2,j}) - w^{1,j}(z + z_n^{2,j})$. By Lemma 2.12, $w^{1,j}(z + z_n^{2,j}) \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$. Moreover, $w_n^{2,j}(z) \rightharpoonup w^{2,j}(z)$ weakly in $H_0^1(\mathbf{A}^r)$, hence $w_n^{1,j}(z + z_n^{2,j}) \rightharpoonup w^{2,j}(z)$ weakly in $H_0^1(\mathbf{A}^r)$, since

$$w_n^{1,j}(z + z_n^{2,j}) = \begin{cases} \xi_n(z + \tilde{z}_n^{2,j}) u_n^1(z + \tilde{z}_n^{2,j}) & \text{if } z \in \Omega^j - z_n^{1,j}, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to (*ii*), $w^{2,j} \ge 0$.
(*iv*) Continuing this process, we obtain $w^{i,j} \ge 0$ for each $i = 1, 2, \ldots, m$. □

Similarly, we have

Theorem 7.3 (Symmetric Palais-Smale Decomposition Theorem in $\mathbb{R}^N$)**.** *Let Ω be a y-symmetric large domain in $\mathbb{R}^N$ separated by a y-symmetric bounded domain, and let $\{u_n\} \subset H_s^1(\Omega)$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J. Then there are a subsequence $\{u_n\}$, a positive integer m, sequences $\{\tilde{z}_n^{i,j}\}_{n=1}^{\infty}$ in $\mathbb{R}^N$, a function $\bar{u} \in H_s^1(\Omega)$, and $0 \ne w^{i,j} \in H^1(\mathbb{R}^N)$ for $1 \le i \le m$, $j = 1, 2$ such that*

$$w^{i,1}(x, y) = w^{i,2}(x, -y),$$

$$|\tilde{z}_n^{i,j}| \to \infty \qquad \text{for } i = 1, 2, \ldots, m,$$

$$-\Delta \bar{u} + \bar{u} = |\, \bar{u}\, |^{p-2}\, \bar{u} \quad in\ \Omega,$$
$$-\Delta w^{i,j} + w^{i,j} = |\, w^{i,j}|^{p-2} w^{i,j} \quad in\ \mathbb{R}^N,$$

and

$$u_n = \bar{u} + \sum_{i=1}^{m} w^{i,1}(\cdot - \widetilde{z}_n^{i,1}) + \sum_{i=1}^{m} w^{i,2}(\cdot - \widetilde{z}_n^{i,2}) + o(1) \quad strongly\ in\ H^1(\mathbb{R}^N),$$
$$a(u_n) = a(\bar{u}) + 2\sum_{i=1}^{m} a(w^{i,j}) + o(1) \quad for\ some\ j = 1, 2,$$
$$b(u_n) = b(\bar{u}) + 2\sum_{i=1}^{m} b(w^{i,j}) + o(1) \quad for\ some\ j = 1, 2,$$
$$J(u_n) = J(\bar{u}) + 2\sum_{i=1}^{m} J(w^{i,j}) + o(1) \quad for\ some\ j = 1, 2.$$

In addition, if $u_n \geq 0$, then $\bar{u} \geq 0$ and $w^{i,j} \geq 0$ for each $1 \leq i \leq m$, $j = 1, 2$.

Corollary 7.4. *If $\Omega = \mathbf{A}^r \backslash \omega$, where ω is an axially symmetric bounded set in $\mathbf{A}^r$ and $\{u_n\}$ is a $(PS)_\beta$-sequence in $H_s^1(\Omega)$ for J and $0 < \beta < 2\alpha(\mathbf{A}^r)$, then the sequence $\{u_n\}$ contains a strongly convergent subsequence and there is a positive solution $\overline{u}$ of (1.1) in Ω.*

Proof. By Theorem 7.2, we have
$$J(u_n) = J(\bar{u}) + 2\sum_{i=1}^{m} J(w^{i,j}) + o(1) \quad \text{for some } j = 1, 2.$$
Note that $J(w^{i,j}) \geq \alpha(\mathbf{A}^r)$ and $J(\bar{u}) \geq 0$. If $m \geq 1$, then we have
$$2\alpha(\mathbf{A}^r) > \beta + o(1) = J(u_n) \geq J(\bar{u}) + 2m\alpha(\mathbf{A}^r),$$
which is a contradiction. Thus, $m = 0$. By Theorem 7.2, we have
$$u_n = \bar{u} + o(1) \text{ strongly in } H_0^1(\mathbf{A}^r).$$
Since $J(\bar{u}) = \beta$ and $\beta > 0$, we have $\bar{u} \neq 0$. □

Corollary 7.5. *If Ω is a y-symmetric strictly large domain in $\mathbb{R}^N$ separated by a y-symmetric bounded domain, and $\{u_n\}$ is a positive $(PS)_\beta$-sequence in $H_s^1(\Omega)$ for J, with $0 < \beta < 3\alpha(\mathbb{R}^N)$ but $\beta \neq 2\alpha(\mathbb{R}^N)$, then $\{u_n\}$ contains a strongly convergent subsequence, and there is a positive solution $\bar{u}$ of Equation (1.1) in Ω. In particular, if $\{u_n\}$ is a $(PS)_\beta$-sequence in $H_s^1(\Omega)$ for J with $0 < \beta < 2\alpha(\mathbb{R}^N)$, then $\{u_n\}$ contains a strongly convergent subsequence.*

Proof. By Theorem 7.3, we have
$$J(u_n) = J(\bar{u}) + 2\sum_{i=1}^{m} J(w^{i,j}) + o(1), \quad \text{for some } j = 1, 2.$$
By the uniqueness of the positive solution for Equation (1.1) in $\mathbb{R}^N$, we have $J(w^{i,j}) = \alpha(\mathbb{R}^N)$ and $J(\bar{u}) \geq 0$. If $m \geq 1$, then
$$3\alpha(\mathbb{R}^N) > \beta + o(1) = J(u_n) = J(\bar{u}) + 2m\alpha(\mathbb{R}^N),$$
thus, $m = 0, 1$. Suppose that $m = 1$, then
$$2\alpha(\mathbb{R}^N) \neq \beta + o(1) = J(u_n) = J(\bar{u}) + 2\alpha(\mathbb{R}^N),$$

which implies that $J(\bar{u}) \neq 0$ or $J(\bar{u}) > 0$, or $J(\bar{u}) \geq \alpha(\mathbb{R}^N)$. Therefore,

$$3\alpha(\mathbb{R}^N) > \beta + o(1) = J(u_n) \geq J(\bar{u}) + 2m\alpha(\mathbb{R}^N) \geq 3m\alpha(\mathbb{R}^N),$$

which is a contradiction. Thus, $m = 0$. By Theorem 7.3 again, we have

$$u_n = \bar{u} + o(1) \quad \text{strongly in } H_0^1(\mathbb{R}^N).$$

Since $J(\bar{u}) = \beta$ and $\beta > 0$, we have $\bar{u} \neq 0$. □

Bibliographical notes: The results of this section are from Wang-Wu [77].

8. Fundamental Properties, Regularity, and Asymptotic Behavior of Solutions

In this section we study the fundamental properties, regularity, and asymptotic behavior of solutions of (1.1).

8.1. Fundamental Properties of Solutions.

Theorem 8.1. *Let $u \in H_0^1(\Omega)$ be a positive symmetric and radially decreasing solution of (1.1). Then $u(0) \geq 1$.*

Proof. Since u is symmetric and radially decreasing, we have $-\Delta u(0) \geq 0$. Thus,

$$u(0) \leq -\Delta u(0) + u(0) = u^{p-1}(0),$$

or $u(0) \geq 1$. □

A ground state solution in $X(\Omega)$ is of constant sign.

Lemma 8.2. *Let u in $X(\Omega)$ be a solution of (1.1) that changes sign , and let $\alpha_X(\Omega)$ be the index of J in Ω. Then $J(u) > 2\alpha_X(\Omega)$.*

Proof. Let $u^- = \max\{-u, 0\}$. Then u^- is nonzero. Multiply (1.1) by u^- and integrate to obtain

$$\int_\Omega \nabla u \nabla u^- + \int_\Omega u u^- = \int_\Omega |u|^{p-2} u u^-.$$

Consequently,

$$\int_\Omega |\nabla u^-|^2 + \int_\Omega |u^-|^2 = \int_\Omega |u^-|^p.$$

Thus, $u^- \in \mathbf{M}(\Omega)$ and hence $J(u^-) \geq \alpha_X(\Omega)$. Suppose that $J(u^-) = \alpha_X(X)$. By Theorem 4.5, u^- is a nonzero solution of (1.1). By the maximum principle, $u = u^-$, which contradicts the sign assumption on u. Thus $J(u^-) > \alpha_X(\Omega)$. Similarly, $J(u^+) > \alpha_X(\Omega)$, where $u^+ = \max\{u, 0\}$. Thus,

$$J(u) = J(u^+) + J(u^-) > 2\alpha_X(\Omega).$$

□

The positive solution of Equation (1.1) in $\mathbb{R}^N$ is unique.

Theorem 8.3. (*i*) *There is a ground state solution of Equation (1.1) in $\mathbb{R}^N$;*
(*ii*) *The only positive solutions of Equation (1.1) in $\mathbb{R}^N$ are ground state solutions;*
(*iii*) *Every positive ground state solution $\bar{u} \in H^1(\mathbb{R}^N)$ of Equation (1.1) is spherically symmetric about some point x_0 in $\mathbb{R}^N$, $\bar{u}'(r) < 0$ for $r = |x - x_0|$, and*

$$\lim_{r \to \infty} r^{\frac{N-1}{2}} e^r \bar{u}(r) = \gamma > 0,$$

$$\lim_{r\to\infty} r^{\frac{N-1}{2}} e^r \bar{u}'(r) = -\gamma;$$

(iv) The positive solution of Equation (1.1) *in* $\mathbb{R}^N$ *is unique.*

Proof. (*i*) By Lien-Tzeng-Wang [47]. For (*ii*) and (*iii*) see Gidas-Ni-Nirenberg [35]. For (*iv*) See Kwong [46]. □

8.2. **Regularity of Solutions.** In addition to the study of (1.1), we also study Equation (1.2), a perturbation of (1.1): associated with (1.2). We consider the energy functionals J_h for $u \in H_0^1(\Omega)$:

$$J_h(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u) - \int_\Omega hu.$$

We let $J_0 = J$.

We first recall some fundamental estimates for elliptic equations. Let us first consider the classical $C^\beta-$setting: Schauder estimates.

Theorem 8.4. *Let* Ω *be a bounded* $C^{2,\beta}-$*domain,* $h \in C^\beta(\overline{\Omega})$*. Then the Dirichlet problem (1.2) has a unique classical solution* $u \in C^{2,\beta}(\overline{\Omega})$*.*

For the proof of the above theorem see Gilbarg-Trudinger [36, Theorem 6.14]. We have the following Kato regularity.

Theorem 8.5. *Let* Ω *be a domain in* $\mathbb{R}^N$ *and let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function such that for almost every* $z \in \Omega$

$$|f(z,u)| \le a(z)(1+|u|)$$

with a nonnegative function $a \in L^{N/2}_{\rm loc}(\Omega)$*. In addition, let* $u \in H^1_{\rm loc}(\Omega)$ *be a weak solution of Equation (1.4). Then* $u \in L^q_{\rm loc}(\Omega)$ *for any* $1 \le q < \infty$*. If* $a \in L^{N/2}(\Omega) \cap L^2(\Omega)$ *and* $u \in H_0^1(\Omega)$*, then* $u \in L^q(\Omega)$ *for any* $2 \le q < \infty$*.*

Proof. Recall that if $u \in H^1_{\rm loc}(\Omega)$ is a weak solution of Equation (1.4) in Ω, then u satisfies

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f(z,u)\varphi$$

for each $\varphi \in C_c^\infty(\Omega)$. Choose $\eta \in C_c^\infty(\Omega)$ and for $s \ge 0$, $L \ge 1$, let

$$\varphi = \varphi_{s,L} = u \min\{|u|^{2s}, L^2\}\eta^2 \in H_0^1(\Omega),$$

with supp $\eta = F \subset\subset \Omega$. Note that

$$\nabla\varphi = \nabla u \min\{|u|^{2s}, L^2\}\eta^2 + 2s\chi_{\{|u|^s\le L\}}|u|^{2s-2}u^2\nabla u\eta^2 + 2u\min\{|u|^{2s}, L^2\}\eta\nabla\eta.$$

Testing Equation (1.4) with φ, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2 + \frac{s}{2}\int_{\{|u(x)|^s\le L\}} |\nabla(|u|^2)|^2|u|^{2s-2}\eta^2\\
&\int_\Omega 2u\nabla u \min\{|u|^{2s}, L^2\}\eta\nabla\eta\\
&= \int_\Omega f(\cdot,u)u\min\{|u|^{2s}, L^2\}\eta^2.
\end{aligned}$$

Note that

$$2xy = 2\sqrt{1/2}x\sqrt{2}y \le \frac{1}{2}x^2 + 2y^2,$$

if $|u| \le 1$ we have

$$(1+|u|)|u|\min\{|u|^{2s},L^2\} \le 2,$$

and if $|u| \ge 1$, we have

$$(1+|u|)|u|\min\{|u|^{2s},L^2\} \le 2|u|^2\min\{|u|^{2s},L^2\}.$$

Thus,

$$|a|(1+|u|)|u|\min\{|u|^{2s},L^2\}\eta^2 \le 2|a||u|^2\min\{|u|^{2s},L^2\}\eta^2 + 2|a|\eta^2.$$

Hence,

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\min\{|u|^{2s},L^2\}\eta^2 + \frac{s}{2}\int_{\{|u(x)|^s\le L\}}|\nabla(|u|^2)|^2|u|^{2s-2}\eta^2\\
&\le -2\int_\Omega u\nabla u\min\{|u|^{2s},L^2\}\eta\nabla\eta + \int_\Omega |a|(1+|u|)|u|\min\{|u|^{2s},L^2\}\eta^2\\
&\le \frac{1}{2}\int_\Omega |\nabla u|^2\min\{|u|^{2s},L^2\}\eta^2 + 2\int_\Omega |u|^2\min\{|u|^{2s},L^2\}|\nabla\eta|^2\\
&\quad + 2\int_\Omega a|u|^2\min\{|u|^{2s},L^2\}\eta^2 + 2\int_\Omega a\eta^2.
\end{aligned} \tag{8.1}$$

Suppose that $u \in L^{2s+2}_{\rm loc}(\Omega)$ for some $s \ge 0$. Let

$$c_s = \int_F |u|^{2s+2} < \infty; \quad d = \max\{1, \max|\nabla\eta|^2\};$$
$$\epsilon(M) = \Big(\int_{\{a\ge M\}}|a|^{N/2}\Big)^{2/N} = o(1).$$

We have

$$\int_\Omega |u|^2\min\{|u|^{2s},L^2\}|\nabla\eta|^2 \le d\int_F |u|^{2s+2} = dc_s,$$

and

$$\begin{aligned}
&\int_\Omega a|u|^2\min\{|u|^{2s},L^2\}\eta^2\\
&= \int_{\{a\ge M\}\cap F} a|u|^2\min\{|u|^{2s},L^2\}\eta^2 + \int_{\{a<M\}\cap F} a|u|^2\min\{|u|^{2s},L^2\}\eta^2\\
&\le \Big(\int_{\{a\ge M\}}|a|^{N/2}\Big)^{2/N}\Big(\int_F ||u|^2\min\{|u|^{2s},L^2\}\eta^2|^{N/(N-2)}\Big)^{(N-2)/N}\\
&\quad + M\int_F |u|^2\min\{|u|^{2s},L^2\}\eta^2\\
&\le \epsilon(M)\Big(\int_F ||u|^2\min\{|u|^{2s},L^2\}\eta^2|^{N/(N-2)}\Big)^{(N-2)/N} + Mcc_s\\
&\le \epsilon(M)\||u\min\{|u|^s,L^2\}\eta|\|^2_{L^{2^*}(F)} + Mcc_s\\
&\le \epsilon(M)\int_F |\nabla(u\min\{|u|^s,L\}\eta)|^2 + cc_s,
\end{aligned}$$

and

$$\int_\Omega a\eta^2 = \int_F |a|\eta^2 \le (\int_F |a|^{N/2})^{2/N}(\int_F |\eta|^{2N/(N-2)})^{(N-2)/N} = c.$$

Since $\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2$ and $\frac{s}{2}\int_{\{|u(x)|^s \le L\}} |\nabla(|u|^2)|^2|u|^{2s-2}\eta^2$ are nonnegative, by (8.1), we have

$$\begin{aligned}&\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2\\ &\le 4\int_\Omega |u|^2 \min\{|u|^{2s}, L^2\}|\nabla\eta|^2 + 4\int_\Omega a|u|^2 \min\{|u|^{2s}, L^2\}\eta^2 + 4\int_\Omega a\eta^2\\ &\le c\epsilon(M)\int_\Omega |\nabla(u \min\{|u|^s, L^2\}\eta)|^2 + cc_s\end{aligned}$$

and

$$\frac{s}{2}\int_{\{|u(x)|^s \le L\}} |\nabla(|u|^2)|^2|u|^{2s-2}\eta^2 \le cc_s + c\epsilon(M)(\int_\Omega |\nabla(u \min\{|u|^s, L\}\eta)|^2). \quad (8.2)$$

Together with (8.2),

$$4\int_\Omega |u \min\{|u|^s, L\}\nabla\eta|^2 \le 4dc_s,$$

and

$$\begin{aligned}|\nabla(u \min\{|u|^s, L\}\eta|^2 \le 3(&|\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2 + \frac{5s^2}{8}\chi_{\{|u|^s \le L\}}|\nabla(|u|^2)|^2|u|^{2s-2}\eta^2\\ &+ 4|u \min\{|u|^s, L\}\nabla\eta|^2,\end{aligned}$$

we have

$$\int_\Omega |\nabla(u \min\{|u|^s, L\}\eta|^2 \le cc_s + c\epsilon(M)\int_\Omega |\nabla(u \min\{|u|^s, L\}\eta)|^2.$$

Choose M so that $c\epsilon(M) < \frac{1}{2}$ to obtain

$$\int_\Omega |\nabla(u \min\{|u|^s, L\}\eta|^2 \le cc_s,$$

or

$$\int_{\{|u|^s \le L\}} |\nabla(|u|^{s+1}\eta)|^2 \le cc_s.$$

Letting $L \to \infty$, we obtain

$$\int_\Omega |\nabla(|u|^{s+1}\eta)|^2 \le cc_s = c. \quad (8.3)$$

Since $|u|^{s+1}\eta \in H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we have $u \in L^{(2s+2)N/N-2}_{\rm loc}(\Omega)$. Letting $s_0 = 0$, $u \in L^2_{\rm loc}(\Omega)$, then by (8.3), we have $u \in L^{2N/N-2}_{\rm loc}(\Omega)$, and by (8.3) again, we have $u \in L^{2N^2/(N-2)^2}_{\rm loc}(\Omega)$. Continuing this way, let $2s_{i+1} + 2 = (2s_i + 2)\frac{N}{N-2}$, for $i \ge 0$ to obtain

$$u \in L^{2s_i+2}_{\rm loc}(\Omega)$$

for each s_i. However, $s_i \to \infty$ as $i \to \infty$. Thus, $u \in L^q_{\rm loc}(\Omega)$ for any $q < \infty$.

If $a \in L^{N/2}(\Omega) \cap L^2(\Omega)$ and $u \in H_0^1(\Omega)$, then take $\eta = 1$. □

We have the following existence and uniqueness theorem for the Dirichlet problem for strong solutions: L^p−theory.

Theorem 8.6. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^N$ and $h \in L^q(\Omega)$ for some $1 < q < \infty$. Then the Dirichlet problem (1.2) has a unique strong solution $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$.*

The proof of the above theorem can be found in Gilbarg-Trudinger [36, Theorem 9.15].

Theorem 8.7. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^N$. Then there is a constant $c > 0$ (independent of u) such that*

$$\|u\|_{W^{2,q}(\Omega)} \leq c\| - \Delta u + u\|_{L^q(\Omega)}$$

for each $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $1 < q < \infty$.

The proof of the above theorem can be found in Gilbarg-Trudinger [36, Lemma 9.17].

Theorem 8.8. *Let $k \geq 0$ be an integer and Ω be a $C^{k+1,1}$ domain in $\mathbb{R}^N$ and $h \in W^{k,q}(\Omega)$ for some q, $1 < q < \infty$. If $u \in W^{2,q}_{\rm loc}(\Omega)$ solves $-\Delta u + u = h(z)$ in Ω, then $u \in W^{k+2,q}(\Omega)$.*

The proof of the above theorem can be found in Gilbarg-Trudinger [36, Theorem 9.19].

Theorem 8.9. *Let $k \geq 1$ be an integer and let Ω be a $C^{k+1,\beta}$ domain in $\mathbb{R}^N$ and $h \in C^{k-1,\beta}(\overline{\mathbf{\Omega}})$ for some q, $1 < q < \infty$. If $u \in W^{2,q}_{\rm loc}(\Omega)$ solves $-\Delta u + u = h(z)$ in Ω, then $u \in C^{k+1,\beta}(\overline{\mathbf{\Omega}})$.*

The proof of the above theorem can be found in Gilbarg-Trudinger [36, Theorem 9.19]. We also have the following existence and uniqueness theorem for the Dirichlet problem for weak solutions: L^2-theory. Its proof can be found in Gilbarg-Trudinger [36, Corollary 8.2].

Theorem 8.10. *Let $u \in H_0^1(\Omega)$ be a weak solution of the equation $-\Delta u + u = 0$ in Ω. Then $u = 0$ in Ω.*

Applying these results, we have the following theorem.

Theorem 8.11. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^N$. Let $u \in H_0^1(\Omega)$ be a weak solution of (1.2) in Ω.*
(i) If $h \in L^{N/2}(\Omega) \cap L^2(\ \Omega)$, then $u \in L^q(\Omega)$ for every $q \in [2, \infty)$. Furthermore,

$$\|u\|_{L^q(\Omega)} \leq p(\|u\|_{H^1(\Omega)}),$$

where $p(\|u\|_{H^1(\Omega)})$ is a polynomial of $\|u\|_{H^1(\Omega)}$ with real powers;
(ii) Let $h \in L^{N/2} \cap L^s(\Omega) \cap L^2(\Omega)$ for some s, $s > N$ and $\theta = 2 - \frac{N}{s} - [2 - \frac{N}{s}]$, then $0 < \theta < 1$, $u \in C^{1,\theta}(\overline{\Omega}) \cap W^{2,s}(\Omega)$, and

$$\|u\|_{L^\infty(\Omega)} \leq \|u\|_{C^{1,\theta}(\overline{\Omega})} \leq c\|u\|_{W^{2,s}(\Omega)};$$

(iii) Let $h \in C^\theta(\overline{\Omega}) \cap L^{N/2}(\Omega) \cap L^s(\Omega) \cap L^2(\Omega)$ for s as defined in (ii), then $u \in C^2(\overline{\Omega})$.

Proof. (i) $N = 1, 2$ follows from the Sobolev inequality. Suppose that $N \geq 3$. For $d \geq 0$, $l \geq 1$, let $\varphi = \varphi_{d,l} = u \min\{|u|^{2d}, l^2\}$, then

$$\nabla\varphi = \min\{|u|^{2d}, l^2\}\nabla u + \chi_{\{|u|^d \leq l\}} 2d|u|^{2d}\nabla u.$$

Since $u \in H_0^1(\Omega)$, we have

$$\int_\Omega |\nabla\varphi|^2 = \int_\Omega | \min\{|u|^{2d}, l^2\}\nabla u + \chi_{\{|u|^d \leq l\}} 2d|u|^{2d}\nabla u|^2$$

$$
\begin{aligned}
&\le 2\Big(\int_\Omega |\min\{|u|^{2d}, l^2\}\nabla u|^2 + |\chi_{\{|u|^d\le l\}} 2d|u|^{2d}\nabla u|^2\Big)\\
&\le 2l^4\int_\Omega |\nabla u|^2 + 8d^2l^4\int_{\{|u|^d\le l\}}|\nabla u|^2\\
&\le c\int_\Omega |\nabla u|^2 < \infty.
\end{aligned}
$$

Clearly, $\varphi \in L^2(\Omega)$. Thus, $\varphi \in H_0^1(\Omega)$. Note that we have $h\varphi \le |h||u|$ for $|u|\le 1$ and $h\varphi \le |h||u|^2\min\{|u|^{2d}, l^2\}$ for $|u|>1$. Suppose $u \in L^{2d+p}$. Multiplying and integrating (1.2) with φ, we have

$$
\begin{aligned}
&\int_\Omega \nabla u\nabla\varphi\\
&= -\int_\Omega u\varphi + \int_\Omega |u|^{p-2}u\varphi + \int_\Omega h\varphi\\
&\le -\int_\Omega |u|^2\min\{|u|^{2d}, l^2\} + \int_\Omega |u|^{2d+p} + \int_\Omega |h||u| + \int_\Omega |h||u|^2\min\{|u|^{2d}, l^2\}\\
&\le \|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + \int_{\{h<M\}}|h||u|^2\min\{|u|^{2d}, l^2\}\\
&+ \int_{\{|h|\ge M\}}|h||u|^2\min\{|u|^{2d}, l^2\}\\
&\le \|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\int_\Omega |u|^{2d+2} + \varepsilon(M)\Big(\int_\Omega \big||u|^2\min\{|u|^{2d}, l^2\}\big|^{\frac{N}{N-2}}\Big)^{\frac{N-2}{N}}\\
&\le \|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^{2d+2}}^{2d+2} + S\varepsilon(M)\|\nabla(u\min\{|u|^d, l\})\|_{L^2}^2,
\end{aligned}
$$

where $\varepsilon(M) = \Big(\int_{\{h\ge M\}} h^{\frac{N}{2}}\Big)^{\frac{2}{N}} = o(1)$ and S is the Sobolev critical constant. Thus we have

$$
\begin{aligned}
&\int_\Omega \nabla u\nabla\varphi\\
&\le \|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^{2d+2}}^{2d+2} + S\varepsilon(M)\|\nabla(u\min\{|u|^d, l\})\|_{L^2}^2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_\Omega \nabla u\nabla\varphi\\
&\le \|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^{2d+2}}^{2d+2} + S\varepsilon(M)\|\nabla(u\min\{|u|^d, l\})\|_{L^2}^2.
\end{aligned}
$$

Since $2 \le 2d+2 < 2d+p$, write $\frac{1}{2d+2} = \frac{\alpha}{2} + \frac{1-\alpha}{2d+p}$. Then

$$
\|u\|_{L^{2d+2}} \le \|u\|_{L^2}^{\alpha}\|u\|_{L^{2d+p}}^{1-\alpha}.
$$

Therefore,

$$
\begin{aligned}
\int_\Omega \nabla u\nabla\varphi \le \|u\|_{L^{2d+p}}^{2d+p} &+ \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^2}^{\alpha(2d+2)}\|u\|_{L^{2d+p}}^{(1-\alpha)(2d+2)}\\
&+ S\varepsilon(M)\|\nabla(u\min\{|u|^d, l\})\|_{L^2}^2.
\end{aligned}
\tag{8.4}
$$

Note that

$$
\int_\Omega \nabla u\nabla\varphi = \int_\Omega |\nabla u|^2\min\{|u|^{2d}, l^2\} + 2d\int_{\{|u|^d\le l\}}|\nabla u|^2|u|^{2d}. \tag{8.5}
$$

Now, we have

$$\begin{aligned}\int_{\{|u|^d\le l\}}\nabla(|u|^{d+1})|^2 &\le \int_{\{|u|^d\le l\}}|\nabla(|u|^{d+1})|^2 + l^2\int_{\{|u|^d\ge l\}}|\nabla u|^2\\ &= \int_\Omega|\nabla(u\min\{|u|^d,l\})|^2\\ &= \int_\Omega|\min\{|u|^d,l\}\nabla u + \chi_{\{|u|^d\le l\}}d|u|^d\nabla u|^2\\ &\le \int_\Omega|\nabla u|^2\min\{|u|^{2d},l^2\} + (2d+d^2)\int_{\{|u|^d\le l\}}|\nabla u|^2|u|^{2d}\\ &\le (1+\frac{d}{2})\int_\Omega\nabla u\nabla\varphi.\end{aligned}$$

In particular, we have

$$\int_\Omega|\nabla(u\min\{|u|^d,l\})|^2 \le (1+\frac{d}{2})\int_\Omega\nabla u\nabla\varphi, \tag{8.6}$$

and

$$\int_{\{|u|^d\le l\}}|\nabla(|u|^{d+1})|^2 \le \int_\Omega\big|\nabla(u\min\{|u|^d,l\})\big|^2. \tag{8.7}$$

Then from (8.4) and (8.6), we have

$$\begin{aligned}&\|\nabla(u\min\{|u|^d,l\})\|_{L^2}^2\\ &\le (1+\frac{d}{2})\Big(\|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^2}^{\alpha(2d+2)}\|u\|_{L^{2d+p}}^{(1-\alpha)(2d+2)}\Big)\\ &\quad + (1+\frac{d}{2})S\varepsilon(M)\|\nabla(u\min\{|u|^d,l\})\|_{L^2}^2.\end{aligned}$$

Let $M>0$ be such that $(1+\frac{d}{2})S\varepsilon(M)<\frac12$. Then we have

$$\begin{aligned}&\|\nabla(u\min\{|u|^d,l\})\|_{L^2}^2\\ &\le (2+d)(\|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^2}^{\alpha(2d+2)}\|u\|_{L^{2d+p}}^{(1-\alpha)(2d+2)}).\end{aligned}$$

By (8.7), we have

$$\begin{aligned}&\int_{\{|u|^d\le l\}}|\nabla(|u|^{d+1})|^2\\ &\le \int_\Omega|\nabla(u\min\{|u|^d,l\})|^2\\ &\le (2+d)(\|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^2}^{\alpha(2d+2)}\|u\|_{L^{2d+p}}^{(1-\alpha)(2d+2)}).\end{aligned}$$

Letting $l\to\infty$, we obtain

$$\begin{aligned}&\|\nabla(|u|^{d+1})\|_{L^2}^2\\ &\le (2+d)\Big(\|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^2}^{\alpha(2d+2)}\|u\|_{L^{2d+p}}^{(1-\alpha)(2d+2)}\Big).\end{aligned}$$

Since

$$\|u\|_{L^{(d+1)2^*}}^{2(d+1)} = \||u|^{d+1}\|_{L^{2^*}}^2 \le S\|\nabla(|u|^{d+1})\|_{L^2}^2,$$

we have

$$\|u\|_{L^{(d+1)2^*}}^{2(d+1)}$$

$$\leq S(2+d)\Big[\|u\|_{L^{2d+p}}^{2d+p} + \|h\|_{L^2}\|u\|_{L^2} + M\|u\|_{L^2}^{\alpha(2d+2)}\|u\|_{L^{2d+p}}^{(1-\alpha)(2d+2)}\Big].$$

Let $d_0 = 0$ and $2d_i + p = (d_{i-1}+1)2^*$ for $i \geq 1$. Since $p < 2^*$, then $d_1 > 0$ and $d_i \geq (\frac{2^*}{2})^{i-1} d_1$. Hence, $\lim_{n\to\infty} d_i = \infty$ and

$$\|u\|_{L^{2d_i+p}} = \|u\|_{L^{(d_{i-1}+1)2^*}} \leq p_{i-1}(\|u\|_{L^{2d_{i-1}+p}}).$$

By iterating, we conclude that

$$\|u\|_{L^{2d_i+p}} \leq p_i(\|u\|_{H^1}).$$

By the interpolation property, for every $q \in [2,\infty)$, we have $u \in L^q$ that satisfies

$$\|u\|_{L^q} \leq p_q(\|u\|_{H^1}).$$

(ii) Let $g(z,u) = |u|^{p-2}u + h(z)$, since $(p-1)s > s > N$, then by *(i)*, $|u|^{p-2}u \in L^s(\Omega) \cap L^2(\Omega)$. Thus, $g(z,u) \in L^s(\Omega) \cap L^2(\Omega)$. By Theorem 8.6, the Dirichlet problem

$$\begin{gathered} -\Delta v + v = g(z,u) \quad \text{in } \Omega, \\ v \in W_0^{1,s}(\Omega) \cap H_0^1(\Omega), \end{gathered}$$

has a unique strong solution $v \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap H_0^1(\Omega)$ and

$$\|v\|_{W^{2,s}(\Omega)} \leq c\|g(z,u)\|_{L^s(\Omega)}.$$

Since $v \in H_0^1(\Omega)$ satisfies Equation (1.3), we have for each $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega (\nabla v \nabla\varphi + v\varphi) = \int_\Omega g(z,u)\varphi.$$

Thus u and v satisfy weakly

$$\begin{gathered} -\Delta v + v = g(z,u) \quad \text{in } \Omega, \\ -\Delta u + u = g(z,u) \quad \text{in } \Omega. \end{gathered}$$

Let $w = v - u$. Then $-\Delta w + w = 0$ in Ω. By Theorem 8.10, $w = 0$, or $u = v \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega) \cap H_0^1(\Omega)$ and

$$\|u\|_{W^{2,s}(\Omega)} \leq c\|g(z,u)\|_{L^s(\Omega)}.$$

Now $s > N$ and $\theta = 2 - \frac{N}{s} - [2 - \frac{N}{s}]$. Then by the Sobolev embedding theorem 2.30, $u \in C^{1,\theta}(\overline{\Omega})$ and

$$\|u\|_{L^\infty(\Omega)} \leq \|u\|_{C^{1,\theta}(\overline{\Omega})} \leq c\|u\|_{W^{2,s}(\Omega)}.$$

(iii) By *(ii)*, we know that $u \in C^{1,\theta}(\overline{\Omega})$. Since u is bounded, then $|u|^{p-2}u \in C^\theta(\overline{\Omega})$. By Theorem 8.9, we have $u \in C^{2,\theta}(\overline{\Omega})$. □

8.3. **Asymptotic Behavior of Solutions.** By Theorem 8.11, we obtain the following three results about asymptotic behavior of solutions. We define the generalized infinite strip $\mathbf{S}^r = B^m(0;r) \times \mathbb{R}^n$, where $N \geq 4$, $m \geq 2$, $n \geq 1$ and $m+n = N$. Let λ_1 be the first eigenvalue of $-\Delta$ in $B^m(0;r)$ with the Dirichlet problem, and ϕ_1 the corresponding positive eigenfunction to λ_1.

Theorem 8.12. *(i) Let $h \in L^{N/2}(\mathbf{S}^r) \cap L^s(\mathbf{S}^r) \cap L^2(\mathbf{S}^r)$ for $s > N$. If u in $H_0^1(\mathbf{S}^r)$ is a weak solution of* (1.2) *in $\mathbf{S}^r$, then $u \in C^1(\overline{\mathbf{S}^r})$ and*

$$\lim_{|y|\to\infty} u(x,y) = 0 \qquad \textit{uniformly in} x \in B^m(0;r);$$

(ii) Let $h \in L^{N/2}(\mathbf{F}_s^r) \cap L^s(\mathbf{F}_s^r) \cap L^2(\mathbf{F}_s^r)$ *for* $s > N$. *If* u *in* $H_0^1(\mathbf{F}_s^r)$ *is a weak solution of* (1.2) *in* $\mathbf{F}_s^r$, *then* $u \in C^1(\overline{\mathbf{F}_s^r})$ *and*

$$\lim_{y\to\infty} u(x,y) = 0 \quad \textit{uniformly in} x \in B^m(0;r);$$

(iii) Let Ω *be an unbounded domain in* $\mathbb{R}^N$, $h \in L^{N/2}(\Omega) \cap L^s(\Omega) \cap L^2(\Omega)$ *for* $s > N$. *If* u *in* $H_0^1(\Omega)$ *is a weak solution of* (1.2) *in* Ω, *then* $u \in C^1(\overline{\Omega})$ *and* $\lim_{|z|\to\infty} u(z) = 0$.

Proof. (i) By Theorem 8.11, $u \in C^1(\overline{\mathbf{S}^r}) \cap W^{2,s}(\mathbf{S}^r)$. For each $t > 0$, apply Theorem 8.11 (ii) to obtain

$$\|u\|_{L^\infty(\mathbf{S}_t^r)} \le c\|u\|_{W^{2,N}(\mathbf{S}_t^r)},$$

where $\mathbf{S}_t^r = \{z = (x,y) \in \mathbf{S}^r : |y| > t\}$. Since $\|u\|_{W^{2,N}(\mathbf{S}_t^r)} = o(1)$ as $t \to \infty$, we obtain

$$\lim_{|y|\to\infty} u(x,y) = 0 \quad \text{uniformly in } x \in B^m(0;r).$$

The proofs of (ii) and (iii) are similar to (i). □

By Lien-Tzeng-Wang [47], there is a positive solution of (1.1) in $\mathbf{S}^r$. Such a solution admits exponential decay in y.

Theorem 8.13. *Let* u *be a positive solution of Equation* (1.1) *in* $\mathbf{S}^r$. *Then for every* $0 < \delta < 1 + \lambda_1$ $\gamma > 0$ *and* $\beta > 0$ *exist such that*

$$\gamma\phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \le u(z) \le \beta\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|y|} \quad \textit{for } z = (x,y) \in \mathbf{S}^r.$$

Proof. By Theorem 8.11 (iii), $u \in C^2(\overline{\Omega})$.
(i) $\gamma\phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \le u(z)$ for $z = (x,y) \in \mathbf{S}^r$: define

$$w_\delta(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \quad \text{for } z = (x,y) \in \overline{B^m(0;r)} \times \mathbb{R}^n = \overline{\mathbf{S}^r}.$$

For $0 < \delta < 1 + \lambda_1$, take $R > 0$ such that $\delta - \frac{\sqrt{1+\lambda_1+\delta}(n-1)}{|y|} \ge 0$ for $|y| \ge R$ (for $n = 1$, take $R = 1$). Set

$$\gamma = \inf_{z\in\mathbf{S}^r,\,|y|\le R} \frac{u(x,y)}{w_\delta(x,y)}.$$

Note that $w_\delta(x,y)$ and $u(x,y)$ are radially symmetric in x and y, and decreasing in $|y|$ for a fixed x. Thus

$$\frac{u(x,y)}{w_\delta(x,y)} = \frac{u(x,y)}{\phi_1(x)}e^{\sqrt{1+\lambda_1+\delta}|y|} \ge \frac{u(x,Re_1)}{\phi_1(x)}, \qquad \text{for } |y| \le R,\ x \in B^m(0;r),$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$. Therefore

$$\gamma = \inf_{z\in\mathbf{S}^r,\,|y|\le R} \frac{u(z)}{w_\delta(z)} \ge \inf_{x\in B^m(0;r)} \frac{u(x,Re_1)}{\phi_1(x)} = \inf_{x\in L} \frac{u(x,Re_1)}{\phi_1(x)},$$

where L is a fixed diameter of $B^m(0;r)$. Note that

$$\frac{u(x,Re_1)}{\phi_1(x)} > 0 \quad \text{for } x \in L.$$

Furthermore, for each $x_0 \in \partial L \subset \partial B^m(0;r)$, take a small ball B^1 in $B^m(0;r)$ such that $x_0 \in \partial B^1$. Note that $\phi_1(x) > 0$ for $x \in B^1$, $\phi_1(x_0) = 0$, and $\phi_1(x) \in C^2(\overline{B^m(0;r)})$. Then by Lemma 9.1 below, $\frac{\partial\phi_1}{\partial\nu}(x_0) < 0$, where ν is the outward unit normal vector of B^1 at x_0. Let $u_1(x) = u(x,Re_1)$, and for each $z_1 = (x_0,Re_1) \in \partial L \times \mathbb{R}^n \subset \partial\mathbf{S}^r$, take a small ball B^2 in $\mathbf{S}^r$ such that $z_1 \in \partial B^2$. Note that $u(z) > 0$

for $z \in B^2$, and $u(z_1) = 0$. By Theorem 8.12, $u(x) \in C^2(\overline{\mathbf{S}^r})$, then by Lemma 9.1 below, $\frac{\partial u}{\partial \bar\nu}(z_1) < 0$, where $\bar\nu = (\nu, 0)$ is the outward unit normal vector of B^2 at z_1. Thus, $\frac{\partial u_1}{\partial \nu}(x_0) = \nabla u_1(x_0) \cdot \nu = \nabla u(z_1) \cdot \bar\nu = \frac{\partial u}{\partial \bar\nu}(z_1) < 0$. By L'Hôpital's rule, we have

$$\lim_{\substack{x \in B^m(0;r) \\ x \to x_0 \quad \text{normally}}} \frac{u(x, Re_1)}{\phi_1(x)} = \lim_{h \to 0^-} \frac{u_1(x_0 + h\nu)}{\phi_1(x_0 + h\nu)} = \frac{\frac{\partial u_1}{\partial \nu}(x_0)}{\frac{\partial \phi_1}{\partial \nu}(x_0)} > 0.$$

Define

$$\frac{u(x, Re_1)}{\phi_1(x_0)} = \lim_{\substack{x \in B^m(0;r) \\ x \to x_0 \quad \text{normally}}} \frac{u(x, Re_1)}{\phi_1(x)}.$$

Thus $\frac{u(x,Re_1)}{\phi_1(x)} > 0$ for $x \in \overline{L}$. Since $\frac{u(x,Re_1)}{\phi_1(x)} : \overline{L} \to \mathbb{R}$ is continuous, we have

$$\gamma \geq \inf_{x \in \bar L} \frac{u(x, Re_1)}{\phi_1(x)} > 0.$$

Let $v(z) = \gamma w_\delta(z)$ for $z \in \overline{\mathbf{S}^r}$, then we have

$$v(z) \leq u(z) \text{ for } z \in \overline{\mathbf{S}^r},\ |y| \leq R.$$

For $z \in \mathbf{S}^r$, $|y| > R$, we have

$$\begin{aligned} -\Delta(u - v)(z) + (u - v)(z) &= (-\Delta u(z) + u(z)) + (\Delta v(z) - v(z)) \\ &= u^{p-1} + v(z)\Big(\delta - \frac{\sqrt{1 + \lambda_1 + \delta}(n - 1)}{|y|}\Big) \geq 0. \end{aligned}$$

(For $n = 1$, we only consider the domain $\{z \in \mathbf{S}^r | \ y > R\}$.) Since $u(z) - v(z) \geq 0$ on $\partial A \cup \partial \mathbf{S}^r$ where $A = \{z \in \mathbf{S}^r : |y| > R\}$, by Lemma 9.3 below, we have $v(z) \leq u(z)$ for $z = (x, y) \in \mathbf{S}^r$, $|y| > R$. Thus, we conclude that $v(z) \leq u(z)$ for $z \in \mathbf{S}^r$, or $\gamma\phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \leq u(z)$.

(ii) $u(z) \leq \beta\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|y|}$, for $z = (x, y) \in \mathbf{S}^r$: for $0 < \delta < 1 + \lambda_1$. By Theorem 8.12, $\lim_{|y|\to\infty} u(x, y) = 0$ uniformly in x. Take $R' > 0$ such that $u^{p-2} \leq \frac{\delta}{2+[\lambda_1]}$ for $|y| \geq R'$, $x \in B^m(0; r)$. Define

$$w^\delta(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|y|} \quad \text{for } z = (x, y) \in \overline{B^m(0; r)} \times \mathbb{R}^n = \overline{\mathbf{S}^r};$$

$$\beta = \sup_{z \in \mathbf{S}^r,\, |y| \leq R'} \frac{u(z)}{w^\delta(z)} > 0;$$

$$\mu(z) = \beta w^\delta(z) \quad \text{for } z \in \overline{\mathbf{S}^r}.$$

Fix a diameter L of $B^m(0; r)$. We have

$$\frac{u(z)}{w^\delta(z)} = \frac{u(x, y)}{\phi_1(x)} e^{\sqrt{1+\lambda_1-\delta}\,|y|} \leq \frac{u(x, 0)}{\phi_1(x)} e^{\sqrt{1+\lambda_1-\delta}\,R'},$$

for $|y| \leq R'$, $x \in B^m(0; r)$, and

$$\beta = \sup_{z \in \mathbf{S}^r,\, |y| \leq R'} \frac{u(z)}{w^\delta(z)} \leq \sup_{x \in B^m(0;r)} \frac{u(x, 0)}{\phi_1(x)} e^{\sqrt{1+\lambda_1-\delta}\,R'} = \sup_{x \in L} \frac{u(x, 0)}{\phi_1(x)} e^{\sqrt{1+\lambda_1-\delta}\,R'}.$$

Similarly to part *(i)*, for $x_0 \in \partial L \subset \partial B^m(0; r)$, we have

$$\lim_{\substack{x \in B^m(0;r) \\ x \to x_0 \quad \text{normally}}} \frac{u(x, 0)}{\phi_1(x)} = \lim_{h \to 0^-} \frac{u_0(x_0 + h\nu)}{\phi_1(x_0 + h\nu)} = \frac{\frac{\partial u_0}{\partial \nu}(x_0)}{\frac{\partial \phi_1}{\partial \nu}(x_0)} < \infty,$$

where ν is the outward unit normal vector of $B^m(0;r)$ at x_0 and $u_0(x) = u(x,0)$. Define

$$\frac{u(x_0,0)}{\phi_1(x_0)} = \lim_{\substack{x\in B^m(0;r)\\ x\to x_0 \quad \text{normally}}} \frac{u(x,0)}{\phi_1(x)} < \infty.$$

Thus,

$$\beta \le \sup_{x\in L} \frac{u(x,0)}{\phi_1(x)} e^{\sqrt{1+\lambda_1-\delta}\,R'} \le \sup_{x\in \bar{L}} \frac{u(x,0)}{\phi_1(x)} e^{\sqrt{1+\lambda_1-\delta}\,R'} < \infty.$$

Therefore, $\mu(z) \ge u(z)$ for $z \in \mathbf{S}^r$, $|y| \le R'$. For $z \in \mathbf{S}^r$, $|y| > R'$ we have

$$\begin{aligned} -\Delta(u-\mu)(z) + (u-\mu)(z) &= (-\Delta u(z) + u(z)) + (\Delta\mu(z) - \mu(z)) \\ &= u^{p-1}(z) + \big(-\delta - \frac{\sqrt{1+\lambda_1-\delta}(n-1)}{|y|}\big)\mu(z) \\ &\le \frac{\delta}{2+[\lambda_1]}(u-\mu)(z), \end{aligned}$$

therefore,

$$-\Delta(u-\mu)(z) + (1 - \frac{\delta}{2+[\lambda_1]})(u-\mu)(z) \le 0.$$

Since $1 - \frac{\delta}{2+[\lambda_1]} > 0$, $u(z) - \mu(z) \le 0$ on ∂B, where $B = \{z \in \mathbf{S}^r : |y| > R'\}$, by Lemma 9.3 below, $u(z) - \mu(z) \le 0$ in B. Thus, we conclude that $u(z) \le \mu(z)$ for $z \in \mathbf{S}^r$. $\square$

We similarly present the asymptotic behavior of each solution of (1.1) in the interior flask domains $\mathbf{F}^r_s$, where $s > r$.

Theorem 8.14. *Let u be a positive solution of* (1.1) *in $\mathbf{F}^r_s$. Then for any $0 < \delta < 1 + \lambda_1$, $\gamma > 0$, $\beta > 0$, and $R > s$ exist such that for $z = (x,y) \in \mathbf{A}^r_R$,*

$$\gamma\phi_1(x)e^{-\sqrt{1+\lambda_1}\,y} \le u(z) \le \beta\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,y}.$$

Proof. (*i*) $\gamma\phi_1(x)e^{-\sqrt{1+\lambda_1}\,y} \le u(z)$ for $z = (x,y) \in \mathbf{A}^r_R$: by Theorem 8.12 (*ii*), $\lim_{y\to\infty} u(x,y) = 0$ uniformly in x, where $(x,y) \in \mathbf{F}^r_s$. For $0 < \delta < 1 + \lambda_1$, take $R > s$ such that $u^{p-2}(x,y) \le \frac{\delta}{2+[\lambda_1]}$ for $y \ge R$. In the remaining proofs, we fix such R. Define

$$w(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1}\,y} \quad \text{for } z = (x,y) \in \overline{\mathbf{A}^r_s}.$$

Set

$$\gamma = \inf_{z\in\mathbf{A}^r_s,\, y=R} \frac{u(x,y)}{w(x,y)}.$$

Similarly to Theorem 8.13, $\gamma > 0$. Let $v(z) = \gamma w(z)$ for $z \in \overline{\mathbf{A}^r_s}$, then we have $v(z) \le u(z)$ for $z \in \overline{\mathbf{A}^r_s}$, $y = R$. For $z \in \mathbf{A}^r_R$, we have

$$-\Delta(u-v)(z) + (u-v)(z) = (-\Delta u(z) + u(z)) + (\Delta v(z) - v(z)) = u^{p-1} \ge 0.$$

Since $u - v \ge 0$ on $\partial\mathbf{A}^r_R$, by the strong maximum principle, we have $v(z) \le u(z)$ for $z = (x,y)$, $z \in \mathbf{A}^r_R$.
(*ii*) $u(z) \le \beta\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,y}$, for $z = (x,y) \in \mathbf{A}^r_R$: define

$$w_\delta(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,y} \quad \text{for } z = (x,y) \in \overline{\mathbf{A}^r_s};$$

$$\beta = \sup_{z\in\mathbf{A}^r_s,\, y=R} \frac{u(z)}{w_\delta(z)} > 0;$$

$$\mu(z) = \beta w_\delta(z) \quad \text{for } z \in \overline{\mathbf{A}_s^r}.$$

Similarly to Theorem 8.13, $\beta < \infty$. Therefore, $u(z) \le \mu(z)$ for $z \in \overline{\mathbf{A}_s^r}$, $y = R$. For $z \in \mathbf{A}_s^r$, $y \ge R$, we have

$$\begin{aligned} -\Delta(u-\mu)(z) + (u-\mu)(z) &= (-\Delta u(z) + u(z)) + (\Delta\mu(z) - \mu(z)) \\ &= u^{p-1}(z) - \delta\mu(z) \\ &\le \frac{\delta}{2+[\lambda_1]}(u-\mu)(z); \end{aligned}$$

therefore,

$$-\Delta(u-\mu)(z) + (1 - \frac{\delta}{2+[\lambda_1]})(u-\mu)(z) \le 0.$$

Since $1 - \frac{\delta}{2+[\lambda_1]} > 0$ and $u - \mu \le 0$ on $\partial\mathbf{A}_R^r$, by the strong maximum principle, we obtain $u(z) \le \mu(z)$ for $z \in \mathbf{A}_R^r$. □

Bibliographical notes: Theorem 8.2 is from Benci-Cerami [11]. Theorem 8.3 is from Lien-Tzeng-Wang [47] and Gidas-Ni-Nirenberg [35].The asymptotic behavior results are from Wang [71] and Chen-Chen-Wang [23].

9. Symmetry of Solutions

We use the asymptotic behavior of solutions developed in Section 8 and apply the "moving plane" method to prove the symmetry of solutions to (1.1) in the infinite strip $\mathbf{A}^r$. Our approach is similar to those in Gidas-Ni-Nirenberg [34, Theorem 1] and [35, Theorem 2] but is more complicated. Before proving our main results, we first establish a version of the Hopf boundary point lemma and the strong maximum principle that will be used in our case.

Lemma 9.1 (Hopf Boundary Point Lemma). *Let Ω be a domain (possibly unbounded) in $\mathbb{R}^n$. Let L be a differential operator given by*

$$Lu = \sum_{i,j=1}^n a^{ij}(x)D_{ij}u + \sum_{i=1}^n b^i(x)D_i u + c(x)u, \;\; a^{ij}(x) = a^{ji}(x),$$

which is uniformly elliptic, $\frac{|b_i(x)|}{\lambda(x)}$ and $\frac{c(x)}{\lambda(x)}$ are uniformly bounded, where $\lambda(x)$ is the minimum eigenvalue of $[a^{ij}(x)]$. Assume $Lu \le 0$. Let $x_0 \in \partial\Omega$ satisfy:
(i) u is continuous at x_0;
(ii) $u(x_0) < u(x)$ for all $x \in \Omega$;
(iii) A ball $B \subset \Omega$ exists with $x_0 \in \partial B$.
Suppose that one of the following conditions holds:
(i) $u(x_0) < 0$ and $c(x) \le 0$;
(ii) $u(x_0) = 0$;
(iii) $c(x) \equiv 0$.
If the outer normal derivative $\frac{\partial u}{\partial \nu}(x_0)$ of u at x_0 exists, then $\frac{\partial u}{\partial \nu}(x_0) < 0$.

For the proof of the above lemma, see Gilbarg-Trudinger [36, Lemma 3.4].

Lemma 9.2 (Hopf Boundary Point Lemma). *Let Ω be a domain (possibly unbounded) in $\mathbb{R}^n$. Let L be a differential operator given by*

$$Lu = \sum_{i,j=1}^n a^{ij}(x)D_{ij}u + \sum_{i=1}^n b^i(x)D_i u + c(x)u, \;\; a^{ij}(x) = a^{ji}(x),$$

which is uniformly elliptic, $\frac{|b_i(x)|}{\lambda(x)}$ *and* $\frac{c(x)}{\lambda(x)}$ *are uniformly bounded, where* $\lambda(x)$ *is the minimum eigenvalue of* $[a^{ij}(x)]$*. Assume* $Lu \geq 0$*. Let* $x_0 \in \partial\Omega$ *satisfy:*
(i) u *is continuous at* x_0*;*
(ii) $u(x_0) > u(x)$ *for all* $x \in \Omega$*;*
(iii) A ball $B \subset \Omega$ *exists with* $x_0 \in \partial B$*.*
Suppose that one of the following conditions holds:
(i) $u(x_0) > 0$ *and* $c(x) \leq 0$*;*
(ii) $u(x_0) = 0$*;*
(iii) $c(x) \equiv 0$*.*
If the outer normal derivative $\frac{\partial u}{\partial \nu}(x_0)$ *of* u *at* x_0 *exists, then* $\frac{\partial u}{\partial \nu}(x_0) > 0$*.*

For the proof of the above lemma, see Gilbarg-Trudinger [36, Lemma 3.4].

Lemma 9.3 (Strong Maximum Principle)**.** *Let* L *be uniformly elliptic,* $c = 0$ *and* $Lu \geq 0$ (≤ 0) *in a domain* Ω *(not necessarily bounded). Then if* u *achieves its maximum (minimum) in the interior of* Ω*,* u *is constant. If* $c \leq 0$ *and* c/λ *is bounded, then* u *cannot achieve a nonnegative maximum (nonpositive minimum) in the interior of* Ω *unless it is constant.*

For the proof of the above lemma, see Gilbarg-Trudinger [36, Theorem 3.5].

We define the generalized infinite strip by $\mathbf{S}^r = B^m(0;r) \times \mathbb{R}^n$, where $m \geq 2$, $n \geq 1$, and $m + n = N$, and suppose that
$(g1)$ $g(u) > 0$ as $u > 0$;
$(g2)$ $g(u) = O(u^p)$ as $u \to 0$ for some $p > 1$.
Now we consider the equation

$$\begin{gathered} -\Delta u + u = g(u) + h(z) \quad \text{in } \mathbf{S}^r, \\ u > 0 \quad \text{in } \mathbf{S}^r, \\ u = 0 \quad \text{on } \partial\mathbf{S}^r, \\ \lim |y| \to \infty u(x,y) = 0 \quad \text{uniformly in } x \in B^m(0;r). \end{gathered} \tag{9.1}$$

We apply the "moving plane" method to prove the symmetry of solutions of (9.1).

Theorem 9.4. *Assume that* $g \in C^1$ *satisfies* $(g1)$ *and* h *is radially symmetric in* x *and* y *and strictly decreasing in* $|x|$ *and* $|y|$ *. Let* $u(x,y)$ *be a* C^2 *solution of Equation (9.1). Then* u *is radially symmetric in* x *and in* y*; that is to say,* $u(x,y) = u(|x|,|y|)$*.*

Part I: u is radially symmetric in y.
Notation:

$$\begin{aligned} S_\theta &= \{(x, y_1, y_2, \dots, y_n) \in \mathbf{S}^r \mid x \in B^m(0;r),\ y_1 = \theta\}; \\ \Gamma_\theta &= \{(x, y_1, y_2, \dots, y_n) \in \mathbf{S}^r \mid x \in B^m(0;r),\ y_1 < \theta\}. \end{aligned}$$

For any $(x,y) \in \mathbf{S}^r$, set $(x, y^\theta) = (x, 2\theta - y_1, y_2, \dots, y_n)$; that is, (x, y^θ) is the reflection of (x,y) with respect to S_θ;

Let Θ be the collection of all $\theta \in \mathbb{R}$ such that the following statements hold:

$$\begin{gathered} u(x,y) < u(x,y^\theta) \quad \text{for all } (x,y) \in \Gamma_\theta, \\ u_{y_1}(x,y) > 0 \quad \text{on } S_\theta. \end{gathered}$$

Lemma 9.5. *There exists* $\theta_0 > 0$ *such that either* $(-\infty, -\theta_0] \subset \Theta$ *or* $u(x,y) \equiv u(x, y^{-\theta_0})$ *in* $\Gamma_{-\theta_0}$*.*

Proof. Given $\theta < 0$, set $w^\theta(x,y) = u(x,y) - u(x,y^\theta)$ for $(x,y) \in \Gamma_\theta$, and $w^\theta(x,y)$ satisfies

$$\Delta w^\theta(x,y) + c_\theta(x,y)w^\theta(x,y) = h(x,y^\theta) - h(x,y) \geq 0, \tag{9.2}$$

where $c_\theta(x,y) = \frac{g(u(x,y))-g(u(x,y^\theta))}{u(x,y)-u(x,y^\theta)} - 1 = g'(\xi_\theta) - 1$ and ξ_θ is in between $u(x,y)$ and $u(x,y^\theta)$.

We claim that $\theta_0 > 0$ exists such that if $\theta \leq -\theta_0$, then $w^\theta(x,y) \leq 0$ in Γ_θ. Otherwise, suppose $w^\theta(x,y) > 0$ for some $(x,y) \in \Gamma_\theta$. Since $\lim |y| \to \infty w^\theta(x,y) = 0$ uniformly in x, $w^\theta(x,y)$ achieves its maximum at $(x_\theta, y_\theta) \in \Gamma_\theta$. Then

$$\nabla w^\theta(x_\theta, y_\theta) = 0, quad\{w^\theta_{ij}(x_\theta, y_\theta)\} \leq 0.$$

Note that by $(g2)$, $\lim t \to 0^+ g'(t) = 0$. Take $t_0 > 0$ such that if $0 < t \leq t_0$, then $g'(t) < 1$. Since $\lim y_1 \to -\infty u(x,y) = 0$, we can choose $\theta_0 > 0$ such that if $y_1 \leq -\theta_0$, then $u(x,y) \leq t_0$, therefore, $g'(u(x,y)) < 1$. For $\theta \leq -\theta_0$, $(x_\theta, y_\theta) \in \Gamma_\theta$, then

$$\Delta w^\theta(x_\theta, y_\theta) \leq 0, \quad c_\theta(x,y)w^\theta(x_\theta, y_\theta) = (g'(\xi_\theta) - 1)w^\theta(x_\theta, y_\theta) < 0,$$

contradicting (9.2). As a consequence of the maximum principle and the Hopf boundary point lemma, either $w^{-\theta_0}(x,y) \equiv 0$ in $\Gamma_{-\theta_0}$ or $w^\theta(x,y) < 0$ in Γ_θ and $w^\theta_{y_1}(x,y) > 0$ for $(x,y) \in S_\theta$ for $\theta \leq -\theta_0$, or $u_{y_1}(x,y) > 0$ for $(x,y) \in S_\theta$. □

Lemma 9.6. *If $(-\infty, \theta] \subset \Theta$, then there exists $\varepsilon > 0$ such that $[\theta, \theta + \varepsilon) \subset \Theta$.*

Proof. Suppose instead that a decreasing sequence $\theta_k \to \theta$ and a sequence $\{(x^k, y^k)\}$ of points in Γ_{θ_k} exist such that $w^{\theta_k}(x^k, y^k) = u(x^k, y^k) - u(x^k, y^{\theta_k}) > 0$, where (x^k, y^{θ_k}) is the reflection of (x^k, y^k) with respect to S_{θ_k}. There is a subsequence $\{(x^k, y^k)\}$ such that $x^k \to \overline{x}$ as $k \to \infty$. Two possibilities may arise, as shown in Case 1 and 2.

Case 1. $|y^k| \to \infty$. As shown in Lemma 9.5, we assume

$$w^{\theta_k}(\widetilde{x^k}, \widetilde{y^k}) = \max{(x,y) \in \overline{\Gamma}_{\theta_k}} w^{\theta_k}(x,y),$$

$$\nabla w^{\theta_k}(\widetilde{x^k}, \widetilde{y^k}) = 0, \ \{w^{\theta_k}_{ij}(\widetilde{x^k}, \widetilde{y^k})\} \leq 0. \tag{9.3}$$

From $\lim |y| \to \infty u(x,y) = 0$, as in Lemma 9.5, we obtain a contradiction.

Case 2. $y^k \to \overline{y}$. We have $(x^k, y^k) \to (\overline{x}, \overline{y}) \in \overline{\Gamma_\theta}$, thus $w^\theta(\overline{x}, \overline{y}) \geq 0$. Clearly $(\overline{x}, \overline{y}) \notin \Gamma_\theta$. If $(\overline{x}, \overline{y}) \in S_\theta$, then $u_{y_1}(\overline{x}, \overline{y}) < 0$, which contradicts $\theta \in \Theta$. Moreover, $(\overline{x}, \overline{y}) \notin \partial \mathbf{S}^r \cap \overline{\Gamma_\theta}$. Note that $w^\theta(x,y)$ satisfies Equation (9.2), and by the Hopf boundary point lemma, we obtain $\frac{\partial}{\partial \nu} w^\theta(\overline{x}, \overline{y}) > 0$. On the other hand, by taking limits in (9.3), we obtain $\nabla w^\theta(\overline{x}, \overline{y}) = 0$, which is a contradiction. We conclude that either Case 1 or Case 2 is impossible. □

Proof of Part I. Let $\sigma = \sup\{\theta \in \mathbb{R} : (-\infty, \theta) \subset \Theta\}$. Then $\sigma \notin \Theta$. If not, by Lemma 9.6 we would have $[\sigma, \sigma + \epsilon) \subset \Theta$, which contradicts the definition of σ. We claim that $\sigma = 0$. Suppose instead that $\sigma \in (-\infty, 0)$. By continuity, $u(x,y) \leq u(x,y^\sigma)$ for all $(x,y) \in \Gamma_\sigma$, then by the maximum principle, we have $u(x,y) \equiv u(x,y^\sigma)$ for all $(x,y) \in \Gamma_\sigma$. This implies that $h(x,y) = h(x,y^\sigma)$ for all $(x,y) \in \Gamma_\sigma$, which is a contradiction. This proves $u(x,y)$ is symmetric with respect to the hyperplane $y_1 = 0$ for all $(x,y) \in \mathbf{S}^r$. By reversing the y_1 axis, we conclude that $u(x,y)$ is symmetric with respect to the hyperplane S_0. Since the y_1 direction can be chosen arbitrarily, we conclude that $u(x,y)$ is radially symmetric in $\mathbb{R}^n$. □

Part II: u is radially symmetric in $B^m(0;r)$.
Notation:

$$T_\lambda = \{(x,y) = (x_1, x_2, \dots, x_{N-1}, y) \in \mathbf{S}^r : x_1 = \lambda\};$$
$$\Sigma_\lambda = \{(x,y) = (x_1, x_2, \dots, x_{N-1}, y) \in \mathbf{S}^r : x_1 < \lambda\}.$$

For any $(x,y) = (x_1, x_2, \dots, x_{N-1}, y) \in \mathbf{S}^r$, set $(x^\lambda, y) = (2\lambda - x_1, x_2, \dots, x_{N-1}, y)$, that is, (x^λ, y) is the reflection of (x,y) with respect to T_λ.

Let Λ be the collection of all $\lambda \in (-r, 0)$ such that the following statements hold:

$$u(x,y) < u(x^\lambda, y) \quad \text{for all } (x,y) \in \Sigma_\lambda,$$
$$u_{x_1}(x,y) > 0 \quad \text{on } T_\lambda.$$

Lemma 9.7. *For some δ such that $0 < \delta < r$, $(-r, -r+\delta) \subset \Lambda$.*

Proof. Given $\lambda \in (-r, 0)$, set $v^\lambda(x,y) = u(x,y) - u(x^\lambda, y)$ for $(x,y) \in \Sigma_\lambda$, then $v^\lambda(x,y) = 0$ for $(x,y) \in T_\lambda$, and $v^\lambda(x,y)$ satisfies

$$\Delta v^\lambda(x,y) + c_\lambda(x,y) v^\lambda(x,y) = h(x^\lambda, y) - h(x,y) \geq 0, \tag{9.4}$$

where $c_\lambda(x,y) = \frac{g(u(x,y)) - g(u(x^\lambda,y))}{u(x,y) - u(x^\lambda,y)} - 1 = g'(\zeta_\lambda) - 1$ where ζ_λ is in between $u(x,y)$ and $u(x^\lambda, y)$.

We claim that if $-r < \lambda < -r + \delta$, then $v^\lambda(x,y) \leq 0$ in Σ_λ. Otherwise, suppose λ exists such that $-r < \lambda < -r+\delta$, $v^\lambda(x,y) > 0$ for some $(x,y) \in \Sigma_\lambda$. Since $\lim |y| \to \infty v^\lambda(x,y) = 0$ uniformly in x, $v^\lambda(x,y)$ achieves its maximum at $(x_\lambda, y_\lambda) \in \Sigma_\lambda$. Then

$$\nabla v^\lambda(x_\lambda, y_\lambda) = 0, \quad \{v^\lambda_{ij}(x_\lambda, y_\lambda)\} \leq 0.$$

Note that by $(g2)$, $\lim t \to 0^+ g'(t) = 0$. Take $t_0 > 0$ such that if $0 < t \leq t_0$, then $g'(t) < 1$. Since $\lim |x| \to r^- u(x,y) = 0$, we can choose δ, $0 < \delta < r$ such that $u(x,y) \leq t_0$ whenever $r - \delta < |x| < r$. For $-r < \lambda < -r + \delta$, we have

$$\Delta v^\lambda(x_\lambda, y_\lambda) \leq 0, \quad c_\lambda(x,y) v^\lambda(x_\lambda, y_\lambda) = (g'(\zeta_\lambda) - 1) v^\lambda(x_\lambda, y_\lambda) < 0,$$

which contradicts (9.4). Therefore, for $-r < \lambda < -r + \delta$, $v^\lambda(x,y) \leq 0$ in Σ_λ. By applying the maximum principle and the Hopf boundary point lemma, for $-r < \lambda < -r + \delta$, we obtain $v^\lambda(x,y) < 0$ in Σ_λ and $v^\lambda_{x_1}(x,y) > 0$ for $(x,y) \in T_\lambda$. Hence, $u_{x_1}(x,y) > 0$ for $(x,y) \in T_\lambda$. Then $(-r, -r+\delta) \subset \Lambda$. ☐

Lemma 9.8. *If $(-r, \lambda] \subset \Lambda$, then there is a $\tau > 0$ such that $[\lambda, \lambda + \tau) \subset \Lambda$.*

Proof. Suppose that a decreasing sequence $\lambda_k \to \lambda$ and a sequence $\{(x^k, y^k)\}$ of points in Σ_{λ_k} exist such that $v^{\lambda_k}(x^k, y^k) = u(x^k, y^k) - u(x^{\lambda_k}, y^k) > 0$, where (x^{λ_k}, y^k) is the reflection of (x^k, y^k) with respect to T_λ. There is a subsequence $\{(x^k, y^k)\}$ such that $x^k \to \overline{x} \in \overline{B^m(0;r)}$. Two possibilities may arise as shown in Case 1 and 2:
Case 1. $|y^k| \to \infty$. As shown in Lemma 9.7, we assume

$$v^{\lambda_k}(\widetilde{x^k}, \widetilde{y^k}) = \max (x,y) \in \overline{\Sigma_{\lambda_k}} v^{\lambda_k}(x,y),$$
$$\nabla v^{\lambda_k}(\widetilde{x^k}, \widetilde{y^k}) = 0, \quad \{v^{\lambda_k}_{ij}(\widetilde{x^k}, \widetilde{y^k})\} \leq 0.$$

From $\lim |y| \to \infty \; u(x,y) = 0$, as in Lemma 9.7, we obtain a contradiction.
Case 2. $y^k \to \overline{y}$. We have $(x^k, y^k) \to (\overline{x}, \overline{y}) \in \overline{\Sigma_\lambda}$. Thus, $v^\lambda(\overline{x}, \overline{y}) \geq 0$. Clearly $(\overline{x}, \overline{y}) \notin \Sigma_\lambda$. If $(\overline{x}, \overline{y}) \in T_\lambda$ then $u_{x_1}(\overline{x}, \overline{y}) < 0$, which contradicts $\lambda \in \Lambda$. Moreover,

$(\overline{x}, \overline{y}) \notin \partial\mathbf{S}^r \cap \overline{\Sigma_\lambda}$, since if $(\overline{x}, \overline{y}) \in \partial\mathbf{S}^r \cap \overline{\Sigma_\lambda}$ then $0 = u(\overline{x}, \overline{y}) \geq u(\overline{x}^\lambda, \overline{y}) > 0$, a contraction. We conclude that either Case 1 or Case 2 is impossible. □

Proof of Part II. Let $\mu = \sup\{\lambda \in (-r, 0) : (-r, \lambda) \subset \Lambda\}$. Then $\mu \notin \Lambda$. If not, by Lemma 9.8, we would have $[\mu, \mu + \epsilon) \subset \Lambda$, which contradicts the definition of μ. We claim that $\mu = 0$. Suppose instead that $\mu \in (-r, 0)$. By continuity we have $u(x, y) \leq u(x^\mu, y)$ for all $(x, y) \in \Sigma_\mu$. Then by the maximum principle we have $u(x, y) \equiv u(x^\mu, y)$ for all $(x, y) \in \Sigma_\mu$, which is impossible. Thus $\mu = 0$. By reversing the x_1 axis, we conclude that $u(x, y)$ is symmetric with respect to the hyperplane T_0 and $u_{x_1}(x, y) < 0$ for $x_1 > 0$. Since the x_1 direction can be chosen arbitrarily, we conclude that $u(x, y)$ is radially symmetric in $B^m(0; r)$. □

Corollary 9.9. *Assume that $g \in C^1$ satisfies property $(g1)$ and $h(x) \equiv 0$. Let $u(x, y)$ be a C^2 solution of Equation* (9.1). *Then u is radially symmetric in x and in y; that is to say, $u(x, y) = u(|x|, |y|)$.*

Proof of Part I. u is radially symmetric in y. Similarly to Theorem 9.4, let $\sigma_1 = \sup\{\theta \in \mathbb{R} \mid (-\infty, \theta) \subset \Theta\}$. Note that σ_1 is not necessarily zero. Similarly, $u(x, y)$ is symmetric with respect to the hyperplane $y_1 = \sigma_1$ for all $(x, y) \in \mathbf{S}^r$. The same conclusion holds for the other coordinate direction, and we conclude that u is symmetric about each of n planes $y_j = \sigma_j$ and $\nabla u = 0$ only at their intersection. We may now take their intersection as the origin.

The same argument may be applied to any unit direction γ and we infer that u is symmetric about some plane

$$B^m(0; r) \times \{y \in \mathbb{R} : y \cdot \gamma = c(\gamma) = \text{const.}\}$$

At the point on this plane where u achieves its maximum we have $\nabla u = 0$ (since the derivative normal to the plane is zero at every point of the plane). It follows that $c(\gamma) = 0$. Thus u is symmetric about every plane through the origin, that is, u is radially symmetric in y. In addition, we also conclude that $u_\rho < 0$ where $\rho = |y|$.
Proof of Part II: Since u is radially symmetric in $B^m(0; r)$: the proof is similarly to the proof of Theorem 9.4. □

9.1. **Open Question:** Are positive solutions of (1.1) in the generalized infinite strip by $\mathbf{S}^r$ unique up to a translation?
Bibliographical notes: The results of this section are from Chen-Chen-Wang [23].

10. Nonachieved Domains and Esteban-Lions Domains

In this section we also characterize Esteban-Lions domains. We prove that proper large domains, Esteban-Lions domains, and some interior flask domains are nonachieved.

Theorem 10.1. *Let Ω_2 be one of $\mathbf{A}^r$, $\mathbf{A}^{r_1,r_2}$, and $\mathbb{R}^N$, and Ω_1 a proper large domain of Ω_2. Then $\alpha_1 = \alpha_2$, J does not satisfy the (PS)$_{\alpha_1}$-condition, and Equation* (1.1) *does not admit any ground state solution in Ω_1. In particular, a proper large domain Ω_1 of Ω_2 is nonachieved.*

Proof. Since Ω_1 is a proper large domain of Ω_2, by Theorem 4.18, $\alpha_1 = \alpha_2$. Then by Theorem 5.7 (i) and (ii), J does not satisfy the (PS)$_{\alpha_1}$-condition, and Equation (1.1) does not admit any ground state solution in Ω_1. □

The only solution in an Esteban-Lions domain is trivial. The following lemmas are required. The first lemma is from Protter [62].

Lemma 10.2. *Let u be a C^2 real-valued function in Ω. Suppose that δ_0, $\rho_0 > 0$ and $z_0 \in \Omega$ exist such that $0 < \delta_0 < \rho_0 < 1$ and*

$$u(z) = 0 \quad \text{in } \{z \in \mathbb{R}^N : \delta_0 \le |z - z_0| \le \rho_0\}^c.$$

Then $m_0 > 0$ and $c > 0$ exist such that if $m \ge m_0$, then

$$\int_\Omega \rho^{-2m-2} e^{2\rho^{-m}} u^2 dz \le \frac{c}{m^4} \int_\Omega \rho^{m+2} e^{2\rho^{-m}} (\Delta u)^2 dz,$$

where $\rho = |z - z_0|$.

Proof. Without loss of generality, we assume that $z_0 = 0$, so that $\rho = |z|$. Let $v = e^{\rho^{-m}} u$. By the inequality $(A + B + C)^2 \ge 2B(A + C)$, we have

$$\begin{aligned}
(\Delta u)^2 &= \big[e^{-\rho^{-m}} \Delta v + 2\nabla v \nabla (e^{-\rho^{-m}}) + v\Delta (e^{-\rho^{-m}})\big]^2 \\
&= [e^{-\rho^{-m}} \Delta v + 2me^{-\rho^{-m}} \rho^{-m-2} \sum_{i=1}^N z_i \frac{\partial v}{\partial z_i} \\
&\quad + mv\rho^{-m-2} e^{-\rho^{-m}} (m\rho^{-m} - m - 2 + N)]^2 \\
&\ge 4me^{-2\rho^{-m}} \rho^{-m-2} \sum_{i=1}^N z_i \frac{\partial v}{\partial z_i} \big[\Delta v + mv\rho^{-m-2}(m\rho^{-m} - m - 2 + N)\big]
\end{aligned}$$

and

$$\begin{aligned}
&\rho^{m+2} e^{2\rho^{-m}} (\Delta u)^2 \\
&\ge 4m \sum_{i=1}^N z_i \frac{\partial v}{\partial z_i} \big[\Delta v + m^2 v\rho^{-2m-2} - (m + 2 - N) mv\rho^{-m-2}\big] .
\end{aligned}$$

Thus,

$$\int_{\{\delta_0 \le |z| \le \rho_0\}} \rho^{m+2} e^{2\rho^{-m}} (\Delta u)^2 dz \ge (I) + (II) - (III),$$

where

$$\begin{aligned}
(I) &= 4m \sum i = 1^N \int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \Delta v \, dz\,, \\
(II) &= 4m^3 \sum i = 1^N \int_{\{\delta_0 \le |z| \le \rho_0\}} \rho^{-2m-2} v z_i \frac{\partial v}{\partial z_i} \, dz\,, \\
(III) &= 4m^2 \sum i = 1^N \int_{\{\delta_0 \le |z| \le \rho_0\}} \rho^{-m-2} (m + 2 - N) v z_i \frac{\partial v}{\partial z_i} \, dz\,.
\end{aligned}$$

We claim that (i)

$$(I) = 2m(N - 2) \int_{\{\delta_0 \le |z| \le \rho_0\}} |\nabla v|^2 dz > 0;$$

(ii) There exists $m_1 > 0$ such that

$$(II) \ge m^4 \int_{\{\delta_0 \le |z| \le \rho_0\}} \rho^{-2m-2} v^2 dz \quad \text{for } m \ge m_1;$$

and (iii) for $0 < \varepsilon < 1$, there exists $m_2 > 0$ such that

$$(III) \le \varepsilon m^4 \Big(\int_{\{\delta_0 \le |z| \le \rho_0\}} \rho^{-2m-2} v^2 dz \Big) \quad \text{for } m \ge m_2.$$

(i) Since $\frac{\partial v}{\partial z_i} = 0$ on $\partial\{\delta_0 \le |z| \le \rho_0\}$ for all $i = 1, 2, \dots, N$ and

$$\sum_{i=1}^N \int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \triangle v dz = \sum_{i=1}^N \sum_{j=1}^N \int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \frac{\partial^2 v}{\partial z_j^2} dz.$$

For $i \ne j$, by integration by parts

$$\begin{aligned} \int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \frac{\partial^2 v}{\partial z_j^2} dz &= -\int_{\{\delta_0 \le |z| \le \rho_0\}} \frac{\partial v}{\partial z_j} \frac{\partial}{\partial z_j} (z_i \frac{\partial v}{\partial z_i}) dz \\ &= -\int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_j} \frac{\partial^2 v}{\partial z_i \partial z_j} dz \\ &= \int_{\{\delta_0 \le |z| \le \rho_0\}} \big[(\frac{\partial v}{\partial z_j})^2 + z_i \frac{\partial v}{\partial z_j} \frac{\partial^2 v}{\partial z_i \partial z_j} \big] dz, \end{aligned}$$

we have

$$\int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_j} \frac{\partial^2 v}{\partial z_i \partial z_j} dz = -\frac{1}{2} \int_{\{\delta_0 \le |z| \le \rho_0\}} (\frac{\partial v}{\partial z_j})^2 dz.$$

Thus,

$$\begin{aligned} \sum_{\substack{j=1 \\ j \ne i}}^N \int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \frac{\partial^2 v}{\partial z_j^2} dz &= \frac{1}{2} \sum_{\substack{j=1 \\ j \ne i}}^N \int_{\{\delta_0 \le |z| \le \rho_0\}} (\frac{\partial v}{\partial z_j})^2 dz \\ &= \frac{1}{2} \int_{\{\delta_0 \le |z| \le \rho_0\}} |\nabla v|^2 - (\frac{\partial v}{\partial z_i})^2 dz. \end{aligned}$$

For $i = j$,

$$\int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \frac{\partial^2 v}{\partial z_j^2} dz = -\int_{\{\delta_0 \le |z| \le \rho_0\}} \big[z_i \frac{\partial v}{\partial z_i} \frac{\partial^2 v}{\partial z_i^2} + (\frac{\partial v}{\partial z_i})^2 \big] \, dz,$$

then we have

$$\int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \frac{\partial^2 v}{\partial z_j^2} dz = -\frac{1}{2} \int_{\{\delta_0 \le |z| \le \rho_0\}} (\frac{\partial v}{\partial z_i})^2 dz.$$

Hence,

$$\begin{aligned} &\sum_{i=1}^N \int_{\{\delta_0 \le |z| \le \rho_0\}} z_i \frac{\partial v}{\partial z_i} \triangle v \, dz \\ &= \sum_{i=1}^N \Big(\frac{1}{2} \int_{\{\delta_0 \le |z| \le \rho_0\}} |\nabla v|^2 dz - \int_{\{\delta_0 \le |z| \le \rho_0\}} (\frac{\partial v}{\partial z_i})^2 dz \Big) \\ &= (\frac{N}{2} - 1) \int |\nabla v|^2 > 0. \end{aligned}$$

(ii) $m_1 > 0$ exists such that $4m^4 + 4m^3 - 2m^3 N \ge m^4$ for $m \ge m_1$. Since $v = 0$ on $\partial\{\delta_0 \le |z| \le \rho_0\}$, we have

$$\int_{\{\delta_0 \le |z| \le \rho_0\}} \rho^{-2m-2} v z_i \frac{\partial v}{\partial z_i} dz$$

$$
\begin{aligned}
&= -\int_{\{\delta_0\le|z|\le\rho_0\}}\frac{\partial}{\partial z_i}(\rho^{-2m-2}vz_i)v\,dz\\
&= (2m+2)\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-4}v^2z_i^2dz\\
&\quad -\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}vz_i\frac{\partial v}{\partial z_i}dz-\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}v^2dz
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}vz_i\frac{\partial v}{\partial z_i}dz\\
&= (m+1)\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-4}v^2z_i^2dz-\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}v^2dz.
\end{aligned}
$$

Thus, for $m\ge m_1$ we have

$$
\begin{aligned}
(II) &= 4m^3\sum_{i=1}^N\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}vz_i\frac{\partial v}{\partial z_i}dz\\
&= 4m^3(m+1)\sum_{i=1}^N\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-4}v^2z_i^2dz-2m^3\sum_{i=1}^N\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}v^2dz\\
&= (4m^4+4m^3-2m^3N)\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}v^2dz\\
&\ge m^4\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-2m-2}v^2dz\quad\text{for } m\ge m_1.
\end{aligned}
$$

(iii) Since $\rho_0<1$, for $0<\varepsilon<1$ there is an $m_2>0$ such that

$$
\begin{gathered}
2m^2(m+2-N)^2\le 2m^4\quad\text{for } m\ge m_2,\\
\int\rho^{-m-2}v^2dz\le\frac{\varepsilon}{2}\int\rho^{-2m-2}v^2dz\quad\text{for } m\ge m_2.
\end{gathered}
$$

Similarly to part (ii), we have

$$
\begin{aligned}
&\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}vz_i\frac{\partial v}{\partial z_i}dz\\
&= (m+2)\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-4}v^2z_i^2dz\\
&\quad -\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}vz_i\frac{\partial v}{\partial z_i}dz-\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}v^2dz
\end{aligned}
$$

and

$$
\sum_{i=1}^N\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}vz_i\frac{\partial v}{\partial z_i}dz=\frac12(m+2-N)\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}v^2dz.
$$

Thus, for $m\ge m_2$ we have

$$
\begin{aligned}
(III) &= 4m^2(m+2-N)\sum_{i=1}^N\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}vz_i\frac{\partial v}{\partial z_i}dz\\
&= 2m^2(m+2-N)^2\int_{\{\delta_0\le|z|\le\rho_0\}}\rho^{-m-2}v^2dz
\end{aligned}
$$

$$
\begin{aligned}
&\le m^4 \int_{\{\delta_0\le |z|\le \rho_0\}} \rho^{-m-2} v^2 dz\\
&\le \varepsilon m^4 \int_{\{\delta_0\le |z|\le \rho_0\}} \rho^{-2m-2} v^2 dz.
\end{aligned}
$$

Take $m_0 = \max\{m_1, m_2\}$. For $m \ge m_0$, since $u \in C^2$ and $u = 0$ on the set $\{z \in \mathbb{R}^N : \delta_0 \le |z| \le \rho_0\}^c$, we have

$$
\begin{aligned}
&\int_\Omega \rho^{m+2} e^{2\rho^{-m}} (\Delta u)^2 dz\\
&= \int_{\{\delta_0\le |z|\le \rho_0\}} \rho^{m+2} e^{2\rho^{-m}} (\Delta u)^2 dz\\
&\ge (I) + (II) - (III)\\
&\ge m^4 (\int_{\{\delta_0\le |z|\le \rho_0\}} \rho^{-2m-2} v^2 dz) - \varepsilon m^4 (\int_{\{\delta_0\le |z|\le \rho_0\}} \rho^{-2m-2} v^2 dz)\\
&= (1-\varepsilon) m^4 (\int_{\{\delta_0\le |z|\le \rho_0\}} \rho^{-2m-2} v^2 dz)\\
&= (1-\varepsilon) m^4 (\int_\Omega \rho^{-2m-2} v^2 dz).
\end{aligned}
$$

Hence, for $m \ge m_0$, we have

$$
\int \rho^{-2m-2} e^{2\rho^{-m}} u^2 dz \le \frac{c}{m^4} \int \rho^{m+2} e^{2\rho^{-m}} (\Delta u)^2 dz.
$$

□

Recall the uniqueness of the analytic function: suppose that f is analytic in a domain Ω in $\mathbb{R}^2$. If $f(z_n) = 0$ for some sequence $\{z_n\}$ of distinct points such that $z_n \to z_0 \in \Omega$, then $f \equiv 0$ in Ω. We know that $f = u + iv$, where u and v are harmonic functions, but the uniqueness of harmonic functions is not elegant, taking the form below. Let $\delta(z_0) = dist(z_0, \partial\Omega)$. Then we have the following uniqueness result (see Heinz [39]).

Theorem 10.3. *Let u be a C^2 real-valued function on Ω. Suppose that $u(z) = 0$ is in the neighborhood of a point $z_0 \in \Omega$ and that $M > 0$ exists such that*

$$
(\Delta u)^2 \le M u^2 \quad \text{for any } z \in \Omega.
$$

Then $u(z) = 0$ for any $z \in \Omega$.

Proof. Let $R = \min\{\frac12, \frac14 \delta(z_0)\}$ and $\Phi(t) \in C_c^2([0,\infty))$ satisfy

$$
\Phi(t) = \begin{cases} 1 & \text{for } 0 \le t \le R,\\ 0 & \text{for } \frac32 R \le t < \infty. \end{cases}
$$

Let $\widetilde{u}(z) = u(z)\Phi(|z - z_0|)$. Note that $2R < \delta(z_0)$ and $B^N(z_0; 2R) \subset \Omega$. Thus, $\widetilde{u}(z)$ is well-defined on $B^N(z_0; 2R)$ and $\widetilde{u}(z) \in C_c^2\big(\overline{B^N(z_0; 2R)}\big)$. We also have

$$
\widetilde{u}(z) = \begin{cases} u(z) & \text{in } \{z \in \mathbb{R}^N : |z - z_0| \le R\},\\ 0 & \text{in } \{z \in \mathbb{R}^N : |z - z_0| \ge \frac32 R\}, \end{cases}
$$

with $\frac32 R < 1$. By Lemma 10.2, $m_0 > 0$ and $c_0 > 0$ exist such that

$$
R^{-2m-2} - c_0 M m^{-4} R^{m+2} > 0,
$$

and for all $m \geq m_0$

$$\int \rho^{-2m-2} e^{2\rho^{-m}} \tilde{u}^2 dz \leq \frac{c_0}{m^4} \int \rho^{m+2} e^{2\rho^{-m}} (\Delta \tilde{u})^2 dz.$$

We have

$$\begin{aligned}
&R^{-2m-2} \int_{\{|z-z_0| \leq R\}} e^{2\rho^{-m}} u^2 dz \\
&\leq \int_{\{|z-z_0| \leq R\}} \rho^{-2m-2} e^{2\rho^{-m}} u^2 dz \\
&\leq \int_{\{|z-z_0| \leq 2R\}} \rho^{-2m-2} e^{2\rho^{-m}} \widetilde{u}^2 dz \\
&\leq c_0 m^{-4} \int_{\{|z-z_0| \leq 2R\}} \rho^{m+2} e^{2\rho^{-m}} (\Delta \widetilde{u})^2 dz \\
&\leq c_0 M m^{-4} \int_{\{|z-z_0| \leq R\}} \rho^{m+2} e^{2\rho^{-m}} u^2 dz + c_0 m^{-4} \int_{\{R \leq |z-z_0| \leq 2R\}} \rho^{m+2} e^{2\rho^{-m}} (\Delta \widetilde{u})^2 dz \\
&\leq c_0 M m^{-4} R^{m+2} \int_{\{|z-z_0| \leq R\}} e^{2\rho^{-m}} u^2 dz \\
&\quad + c_0 m^{-4} (2R)^{m+2} e^{2R^{-m}} \int_{\{R \leq |z-z_0| \leq 2R\}} (\Delta \widetilde{u})^2 dz,
\end{aligned}$$

or

$$\begin{aligned}
&(R^{-2m-2} - c_0 M m^{-4} R^{m+2}) \int_{\{|z-z_0| \leq R\}} e^{2\rho^{-m}} u^2 dz \\
&\leq c_0 m^{-4} (2R)^{m+2} e^{2R^{-m}} \int_{\{R \leq |z-z_0| \leq 2R\}} (\Delta \widetilde{u})^2 dz.
\end{aligned}$$

Since $\widetilde{u}(z) \in C_c^2(\overline{B^N(z_0; 2R)})$, we have

$$\begin{aligned}
&(R^{-2m-2} - c_0 M m^{-4} R^{m+2}) e^{2R^{-m}} \int_{\{|z-z_0| \leq R\}} u^2 dz \\
&\leq (R^{-2m-2} - c_0 M m^{-4} R^{m+2}) \int_{\{|z-z_0| \leq R\}} e^{2\rho^{-m}} u^2 dz \\
&\leq c_0 m^{-4} (2R)^{m+2} e^{2R^{-m}} \int_{\{R \leq |z-z_0| \leq 2R\}} (\Delta \widetilde{u})^2 dz \\
&\leq c_0 m^{-4} (2R)^{m+2} e^{2R^{-m}} C.
\end{aligned}$$

Thus,

$$\int_{\{|z-a| \leq R\}} u^2(z) dz \leq \frac{c_0 m^{-4} (2R)^{m+2} C}{R^{-2m-2} - c_0 M m^{-4} R^{m+2}}.$$

Let $m \to \infty$, then $u(z) = 0$ for any $z \in B^N(z_0; R)$. We claim that $u(\tilde{z}) = 0$ for any $\tilde{z} \in \Omega$. For $\tilde{z} \in \Omega$, let $h : [0,1] \to \Omega$ be a path satisfying $h(0) = z_0$, $h(1) = \tilde{z}$. Since h and δ are continuous functions, then

$$A = \min \big\{ \frac{1}{2}, \frac{1}{4} \inf_{0 \leq t \leq 1} \delta(h(t)) \big\} > 0.$$

Since $\{h(t) : 0 \le t \le 1\}$ is a compact set, $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ exist such that

$$\{h(t) | 0 \le t \le 1\} \subset \bigcup_{i=0}^{n} B^N(h(t_i); \frac{1}{3}A)$$

and

$$B^N\big(h(t_i); \frac{1}{3}A\big) \cap B^N\big(h(t_{i+1}); \frac{1}{3}A\big) \ne \emptyset.$$

For $i = 1, 2, \dots, n$, let $z_i = h(t_i)$ and $R_i = \min\{\frac{1}{2}, \frac{1}{4}\delta(z_i)\}$, then $z_i \in B^N(z_{i-1}; A) \subset B^N(z_{i-1}; R_i)$. Applying the same process to z_0, z_1 ,..., z_n, we conclude that $u(z) = 0$ on $\cup_{i=0}^{n} B^N(z_i; R_i)$, in particular, $u(\tilde{z}) = 0$. □

Let us denote the linear space of k times weakly differentiable functions by $W^k(\Omega)$. Clearly $C^k(\Omega) \subset W^k(\Omega)$. For $p \ge 1$ and a nonnegative integer k, we define $W^{k,p}(\Omega) = \{u \in W^k(\Omega) : D^\alpha u \in L^p(\Omega) \quad \text{for all } |\alpha| \le k\}$. We have the following lemma.

Lemma 10.4. *Let $1 \le p < \infty$, $R > 0$, and $u \in W^{2,p}(\Omega \cap B^N(0; R))$ be a solution of*

$$\begin{gathered} -\triangle u = f(u) \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{10.1}$$

where f is locally Lipschitz continuous on $\mathbb{R}$, and Ω is a smooth unbounded domain. Let $F(t) = \int_0^t f(s)ds$, and assume that $\nabla u \in L^2(\Omega)$ and $F(u) \in L^1(\Omega)$. Then a sequence $\{R_k\} \nearrow \infty$ exists such that
(i) (Pohozaev identity)

$$\int_\Omega \{NF(u) + (1 - \frac{N}{2})|\nabla u|^2\}dz = \lim_{k\to\infty} \frac{1}{2} \int_{\partial\Omega \cap B^N(0;R_k)} (z \cdot \nu(z))|\nabla u|^2 ds$$

(ii)

$$\lim_{k\to\infty} \int_{\partial\Omega \cap B^N(0;R_k)} \nu_i(z)|\nabla u|^2 ds = 0 \quad \text{for each } 1 \le i \le N,$$

where $\nu(z) = (\nu_1(z), \nu_2(z), \dots, \nu_N(z))$ is the outward unit normal vector at z.

Proof. (i) See Esteban-Lions [33, Proposition I.1] .
(ii) Let $B_R = B^N(0; R)$. Then we multiply (10.1) by $\frac{\partial u}{\partial z_i}$ and use integration by parts over $\Omega \cap B_R$ to obtain

$$\begin{aligned} \int_{\Omega\cap B_R} (-\triangle u)\frac{\partial u}{\partial z_i} &= \int_{\Omega\cap B_R} f(u)\frac{\partial u}{\partial z_i} = \int_{\Omega\cap B_R} \frac{\partial F(u)}{\partial z_i} \\ &= \int_{\partial(\Omega\cap B_R)} F(u)\nu_i ds \\ &= \int_{\Omega\cap\partial B_R} F(u)\frac{z_i}{|z|} ds. \end{aligned}$$

Note that $\nabla u = \frac{\partial u}{\partial \nu}\nu$ on $\partial\Omega$ and $\nu = \frac{z}{|z|}$ on ∂B_R. We use the Green first identity and integrate by parts to obtain

$$\begin{aligned} \int_{\Omega\cap B_R} (-\triangle u)\frac{\partial u}{\partial z_i} &= \int_{\Omega\cap B_R} \nabla u \nabla(\frac{\partial u}{\partial z_i}) - \int_{\partial(\Omega\cap B_R)} \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i} ds \\ &= \sum j = 1^N \int_{\Omega\cap B_R} \frac{\partial u}{\partial z_j}\frac{\partial^2 u}{\partial z_i \partial z_j} - \int_{\partial(\Omega\cap B_R)} \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i} ds \end{aligned}$$

$$
\begin{aligned}
&= \sum j = 1^N \frac{1}{2} \int_{\Omega\cap B_R} \frac{\partial}{\partial z_i} (\frac{\partial u}{\partial z_j})^2 - \int_{\partial(\Omega\cap B_R)} \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i} ds \\
&= \sum j = 1^N \frac{1}{2} \int_{\partial(\Omega\cap B_R)} (\frac{\partial u}{\partial z_j})^2 \nu_i ds - \int_{\partial(\Omega\cap B_R)} \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i} ds \\
&= \frac{1}{2}\int_{\partial\Omega\cap B_R} |\nabla u|^2 \nu_i ds + \frac{1}{2}\int_{\Omega\cap\partial B_R} |\nabla u|^2 \frac{z_i}{|z|} ds \\
&- \int_{\partial\Omega\cap B_R} (\frac{\partial u}{\partial \nu})^2 \nu_i ds - \int_{\Omega\cap\partial B_R} \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i} ds \\
&= -\frac{1}{2}\int_{\partial\Omega\cap B_R} |\nabla u|^2 \nu_i ds + \frac{1}{2}\int_{\Omega\cap\partial B_R} |\nabla u|^2 \frac{z_i}{|z|} ds \\
&- \int_{\Omega\cap\partial B_R} \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i} ds.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\big| -\frac{1}{2}\int_{\partial\Omega\cap B_R} |\nabla u|^2 \nu_i ds\big| &= \big| \int_{\Omega\cap\partial B_R} \big[F(u)\frac{z_i}{|z|} - \frac{1}{2}|\nabla u|^2 \frac{z_i}{|z|} + \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial z_i}\big] ds\big| \\
&\le \int_{\Omega\cap\partial B_R} \big(|F(u)| + \frac{1}{2}|\nabla u|^2 + |\nabla u|^2\big) ds.
\end{aligned}
$$

Since $F(u) \in L^1(\Omega)$ and $\nabla u \in L^2(\Omega)$,

$$
\begin{aligned}
\infty > \int_\Omega (|F(u)| + \frac{3}{2}|\nabla u|^2) dz &= \lim_{R\to\infty} \int_{\Omega\cap B_R} (|F(u)| + \frac{3}{2}|\nabla u|^2) dz \\
&= \int_0^\infty r^{N-1} \big[\int_{\Omega\cap\partial B_r} (|F(u)| + \frac{3}{2}|\nabla u|^2) ds\big] dr \\
&= \int_0^\infty r^{N-1} M(r) dr,
\end{aligned}
$$

where $M(r) = \int_{\Omega\cap\partial B_r} (|F(u)| + \frac{3}{2}|\nabla u|^2) ds$. Suppose that for every sequence $R_k \to \infty$ we have $M(R_k) \nrightarrow 0$ as $k \to \infty$, that is, a subsequence $\{R_k\}$ and $c > 0$ exist such that $M(R_k) \to c$ as $n \to \infty$. Then for sufficiently large k we obtain $M(R_k) \ge \frac{c}{2} > 0$. Thus,

$$
\int_0^\infty r^{N-1} M(r) dr \ge \frac{c}{2}\int_k^\infty r^{N-1} dr = \infty.
$$

This is a contradiction. We conclude that $R_k \to \infty$ exists such that

$$
\lim_{k\to\infty} \int_{\Omega\cap\partial B_{R_k}} (|F(u)| + \frac{3}{2}|\nabla u|^2) ds = 0,
$$

that is,

$$
\lim_{k\to\infty} \int_{\Omega\cap\partial B_{R_k}} |\nabla u|^2 \nu_i ds = 0 \quad \text{for } 1 \le i \le N.
$$

□

Lemma 10.5. *Let $Lu = a^{ij}(z)D_{ij}u + b^i(z)D_i u + c(z)u$ and $u \in W^{2,p}_{\rm loc}(\Omega)$ be a solution of the elliptic equation $Lu = f$ in a domain Ω, where the coefficients of L belong to $C^{k-1,\alpha}(\Omega)$, $f \in C^{k-1,\alpha}(\Omega)$ with $1 < p < \infty$, $k \ge 1$, $0 < \alpha < 1$. Then $u \in C^{k+1,\alpha}(\Omega)$.*

The proof of the lemma above can be found in Gilbarg-Trudinger [36, Theorem 9.19].

Theorem 10.6. *Let Ω be an Esteban-Lions domain in $\mathbb{R}^N$ with χ as in Definition 2.6, and let f be a locally Lipschitz-continuous functional on $\mathbb{R}$ with $f(0) = 0$. If $u \in C^2(\overline{\Omega})$ is a solution of*

$$\begin{gathered} -\triangle u = f(u) \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

and satisfies $\nabla u \in L^2(\Omega)$, $F(u) \in L^1(\Omega)$ (with $F(t) = \int_0^t f(s)ds$), then $u \equiv 0$ in Ω.

Proof. By Lemma 10.4 (ii), a sequence $R_k \to \infty$ as $k \to \infty$ exists such that

$$\lim_{k\to\infty} \int_{\partial\Omega\cap B^N(0;R_k)} (\nu(z)\cdot\chi)|\nabla u|^2 ds = 0.$$

This immediately yields $(\nu(z)\cdot\chi)|\nabla u| = 0$ on $\partial\Omega$. Since Ω is an Esteban-Lions domain, then $z_0 \in \partial\Omega$ exists such that $\nu(z_0)\cdot\chi > 0$. Thus, $\delta > 0$ exists such that $\nu(z)\cdot\chi > 0$ for $z \in \partial\Omega \cap B^N(z_0;\delta)$. Then we have $\nabla u = 0$ on $\partial\Omega \cap B^N(z_0;\delta)$. Let

$$\widetilde{u}(z) = \begin{cases} u(z) & \text{for } z \in \overline{\Omega}; \\ 0 & \text{for } z \in B^N(z_0;\delta)\backslash\overline{\Omega}. \end{cases}$$

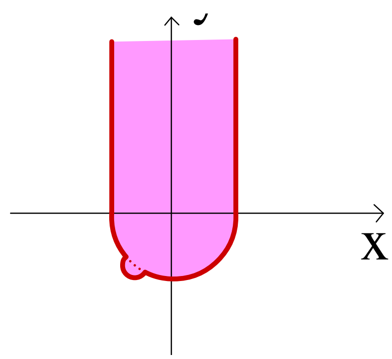

FIGURE 7. Esteban-Lions domain with a small ball.

We claim that $\widetilde{u}$ is twice weakly differentiable. Since $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, for each $\varphi \in C_c^1(\Omega \cup B^N(z_0;\delta))$, we have

$$\begin{aligned} \int_{\Omega\cup B^N(z_0;\delta)} \widetilde{u}\frac{\partial\varphi}{\partial z_i} &= \int_\Omega u\frac{\partial\varphi}{\partial z_i} \\ &= -\int_\Omega \frac{\partial u}{\partial z_i}\varphi + \int_{\partial\Omega} u\varphi\nu_i ds \\ &= -\int_{\Omega\cup B^N(z_0;\delta)} v_i\varphi, \end{aligned}$$

where

$$v_i(z) = \begin{cases} \frac{\partial u}{\partial z_i} & \text{for } z \in \Omega; \\ 0 & \text{for } z \in B^N(z_0;\delta)\backslash\Omega. \end{cases}$$

Thus, $v_i = \frac{\partial \widetilde{u}}{\partial z_i}$. Similarly, we have $w_{ij} = \frac{\partial^2 \widetilde{u}}{\partial z_i \partial z_j}$, where

$$w_{ij}(z) = \begin{cases} \frac{\partial^2 u}{\partial z_i \partial z_j} & \text{for } z \in \Omega; \\ 0 & \text{for } z \in B^N(z_0;\delta) \backslash \Omega. \end{cases}$$

Therefore, $\widetilde{u} \in W^{2,2}_{\rm loc}(\Omega)$. Next, since $u \in C^2(\overline{\Omega})$ and by the Green first identity, for each $\varphi \in C_c^\infty(\Omega \cup B^N(z_0;\delta))$, we have

$$\begin{aligned}\int_{\Omega \cup B^N(z_0;\delta)} \nabla \widetilde{u} \nabla \varphi &= \int_\Omega \nabla u \nabla \varphi \\ &= -\int_\Omega (\triangle u)\varphi + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \varphi ds \\ &= \int_\Omega f(u)\varphi + \int_{\partial\Omega \cap B^N(z_0;\delta)} \frac{\partial u}{\partial \nu}\varphi ds + \int_{\partial\Omega \backslash B^N(z_0;\delta)} \frac{\partial u}{\partial \nu}\varphi ds \\ &= \int_{\Omega \cup B^N(z_0;\delta)} f(\widetilde{u})\varphi. \end{aligned}$$

Hence, $\widetilde{u}$ is a weak solution of

$$\begin{gathered} -\triangle \widetilde{u} = f(\widetilde{u}) \quad \text{in } \Omega \cup B^N(z_0;\delta); \\ \widetilde{u} = 0 \quad \text{on } \partial(\Omega \cup B^N(z_0;\delta)). \end{gathered}$$

We claim that $f \circ \widetilde{u}$ is locally Hölder continuous with exponent α in $\Omega \cup B^N(z_0;\delta)$, where $0 < \alpha < 1$. That is, for each compact set $K \subset \Omega \cup B^N(z_0;\delta)$, we must show that a constant $C(K) > 0$ exists such that

$$|f \circ \widetilde{u}(z) - f \circ \widetilde{u}(\overline{z})| \le C(K)|z - \overline{z}|^\alpha \tag{10.2}$$

for all $z, \overline{z} \in K$. Since f is locally Lipschitz with $f(0) = 0$, and $u \in C^2(\overline{\Omega})$, for $z, \overline{z} \in K$, we have

(i) (10.2) holds for $z, \overline{z} \in \{B^N(z_0;\delta)\backslash\Omega\} \cap K$;

(ii) if $z \in \Omega$ and $\overline{z} \in \{B^N(z_0;\delta)\backslash\Omega\} \cap K$, then $\widehat{z} \in \partial\Omega$ exists such that $|z - \widehat{z}| \le |z - \overline{z}|$. Thus,

$$\begin{aligned} |f \circ \widetilde{u}(z) - f \circ \widetilde{u}(\overline{z})| &\le |f \circ \widetilde{u}(z) - f \circ \widetilde{u}(\widehat{z})| + |f \circ \widetilde{u}(\widehat{z}) - f \circ \widetilde{u}(\overline{z})| \\ &\le C_1(K)|\widetilde{u}(z) - \widetilde{u}(\widehat{z})| \\ &\le C_2(K)|z - \widehat{z}| \le C_2(K)|z - \overline{z}| \\ &\le C_2(K)|z - \overline{z}|^{1-\alpha}|z - \overline{z}|^\alpha \\ &\le C_3(K)|z - \overline{z}|^\alpha; \end{aligned}$$

(iii) it is clear that both z and $\overline{z}$ are in $\Omega \cap K$.

By Lemma 10.5, we have $\widetilde{u} \in C^2(\Omega \cup B^N(z_0;\delta))$. Finally, since f is locally Lipschitz and $f(0) = 0$, then $(\triangle \widetilde{u})^2 = |f(\widetilde{u})|^2 \le C(K)(\widetilde{u}(z))^2$ on each compact subset K of $\Omega \cup B^N(z_0;\delta)$. By Theorem 10.3, $\widetilde{u} \equiv 0$ on K, and $\widetilde{u} \equiv 0$ on $\Omega \cup B^N(z_0;\delta)$. Otherwise, if there is a $z \in \Omega$ such that $\widetilde{u}(z) \neq 0$, then a bounded domain Ω_1 exists such that $z \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega \cup B^N(z_0;\delta)$, and by the previous argument, $\widetilde{u} \equiv 0$ on $\overline{\Omega}_1$, which is a contradiction. Hence, $u \equiv 0$. □

Esteban-Lions [33, Theorem I.1] proved the following result.

Theorem 10.7. *Equation* (1.1) *in an Esteban-Lions domain* Ω *does not admit any nontrivial solution. In particular, an Esteban-Lions domain is a nonachieved domain.*

We have the following lemma.

Lemma 10.8. (i) *Esteban-Lions domains are invariant under any rigid motions;* (ii) *If* Ω *is an Esteban-Lions domain, then* $\overline{\Omega}^c$ *is also an Esteban-Lions domain.*

Proof. (i) Clearly.
(ii) Note that $n(x) \cdot \chi = -n(x) \cdot -\chi$. □

In fact, in a star-shaped domain $\Omega \subset \mathbb{R}^N$, there is a fixed point $z_0 \in \Omega$ such that the segment $\overline{zz_0}$ is contained in Ω for each $z \in \Omega$. We assert that an Esteban-Lions domain Ω is an infinite star-shaped domain in the sense that it is $v-$convex for some direction v: if $z_1, z_2 \in \Omega$ with $z_1 - z_2 = tv$ for some $t \in \mathbb{R}$, then the segment $\overline{z_1 z_2}$ is contained in Ω. This is a consequence of the following lemma.

Lemma 10.9. *An Esteban-Lions domain* Ω *with* χ *as in Definition 2.6 in* $\mathbb{R}^N$ *is* $\chi-$*convex.*

Proof. Let Ω be an Esteban-Lions domain in $\mathbb{R}^N$. Without loss of generality, we may assume that $\chi = (0, \dots, 0, -1) \in \mathbb{R}^N$ satisfying $n(z) \cdot \chi \geq 0$, and $n(z) \cdot \chi \not\equiv 0$ for each $z \in \partial\Omega$. For each $z_1 \in \Omega$, we claim that $\{z_1 - \lambda\chi | \lambda \geq 0\} \subset \Omega$. Otherwise, set $\lambda_0 = \inf\{\lambda \geq 0 | z_1 - \lambda\chi \notin \Omega\}$. Then $\lambda_0 > 0$ and the point $z_0 = z_1 - \lambda_0\chi \in \partial\Omega$. There are only two possibilities:
(i) the curve $\partial\Omega$ is transverse with the x_N-axis at z_0, and then $n(z_0) \cdot \chi < 0$;
(ii) the curve $\partial\Omega$ is tangent with the x_N-axis at z_0, and then $z \in \partial\Omega$ near z_0 exists such that $n(z) \cdot \chi < 0$.
Both (i) and (ii) contradict the definition of χ. We conclude that an Esteban-Lions domain Ω in $\mathbb{R}^N$ is $\chi-$convex. □

In $\mathbb{R}^2$, let an upper semi-strip Ω_1, a first quadrant Ω_2, a second quadrant Ω_3, and an upper half plane Ω_4 be defined as follows:

$$\Omega_1 = \{z = (x, y) \in \mathbb{R}^2 : a < x < b,\ f_1(x) < y\};$$
$$\Omega_2 = \{z = (x, y) \in \mathbb{R}^2 : a < x < \infty,\ f_2(x) < y\};$$
$$\Omega_3 = \{z = (x, y) \in \mathbb{R}^2 : -\infty < x < b,\ f_3(x) < y\};$$
$$\Omega_4 = \{z = (x, y) \in \mathbb{R}^2 : -\infty < x < \infty, f_4(x) < y\},$$

where $f_1 : (a, b) \to \mathbb{R}$, $f_2 : (a, \infty) \to \mathbb{R}$, $f_3 : (-\infty, b) \to \mathbb{R}$, and $f_4 : (-\infty, \infty) \to \mathbb{R}$ are smooth functions with single values.

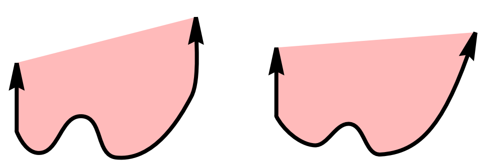

Figure 8. Esteban-Lions domains 1.

As a consequence of Lemma 10.9, we have the following two lemmas.

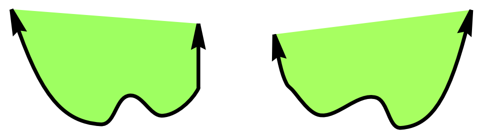

FIGURE 9. Esteban-Lions domains 2.

Lemma 10.10. *An Esteban-Lions domain in $\mathbb{R}^2$ must be one of the following types: an upper half strip Ω_1, a first quadrant Ω_2, a second quadrant Ω_3, or an upper half plane Ω_4.*

Lemma 10.11. *An Esteban-Lions domain Ω with χ as in Definition 2.6 in $\mathbb{R}^N$ is a large domain if and only if the projection $\tilde{\Omega}$ of Ω in the hyperplane $\mathbb{R}^{N-1}$ that is perpendicular to χ, is a large domain in $\mathbb{R}^{N-1}$.*

Proof. Suppose that Ω is a large domain in $\mathbb{R}^N$. Then for $r > 0$, $x \in \Omega$ exists such that $B^N(x,r) \subset \Omega$ which implies $\tilde{B}^N(x,r) \subset \tilde{\Omega}$, where $\tilde{B}^N(x,r)$ and $\tilde{\Omega}$ are the projections of $B^N(x,r)$ and Ω, respectively. On the other hand, suppose $\tilde{\Omega}$ is a large domain in $\mathbb{R}^{N-1}$. Then for $r > 0$, $x \in \Omega$ exists such that $\tilde{B}^N(x,r) \subset \tilde{\Omega}$, which means that for any $\tilde{y} \in \tilde{B}^N(x,r)$, $y \in \Omega$ exists and $\tilde{y}$ is the projection of y. By Lemma 10.9, $\lambda > 0$ exists such that $\{\tilde{y} - \chi t : t \geq \lambda\} \subset \Omega$. Let

$$\bar{\lambda} = \inf \tilde{y} \in \tilde{B}^N(x,r)\{\lambda > 0 : \tilde{y} - \chi t \in \Omega \quad \text{for } t \geq \lambda\},$$

then $B^N(z,r) \subset \Omega$, where $z = \tilde{x} - (r + 1 + \bar{\lambda})\chi$. Thus, Ω is a large domain in $\mathbb{R}^N$. □

In $\mathbb{R}^3$, there is an Esteban-Lions domain that is not a large domain.

Example 10.12. Let Ω be a domain in $\mathbb{R}^3$ such that Ω contains the point $(0,0,1)$ with the boundary

$$\partial\Omega = \{((u+2)\cos v, \quad \sin v, \quad u) \in \mathbb{R}^3 \mid 0 \leq v \leq 2\pi, \quad u \geq 0\}.$$

Then Ω is an Esteban-Lions domain in $\mathbb{R}^3$ with $\chi = (0,0,-1)$, but it is not a large domain in either $\mathbb{R}^3$ or $\mathbf{A}^r$.

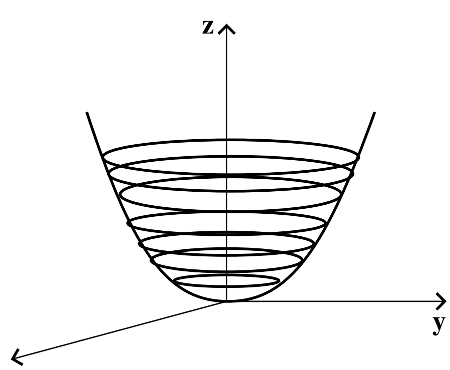

FIGURE 10. Esteban-Lion domain but not a large domain.

Proof. Let $p(u,v) = ((u+2)\cos v, \sin v, u)$. Then

$$p_v = (-(u+2)\sin v, \cos v, 0)$$

and $p_u = (\cos v, 0, 1)$. For $x \in \partial\Omega$, we have the outer normal vector $n(x) = p_v \times p_u = (\cos v, (u+2)\sin v, -\cos^2 v)$, and $\chi \cdot n(x) = \cos^2 v \geq 0$. Therefore, we have $\chi \cdot n(x) \geq 0$. Clearly, $B(x;2) \nsubseteq \Omega$ for any $x \in \Omega$, and thus, Ω is not a large domain in $\mathbb{R}^3$. □

Let $\mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0;s)$ be an interior flask domains. Interior flask domains are achieved for large s, but are nonachieved for small s. By Theorem 12.7 below, we have the following result.

Theorem 10.13. *$s_0 > 0$ exists such that the interior flask domains $\mathbf{F}_s^r$ nonachieved if $s < s_0$.*

Bibliographical notes: Theorem 10.6 is from Esteban-Lions [33].

11. Higher Energy Solutions

Nonachieved domains may admit higher energy solutions. The Berestycki conjecture states: there is a positive solution of Equation (1.1) in an Esteban-Lions domain with a hole. We answer this conjecture affirmatively.

The Berestycki conjecture is based on some historical and physical reasons: suppose that no solution of an equation exists in a domain $\mathbf{\Theta}$. If we break the symmetry of the domain $\mathbf{\Theta}$ by adding a ball to it or by removing a ball, then the same equation in the perturbed domain admits a solution.

(i) Pohozaev [61] proved that the Dirichlet problem $\Delta u + u^{\frac{N+2}{N-2}} = 0$ in a ball does not have any nontrivial solution. However, if we remove a small ball, then Coron [28] proved that there is a positive solution.

(ii) Some turbulence equations in a ball do not admit any nontrivial solutions. Lions-Zuazua [48] added a small ball on the boundary to break the symmetry, and proved that the equation then has a nontrivial solution. One description of the phenomenon is that if we add a small bump to the surface of a plane, then the turbulence will be controlled.

We assert the existence of higher energy solutions of Equation (1.1) in Ω, thus answering the Berestycki conjecture affirmatively. Then we study the dynamic systems of solutions.

Our results in this paper are still true for any one of the above known four Esteban-Lions domains. For simplicity, however,we study only the upper half strip with a hole Ω. We also believe that the analyses and the results in this paper will be helpful for studying the existence of solutions of equations in unbounded domains.

11.1. **Existence Results.** For $h > r$ and $\mathbf{B} = B^N((0,h); r/2)$, let $\Omega = \Omega_h = (\mathbf{A}_0^r \cup B^N(0;r)) \backslash \overline{\mathbf{B}}$ be the upper half strip with a hole. By Theorem 10.1, there are no ground state solutions of Equation (1.1) in Ω. However, in this section, we prove that a positive higher energy solution of Equation (1.1) exists in Ω.

Let $\overline{u}$ be a ground state solution of Equation (1.1) in $\mathbf{A}^r$, $\overline{h} = (0,h) \in \mathbf{A}^r$ and $\phi : \mathbf{A}^r \to [0,1]$, a C^∞ cut-off function such that $0 \leq \phi \leq 1$ and

$$\phi(z) = \begin{cases} 0 & textfor z \in \mathbf{B} \cup (\mathbf{A}^r \backslash (\mathbf{A}_0^r \cup B^N(0;r))), \\ 1 & \text{for } z \in (\mathbf{A}_0^r \cup B^N(0;r)) \backslash (B^N(0;\frac{2}{3}r) \\ & \qquad \cup \{z = (x,y) \in \mathbf{A}^r | y \leq r\}), \end{cases}$$

$$I = \{0\} \times [-\frac{r}{2}, \frac{r}{2}], \quad I_h = \overline{h} + I,$$
$$v_t(z) = \phi(z)\bar{u}(z - t - 2\overline{h}) \quad \text{for } z \in \mathbf{A}^r, \quad t \in I.$$

Then $v_t \in H_0^1(\Omega)$. Furthermore, we have the following lemma.

Lemma 11.1. *For $t \in I$ or $\bar{t} \in I_h$, where $\bar{t} = \overline{h} + t$, then*
(i) $\|v_t(z) - \bar{u}(z - t - 2\overline{h})\|_{L^p(\mathbf{A}^r)} = o(1)$ as $h \to \infty$;
(ii) $\|v_t(z) - \bar{u}(z - t - 2\overline{h})\|_{H^1(\mathbf{A}^r)} = o(1)$ as $h \to \infty$;
(iii) $J(v_t) = \alpha(\mathbf{A}^r) + o(1)$ as $h \to \infty$.

Proof. (i)

$$\begin{aligned}\|v_t(z) - \bar{u}(z - t - 2\overline{h})\|^p_{L^p(\mathbf{A}^r)} &= \int_{\mathbf{A}^r} |\phi(z) - 1|^p |\bar{u}(z - t - 2\overline{h})|^p \\ &\le \int_{(\mathbf{A}^r_{h+r})^c} |\bar{u}(z - t - 2\overline{h})|^p \\ &= o(1) \quad \text{as } h \to \infty.\end{aligned}$$

(ii) We have

$$\begin{aligned}&\|v_t(z) - \bar{u}(z - t - 2\overline{h})\|^2_{H^1(\mathbf{A}^r)} \\ &= \|(\phi(z) - 1)\bar{u}(z - t - 2\overline{h})\|^2_{H^1(\mathbf{A}^r)} \\ &\le c(\frac{1}{r^2} + 1)\int_{(\mathbf{A}^r_{h+r})^c} (|\nabla\bar{u}(z - t - 2\overline{h})|^2 + |\bar{u}(z - t - 2\overline{h})|^2) \\ &= o(1) \quad \text{as } h \to \infty.\end{aligned}$$

(iii) This follows from (i), (ii), Theorem 12.5, and

$$\alpha(\mathbf{A}^r) = J(\overline{u}) = \frac{1}{2}a(\bar{u}) - \frac{1}{p}b(\bar{u}) = \frac{1}{2}a(v_t) - \frac{1}{p}b(v_t) + o(1) = J(v_t) + o(1).$$

□

From Lemma 11.1, since $\|\overline{u}\|^2_{H^1(\mathbf{A}^r)} = \|\overline{u}\|^p_{L^p(\mathbf{A}^r)}$, we have

$$\|v_t\|^2_{H^1(\mathbf{A}^r)} = \|\overline{u}\|^2_{H^1(\mathbf{A}^r)} + o(1) \quad \text{as } h \to \infty,$$
$$\|v_t\|^p_{L^p(\mathbf{A}^r)} = \|\overline{u}\|^p_{L^p(\mathbf{A}^r)} + o(1) \quad \text{as } h \to \infty.$$

Therefore, $\|v_t\|^2_{H^1(\mathbf{A}^r)} = \|v_t\|^p_{L^p(\mathbf{A}^r)} + o(1)$ as $h \to \infty$. By Lemma 4.2, there exists $\lambda_t > 0$ such that $u_t = \lambda_t v_t$ in $\mathbf{M}$: $\|u_t\|^2_{H^1(\mathbf{A}^r)} = \|u_t\|^p_{L^p(\mathbf{A}^r)}$. Therefore, $\lambda_t \to 1$ as $h \to \infty$, or $J(u_t) = \alpha(\mathbf{A}^r) + o(1)$ as $h \to \infty$.

For $u \in H_0^1(\Omega)$, define the center of mass function by

$$j(u) = \|u\|^{-p}_{L^p(\mathbf{A}^r)} \int_{\mathbf{A}^r} (\overline{h} + \frac{r}{2}\frac{z}{|z|})|u(x,y)|^p dxdy.$$

Let

$$\beta_0 = \inf\{J(u) : u \in \mathbf{M}(\Omega),\, u \ge 0,\, j(u) = \overline{h}\}.$$

Proposition 11.2. $\alpha(\mathbf{A}^r) = \alpha(\Omega) < \beta_0$.

Proof. By Theorem 10.1, $\alpha(\mathbf{A}^r)=\alpha(\Omega)$. Clearly $\alpha(\mathbf{A}^r)\le\beta_0$. Suppose $\alpha(\mathbf{A}^r)=\beta_0$. By Theorem 4.4, there is a sequence $\{u_k\}$ in $\mathbf{M}(\Omega)$, $u_k\ge 0$, $j(u_k)=\overline{h}$ for each k, such that

$$J(u_k)=\alpha(\mathbf{A}^r)+o(1)\quad\text{as } k\to\infty,$$
$$J'(u_k)=o(1)\quad\text{strongly}\quad\text{in } H^{-1}(\Omega)\quad\text{as } k\to\infty.$$

By Theorem 3.5, there is an unbounded sequence $\{(0,y_k)\}$ in $\mathbf{A}^r$ such that

$$u_k(x,y)=\overline{u}(x,y-y_k)+o(1)\quad\text{strongly in } H_0^1(\mathbf{A}^r),$$

where $\overline{u}$ is a ground state solution of Equation (1.1) in $\mathbf{A}^r$. Assume $(\overline{h}+\frac{r}{2}\frac{(0,y_k)}{|(0,y_k)|})=\varsigma+o(1)$ as $k\to\infty$, where $\varsigma\in\partial I_h$. Then by the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\overline{h}=j(u_k)&=\|u_k\|^{-p}_{L^p(\mathbf{A}^r)}\int_{\mathbf{A}^r}(\overline{h}+\frac{r}{2}\frac{z}{|z|})|u_k(x,y)|^p dxdy\\&=\|\overline{u}\|^{-p}_{L^p(\mathbf{A}^r)}\int_{\mathbf{A}^r}(\overline{h}+\frac{r}{2}\frac{(x,y+y_k)}{|(x,y+y_k)|})|\overline{u}(x,y)|^p dxdy+o(1)\\&=\varsigma+o(1)\quad\text{as } k\to\infty,\end{aligned}$$

which is a contradiction. Therefore $\alpha(\mathbf{A}^r)=\alpha(\Omega)<\beta_0$. □

Let

$$V=\{u\in\mathbf{M}(\Omega):u\ge 0\};$$
$$\Gamma=\{k:I_h\to V\text{ continuous}:k(\overline{t})=u_t\ \text{ for }\ t\in\partial I\};$$
$$\beta_1=\inf_{k\in\Gamma}\max_{\overline{t}\in I_h}J(k(\overline{t})).$$

Proposition 11.3. *There is an $h_0>0$ such that for $h\ge h_0$,*
(i) $\alpha(\mathbf{A}^r)<J(u_t)<\frac{\beta_0+\alpha(\mathbf{A}^r)}{2}<\beta_0$, *for $t\in I$;*
(ii) $\alpha(\mathbf{A}^r)<J(u_t)<2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$, *for $t\in I$;*
(iii) $(j\circ u_t,\overline{t})>0$, *for $t\in\partial I$.*

Proof. *(i)* and *(ii)* follow from Theorem 10.1, Proposition 11.2, and Lemma 11.1.
(iii) $c_1,c_2>0$ exist such that $c_1\le\|\phi(z)\bar{u}(z-t-2\overline{h})\|_{L^p(\mathbf{A}^r)}\le c_2$. For $t\in\partial I$ with $z+t+2\overline{h}\ne 0$, we have

$$\begin{aligned}(\frac{z+t+2\overline{h}}{|z+t+2\overline{h}|},t)&=|z+t+2\overline{h}|-\frac{1}{|z+t+2\overline{h}|}(z+t+2\overline{h},z+2\overline{h})\\&\ge|z+t+2\overline{h}|-|z+2\overline{h}|\ge|t|-2|z+2\overline{h}|=\frac{r}{2}-2|z+2\overline{h}|.\end{aligned}$$

Then there are constants c_3, c_4, c_5, c_6, $h_0>0$ such that for $h\ge h_0$

$$\begin{aligned}(j(u_t),\overline{h}+t)&=\|u_t(z)\|^{-p}_{L^p(\mathbf{A}^r)}\int_{\mathbf{A}^r}(\overline{h}+\frac{r}{2}\frac{z}{|z|},\overline{h}+t)|u_t(z)|^p dz\\&=c_3\int_{\mathbf{A}^r}(\overline{h}+\frac{r}{2}\frac{z}{|z|},\overline{h}+t)|\phi(z)\bar{u}(z-t-2\overline{h})|^p dz\\&=c_3\int_{\mathbf{A}^r}(\overline{h}+\frac{r}{2}\frac{z+t+2\overline{h}}{|z+t+2\overline{h}|},\overline{h}+t)|\phi(z+t+2\overline{h})\bar{u}(z)|^p dz\\&\ge c_3(h^2c_5-hc_4c_5-hc_5-2c_6-4hc_5)>0\quad\text{since } h\ge h_0,\end{aligned}$$

where $\int_{\mathbf{A}^r}|\phi(z+t+2\overline{h})\bar{u}(z)|^p dz \geq c_5$ and $\int_{\mathbf{A}^r}|z||\phi(z+t+2\overline{h})\bar{u}(z)|^p dz \geq c_6$. By Theorem 8.13, $c_4 < \infty$. □

Proposition 11.4. *For $h \geq h_0$, we have $\alpha(\mathbf{A}^r) < \beta_0 = \beta_1 < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$.*

Proof. We claim that: (i) $\beta_0 = \beta_1$: for any $k \in \Gamma$, consider the homotopy $H(\lambda, \bar{t}) : [0,1] \times I_h \to \mathbb{R}^N$ defined by

$$H(\lambda, \bar{t}) = (1-\lambda) j(k(\bar{t})) + \lambda i(\bar{t}),$$

where i denotes the identity map. Note that $j(k(\bar{t})) = j(u_t)$ for $\bar{t} \in \partial I_h$. By Proposition 11.3 (iii), $H(\lambda, \bar{t}) \neq \overline{h}$ for $\bar{t} \in \partial I_h$ and $\lambda \in [0,1]$. Therefore

$$\deg(j \circ k, I_h, \overline{h}) = \deg(i, I_h, \overline{h}) = 1.$$

$t_0 \in I_h$ exists such that

$$j(k(t_0)) = \overline{h}.$$

Hence, for each $k \in \Gamma$,

$$\begin{aligned} \beta_0 &= \inf\{J(u) : u \in \mathbf{M}, \quad u \geq 0, \quad j(u) = \overline{h}\} \\ &\leq J(k(t_0)) \\ &\leq \max_{\bar{t} \in I_h} J(k(\bar{t})). \end{aligned}$$

We have $\beta_0 \leq \beta_1$. On the other hand, by Proposition 11.3 (i), for $t \in I$, we have $u_t \in V$ and $J(u_t) < \beta_0$. Thus, $\max_{t \in I} J(u_t) \leq \beta_0$, or $\beta_1 \leq \beta_0$.
(ii) $\beta_1 < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$: by Proposition 11.3 (ii), $J(u_t) < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$ for $t \in I$. Thus

$$\max_{t \in I} J(u_t) < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r).$$

We have $\beta_1 < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$. By Proposition 11.3 (i), we have

$$\alpha(\mathbf{A}^r) < \beta_0 = \beta_1 < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r).$$

□

Now we assert that there is a higher energy solution of Equation (1.1) in Ω.

Theorem 11.5. *Suppose that the positive solution of Equation* (1.1) *in the infinite strip $\mathbf{A}^r$ is unique up to y-translations. $h_0 > 0$ exists such that if $h \geq h_0$, then there is a positive higher energy solution v of Equation* (1.1) *in the upper half strip with a hole Ω such that $\alpha(\mathbf{A}^r) < J(v) < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$.*

Proof. Note that $\beta_0 = \inf\{J(u) : u \in \mathbf{M}, \quad u \geq 0, \quad j(u) = \overline{h}\}$. Take a minimizing sequence $\{u_k\}$ in $\mathbf{M}$: $J(u_k) \to \beta_0$ as $k \to \infty$. By Theorem 4.4, $\{u_k\}$ is a $(PS)_{\beta_0}$-sequence for J : $J(u_k) \to \beta_0$ and $J'(u_k) \to 0$ as $k \to \infty$. By Theorem 3.5, an integer $\ell \geq 0$, and sequences $\{z_k^i\}$, where $z_k^i = (0, y_k^i) \in \mathbf{A}^r$ for $1 \leq i \leq \ell$ exist, such that for some subsequence $\{u_k\}$, there are $u^0 \in H_0^1(\Omega)$, $u^0 \geq 0$ in Ω, $u^i \in H_0^1(\mathbf{A}^r)$, and $u^i > 0$ in $\mathbf{A}^r$, $1 \leq i \leq \ell$, satisfying

$$\begin{gathered} u_k(z) = u^0(z) + [u^1(z - z_k^1) + u^2(z - z_k^2) + \dots \\ + u^\ell(z - z_k^\ell)] + o(1) \quad \text{strongly in} H_0^1(\mathbf{A}^r), \\ -\triangle u^0 + u^0 = (u^0)^{p-1} \quad \text{in } \Omega, \\ -\triangle u^i + u^i = (u^i)^{p-1} \quad \text{in } \mathbf{A}^r,\ 1 \leq i \leq \ell, \end{gathered}$$

$$J(u_k) = J(u^0) + \sum_{i=1}^{\ell} J(u^i) + o(1) \quad \text{as } k \to \infty.$$

Suppose that the solution of (1.1) in the infinite strip $\mathbf{A}^r$ is unique up to y-translations. From Theorems 3.5 and 9.4, we find that u^i are the same and $J(u^i) = \alpha(\mathbf{A}^r)$ for $i = 1, 2, \dots, l$. Therefore

$$\beta_0 = J(u^0) + l\alpha(\mathbf{A}^r).$$

Since $\alpha(\mathbf{A}^r) < \beta_0 < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$. We conclude that u^0 is nonzero and $l = 0$. Thus, there is a positive higher energy solution $v = u^0$ of Equation (1.1) in the upper half strip with a hole Ω such that $\alpha(\mathbf{A}^r) < J(v) = \beta_0 < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)$. □

11.2. **Dynamic Systems of Solutions.** As in Subsection 1, for $k = 1, 2, \dots$, define $\Omega_k = (\mathbf{A}_0^r \cup B^N(0;r)) \setminus \overline{B^N((0,h);\frac{1}{k})}$, where $h \geq 2h_0$, $\frac{1}{k} < r$. Then Ω_k is an increasing sequence such that

$$(\mathbf{A}_0^r \cup B^N(0;r)) \setminus \{0\} = \cup_{k=1}^{\infty}\Omega_k.$$

By Theorem 11.5, we have, for each k, a positive solution $u_k \in H_0^1(\Omega_k)$ of $-\Delta u_k + u_k = u_k^{p-1}$ in Ω_k satisfying

$$\alpha(\mathbf{A}^r) < J(u_k) < 2^{\frac{p-2}{p}}\alpha(\mathbf{A}^r)\,.$$

Lemma 11.6. *If $u_k \rightharpoonup u$ weakly in $H_0^1((\mathbf{A}_0^r \cup B^N(0;r)))$ as $k \to \infty$, then $u \equiv 0$.*

Proof. For $\varphi \in C_0^\infty((\mathbf{A}_0^r \cup B^N(0;r)))$, we have

$$\begin{aligned}\int_{(\mathbf{A}_0^r \cup B^N(0;r))} u_k(-\Delta\varphi + \varphi) &= \int_{(\mathbf{A}_0^r \cup B^N(0;r))} (-\Delta u_k + u_k)\varphi \\ &= \int_{(\mathbf{A}_0^r \cup B^N(0;r))} u_k^{p-1}\varphi.\end{aligned}$$

Let $k \to \infty$, and we obtain

$$\int_{(\mathbf{A}_0^r \cup B^N(0;r))} u(-\Delta\varphi + \varphi) = \int_{(\mathbf{A}_0^r \cup B^N(0;r))} u^{p-1}\varphi.$$

Thus, $-\Delta u + u = u^{p-1}$ in $(\mathbf{A}_0^r \cup B^N(0;r))$. By Theorems 3.5 and 9.4, $u \equiv 0$, or $u_k \rightharpoonup 0$ weakly in $H_0^1((\mathbf{A}_0^r \cup B^N(0;r)))$ as $k \to \infty$. □

We have the following dynamic systems of solutions $\{u_k\}$:

Theorem 11.7. *$|\nabla u_k|^2 dz = c\delta_0 + o(1)$ for some positive number c.*

Proof. Let $u_k \rightharpoonup u$ weakly in $H_0^1((\mathbf{A}_0^r \cup B^N(0;r)))$ as $k \to \infty$, $\mu_k = |\nabla u_k|^2 dz = \mu + o(1)$ weak*, and $\nu_k = |u_k|^p dz = \nu + o(1)$ weak*. Then by the second concentration lemma (see Lions [50, Lemma I.1, p.24]), there exist $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty$ in $\mathbb{R}^+$ such that

$$\begin{aligned}m^{p/(p-2)} + o(1) &= \|u_k\|_{H^1(\Omega_k)}^2 \\ &= \int_{(\mathbf{A}_0^r \cup B^N(0;r))} d\mu_k = \int_{(\mathbf{A}_0^r \cup B^N(0;r))} d\mu + o(1) \\ &\geq \|u\|_{H^1(\Omega_k)}^2 + \sum_j a_j + o(1)\end{aligned}$$

$$
\begin{aligned}
&\geq m(\|u\|_{L^p}^2 + \sum_j b_j{}^{\frac{2}{p}}) + o(1) \\
&\geq m(\|u\|_{L^p}^p + \sum_j b_j)^{\frac{2}{p}} + o(1) \\
&= m\Big(\int_{(\mathbf{A}_0^r \cup B^N(0;r))} d\nu\Big)^{2/p} + o(1) = m^{p/(p-2)} + o(1).
\end{aligned}
$$

By Lemma 11.6, $u = 0$. Thus, only one of the a_j is different from 0, say $a_1 = c > 0$, $a_j = 0$, $j = 2, 3, \dots$. Thus, $|\nabla u_k|^2 dz = c\delta_{z_1} + o(1)$. Clearly, $z_1 = 0$. ☐

Similarly, let $\{w_k\}$ be the solutions of Equation (1.1) in the interior flask domains $\mathbf{F}_s^r$, where $s > s_0$. Then we have dynamic systems of $\{w_k\}$ as follows:

Theorem 11.8. *Let $\overline{w}$ be a ground state solution of Equation (1.1) in $\mathbb{R}^N$. Then $w_k \to \overline{w}$ strongly in $H^1(\mathbb{R}^N)$ as $k \to \infty$.*

Proof. Note that

$$
\begin{gathered}
J(w_k) = \alpha(\mathbb{R}^N) + o(1), \\
J'(w_k) = o(1) \quad \text{as } k \to \infty.
\end{gathered}
$$

By Theorem 3.1, we prove that an integer $\ell \geq 0$ and sequences $\{z_k^i\} \subset \mathbb{R}^N$ for $0 \leq i \leq \ell$ exist such that for some subsequence $\{w_k\}$, there are $w^i \in H^1(\mathbb{R}^N)$, $w^i > 0$ in $\mathbb{R}^N$ for $0 \leq i \leq \ell$, satisfying

$$
\begin{gathered}
w_k(z) = w^0(z) + [w^1(z - z_k^1) + w^2(z - z_k^2) + \dots \\
+ w^\ell(z - z_k^\ell)] + o(1) \quad \text{strongly in} H^1(\mathbb{R}^N), \\
-\triangle w^i + w^i = (w^i)^{p-1} \quad textin \mathbb{R}^N,\ 0 \leq i \leq \ell, \\
J(w_k) = \sum_{i=0}^{\ell} J(w^i) + o(1) \quad \text{as } k \to \infty.
\end{gathered}
$$

Then, since $J(w_k) = \alpha(\mathbb{R}^N) + o(1)$, we conclude that $w_k(z) = \overline{w}(z) + o(1)$ strongly in $H^1(\mathbb{R}^N)$. ☐

Bibliographical notes: The results of this section are from Wang [71].

12. Achieved Domains

In this section we assert that the bounded domains, the quasibounded domains, the periodic domains, some interior flask domains, some flat interior flask domains, some canal domains, and some manger domains are achieved.

We begin with the following lemma.

Lemma 12.1. *$\gamma(\Omega)$ is achieved if and only if $\alpha(\Omega)$ is achieved.*

Proof. Recall that $\alpha(\Omega) = (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{\frac{2p}{2-p}}$. Suppose that there is a $u \in H_0^1(\Omega)$ such that

$$
J(u) = \alpha(\Omega),\ \langle J'(u), u\rangle = a(u) - b(u) = 0.
$$

Then we have $a(u)^{(\frac{1}{p} - \frac{1}{2})} = \gamma(\Omega)$. Let $v = \frac{u}{\|u\|_{H^1}}$. Then

$$
\|v\|_{L^p} = \frac{b(u)^{1/p}}{a(u)^{1/2}} = a(u)^{\frac{1}{p} - \frac{1}{2}} = \gamma(\Omega).
$$

Thus, $\gamma(\Omega)$ is achieved by v. On the other hand, let $\gamma(\Omega)$ be achieved by some function u where $a(u) = \|u\|_{H^1}^2 = 1$ and $b(u) = \|u\|_{L^p}^p = \gamma(\Omega)^p$. By the Lagrange multiplier theorem there is a λ such that $b'(u) = \lambda a'(u)$. It is easy to see that $\lambda = \frac{p}{2}\gamma(\Omega)^p$, so we have

$$b'(u) = \frac{p}{2}\gamma(\Omega)^p a'(u).$$

This implies

$$\gamma(\Omega)^{-p}(\int |u|^{p-2}u\varphi) = (\int \nabla u \nabla \varphi + u\varphi).$$

Thus, u is a weak solution of

$$-\Delta u + u = \gamma(\Omega)^{-p}|u|^{p-2}u.$$

Let $v = \gamma(\Omega)^{\frac{p}{2-p}}u$. Then

$$-\Delta v + v = |v|^{p-2}v.$$

We have $a(v) = b(v) = \gamma(\Omega)^{\frac{2p}{2-p}}$, $\langle J'(v), \varphi\rangle = 0$ for each $\varphi \in C_c^\infty(\Omega)$, and

$$J(v) = (\frac{1}{2} - \frac{1}{p})\gamma(\Omega)^{\frac{2p}{2-p}} = \alpha(\Omega).$$

□

Remark 12.2. Note that if u is a ground state solution for J in Ω, then u solves the semilinear elliptic (1.1) and $J(u) = \alpha(\Omega)$. By the Kato regularity, L^p-regularity and Schauder regularity, the ground state solution u of (1.1) is classical.

Theorem 12.3. *A bounded domain Ω is an achieved domain.*

For the proof of this theorem follows from Theorem 5.1.

An unbounded domain may be achieved.

Theorem 12.4. *A C^1 quasibounded domain is achieved.*

The statment of this theorem follows from Theorem 5.4.

A periodic domain in $\mathbb{R}^N$ is achieved. In Theorem 9.1, we proved that if a $(\mathrm{PS})_\alpha$-sequence for J admits a nonzero weak limit u, then u is a ground state solution for J. However, even though the weak limit is zero we can still obtain a ground state solution for J if the domain is periodic.

Theorem 12.5. *A periodic domain in $\mathbb{R}^N$ is achieved. In particular, there is a ground state solution of Equation (1.1) in $\mathbf{A}^r$, $\mathbf{A}^{r_1,r_2}$, and $\mathbb{R}^N$.*

Proof. It suffices to prove the case $\Omega = \mathbf{A}^r$. Let $\{u_n\}$ be a $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-sequence such that

$$J(u_n) = \alpha(\mathbf{A}^r) + o(1),\ J'(u_n) = o(1).$$

By Lemma 2.38, there are a subsequence $\{u_n\}$ and a $u \in H_0^1(\mathbf{A}^r)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\mathbf{A}^r).$$

Suppose that u is nonzero, then by Theorem 5.6, we are done. Suppose that $u_n \rightharpoonup 0$ weakly in $H_0^1(\mathbf{A}^r)$. Since $\alpha(\Omega)$ is positive, we have $u_n \not\to 0$ strongly in $H_0^1(\Omega)$. By Lemma 2.16, there is a subsequence $\{u_n\}$, and a constant $\alpha > 0$ such that for $n = 1, 2, \dots$,

$$Q_n = \sup_{y\in\mathbb{R}} \int_{(0,y)+\mathbf{A}^r_{-2,2}} |u_n(z)|^2 dz > \alpha > 0.$$

Take $\{z_n\}$ in $\mathbf{A}^r$, where $z_n = (0, y_n)$ such that $\int_{z_n+\mathbf{A}^r_{-2,2}} |u_n(z)|^2 dz \geq \alpha/2$, and let $w_n(z) = u_n(z + z_n)$. Then for $n = 1, 2, \dots,$

$$\int_{\mathbf{A}^r_{-2,2}} |w_n(z)|^2 dz = \int_{z_n+\mathbf{A}^r_{-2,2}} |u_n(z)|^2 dz \geq \alpha/2,$$

$$\|w_n\|_{H^1(\mathbf{A}^r)} = \|u_n\|_{H^1(\mathbf{A}^r)} \leq c,$$

so $w \in H^1_0(\mathbf{A}^r)$ exists such that $w_n \rightharpoonup w$ weakly in $H^1_0(\mathbf{A}^r)$. Clearly, $\{w_n\}$ is a (PS)-sequence in $H^1_0(\mathbf{A}^r)$ for J. By Theorem 2.31,

$$\int_{\mathbf{A}^r_{-2,2}} |w|^2 = \lim_{n\to\infty} \int_{\mathbf{A}^r_{-2,2}} |w_n|^2 \geq \alpha/2,$$

so $w \not\equiv 0$. By Theorem 5.6, there is a ground state solution of Equation (1.1) in $H^1_0(\Omega)$. $\square$

Moreover, $\mathbb{R}$ is an achieved domain: there is a classical solution u of the Equation

$$u'' = u - \gamma(\mathbb{R})^{-1}|u|^{p-2}u. \tag{12.1}$$

By Berestycki-Lions [13], such a solution is unique. The solution can be constructed as follows by routine computations.

Theorem 12.6. *With $\mu = 2/(p-2)$, we have*

$$u(r) = \Big(\frac{p\gamma(\mathbb{R})}{2}\Big)^{\frac{\mu}{2}}\{\cosh\,(r/\mu)\}^{-\mu};$$

$$\gamma(\mathbb{R}) = \Big[\frac{(2\mu+1)\Gamma(2\mu)}{\mu\Gamma(\mu)^2}\Big]^{\mu^{-1}}\Big(\frac{\mu}{4}\Big)(\mu+1)^{-\frac{p}{2}},$$

to solve Equation (12.1). In particular, $\mathbb{R}$ is an achieved domain.

Next we present achieved domains from the perturbations of nonachieved domains. By Theorem 10.7, the upper half strip $\mathbf{A}^r_0$ and the upper half hollow strip $\mathbf{A}^{r_1,r_2}_0$ are nonachieved. However, the perturbed domains of $\mathbf{A}^r_0$ and $\mathbf{A}^{r_1,r_2}_0$ may be achieved. Let $\mathbf{F}^r_s = \mathbf{A}^r_0 \cup B^N(0;s)$ be an interior flask domain. Interior flask domains are achieved for large s, but are nonachieved for small s.

Theorem 12.7. *$s_0 > 0$ exists such that Equation (1.1) has a ground state solution in $\mathbf{F}^r_s$ if $s > s_0$, but does not have any ground state solution if $s < s_0$. In particular, the interior flask domains $\mathbf{F}^r_s$ are achieved if $s > s_0$, while $\mathbf{F}^r_s$ are nonachieved if $s < s_0$.*

Proof. By Theorem 12.5, the infinite strip $\mathbf{A}^r$ admits a ground state solution. Then by Theorem 5.7 (ii), $\alpha(\mathbf{A}^r) > \alpha(\mathbb{R}^N)$. By Theorem 4.18 (ii), we have $\alpha(\mathbf{A}^r) = \alpha(\mathbf{A}^r_0)$ and by Theorem 4.27, $\lim_{s\to\infty} \alpha(B^N(0;s)) = \alpha(\mathbb{R}^N)$. Take s large enough so that

$$\alpha(B^N(0;s)) < \alpha(\mathbf{A}^r) = \alpha(\mathbf{A}^r_0).$$

By Theorem 5.1, there is a ground state solution of Equation (1.1) in $B^N(0;s)$. Then by Theorem 5.7 (ii), we have

$$\alpha(\mathbf{F}^r_s) < \alpha(B^N(0;s)).$$

We conclude that

$$\alpha(\mathbf{F}^r_s) < \alpha(B^N(0;s)) < \alpha(\mathbf{A}^r_0),$$

or

$$\alpha(\mathbf{F}_s^r) < \min\{\alpha(B^N(0;s)), \alpha(\mathbf{A}_0^r)\}.$$

By the equivalence of (i) and (vi) in Theorem 5.12, Equation (1.1) has a ground state solution in $\mathbf{F}_s^r$ for large s. If Equation (1.1) has a ground state solution in $\mathbf{F}_{s_1}^r$ and $s_1 < s_2$, then $\mathbf{F}_{s_2}^r = \mathbf{F}_{s_1}^r \cup B^N(0;s_2)$. By Theorem 5.1 and Theorem 5.7 (ii), $\alpha(\mathbf{F}_{s_2}^r) < \alpha(B^N(0;s_2))$ and $\alpha(\mathbf{F}_{s_2}^r) < \alpha(\mathbf{F}_{s_1}^r)$. By the equivalence of (i) and (vi) in Theorem 5.12, Equation (1.1) has a ground state solution in $\mathbf{F}_{s_2}^r$. Let

$$s_0 = \inf\{s > r : \text{Equation (1.1) has a ground state solution in } \mathbf{F}_s^r\}.$$

We then conclude that Equation (1.1) has a ground state solution in $\mathbf{F}_s^r$ if $s > s_0$, and Equation (1.1) does not have any ground state solution in $\mathbf{F}_s^r$ if $s < s_0$. □

Remark 12.8. In Theorem 12.7, we have asserted that the interior flask domains $\mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0;s)$ are achieved if $s > s_0$. In fact, if we replace $\mathbf{A}_0^r \cup B^N(0;s)$ by $\mathbf{A}_0^r \cup \Omega$, where Ω is a bounded domain containing $B^N(0;s)$, the theorem still holds.

For $\delta > 0$, there is a $\varepsilon(\delta) > 0$ such that a flat interior flask domain Ω_ε is an achieved domain, where

$$E_\varepsilon = \{(x,y) \in \mathbb{R}^N : (x, \varepsilon y) \in \mathbf{B}(0; r+\delta)\}; \quad \Omega_\varepsilon = \mathbf{A}_0^r \cup E_\varepsilon.$$

Theorem 12.9. *Given $\delta > 0$, $\varepsilon_0 > 0$ exists such that if $\varepsilon \le \varepsilon_0$, then the flat interior flask domain Ω_ε is an achieved domain.*

Proof. By Theorem 12.5, the infinite strip $\mathbf{A}^r$ admits a ground state solution. Since $\mathbf{A}^r \subsetneq \mathbf{A}^{r+\delta}$, by Theorem 9.4 we have $\alpha(\mathbf{A}^{r+\delta}) < \alpha(\mathbf{A}^r)$. Since $E_\varepsilon \subset \mathbf{A}^{r+\delta}$ and $\lim_{\varepsilon\to 0} \alpha(E_\varepsilon) = \alpha(\mathbf{A}^{r+\delta})$, there is an $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$, then $\alpha(E_\varepsilon) < \alpha(\mathbf{A}^r)$. If ε, $\varepsilon \le \varepsilon_0$ is fixed, a large $N \in \mathbb{N}$ exists such that

$$\alpha((\widetilde{\Omega_\varepsilon})_N) = \alpha(\mathbf{A}_N^r) = \alpha(\mathbf{A}^r).$$

Thus,

$$\alpha(\Omega_\varepsilon) \le \alpha(E_\varepsilon) < \alpha(\mathbf{A}^r) = \alpha((\widetilde{\Omega_\varepsilon})_N).$$

By Theorem 9.5, a ground state solution u of (1.1) exists. By Theorem 12.1, Ω_ε is an achieved domain. □

Fix a number $1 \leqq l \leqq N-1$ and write $\mathbb{R}^N = \mathbb{R}^l \times \mathbb{R}^{N-l}$, so that a generic $z \in \mathbb{R}^N$ is written as $z = (x,y)$ with $x \in \mathbb{R}^l$ and $y \in \mathbb{R}^{N-l}$. Let Ω be a domain in $\mathbb{R}^N$. For $y \in \mathbb{R}^{N-l}$, we denote by $\Omega^y \subset \mathbb{R}^l$ the y-section of Ω, that is,

$$\Omega^y = \{x \in \mathbb{R}^l | \quad (x,y) \in \Omega\}.$$

We consider the following canal properties:

(Ω1) Ω is a smooth domain in $\mathbb{R}^N$ and the sections Ω^y are contained in a bounded set for each $y \in \mathbb{R}^{N-l}$;

(Ω2) there is a smooth domain O in $\mathbb{R}^l$ such that

$$O \subset \Omega^y \quad \text{for each } y \in \mathbb{R}^{N-l};$$

(Ω3) for each $\delta > 0$ there is an $M > 0$ such that

$$\Omega^y \subset \{x \in \mathbb{R}^l | \operatorname{dist}(x, O) < \delta\} \quad \text{for each } |y| \geqq M.$$

Theorem 12.10. *Assume that Ω satisfies (Ω_1), (Ω_2) and (Ω_3). Then Equation (1.1) admits a ground state solution in the canal domain Ω.*

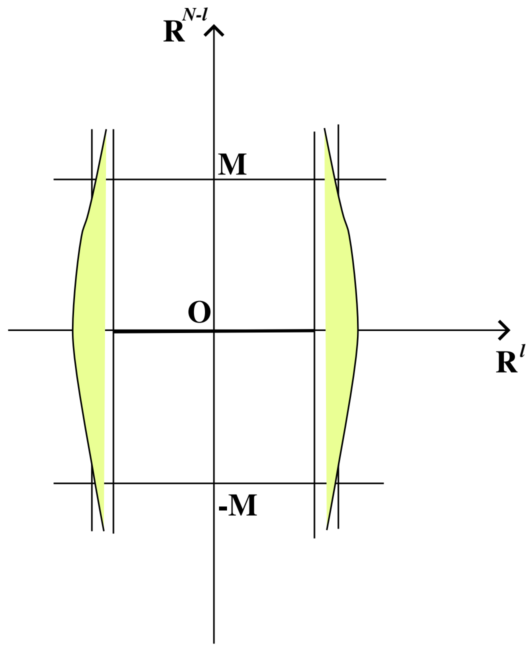

FIGURE 11. perturbed infinite strip domain.

Proof. (i) $N-l=1:(\Omega_2)$ gives

$$\widehat{O}=O\times\mathbb{R}^{N-l}\subsetneqq\Omega.$$

By Theorem 12.5, there is a ground state solution of (1.1) in $\widehat{O}$. By Theorem 5.7 (ii), we have

$$\alpha(\Omega)<\alpha(\widehat{O}). \tag{12.2}$$

Let

$$\begin{gathered}\Omega_+=\{z=(x,y)\in\Omega:y>-1\},\\ \Omega_-=\{z=(x,y)\in\Omega:y<1\},\\ \widehat{O}_+=\{z=(x,y)\in\widehat{O}:y>-1\},\\ \widehat{O}_-=\{z=(x,y)\in\widehat{O}:y<1\}.\end{gathered}$$

Then $\Omega=\Omega_+\cup\Omega$ and$_-\widehat{O}=\widehat{O}_+\cup\widehat{O}$. Moreover, both $\Omega_+\cap\Omega_-$ and $T,\widehat{O}_+\cap\widehat{O}$ are bounded. Since $\widehat{O}_+\subset\Omega_+$, then $\alpha(\Omega_+)\le\alpha(\widehat{O}_+)=\alpha(\widehat{O})$.
(a) Suppose that $\alpha(\Omega_+)=\alpha(\widehat{O}_+)=\alpha(\widehat{O})$. By (12.2), we have $\alpha(\Omega)<\alpha(\Omega_+)$.
(b) Suppose that $\alpha(\Omega_+)<\alpha(\widehat{O}_+)=\alpha(\widehat{O})$. By Theorem 4.30,

$$\lim_{\delta\to1}\alpha(\delta\widehat{O})=\alpha(\widehat{O}),$$

and a $\delta_0>1$ exists such that $\alpha(\Omega_+)<\alpha(\delta_0\widehat{O})$. From (Ω_3), there is $n_0>0$ such that $\Omega_+\backslash\overline{B^N(0;n_0)}\subset\delta_0\widehat{O}$. Thus, $\alpha(\delta_0\widehat{O})\le\alpha(\Omega_+\backslash\overline{B^N(0;n)})$ for $n\ge n_0$. Therefore, $\alpha(\Omega_+)<\alpha(\Omega_+\backslash\overline{B^N(0;n)})$ for $n\ge n_0$. From the proof of Theorem 5.12, if we

assume that (v) holds for each $n \geq n_0$, then we obtain (i). Thus J satisfies the $(PS)_{\alpha(\Omega_+)}$-condition. By Theorem 5.7 (i), $\alpha(\Omega) < \alpha(\Omega_+)$.
From Case (i) and Case (ii), we conclude that $\alpha(\Omega) < \alpha(\Omega_+)$. Similarly, we have $\alpha(\Omega) < \alpha(\Omega_-)$. Finally, we have

$$\alpha(\Omega) < \min\{\alpha(\Omega_+), \alpha(\Omega_-)\}.$$

By the equivalence of (i) and (vi) in Theorem 5.12, J satisfies the $(PS)_{\alpha(\Omega)}$-condition. By Theorem 5.6 (iii), (1.1) admits a ground state solution in Ω.
(ii) $N - l \geq 2$: $(\Omega 2)$ gives

$$\widehat{O} = O \times \mathbb{R}^{N-l} \subsetneqq \Omega,$$

and by Theorem 12.5 and 5.7 (ii), we have $\alpha(\Omega) < \alpha(\widehat{O})$. By Lemma 4.30,

$$\lim_{\gamma \to 1} \alpha(\gamma\widehat{O}) = \alpha(\widehat{O}).$$

Thus, take a γ close to 1 such that $\alpha(\Omega) < \alpha(\gamma\widehat{O})$. By $(\Omega 3)$, there is an $n_0 > 0$ such that $\Omega \backslash B^N(0;n) \subset \gamma\widehat{O}$ for $n \geq n_0$. Hence, $\alpha(\gamma\widehat{O}) \leq \tilde{\alpha}_n$ for $n \geq n_0$. Thus $\alpha(\Omega) < \tilde{\alpha}_n$ for each $n \geq n_0$. The result follows from the proof of (v) to (i) in Theorem 5.12. □

Assume $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ for $N \geq 3$. For $r_2 > r_1 > 0$ and $t > 0$, we consider a manger domain $D_t = \mathbf{A}_0^{r_1,r_2} \cup \mathbf{A}_{0,t}^{r_2}$. We have the following result.

Theorem 12.11. $\widetilde{t} \geq 0$ *exists such that Equation* (1.1) *admits a ground state solution in the manger domain* D_t *if* $t > \widetilde{t}$, *and does not admit any ground state solution in* D_t *if* $t < \widetilde{t}$.

Proof. (i) We claim that there is a $t_0 > 0$ such that Equation (1.1) admits a ground state solution in D_t if $t \geq t_0$.
Method (I): let $\{u_n\}$ be a $(\mathrm{PS})_{\alpha(D_t)}$-sequence in $H_0^1(D_t)$ for J. By Lemma 5.10 (iii), we have $a_\infty = b_\infty$ and $J_\infty \leq \alpha(D_t)$, where $J_\infty = \frac{p-2}{2p} b_\infty$. We claim that there is a $t_0 > 0$ such that $J_\infty < \alpha(D_t)$ for $t \geq t_0$. On the contrary, suppose that $J_\infty = \alpha(D_{t_n})$ for a sequence $\{t_n\}$ such that $t_n \to \infty$ as $t \to \infty$. Then $J_\infty = \alpha(D_t)$ for each $t \geq t_1$. Let $\xi(z)$ be as in (2.1) and $\xi_R(z) = \xi(\frac{2|z|}{R})$. Then there is an $R_0 > 0$ such that $\xi_R u_n \in H_0^1(\mathbf{A}_0^{r_1,r_2})$ for $R \geq R_0$. Let

$$\lambda_n = (\frac{a(\xi_R u_n)}{b(\xi_R u_n)})^{1/(p-2)}.$$

Then we have $a(\lambda_n \xi_R u_n) = b(\lambda_n \xi_R u_n)$. For $R \geq R_0$, we have

$$J(\lambda_n \xi_R u_n) \geq \alpha(\mathbf{A}_0^{r_1,r_2}),$$

or

$$(\frac{1}{2} - \frac{1}{p}) \frac{a(\xi_R u_n)^{p/(p-2)}}{b(\xi_R u_n)^{2/(p-2)}} \geq \alpha(\mathbf{A}_0^{r_1,r_2}).$$

Letting $R \to \infty$ and $n \to \infty$ and using $a_\infty = b_\infty$, we obtain

$$\alpha(D_t) = J_\infty = \frac{p-2}{2p} b_\infty \geq \alpha(\mathbf{A}_0^{r_1,r_2}).$$

Thus we have

$$\alpha(D_t) \geq \alpha(\mathbf{A}_0^{r_1,r_2}) \quad \text{for each } t > t_1. \tag{12.3}$$

Since $\mathbf{A}_0^{r_2}$ and $\mathbf{A}_0^{r_1,r_2}$ are a large domain of $\mathbf{A}^{r_2}$ and $\mathbf{A}^{r_1,r_2}$, respectively, then by Theorem 4.18, we have $\alpha(\mathbf{A}_0^{r_2}) = \alpha(\mathbf{A}^{r_2})$ and $\alpha(\mathbf{A}_0^{r_1,r_2}) = \alpha(\mathbf{A}^{r_1,r_2})$. Moreover, by Theorem 12.5, $\alpha(\mathbf{A}^{r_2}) < \alpha(\mathbf{A}^{r_1,r_2})$. We conclude that

$$\alpha(\mathbf{A}_0^{r_2}) < \alpha(\mathbf{A}_0^{r_1,r_2}). \tag{12.4}$$

By Theorem 4.25, we have $\alpha((\mathbf{A}_0^{r_2})_n) = \alpha(\mathbf{A}_0^{r_2}) + o(1)$. Hence, there is an $n_0 > 0$ such that

$$\alpha(\mathbf{A}_0^{r_2}) < \alpha((\mathbf{A}_0^{r_2})_n) < \alpha(\mathbf{A}_0^{r_1,r_2}) \quad \text{for } n \geq n_0.$$

Then we have

$$\alpha(D_n) \leq \alpha((\mathbf{A}_0^{r_2})_n) < \alpha(\mathbf{A}_0^{r_1,r_2}) \quad \text{for } n \geq n_0.$$

This contradicts (12.3). Thus, there is a $t_0 > 0$ such that $J_\infty < \alpha(D_t)$ for $t \geq t_0$. By the equivalence of (i) and (vii) in Theorem 5.12, Equation (1.1) admits a ground state solution in D_t for $t \geq t_0$.

Method (II): Since $\mathbf{A}_0^{r_2}$ and $\mathbf{A}_0^{r_1,r_2}$ are large domains of $\mathbf{A}^{r_2}$ and $\mathbf{A}^{r_1,r_2}$, respectively, then by Theorem 4.18, we have $\alpha(\mathbf{A}_0^{r_2}) = \alpha(\mathbf{A}^{r_2})$ and $\alpha(\mathbf{A}_0^{r_1,r_2}) = \alpha(\mathbf{A}^{r_1,r_2})$. Moreover, by Theorem 12.5, $\alpha(\mathbf{A}^{r_2}) < \alpha(\mathbf{A}^{r_1,r_2})$. We conclude that

$$\alpha(\mathbf{A}_0^{r_2}) < \alpha(\mathbf{A}_0^{r_1,r_2}).$$

By Theorem 4.25, we have $\alpha((\mathbf{A}_0^{r_2})_n) = \alpha(\mathbf{A}_0^{r_2}) + o(1)$. Hence, there is an $n_0 > 0$ such that

$$\alpha(\mathbf{A}_0^{r_2}) < \alpha((\mathbf{A}_0^{r_2})_n) < \alpha(\mathbf{A}_0^{r_1,r_2}) \quad \text{for } n \geq n_0.$$

Then we have

$$\alpha(D_n) < \alpha((\mathbf{A}_0^{r_2})_n) < \alpha(\mathbf{A}_0^{r_1,r_2}) \quad \text{for } n \geq n_0.$$

By the equivalence of (i) and (vi) in Theorem 5.12, Equation (1.1) admits a ground state solution in D_n for $n \geq n_0$.

(ii) If Equation (1.1) admits a ground state solution in D_{t_2} and $t_2 < t_3$, then $D_{t_3} = D_{t_2} \cup \mathbf{A}_{0,t_3}^{r_2}$. By Theorem 5.1 and Theorem 5.6 (ii), $\alpha(D_{t_3}) < \alpha(\mathbf{A}_{0,t_3}^{r_2})$ and $\alpha(D_{t_3}) < \alpha(D_{t_2})$. By the equivalence of (i) and (vi) in Theorem 5.12, Equation (1.1) admits a ground state solution in D_{t_3}. Let

$$\widetilde{t} = \inf\{t > 0 : \text{Equation (1.1) has a ground state solution in } D_t\}.$$

Then $\widetilde{t} \geq 0$ such that Equation (1.1) admits a ground state solution in D_t if $t > \widetilde{t}$ and does not admit any ground state solution in D_t if $t < \widetilde{t}$. $\square$

12.1. **Open Question:** in Theorem 12.7, is $s_0 = r$?

Bibliographical notes: Theorem 12.5 is from Lien-Tzeng-Wang [47]. Theorem 12.7 is from Chen-Lee-Wang [24] and Chen-Wang [26]. Theorem 12.7 is from Lien-Tzeng-Wang [47], Chen- Lee-Wang [24, Lemma 19], and Chen-Wang [26, Proposition 2.10]. Theorem 12.10 is from del Pino-Felmer [31]. Theorem 12.11 is from Chabrowski [20].

13. Multiple Solutions

In Section 12 we prove that there is a ground state solution in an achieved domain. In this section we prove that if we perturb (1.1) or perturb the achieved domain by adding or taking out a domain, then we obtain multiple solutions.

13.1. **Multiple Solutions for a Perturbed Equation.**

13.1.1. *Introduction.* Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic Equation (1.2)

$$-\Delta u + u = |u|^{p-2}u + h(z) \quad \text{in } \Omega;$$
$$u \in H_0^1(\Omega),$$

where Ω is a domain in $\mathbb{R}^N$ and $0 \not\equiv h(z) \in L^2(\Omega)$. Associated with (1.2), we consider the functionals a, b, and J_h, for $u \in H_0^1(\Omega)$,

$$a(u) = \int_\Omega (|\nabla u|^2 + u^2);$$
$$b(u) = \int_\Omega |u|^p;$$
$$J_h(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u) - \int_\Omega hu.$$

By Rabinowitz [64, Proposition B.10.], a, b, and J_h are from C^2. It is well-known that the solutions of (1.2) and the critical points of the energy functional J_h are the same.

When Ω is bounded, Equation (1.1) has been studied by many mathematicians: see, for instance, Bahri-Berestycki [7], Bahri-Lions [10], Rabinowitz [63], and Tanaka [69]. Suppose that h is nonnegative, small and exponential decay. Using the concentration-compactness principle of P. L. Lions [49] and [50] to prove that Equation (1.2) has at least two positive solutions, Cao-Zhu [17], Hirano [40], and Zhu [80] studied Equation (1.2) in $\mathbb{R}^N$, Hsu-Wang [41] in an exterior strip domain, and Wang [71] in an upper semi-strip with hole.

In this section, we generalize the results of Zhu [80] and Cao-Zhu [17], by relaxing the assumptions of the function h and the domain Ω, to obtain two nonzero solutions of (1.1). By adding the exponential decay of the function h, we obtain three nonzero solutions of (1.1). The main ingredients of the proofs are from Adachi-Tanaka [1], Cao-Zhu [17], Tarantello [70], and Wang [71].

13.1.2. *Existence of (PS)-Sequences.* We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_h as follows.

Definition 13.1. *(i)* For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\text{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h if $J_h(u_n) = \beta + o(1)$ and $J_h'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;
(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $H_0^1(\Omega)$ for J_h if there is a $(\text{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h;
(iii) J_h satisfies the $(\text{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\text{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h contains a convergent subsequence;
(iv) J_h satisfies the (PS)-condition in $H_0^1(\Omega)$ if for every $\beta \in \mathbb{R}$, J_h satisfies the $(\text{PS})_\beta$-condition in $H_0^1(\Omega)$.

Lemma 13.2. *If $u \in H_0^1(\Omega) \backslash \{0\}$, then*

$$\Big(\frac{a(u)^{p/2}}{b(u)}\Big)^{\frac{1}{p-2}} \geq \big(\frac{2p}{p-2}\big)^{1/2} \alpha(\Omega)^{1/2}.$$

Proof. By Lemma 4.2, we can take $\lambda > 0$ such that $\lambda u \in \mathbf{M}(\Omega)$, and then the computations is routine. $\square$

Throughout this section, we assume that $h(z)$ satisfies $0 < \|h\|_{L^2} < d(p,\alpha)$, where

$$d(p,\alpha) = (p-2)(\frac{1}{p-1})^{\frac{p-1}{p-2}}(\frac{2p}{p-2})^{1/2}\alpha(\Omega)^{1/2}. \tag{13.1}$$

Let

$$\mathbf{M}_h = \{u \in H_0^1(\Omega)\backslash\{0\} : \langle J_h'(u), u\rangle = 0\},$$

and $\mathbf{M}_0 = \mathbf{M}$. Define $\psi(u) = \langle J_h'(u), u\rangle = a(u) - b(u) - \int_\Omega hu$. Then

Lemma 13.3. *For each $u \in \mathbf{M}_h$, we have $\langle \psi'(u), u\rangle = a(u) - (p-1)b(u) \neq 0$.*

Proof. For $u \in \mathbf{M}_h$, we have $a(u) - b(u) - \int_\Omega hu = 0$. Then $\langle \psi'(u), u\rangle = 2a(u) - pb(u) - \int_\Omega hu = a(u) - (p-1)b(u)$. We claim that $\langle \psi'(u), u\rangle \neq 0$ for $u \in \mathbf{M}_h$. Let $I : \mathbf{M}_h \to \mathbb{R}$ be given by

$$I(u) = K(p)(\frac{a(u)^{p-1}}{b(u)})^{\frac{1}{p-2}} - \int_\Omega hu,$$

where $K(p) = (p-2)(\frac{1}{p-1})^{\frac{p-1}{p-2}}$. Then by Lemma 13.2, we have

$$\begin{aligned}
I(u) &= K(p)\big(\frac{a(u)^{p-1}}{b(u)}\big)^{\frac{1}{p-2}} - \int_\Omega hu \\
&\geq K(p)\big(\frac{a(u)^{p-1}}{b(u)}\big)^{\frac{1}{p-2}} - \|h\|_{L^2}\|u\|_{H^1} \\
&= \|u\|_{H^1}(K(p)\big(\frac{a(u)^{p/2}}{b(u)}\big)^{\frac{1}{p-2}} - \|h\|_{L^2}) \\
&\geq \|u\|_{H^1}(K(p)\big(\frac{2p}{p-2}\big)^{1/2}\alpha(\Omega)^{\frac{1}{2}} - \|h\|_{L^2}) \\
&= \|u\|_{H^1}(d(p,\alpha) - \|h\|_{L^2}) \quad \text{for } u \in \mathbf{M}_h.
\end{aligned}$$

Thus,

$$I(u) \geq \|u\|_{H^1}(d(p,\alpha) - \|h\|_{L^2}) > 0 \quad \text{for each } u \in \mathbf{M}_h. \tag{13.2}$$

Suppose that there is a $w \in \mathbf{M}_h$ such that $\langle \psi'(w), w\rangle = 0$. Then we have $a(w) = (p-1)b(w)$ and $\int_\Omega hw = a(w) - b(w) = (p-2)b(w)$. Now

$$\begin{aligned}
0 < I(w) &= K(p)\big(\frac{a(w)^{p-1}}{b(w)}\big)^{\frac{1}{p-2}} - \int_\Omega hw \\
&= \big(\frac{1}{p-1}\big)^{\frac{p-1}{p-2}}(p-2)\big(\frac{(p-1)^{p-1}b(w)^{p-1}}{b(w)}\big)^{\frac{1}{p-2}} - (p-2)b(w) \\
&= 0,
\end{aligned}$$

which contradicts (13.2). Thus, we conclude that $\langle \psi'(u), u\rangle \neq 0$ for each $u \in \mathbf{M}_h$. □

By Lemma 13.3, we can decompose $\mathbf{M}_h$ into $\mathbf{M}_h^+$ and $\mathbf{M}_h^-$, where

$$\begin{aligned}
\mathbf{M}_h^+ &= \{u \in \mathbf{M}_h : a(u) - (p-1)b(u) > 0\}; \\
\mathbf{M}_h^- &= \{u \in \mathbf{M}_h : a(u) - (p-1)b(u) < 0\}.
\end{aligned}$$

Consider the Nehari minimization problems for Equation (1.2). Let

$$\alpha_h(\Omega) = \inf_{u \in \mathbf{M}_h} J_h(u);$$

$$\alpha_h^+(\Omega) = \inf_{u\in \mathbf{M}_h^+} J_h(u);$$
$$\alpha_h^-(\Omega) = \inf_{u\in \mathbf{M}_h^-} J_h(u).$$

Let $\alpha_h(\Omega) = \alpha(\Omega)$, $\mathbf{M}_h = \mathbf{M}(\Omega)$ and $J_h(u) = J(u)$ for $h = 0$. For each $u \in H_0^1(\Omega)\backslash\{0\}$, we write

$$t_{\max} = \big[\frac{a(u)}{(p-1)b(u)}\big]^{\frac{1}{p-2}}.$$

Clearly, $t_{\max} > 0$. Note that

$$\begin{aligned}
J_h(tu) &= \frac{1}{2}t^2 a(u) - \frac{1}{p}t^p b(u) - t\int_\Omega hu;\\
\frac{d}{dt}J_h(tu) &= ta(u) - t^{p-1}b(u) - \int_\Omega hu;\\
\frac{d^2}{dt^2}J_h(tu) &= a(u) - (p-1)t^{p-2}b(u);\\
\langle J_h'(tu), tu\rangle &= t^2 a(u) - t^p b(u) - t\int_\Omega hu.
\end{aligned} \tag{13.3}$$

The following lemma is required.

Lemma 13.4. *For each $u \in H_0^1(\Omega)\backslash\{0\}$,*
(i) there is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-u \in \mathbf{M}_h^-$ and $J_h(t^-u) = \max_{t\geq t_{\max}} J_h(tu)$;
(ii) $t^-(u)$ is a continuous function for nonzero u ;
(iii) $\mathbf{M}_h^- = \{u \in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}}t^-(\frac{u}{\|u\|_{H^1}}) = 1\}$;
(iv) if $\int_\Omega hu > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in \mathbf{M}_h^+$ and $J_h(t^+u) = \min_{0\leq t\leq t^-} J_h(tu)$.

Proof. (*i*) Fix $u \in H_0^1(\Omega)\backslash\{0\}$. Let

$$s(t) = ta(u) - t^{p-1}b(u) \quad \text{for } t \geq 0.$$

Then $s(0) = 0$, $s(t) \to -\infty$ as $t \to \infty$, and $s(t)$ is concave and achieves its maximum at $t_{\max}$. Furthermore, by Lemma 13.2, we have

$$\begin{aligned}
s(t_{\max}) &= \Big(\frac{a(u)^{p-1}}{(p-1)b(u)}\big)^{\frac{1}{p-2}} - \big(\frac{a(u)}{(p-1)b(u)^{\frac{1}{p-1}}}\Big)^{\frac{p-1}{p-2}}\\
&= \|u\|_{H^1}(p-2)(\frac{1}{p-1})^{\frac{p-1}{p-2}}\Big[\frac{a(u)^{\frac{p}{2}}}{b(u)}\Big]^{\frac{1}{p-2}}\\
&\geq \|u\|_{H^1}(p-2)(\frac{1}{p-1})^{\frac{p-1}{p-2}}(\frac{2p}{p-2})^{\frac{1}{2}}\alpha(\Omega)^{1/2}\\
&= \|u\|_{H^1}d(p,\alpha),
\end{aligned}$$

or

$$s(t_{\max}) \geq \|u\|_{H^1}d(p,\alpha) \tag{13.4}$$

Case (*a*) : $\int_\Omega hu \leq 0$. There is a unique $t^- > t_{\max}$ such that $s(t^-) = \int_\Omega hu$ and $s'(t^-) < 0$. Now,

$$\begin{aligned}
a(t^-u) - (p-1)b(t^-u) &= (t^-)^2\Big[a(u) - (p-1)(t^-)^{p-2}b(u)\Big]\\
&= (t^-)^2 s'(t^-) < 0,
\end{aligned}$$

and

$$\begin{aligned}\langle J_h'(t^-u), t^-u\rangle &= (t^-)^2 a(u) - (t^-)^p b(u) - t^- \int_\Omega hu \\ &= t^-\Big[t^- a(u) - (t^-)^{p-1} b(u) - \int_\Omega hu\Big] \\ &= t^-\Big[s(t^-) - \int_\Omega hu\Big] = 0.\end{aligned}$$

Thus, $t^-u \in \mathbf{M}_h^-$, since for $t > t_{\max}$, we have

$$a(tu) - (p-1)b(tu) < 0;$$
$$\frac{d^2}{dt^2} J_h(tu) = \frac{1}{t^2}[a(tu) - (p-1)b(tu)] < 0;$$
$$\frac{d}{dt} J_h(tu) = ta(u) - t^{p-1}b(u) - \int_\Omega hu = 0 \qquad \text{if } t = t^-.$$

Thus, $J_h(t^-u) = \max_{t\ge t_{\max}} J_h(tu)$.
Case (b) : $\int_\Omega hu > 0$. By (13.4),

$$s(0) = 0 < \int_\Omega hu \le \|h\|_{L^2}\|u\|_{H^1} < \|u\|_{H^1} d(p,\alpha) \le s(t_{\max}),$$

there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$s(t^+) = \int_\Omega hu = s(t^-)$$
$$s'(t^+) > 0 > s'(t^-).$$

This case is similar to Case (a): We have $t^+u \in \mathbf{M}_h^+$, $t^-u \in \mathbf{M}_h^-$, and $J_h(t^-u) \ge J_h(tu) \ge J_h(t^+u)$ for each $t \in [t^+, t^-]$, and $J_h(t^+u) \le J_h(tu)$ for each $t \in [0, t^+]$. Thus,

$$J_h(t^-u) = \max_{t\ge t_{\max}} J_h(tu),$$
$$J_h(t^+u) = \min_{0\le t\le t^-} J_h(tu).$$

(ii) By the uniqueness of $t^-(u)$ and the extremity property of $t^-(u)$, $t^-(u)$ is a continuous function for nonzero u.
(iii) For $u \in \mathbf{M}_h^-$, let $v = \frac{u}{\|u\|_{H^1}}$. By part (i), there is a unique $t^-(v) > 0$ such that $t^-(v)v \in \mathbf{M}_h^-$ or $t^-(\frac{u}{\|u\|_{H^1}})\frac{1}{\|u\|_{H^1}}u \in \mathbf{M}_h^-$. Since $u \in \mathbf{M}_h^-$, we have $t^-(\frac{u}{\|u\|_{H^1}})\frac{1}{\|u\|_{H^1}} = 1$, implying

$$\mathbf{M}_h^- \subset \big\{u \in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1\big\}.$$

Conversely, let $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1$. Then

$$t^-(\frac{u}{\|u\|_{H^1}})\frac{u}{\|u\|_{H^1}} \in \mathbf{M}_h^-.$$

Thus,

$$\mathbf{M}_h^- = \big\{u \in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1\big\}.$$

(iv) By Case (b) of part (i). □

We have the following results.

Lemma 13.5. *(i) For each $u \in \mathbf{M}_h^+$, we have $\int_\Omega hu > 0$ and $J_h(u) < 0$. In particular, $\alpha_h(\Omega) \le \alpha_h^+(\Omega) < 0$;*
(ii) J_h is coercive and bounded below on $\mathbf{M}_h$;
(iii) For each minimizing sequence $\{u_n\}$ in $\mathbf{M}_h$ for J_h, we have

$$0 < \limsup_{n\to\infty} |\langle \psi'(u_n), u_n\rangle| < \infty.$$

Proof. (i) For each $u \in \mathbf{M}_h^+$, $a(u) - (p-1)b(u) > 0$ and $a(u) = b(u) + \int_\Omega hu$. Thus,

$$\int_\Omega hu = a(u) - b(u) > (p-2)b(u) > 0,$$

and

$$\begin{aligned} J_h(u) &= \frac{1}{2}a(u) - \frac{1}{p}b(u) - \int_\Omega hu \\ &= (\frac{1}{2} - \frac{1}{p})b(u) - \frac{1}{2}\int_\Omega hu \\ &< \frac{p-2}{2p}b(u) - \frac{p-2}{2}b(u) \\ &= -\frac{(p-1)(p-2)}{2p}b(u) < 0. \end{aligned}$$

Then $\alpha_h(\Omega) = \inf_{u\in\mathbf{M}_h} J_h(u) \le \inf_{u\in\mathbf{M}_h^+} J_h(u) = \alpha_h^+(\Omega) < 0$.
(ii) For $u \in \mathbf{M}_h$, we have $a(u) - b(u) - \int_\Omega hu = 0$. Then

$$\begin{aligned} J_h(u) &= (\frac{1}{2} - \frac{1}{p})a(u) - (1 - \frac{1}{p})\int_\Omega hu \\ &\ge (\frac{1}{2} - \frac{1}{p})\|u\|_{H^1}^2 - (1 - \frac{1}{p})\|h\|_{L^2}\|u\|_{H^1} \\ &= (\frac{1}{2} - \frac{1}{p})\big(\|u\|_{H^1} - \frac{p-1}{p-2}\|h\|_{L^2}\big)^2 - \frac{1}{2p(p-2)}\left[(p-1)\|h\|_{L^2}\right]^2 \\ &\ge -\frac{1}{2p(p-2)}\left[(p-1)\|h\|_{L^2}\right]^2. \end{aligned}$$

Thus, J_h is coercive and bounded below on $\mathbf{M}_h$.
(iii) Let $\{u_n\}$ be a minimizing sequence in $\mathbf{M}_h$ for J_h. Since J_h is coercive on $\mathbf{M}_h$, we can assume $\{u_n\}$ is bounded in $\mathbf{M}_h$. By the Sobolev embedding theorem, a $c > 0$ exists such that $|\langle \psi'(u_n), u_n\rangle| = |a(u_n) - (p-1)b(u_n)| \le c$. Thus, $\limsup_{n\to\infty} |\langle \psi'(u_n), u_n\rangle| < \infty$. Suppose that there is a minimizing sequence $\{w_n\}$ in $\mathbf{M}_h$ for J_h such that $\langle \psi'(w_n), w_n\rangle = o(1)$. Since J_h is a continuous function with $J_h(0) = 0$, by part (i), $\alpha_h(\Omega) < 0$. We claim that there is a $\delta > 0$ such that $\|w_n\|_{H^1} > \delta$ for each n. Otherwise, a subsequence $\{w_n\}$ exists such that $\|w_n\|_{H^1} = o(1)$. Then $J_h(w_n) = o(1)$, which is a contradiction. Since $\|w_n\|_{H^1} > \delta$ for each n and

$$o(1) = \langle \psi'(w_n), w_n\rangle = a(w_n) - (p-1)b(w_n), \tag{13.5}$$

there is a $\gamma > 0$ such that $b(w_n) \ge \gamma$ for each n, and

$$\big(\frac{a(w_n)^{p-1}}{b(w_n)}\big)^{\frac{1}{p-2}} = (p-1)^{\frac{p-1}{p-2}} b(w_n) + o(1).$$

Since $w_n \in \mathbf{M}_h$, by (13.2) and (13.5), we have

$$I(w_n) \geq \|w_n\|_{H^1}(d(p,\alpha) - \|h\|_{L^2}) \geq \delta(d(p,\alpha) - \|h\|_{L^2})$$

and

$$\int_\Omega hw_n = a(w_n) - b(w_n) = (p-2)b(w_n) + o(1).$$

Now we have

$$\begin{aligned}
0 &< \delta(d(p,\alpha) - \|h\|_{L^2}) \leq I(w_n) \\
&= (\frac{1}{p-1})^{\frac{p-1}{p-2}}(p-2)\big(\frac{a(w_n)^{p-1}}{b(w_n)}\big)^{\frac{1}{p-2}} - \int_\Omega hw_n \\
&= (\frac{1}{p-1})^{\frac{p-1}{p-2}}(p-2)(p-1)^{\frac{p-1}{p-2}}b(w_n) - (p-2)b(w_n) + o(1) \\
&= o(1),
\end{aligned}$$

which is a contradiction. □

Lemma 13.6. *Let u be in $\mathbf{M}_h$ such that $J_h(u) = \min_{v\in\mathbf{M}_h} J_h(v) = \alpha_h(\Omega)$. Then*
(i) $\int_\Omega hu > 0$;
(ii) u is a solution of Equation (1.2) in Ω.

Proof. (*i*) By Lemma 13.5 (*i*), we have

$$0 > J_h(u) = (\frac{1}{2} - \frac{1}{p})a(u) - (1 - \frac{1}{p})\int_\Omega hu.$$

Thus,

$$\int_\Omega hu > 0.$$

(*ii*) By Lemma 13.3

$$\langle \psi'(v), v\rangle = a(v) - (p-1)b(v) \neq 0 \quad \text{for each } v \in \mathbf{M}_h.$$

Since $J_h(u) = \min_{v\in\mathbf{M}_h} J_h(v)$, by the Lagrange multiplier theorem, there is a $\lambda \in \mathbb{R}^N$ such that $J'(u) = \lambda\psi'(u)$ in $H^{-1}(\Omega)$. Then we have

$$0 = \langle J_h'(u), u\rangle = \lambda\langle \psi'(u), u\rangle.$$

Thus, $\lambda = 0$ and $J_h'(u) = 0$ in $H^{-1}(\Omega)$. Therefore, u is a solution of Equation (1.2) in Ω with $J_h(u) = \alpha_h(\Omega)$. □

The following Lemma is required to prove the existence of the $(\mathrm{PS})_{\alpha_h(\Omega)}$- sequence for J_h.

Lemma 13.7. *Given $u \in \mathbf{M}_h$, then a $\delta > 0$ and a differentiable functional $l : B(0;\delta) \subset H_0^1(\Omega) \to \mathbb{R}^+$ exist such that $l(0) = 1$, $l(v)(u - v) \in \mathbf{M}_h$ for $v \in B(0;\delta)$ and*

$$\langle l'(v), \varphi\rangle\big|_{(l,v)=(1,0)} = \frac{\langle \psi'(u), \varphi\rangle}{\langle \psi'(u), u\rangle} \quad \text{for } \varphi \in C_c^\infty(\Omega).$$

Proof. For $u \in \mathbf{M}_h$, let $G : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ be given by

$$G(l,v) = \psi(l(u-v)).$$

Note that $G(1,0) = \psi(u) = \langle J_h'(u), u\rangle = 0$. Then by Lemma 13.3

$$D_lG(1,0) = \frac{\partial}{\partial l}\big[l^2a(u-v) - |l|^pb(u-v) - l\int_\Omega h(u-v)\big]\big|_{(1,0)}$$

$$\begin{aligned}&= \big[2la(u-v) - p|l|^{p-2}lb(u-v) - \int_\Omega h(u-v)\big]\big|_{(1,0)}\\&= 2a(u) - pb(u) - (a(u)-b(u))\\&= a(u) - (p-1)b(u) \neq 0.\end{aligned}$$

By the implicit function theorem, there exist $\delta > 0$ and a differentiable functional $l : B(0;\delta) \subset H_0^1(\Omega) \to \mathbb{R}$ such that $l(0) = 1$ and

$$G(l(v), v) = 0 \text{ for } v \in B(0;\delta),$$

Thus, $l(v)(u-v) \in \mathbf{M}_h$ for $v \in B(0;\delta)$. Moreover, $\varphi \in C_c^\infty(\Omega)$

$$\begin{aligned}D_\varphi l(v)|_{(1,0)} = \langle l'(v), \varphi\rangle|_{(1,0)} &= -\frac{G_v(l,v)}{G_l(l,v)}|_{(1,0)}\\&= -\frac{-2\int_\Omega \nabla u\nabla\varphi + u\varphi + p\int_\Omega |u|^{p-2}u\varphi + \int_\Omega h\varphi}{a(u)-(p-1)b(u)}\\&= \frac{\langle\psi'(u),\varphi\rangle}{\langle\psi'(u),u\rangle}.\end{aligned}$$

□

Proposition 13.8. *(i) A $(PS)_{\alpha_h(\Omega)}$-sequence $\{u_n\}$ exists in $\mathbf{M}_h$ for J_h;*
(ii) A $(PS)_{\alpha_h^+(\Omega)}$-sequence $\{u_n\}$ exists in $\mathbf{M}_h^+$ for J_h;
(iii) A $(PS)_{\alpha_h^-(\Omega)}$-sequence $\{u_n\}$ exists in $\mathbf{M}_h^-$ for J_h.

Proof. (*i*) Let $\{v_n\}$ be a minimizing sequence in $\mathbf{M}_h$ for J_h. Since J_h is continuous and bounded below on $\mathbf{M}_h$, by the Ekeland variational principle we have a minimizing sequence $\{u_n\}$ in $\mathbf{M}_h$ such that
(*a*) $J_h(u_n) \le J_h(v_n) < \alpha_h(\Omega) + \frac{1}{n^2}$;
(*b*) $\|u_n - v_n\|_{H^1} = o(1)$;
(*c*) $J_h(w) \ge J_h(u_n) - \frac{1}{n}\|u_n - w\|_{H^1}$ for each $w \in \mathbf{M}_h$.
Assume that there is an $n_0 > 0$ such that $\|J_h'(u_n)\|_{H^{-1}} > 0$ for $n \ge n_0$, otherwise we are done. For $n \ge n_0$, by the Riesz representation theorem, a unique $\phi_n \in H_0^1(\Omega)$ exists such that $\|\phi_n\|_{H^1} = 1$ and

$$\langle \frac{J_h'(u_n)}{\|J_h'(u_n)\|_{H^{-1}}}, \varphi\rangle = \langle\phi_n, \varphi\rangle_{H^1} \quad \text{for each } \varphi \in H_0^1(\Omega).$$

Let $t_n(\varepsilon) = l_n(\varepsilon\phi_n)$. Applying Lemma 13.7, we have

$$w_\varepsilon = t_n(\varepsilon)\,[u_n - \varepsilon\phi_n] \in \mathbf{M}_h.$$

Now,

$$t_n'(0) = \lim_{\varepsilon\to 0}\frac{l_n(\varepsilon\phi_n) - l_n(0)}{\varepsilon} = \langle l_n'(0), \phi_n\rangle = \frac{\langle\psi'(u_n),\phi_n\rangle}{\langle\psi'(u_n),u_n\rangle}.$$

By Lemma 13.5 (*iii*), we have $0 < \limsup_{n\to\infty} |\langle\psi'(u_n), u_n\rangle| < \infty$. Thus there is a subsequence $\{u_n\}$ and $c_1 > 0$ such that

$$|\langle\psi'(u_n), u_n\rangle| \ge c_1.$$

By the Hölder inequality and $\|\phi_n\|_{H^1} = 1$, we obtain

$$\begin{aligned}|\langle\psi'(u_n),\phi_n\rangle| &= |2\langle u_n,\phi_n\rangle_{H^1} - p\int_\Omega |u_n|^{p-2}u_n\phi_n - \int_\Omega h\phi_n|\\&\le 2\|u_n\|_{H^1} + p\|u_n\|_{L^p}^{p-1}\|\phi_n\|_{L^p} + \|h\|_{L^2}\end{aligned}$$

$$\leq c_1 + \|h\|_{L^2}.$$

Thus, $|t_n'(0)| \leq c_2$ for each $n \geq n_0$. Moreover, ε_0 and $c_3 > 0$ exist such that for $\varepsilon \leq \varepsilon_0$

$$\begin{aligned}
\frac{\|u_n - w_\varepsilon\|_{H^1}}{\varepsilon} &= \frac{1}{\varepsilon}\|(1-t_n(\varepsilon))u_n + \varepsilon t_n(\varepsilon)\phi_n\|_{H^1} \\
&\leq \Big[\frac{|t_n(0) - t_n(\varepsilon)|}{\varepsilon}\|u_n\|_{H^1} + t_n(\varepsilon)\|\phi_n\|_{H^1}\Big] \\
&= |t_n'(0)|\|u_n\|_{H^1} + 1 + o(1) \qquad \text{as } \varepsilon \to 0. \\
&\leq c_3.
\end{aligned}$$

Note that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, so

$$\begin{aligned}
&\|t_0 u_n + (1-t_0)w_\varepsilon - u_n\|_{H^1} \\
&= \|(1-t_0)(t_n(\varepsilon) - 1)u_n - \varepsilon(1-t_0)t_n(\varepsilon)\phi_n\|_{H^1} \\
&\leq (1-t_0)|t_n(\varepsilon) - 1|\|u_n\|_{H^1} + \varepsilon(1-t_0)t_n(\varepsilon) \\
&= o(1) \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

Since $J_h \in C^1$, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}|\langle J_h'(t_0 u_n + (1-t_0)w_\varepsilon), (u_n - w_\varepsilon)\rangle - \langle J_h'(u_n), (u_n - w_\varepsilon)\rangle| \\
&\leq \|J_h'(t_0 u_n + (1-t_0)w_\varepsilon) - J_h'(u_n)\|_{H^{-1}} \frac{\|u_n - w_\varepsilon\|_{H^1}}{\varepsilon} \\
&= o(1) \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Thus,

$$\langle J_h'(t_0 u_n + (1-t_0)w_\varepsilon), (u_n - w_\varepsilon)\rangle = \langle J_h'(u_n), (u_n - w_\varepsilon)\rangle + o(\varepsilon),$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. By condition (c) and the mean value theorem, we have

$$\begin{aligned}
\frac{1}{n}\|u_n - w_\varepsilon\|_{H^1} &\geq J_h(u_n) - J_h(w_\varepsilon) \\
&= \langle J_h'(t_0 u_n + (1-t_0)w_\varepsilon), (u_n - w_\varepsilon)\rangle,
\end{aligned}$$

where $t_0 \in (0,1)$. Then we obtain

$$\frac{1}{n}\|u_n - w_\varepsilon\|_{H^1} \geq \langle J_h'(u_n), (u_n - w_\varepsilon)\rangle + o(\varepsilon). \tag{13.6}$$

We divide (13.6) by $\varepsilon > 0$ and obtain

$$\begin{aligned}
\frac{c_3}{n} &\geq \frac{1}{n\varepsilon}\|u_n - w_\varepsilon\|_{H^1} \\
&\geq \frac{(1-t_n(\varepsilon))}{\varepsilon}\langle J_h'(u_n), u_n\rangle + t_n(\varepsilon)\langle J_h'(u_n), \phi_n\rangle + \frac{o(\varepsilon)}{\varepsilon} \\
&= \frac{(1-t_n(\varepsilon))}{\varepsilon}\langle J_h'(u_n), u_n\rangle + t_n(\varepsilon)\|J_h'(u_n)\|_{H^{-1}} + \frac{o(\varepsilon)}{\varepsilon}.
\end{aligned}$$

Since $\{u_n\} \subset \mathbf{M}_h$, let $\varepsilon \to 0$ to obtain

$$0 \geq -t_n'(0)\langle J_h'(u_n), u_n\rangle + \|J_h'(u_n)\|_{H^{-1}} = \|J_h'(u_n)\|_{H^{-1}},$$

which is a contradiction. Therefore, $\|J_h'(u_n)\|_{H^{-1}} \to 0$ as $n \to \infty$.
(ii) and (iii) can be proved similarly. □

13.1.3. *Existence of a Local Minimum.* By Proposition 13.8 (i), there is a $(\mathrm{PS})_{\alpha_h(\Omega)}$-sequence $\{u_n\}$ in $\mathbf{M}_h$ for J_h. Then we have the following $(\mathrm{PS})_{\alpha_h(\Omega)}$-condition.

Proposition 13.9. *Let $\{u_n\}$ in $\mathbf{M}_h$ be a $(PS)_{\alpha_h(\Omega)}$-sequence for J_h. Then a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ exist such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$. Furthermore, u_0 is a solution of Equation* (1.2) *such that $J_h(u_0) = \alpha_h(\Omega)$.*

Proof. By Lemma 13.5 (ii), $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Take a subsequence $\{u_n\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. By Lemma 2.11, u_0 is a nonzero solution of Equation (1.2) in Ω. Since

$$\begin{gathered}
J_h(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) - \int_\Omega hu_n = \alpha_h(\Omega) + o(1), \\
\langle J_h'(u_n), u_n\rangle = a(u_n) - b(u_n) - \int_\Omega hu_n = o(1).
\end{gathered} \tag{13.7}$$

By (13.7), we have

$$(\frac{1}{2} - \frac{1}{p})a(u_n) - (1 - \frac{1}{p})\int_\Omega hu_n = \alpha_h(\Omega) + o(1).$$

Since the functional a is weakly lower semicontinuous and $\int_\Omega hu_n \to \int_\Omega hu_0$ as $n \to \infty$, we have

$$\begin{aligned}
\alpha_h(\Omega) \le J_h(u_0) &= (\frac{1}{2} - \frac{1}{p})a(u_0) - (1 - \frac{1}{p})\int_\Omega hu_0 \\
&\le (\frac{1}{2} - \frac{1}{p})\liminf_{n\to\infty} a(u_n) - (1 - \frac{1}{p})\lim_{n\to\infty}\int_\Omega hu_n \\
&= \liminf_{n\to\infty}\big[(\frac{1}{2} - \frac{1}{p})a(u_n) - (1 - \frac{1}{p})\int_\Omega hu_n\big] \\
&= \alpha_h(\Omega)
\end{aligned}$$

or $J_h(u_0) = \alpha_h(\Omega)$. Let $p_n = u_n - u_0$. By Lemma 2.11 and 2.14, we have

$$\begin{aligned}
J_h(p_n) &= \frac{1}{2}a(p_n) - \frac{1}{p}b(p_n) - \int_\Omega hp_n \\
&= \frac{1}{2}a(u_n) - \frac{1}{2}a(u_0) - \frac{1}{p}b(u_n) + \frac{1}{p}b(u_0) - \int_\Omega hu_n + \int_\Omega hu_0 + o(1) \\
&= J_h(u_n) - J_h(u_0) + o(1) = o(1)
\end{aligned} \tag{13.8}$$

By Lemma 2.11, 2.14, $\int_\Omega hp_n = o(1)$ and u_0 is a solution of Equation (1.2), so

$$\begin{aligned}
\langle J_h'(p_n), p_n\rangle &= a(p_n) - b(p_n) - \int_\Omega hp_n \\
&= a(u_n) - a(u_0) - b(u_n) + b(u_0) - \int_\Omega hu_n + \int_\Omega hu_0 + o(1) \\
&= \langle J_h'(u_n), u_n\rangle - \langle J_h'(u_0), u_0\rangle = o(1)
\end{aligned} \tag{13.9}$$

Thus, by (13.8), (13.9) and $\int_\Omega hp_n = o(1)$, we have

$$\frac{p-2}{2p}a(p_n) = o(1)$$

or $u_n \to u_0$ strongly in $H_0^1(\Omega)$. □

The following result is required to prove that u_0 is the unique critical point of $J_h(u)$ in $B(r_0)$.

Lemma 13.10. *Let* $r_0 = (\frac{1}{p-1})^{\frac{1}{p-2}}(\frac{2p}{p-2})^{1/2}\alpha(\Omega)^{\frac{1}{2}}$. *Then*
(i) $\mathbf{M}_h^+ \subset B(r_0) = \{u \in H_0^1(\Omega) : \|u\|_{H^1} < r_0\}$*;*
(ii) $J_h(u)$ *is strictly convex in* $B(r_0)$*.*

Proof. (i) If $u \in \mathbf{M}_h^+$, then $a(u) > (p-1)b(u)$ and $a(u) = b(u) + \int_\Omega hu$. Thus,

$$a(u) < \frac{1}{p-1}a(u) + \|h\|_{L^2}\|u\|_{H^1}.$$

This implies

$$\begin{aligned}
\|u\|_{H^1} &< (\frac{p-1}{p-2})\|h\|_{L^2} \\
&< (\frac{p-1}{p-2})(p-2)(\frac{1}{p-1})^{\frac{p-1}{p-2}}(\frac{2p}{p-2})^{\frac{1}{2}}\alpha(\Omega)^{1/2} \\
&= (\frac{1}{p-1})^{\frac{1}{p-2}}(\frac{2p}{p-2})^{1/2}\alpha(\Omega)^{1/2} = r_0.
\end{aligned}$$

(ii) Similarly to Adachi-Tanaka [1], we have

$$J_h''(u)(v,v) = a(v) - (p-1)\int_\Omega |u|^{p-2}v^2 \quad \text{for all } v \in H_0^1(\Omega).$$

Thus, by Lemma 13.2 for $u \in H_0^1(\Omega)\backslash\{0\}$

$$\Big[\frac{a(u)^{p/2}}{b(u)}\Big]^{\frac{1}{p-2}} \geq (\frac{2p}{p-2})^{1/2}\alpha(\Omega)^{1/2},$$

then

$$\begin{aligned}
J_h''(u)(v,v) &\geq a(v) - (p-1)\|u\|_{L^p}^{p-2}\|v\|_{L^p}^2 \\
&\geq a(v) - (p-1)\big[a(u)^{\frac{p-2}{2}}(\frac{p-2}{2p})^{\frac{p-2}{2}}\alpha(\Omega)^{-\frac{(p-2)^2}{2p}}\big] \\
&\quad \times \big[a(v)(\frac{p-2}{2p})^{\frac{p-2}{p}}\alpha(\Omega)^{\frac{-(p-2)}{p}}\big] \\
&\geq a(v)\big[1 - (p-1)(\frac{2p}{p-2}\alpha(\Omega))^{\frac{2-p}{2}}\|u\|_{H^1}^{p-2}\big] \\
&> 0 \quad \text{for } u \in B(r_0).
\end{aligned}$$

Thus, $J_h''(u)$ is positive definite for $u \in B(r_0)$ and J_h is strictly convex in $B(r_0)$. □

By Proposition 13.9, a solution $u_0 \in \mathbf{M}_h$ of Equation (1.2) exists such that $J_h(u_0) = \alpha_h(\Omega)$. Furthermore, we have the following theorem.

Theorem 13.11. (i) $u_0 \in \mathbf{M}_h^+$ *and* $J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega)$*;*
(ii) u_0 *is the unique critical point of* $J_h(u)$ *in* $B(r_0)$*, where* r_0 *is as in Lemma 13.10;*
(iii) $J_h(u_0)$ *is a local minimum in* $H_0^1(\Omega)$*.*

Proof. (i) By Lemma 13.6 (i), $\int_\Omega hu_0 > 0$. We claim that $u_0 \in \mathbf{M}_h^+$. Otherwise, if $u_0 \in \mathbf{M}_h^-$, then by Lemma 13.4 a unique $t^-(u_0) = 1 > t^+(u_0) > 0$ exists such that $t^+(u_0)u_0 \in \mathbf{M}_h^+$ and

$$\alpha_h(\Omega) \leq \alpha_h^+(\Omega) \leq J_h(t^+(u_0)u_0) < J_h(u_0) = \alpha_h(\Omega),$$

which is a contradiction. Since $u_0 \in \mathbf{M}_h^+$, $\alpha_h^+(\Omega) \le J_h(u_0) = \alpha_h(\Omega) \le \alpha_h^+(\Omega)$, that is, $J_h(u_0) = \alpha_h^+(\Omega)$.
(*ii*) By part (*i*) and Lemma 13.10.
(*iii*) Since $u_0 \in \mathbf{M}_h$, by Lemma 13.7, $\delta_1 > 0$ and a differentiable functional $l(w) > 0$ exist such that $l(0) = 1$ and

$$l(w)(u_0 + w) \in \mathbf{M}_h \quad \text{for } \|w\|_{H^1} < \delta_1. \tag{13.10}$$

By Lemma 13.4,

$$1 = t^+(u_0) < t_{\max}(u_0), \tag{13.11}$$

$l(0) = 1$, and the continuity of $t_{\max}$, δ_2 exists with $\delta_1 > \delta_2 > 0$, such that

$$l(w) < t_{\max}(u_0 + w) \quad \text{for } \|w\|_{H^1} < \delta_2. \tag{13.12}$$

By (13.10) and (13.12), we have $l(w)(u_0 + w) \in \mathbf{M}_h^+$ for $\|w\|_{H^1} < \delta_2$. By Lemma 13.4, for $0 < s < t_{\max}(u_0 + w)$, we have

$$J_h(u_0) = \alpha_h^+(\Omega) \le J_h(l(w)(u_0 + w)) \le J_h(s(u_0 + w)).$$

By the continuity of $t_{\max}$ and (13.11), δ exists with $\delta_2 > \delta > 0$ such that

$$1 < t_{\max}(u_0 + w) \quad \text{for } \|w\|_{H^1} < \delta.$$

Thus, we can take $s = 1$ to obtain $J_h(u_0 + w) \ge J_h(u_0)$ for $\|w\|_{H^1} < \delta$. Hence, $J_h(u_0)$ is a local minimum in $H_0^1(\Omega)$. □

Theorem 13.12. (*i*) *If* $\int_\Omega h|u_0| > 0$*, then* u_0 *is a nonnegative solution of Equation* (1.1) *in* Ω*. Moreover, a positive solution of equation* (*1.2*) *exists for* $h \gneqq 0$*;*
(*ii*) *If* $\int_\Omega h|u_0| < 0$*, then* u_0 *is a nonpositive solution of Equation* (*1.2*) *in* Ω*. Moreover, a negative solution of Equation* (*1.2*) *exists for* $h \lneqq 0$*;*
(*iii*) *If* $\int_\Omega h|u_0| = 0$*, then* u_0 *is a solution of Equation* (*1.2*) *in* Ω *that changes sign.*

Proof. (*i*) If $\int_\Omega h|u_0| > 0$, then by Lemma 13.4 $t_{\max}(|u_0|) > t^+(|u_0|) > 0$ exists such that $t^+(|u_0|)|u_0| \in \mathbf{M}_h^+$. Since $t_{\max}(|u_0|) = t_{\max}(u_0) > 1$, we have

$$\alpha_h(\Omega) = \alpha_h^+(\Omega) \le J_h(t^+(|u_0|)|u_0|) \le J_h(|u_0|) \le J_h(u_0) = \alpha_h(\Omega),$$

or $J_h(|u_0|) = \alpha_h(\Omega)$. By Lemma 13.6 (*ii*) and Lemma 13.11 (*ii*), $u_0 = |u_0|$. Thus, we can take $u_0 \ge 0$. Moreover, if $h \gneqq 0$, we apply the maximum principle and obtain $u_0 > 0$.
(*ii*) The proof is similar to (*i*).
(*iii*) Let $u_0^+ = \max\{u_0, 0\}$ and $u_0^- = \max\{-u_0, 0\}$. Since $\int_\Omega h|u_0| = 0$, then

$$\int_\Omega hu_0^+ + \int_\Omega hu_0^- = 0.$$

By Lemma 13.6 (*i*), we have $\int_\Omega hu_0^+ > 0$ and $\int_\Omega hu_0^- < 0$. Thus, $u_0^+ \gneqq 0$ and $u_0^- \gneqq 0$. Hence, u_0 is a solution of equation (1.2) in Ω that changes sign. □

13.1.4. *Existence of Two Solutions.* Let u_0 be the local minimum for J_h in $H_0^1(\Omega)$ in Theorem 13.11. Then we have the following restricted (PS)-condition.

Proposition 13.13. *If* $\{u_n\}$ *is a (PS)*$_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h *with* $\beta < \alpha_h(\Omega) + \alpha(\Omega)$*, then a subsequence* $\{u_n\}$ *and* u^0 *in* $H_0^1(\Omega)$ *exist such that* $u_n \to u^0$ *strongly in* $H_0^1(\Omega)$ *and* $J_h(u^0) = \beta$*.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h. By Lemma 13.5 (*ii*), $\{u_n\}$ is bounded. As in the proof of Proposition 13.9, a subsequence $\{u_n\}$ and a solution u^0 of Equation (1.2) exist such that $u_n \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$. Suppose that $u_n \not\to u^0$ strongly in $H_0^1(\Omega)$. Let $p_n = u_n - u^0$ for $n = 1, 2, \dots$. By Lemma 2.11 and 2.14, we have

$$\begin{aligned} a(p_n) &= a(u_n) - a(u^0) + 0(1), \\ b(p_n) &= b(u_n) - b(u^0) + 0(1). \end{aligned} \tag{13.13}$$

Since $a(u_n) - b(u_n) - \int_\Omega hu_n = o(1)$, $a(u^0) - b(u^0) - \int_\Omega hu^0 = 0$ and $u_n \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$, we have

$$\begin{aligned} a(p_n) &= a(u_n) - a(u^0) + 0(1) \\ &= b(u_n) + \int_\Omega hu_n - b(u^0) - \int_\Omega hu^0 + o(1) \\ &= b(p_n) + o(1). \end{aligned}$$

Since $p_n \not\to 0$, we have

$$J(p_n) = \frac{1}{2}a(p_n) - \frac{1}{p}b(p_n) = (\frac{1}{2} - \frac{1}{p})a(p_n) + o(1) > 0.$$

By Theorem 4.3, a sequence $\{s_n\}$ in $\mathbb{R}^+$ exists such that $\{s_n p_n\}$ in $\mathbf{M}(\Omega)$ and $J(s_n p_n) = J(p_n) + o(1)$. Thus, by (13.13) and $u_n \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$, we have

$$\begin{aligned} \alpha(\Omega) \le J(s_n p_n) &= J(p_n) + o(1) \\ &= J_h(p_n) + o(1) \\ &= J_h(u_n) - J_h(u^0) + o(1) \\ &= \beta - J_h(u^0) + o(1) \\ &< \alpha_h(\Omega) + \alpha(\Omega) - J_h(u^0) + o(1). \end{aligned}$$

Then $\alpha_h(\Omega) > J_h(u^0) \ge \alpha_h(\Omega)$, which is a contradiction. Thus, $u_n \to u^0$ strongly in $H_0^1(\Omega)$. □

Throughout this section, let Ω be an achieved domain in $\mathbb{R}^N$.

Lemma 13.14. *Let $\overline{u}$ be a positive solution of Equation (1.1) in Ω such that $J(\overline{u}) = \alpha(\Omega)$ and u_0 is the local minimum in Theorem 13.11. Then*
(*i*) *If $\int_\Omega h|u_0| > 0$, then we have*

$$\sup_{t\ge 0} J_h(u_0 + t\overline{u}) < J_h(u_0) + \alpha(\Omega).$$

(*ii*) *If $\int_\Omega h|u_0| < 0$, then we have*

$$\sup_{t\ge 0} J_h(u_0 + t(-\overline{u})) < J_h(u_0) + \alpha(\Omega).$$

Proof. (*i*) Since $\int_\Omega h|u_0| > 0$, by Theorem 13.12 (*i*), u_0 is a nonnegative solution of Equation (1.2). Let $f(s) = s^{p-1}$ and $F(u) = \int_0^u f(s)ds = \frac{1}{p}b(u)$, then

$$\begin{aligned}
&J_h(u_0 + t\overline{u}) \\
&= \frac{1}{2}a(u_0 + t\overline{u}) - \frac{1}{p}b(u_0 + t\overline{u}) - \int_\Omega h(u_0 + t\overline{u}) \\
&= \frac{1}{2}\left[a(u_0) + a(t\overline{u}) + 2\langle u_0, t\overline{u}\rangle_{H^1}\right] - \frac{1}{p}b(u_0 + t\overline{u}) - \int_\Omega h(u_0 + t\overline{u}) \\
&= J_h(u_0) + J(t\overline{u}) + \langle u_0, t\overline{u}\rangle_{H^1} + \frac{1}{p}\left[b(u_0) + b(t\overline{u}) - b(u_0 + t\overline{u})\right] - \int_\Omega ht\overline{u} \\
&= J_h(u_0) + J(t\overline{u}) + t(\int_\Omega u_0^{p-1}\overline{u} + h\overline{u}) - \int_\Omega ht\overline{u} \\
&+ \frac{1}{p}\left[b(u_0) + b(t\overline{u}) - b(u_0 + t\overline{u})\right] \\
&= J_h(u_0) + J(t\overline{u}) - \int_\Omega \Big\{ \int_0^{t\overline{u}} [f(u_0 + s) - f(s) - f(u_0)]ds\Big\}.
\end{aligned} \tag{13.14}$$

For $v > 0$ and $w > 0$, we have

$$f(v+w) = (v+w)^{p-1} = (v+w)^{p-2}v + (v+w)^{p-2}w > v^{p-1} + w^{p-1} = f(v) + f(w). \tag{13.15}$$

Thus, $J_h(u_0 + t\overline{u}) \le J_h(u_0) + J(t\overline{u})$. Since $J(t\overline{u}) \to -\infty$ as $t \to \infty$, there is a $t_0 > 0$ such that $J_h(u_0 + t\overline{u}) < J_h(u_0)$ for $t \ge t_0$. Hence,

$$\sup_{t\ge 0} J_h(u_0 + t\overline{u}) = \sup_{0\le t\le t_0} J_h(u_0 + t\overline{u}).$$

Let $g_1(t) = J_h(u_0 + t\overline{u})$ for $t \ge 0$. By the continuity of $g_1(t)$, given $\varepsilon = \frac{1}{2}\alpha(\Omega) > 0$ there is a $t_0 > t_1 > 0$ such that $g_1(t) < g_1(0) + \frac{1}{2}\alpha(\Omega)$ for $2t_1 > t \ge 0$. Then

$$\sup_{0\le t\le t_1} J_h(u_0 + t\overline{u}) \le J_h(u_0) + \frac{1}{2}\alpha(\Omega) < J_h(u_0) + \alpha(\Omega).$$

Now, it only remains to show that

$$\sup_{t_1\le t\le t_0} J_h(u_0 + t\overline{u}) < J_h(u_0) + \alpha(\Omega).$$

Let $g_2(t) = J(t\overline{u})$ for $t \ge 0$, then

$$g_2'(t) = ta(\overline{u}) - t^{p-1}b(\overline{u}) \quad \text{and } g_2''(t) = a(\overline{u}) - (p-1)t^{p-2}b(\overline{u}).$$

There is a unique $\bar{t} = \left[\frac{a(\overline{u})}{b(\overline{u})}\right]^{1/(p-2)} = 1$ such that $g_2'(\bar{t}) = 0$ and $g_2''(\bar{t}) < 0$. Thus, $g_2(t)$ has an absolute maximum at $t = 1$. Therefore,

$$\sup_{t\ge 0} J(t\overline{u}) = J(\overline{u}) = \alpha(\Omega).$$

By (13.14), (13.15), we obtain

$$\begin{aligned}
&\sup_{t_1\le t\le t_0} J_h(u_0 + t\overline{u}) \\
&\le J_h(u_0) + \alpha(\Omega) - \inf_{t_1\le t\le t_0} \int_\Omega \Big\{ \int_0^{t\overline{u}} \left[f(u_0 + s) - f(s) - f(u_0)\right] ds\Big\} \\
&< J_h(u_0) + \alpha(\Omega).
\end{aligned}$$

Thus, $\sup_{t\geq 0} J_h(u_0+t\overline{u}) < J_h(u_0)+\alpha(\Omega)$.

(ii) Since $\int_\Omega h|u_0| < 0$, by Theorem 13.12 (ii), u_0 is a nonpositive solution of Equation (1.2). Let $f(s)=|s|^{p-2}s$ and $F(u)=\int_0^u f(s)ds = \frac{1}{p}b(u)$. Then

$$\begin{aligned}
J_h(u_0-t\overline{u}) &= \frac{1}{2}a(u_0-t\overline{u}) - \frac{1}{p}b(u_0-t\overline{u}) - \int_\Omega h(u_0-t\overline{u}) \\
&= \frac{1}{2}\left[a(u_0)+a(t\overline{u})+2\langle u_0,-t\overline{u}\rangle_{H^1}\right] \\
&\quad - \frac{1}{p}b(-u_0+t\overline{u}) - \int_\Omega h(u_0-t\overline{u}) \\
&= J_h(u_0)+J(t\overline{u})+\langle u_0,-t\overline{u}\rangle_{H^1} \\
&\quad + \frac{1}{p}\left[b(u_0)+b(t\overline{u})-b(-u_0+t\overline{u})\right] + \int_\Omega ht\overline{u} \\
&= J_h(u_0)+J(t\overline{u}) - t\Big(\int_\Omega |u_0|^{p-2}u_0\overline{u}+h\overline{u}\Big) + \int_\Omega ht\overline{u} \\
&\quad + \frac{1}{p}\left[b(-u_0)+b(t\overline{u})-b(-u_0+t\overline{u})\right] \\
&= J_h(-u_0)+J(t\overline{u}) + \int_\Omega |-u_0|^{p-2}(-u_0)t\overline{u} \\
&\quad + \frac{1}{p}\left[b(u_0)+b(t\overline{u})-b(-u_0+t\overline{u})\right] \\
&= J_h(u_0)+J(t\overline{u}) - \int_\Omega \Big\{\int_0^{t\overline{u}} \left[f(-u_0+s)-f(s)-f(-u_0)\right]ds\Big\}.
\end{aligned}$$

Similarly to part (i), we have $\sup_{t\geq 0} J_h(u_0+t(-\overline{u})) < J_h(u_0)+\alpha(\Omega)$. □

Theorem 13.15. *If $\int_\Omega h|u_0| \neq 0$, where u_0 is the local minimum in Theorem 13.11, then Equation* (1.2) *has two solutions $u_0 \in \mathbf{M}_h^+$, $u^0 \in \mathbf{M}_h^-$ such that $J_h(u_0) = \alpha_h^+ < \alpha_h^- = J_h(u^0)$. Moreover, if $h \gneqq 0$ ($\lneqq 0$), then Equation* (1.2) *has at least two positive (negative) solutions in Ω.*

Proof. For $u \in H_0^1(\Omega)$ with $\|u\|_{H^1}=1$, by Lemma 13.4 there is a unique $t^-(u)>0$ such that $t^-(u)u \in \mathbf{M}_h^-$ and

$$J_h(t^-(u)u) = \max_{t\geq t_{\max}} J_h(tu).$$

By Lemma 13.4 (ii) and (iii), we have that $t^-(u)$ is a continuous function for nonzero u and

$$\mathbf{M}_h^- = \Big\{u\in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}}t^-\Big(\frac{u}{\|u\|_{H^1}}\Big)=1\Big\}.$$

Let

$$\begin{gathered}
A_1 = \Big\{u\in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}}t^-\Big(\frac{u}{\|u\|_{H^1}}\Big)>1\Big\}\cup\{0\} \\
A_2 = \Big\{u\in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}}t^-\Big(\frac{u}{\|u\|_{H^1}}\Big)<1\Big\}.
\end{gathered}$$

Then $H_0^1(\Omega)\backslash\mathbf{M}_h^- = A_1\cup A_2$. For each $u\in\mathbf{M}_h^+$, we have

$$1 < t_{\max}(u) < t^-(u).$$

Since $t^-(u) = \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$, then $\mathbf{M}_h^+ \subset A_1$. In particular, $u_0 \in A_1$. We claim that a $t_0 > 0$ exists such that $u_0 + t_0\overline{u} \in A_2$. First, we find a constant $c > 0$ such that $0 < t^-(\frac{u_0+t\overline{u}}{\|u_0+t\overline{u}\|_{H^1}}) < c$ for each $t \geq 0$. Otherwise, a sequence $\{t_n\}$ exists such that $t_n \to \infty$ and $t^-(\frac{u_0+t_n\overline{u}}{\|u_0+t_n\overline{u}\|_{H^1}}) \to \infty$ as $n \to \infty$. Let $v_n = \frac{u_0+t_n\overline{u}}{\|u_0+t_n\overline{u}\|_{H^1}}$. Since $t^-(v_n)v_n \in \mathbf{M}_h^- \subset \mathbf{M}_h$ and by the Lebesgue dominated convergence theorem,

$$\begin{aligned} b(v_n) &= \frac{1}{\|u_0 + t_n\overline{u}\|_{H^1}^p} \int_\Omega (u_0 + t_n\overline{u})^p \\ &= \frac{1}{\|\frac{u_0}{t_n} + \overline{u}\|_{H^1}^p} \int_\Omega (\frac{u_0}{t_n} + \overline{u})^p \\ &\to \frac{\int_\Omega \overline{u}^p}{\|\overline{u}\|_{H^1}^p} \quad \text{as } n \to \infty. \end{aligned}$$

We have

$$\begin{aligned} J_h(t^-(v_n)v_n) &= \frac{1}{2}\left[t^-(v_n)\right]^2 - \frac{1}{p}\left[t^-(v_n)\right]^p b(v_n) - t^-(v_n)\int_\Omega hv_n \\ &\to -\infty \qquad \text{as } n \to \infty. \end{aligned}$$

However, J_h is bounded below on $\mathbf{M}_h$, which is a contradiction. Let

$$t_0 = \Big(\frac{2p}{(p-2)\alpha(\Omega)}|c^2 - a(u_0)|\Big)^{1/2} + 1.$$

Then

$$\|u_0 + t_0\overline{u}\|_{H^1}^2 = a(u_0) + t_0^2(\frac{p-2}{2p})\alpha(\Omega) + o(1) > c^2 > \left[t^-(\frac{u_0 + t_0\overline{u}}{\|u_0 + t_0\overline{u}\|_{H^1}})\right]^2,$$

that is, $u_0 + t_0\overline{u} \in A_2$. Define a path $\gamma(s) = u_0 + st_0\overline{u}$ for $s \in [0,1]$ where $t_0 > 1$, then

$$\gamma(0) = u_0 \in A_1, \quad \gamma(1) = u_0 + t_0\overline{u} \in A_2.$$

Since $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$ is a continuous function for nonzero u and $\gamma([0,1])$ is connected, a $s_0 \in (0,1)$ exists such that $u_0 + s_0t_0\overline{u} \in \mathbf{M}_h^-$. Thus, by Lemma 13.12 and Theorem 13.14 we have

$$\alpha_h^- \leq J_h(u_0 + s_0t_0\overline{u}) \leq \max_{s\in[0,1]} J_h(\gamma(s)) < J_h(u_0) + \alpha(\Omega) \quad \text{for } \int_\Omega h|u_0| > 0.$$

Similarly, we also have

$$\alpha_h^- < J_h(u_0) + \alpha(\Omega) \quad \text{for } \int_\Omega h|u_0| < 0.$$

By Proposition 13.8 (*iii*), a sequence $\{u_n\}$ in $\mathbf{M}_h^-$ exists such that

$$\begin{gathered} J_h(u_n) = \alpha_h^-(\Omega) + o(1), \\ J_h'(u_n) = o(1) \quad \text{strongly in } H^{-1}(\Omega). \end{gathered}$$

Then by Proposition 13.13, a subsequence $\{u_n\}$ and $u^0 \in \mathbf{M}_h$ exist such that $u_n \to u^0$ strongly in $H_0^1(\Omega)$, u^0 is a solution of Equation (1.2), and $J_h(u^0) = \alpha_h^-(\Omega)$. By the Sobolev continuous embedding theorem, we have $u_n \to u^0$ in $L^p(\Omega)$. Thus,

$$a(u^0) - (p-1)b(u^0) \leq 0.$$

Then $u^0 \in \mathbf{M}_h^-$. This implies $u^0 \neq u_0$. We must now show that

$$J_h(u_0) = \alpha_h < \alpha_h^- = J_h(u^0).$$

Otherwise, assume that $J_h(u^0) = \alpha_h^- = J_h(u_0) = \alpha_h$. By Lemma 13.6 (i) we have $\int_\Omega hu^0 > 0$. By Lemma 13.4, $t^+(u^0) > 0$ exists such that $t^+(u^0)u^0 \in \mathbf{M}_h^+$ and

$$\alpha_h^+(\Omega) \leq J_h(t^+(u^0)u^0) < J_h(u^0) \leq \alpha_h \leq \alpha_h^-,$$

which is a contradiction.

Finally, if $h \gneqq 0$, by Lemma 13.4 $t^-(|u^0|) > 0$ exists such that

$$t^-(|u^0|)|u^0| \in \mathbf{M}_h^-,\ t^-(|u^0|) > t_{\max}(|u^0|) = t_{\max}(u^0)$$

and

$$\begin{aligned} \alpha_h^-(\Omega) &\leq J_h(t^-(|u^0|)|u^0|) \leq J_h(t^-(|u^0|)u^0) \\ &\leq \max_{t \geq t_{\max}(u^0)} J_h(tu^0) = J_h(u^0) = \alpha_h^-(\Omega). \end{aligned}$$

Thus,

$$J_h(t^-(|u^0|)|u^0|) = J_h(t^-(|u^0|)u^0) = \alpha_h^-(\Omega).$$

We conclude that $\int_\Omega hu^0 = \int_\Omega h|u^0|$. Let $u_+^0 = \max\{u^0, 0\}$ and $u_-^0 = \max\{-u^0, 0\}$, then $\int_\Omega hu_-^0 = 0$. Since $h \gneqq 0$ and $u_-^0 \geq 0$, we have $u_-^0 = 0$. Hence, $u^0 \geq 0$. By the maximum principle, $u^0 > 0$. □

Remark 13.16. By Theorems 13.11 and 13.15, there is a unique solution u_0 of Equation (1.2) in Ω such that $J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega)$.

Bibliographical notes: The results of this section are from Lin-Wang-Wu [55].

13.1.5. *Three Solutions.* Throughout this section, we consider a $C^{1,1}$ domain Ω to be $\mathbf{A}^r$, $\mathbb{R}^N$, $\mathbf{A}^r \backslash \overline{D}$, or $\mathbb{R}^N \backslash \overline{D}$, where D is a bounded domain in $\mathbb{R}^N$ and assume that $h \in L^{\frac{N}{2}}(\Omega) \cap L^s(\Omega) \cap L^2(\Omega)$ for some $s > N$, $0 < \|h\|_{L^2} < d(p,\alpha)$, and

$$0 \leq h(z) \leq c\exp(-(1+\varepsilon)|z|) \quad \text{for any } z \in \Omega,$$

for some positive constants c and ε, where $d(p,\alpha)$ is defined as in (13.1). Then we have the following lemma.

Lemma 13.17. *Let u be a positive solution of the Equation (1.2). Then for any $0 < \delta < \min\{\varepsilon, 1\}$, positive constants c_δ^1, c_δ^2 and R exist such that for $|z| \geq R$*

$$c_\delta^1 \exp(-(1+\delta)|z|) \leq u(z) \leq c_\delta^2 \exp(-(1-\delta)|z|).$$

Proof. By Lemma 8.11 and 8.12, we have $u \in W^{2,s}(\Omega) \cap C^{1,\theta}(\overline{\Omega})$ for some θ, $0 < \theta < 1$ and $\lim_{|z|\to\infty} u(z) = 0$. Take $R_1 > 0$ such that $D \subset B^N(0;R_1)$. For any $0 < \delta < \min\{\varepsilon, 1\}$, we choose $R_2 > R_1 > 0$ such that

$$(1+\delta) - \frac{\sqrt{1+\delta}(N-1)}{|z|} \geq 1 \quad \text{for } |z| \geq R_2. \tag{13.16}$$

Let $\beta = \sqrt{1+\delta}$ and $v_1(z) = \mu\exp(-\beta(|z| - R_2))$, where $\mu = \min_{|z|=R_2} u(z) > 0$. Then $\min_{|z|=R_2}(u - v_1)(z) \geq 0$. By (13.16), for $|z| > R_2$

$$\triangle(u - v_1)(z) = u - |u|^{p-2}u - h(z) - \big(\beta^2 - \frac{\beta(N-1)}{|z|}\big)v_1$$

$$\begin{aligned}&\le u - \big(\beta^2 - \frac{\beta(N-1)}{|z|}\big)v_1\\&\le (u-v_1)(z).\end{aligned}$$

By Lemma 9.3, for $|z| > R_2$,

$$u(z) - v_1(z) \ge \min_{|z|=R_2}(u-v_1)(z) \ge 0.$$

Thus, we have

$$\begin{aligned}u(z) &\ge v_1(z)\\&= \mu\exp(-\beta(|z|-R_2))\\&= \mu\exp(R_2\sqrt{1+\delta})\exp(-\beta|z|)\\&\ge c_\delta^1\exp(-(1+\delta)|z|) \quad \text{for } |z| \ge R_2.\end{aligned} \tag{13.17}$$

We know that there exist positive numbers ε, c such that

$$0 \le h(z) \le c\exp(-(1+\varepsilon)|z|) \quad \text{for any } z \in \Omega.$$

For any $0 < \delta < \min\{\varepsilon, 1\}$, by (13.17), there is $R_3 > R_2 > 0$ such that

$$\frac{\delta}{2}u(z) \ge h(z) \quad \text{for } |z| \ge R_3. \tag{13.18}$$

Since $\lim_{|z|\to\infty} u(z) = 0$, there is $R > R_3 > 0$ such that

$$1 - u^{p-2} \ge 1 - \frac{\delta}{2} \quad \text{for } |z| \ge R. \tag{13.19}$$

Let $\gamma = \sqrt{1-\delta}$ and $v_2(z) = \nu\exp(-\gamma(|z|-R))$, where $\nu = \max_{|z|=R} u(z) > 0$. Thus $\min|z| = R(v_2-u)(z) \ge 0$. By (13.18) and (13.19), for $|z| > R$

$$\begin{aligned}\triangle(v_2-u)(z) &= (\gamma^2 - \frac{\gamma(N-1)}{|z|})v_2(z) - u + |u|^{p-2}u + h(z)\\&\le \gamma^2 v_2 - (1-\frac{\delta}{2})u + h(z)\\&= (1-\delta)(v_2(z)-u(z)) - \frac{\delta}{2}u + h(z)\\&\le (1-\delta)(v_2(z)-u(z)).\end{aligned}$$

By Lemma 9.3, for $|z| > R$

$$v_2(z) - u(z) \ge \min_{|z|=R}(v_2-u)(z) \ge 0.$$

Thus, we have

$$\begin{aligned}u(z) &\le v_2(z) = \nu\exp(-\gamma(|z|-R))\\&= \nu\exp(R\sqrt{1-\delta})\exp(-\gamma|z|)\\&\le c_\delta^2\exp(-(1-\delta)|z|) \quad \text{for } |z| \ge R.\end{aligned}$$

□

By Lien-Tzeng-Wang [47], there is a positive ground state solution $\bar{u}$ of the Equation (1.1) in $\mathbb{R}^N$ such that $J(\bar{u}) = \alpha(\mathbb{R}^N)$. By Gidas-Ni-Nirenberg [35], we have that $\overline{u}$ is radially symmetric about 0 in $\mathbb{R}^N$. Similarly to Lemma 13.17, for any $\delta' > 0$, positive constants $c_{\delta'}^1$ and $c_{\delta'}^2$ exist such that

$$c_{\delta'}^1\exp(-(1+\delta')|z|) \le \overline{u}(z) \le c_{\delta'}^2\exp(-(1-\delta')|z|) \quad \text{for } z \in \mathbb{R}^N.$$

By Lemma 13.17, there is a $R > 0$ such that $D \subset B(0;R)$. For such R, let $\psi_R : \mathbb{R}^N \to [0,1]$ be a $C^\infty-$function on $\mathbb{R}^N$ such that $0 \le \psi_R \le 1$,

$$\psi_R(z) = \begin{cases} 1 & \text{for } |z| \ge R+1; \\ 0 & \text{for } |z| \le R. \end{cases}$$

For $\overline{z} \in \mathbb{R}^N$, we define

$$v_{\overline{z}}(z) = \psi_R(z)\bar{u}(z-\overline{z}).$$

Clearly, $v_{\overline{z}}(z) \in H_0^1(\Omega)$.

Lemma 13.18. *(i) $a(v_{\overline{z}}) = b(v_{\overline{z}}) + o(1)$ as $|\overline{z}| \to \infty$;*
(ii) $J(v_{\overline{z}}) = \alpha(\Omega) + o(1)$ as $|\overline{z}| \to \infty$;
(iii) $v_{\overline{z}} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $|\overline{z}| \to \infty$.

Proof. (*i*-1) $a(v_{\overline{z}}) = a(\bar{u}) + o(1)$ as $|\overline{z}| \to \infty$: since $\overline{u} \in H_0^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\|v_{\overline{z}}(z) - \bar{u}(z-\overline{z})\|_{H^1}^2 \\
&= \int_{\mathbb{R}^N} |\nabla[(\psi_R(z)-1)\bar{u}(z-\overline{z})]|^2 dz + \int_{\mathbb{R}^N} |(\psi_R(z)-1)\bar{u}(z-\overline{z})|^2 dz \\
&\le 2\int_{\mathbb{R}^N} |\nabla(\psi_R(z)-1)|^2 |\bar{u}(z-\overline{z})|^2 dz + 2\int_{\mathbb{R}^N} |(\psi_R(z)-1)|^2 |\nabla\bar{u}(z-\overline{z})|^2 dz \\
&\quad + \int_{\{|z| \le R+1\}} |\bar{u}(z-\overline{z})|^2 dz \\
&\le 2\int_{\{R \le |z| \le R+1\}} |\bar{u}(z-\overline{z})|^2 dz + 2\int_{\{|z| \le R+1\}} (|\nabla\bar{u}(z-\overline{z})|^2 + |\bar{u}(z-\overline{z})|^2) dz = o(1).
\end{aligned}$$

Thus, $a(v_{\overline{z}}) = a(\bar{u}(z-\overline{z})) + o(1) = a(\bar{u}) + o(1)$ as $|\overline{z}| \to \infty$.
(*i*-2) $b(v_{\overline{z}}) = b(\bar{u}) + o(1)$ as $|\overline{z}| \to \infty$: since $\overline{u} \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\|v_{\overline{z}}(z) - \bar{u}(z-\overline{z})\|_{L^p}^p &= \int_{\mathbb{R}^N} |(\psi_R(z)-1)|^p |\bar{u}(z-\overline{z})|^p dz \\
&\le \int_{\{|z| \le R+1\}} |\bar{u}(z-\overline{z})|^p dz = o(1).
\end{aligned}$$

Thus, $b(v_{\overline{z}}) = b(\bar{u}(z-\overline{z})) + o(1) = b(\bar{u}) + o(1)$ as $|\overline{z}| \to \infty$. By (*i*-1), (*i*-2)and that $\bar{u}$ is a solution of the Equation (1.1), we have

$$a(v_{\overline{z}}) = a(\bar{u}) + o(1) = b(\bar{u}) + o(1) = b(v_{\overline{z}}) + o(1) \quad \text{as } |\overline{z}| \to \infty.$$

(*ii*) By Lemma 4.18, $J(v_{\overline{z}}) = J(\bar{u}) + o(1) = \alpha(\mathbb{R}^N) + o(1) = \alpha(\Omega) + o(1)$ as $|\overline{z}| \to \infty$.
(*iii*) For $\phi \in C_c^1(\Omega)$ with $K = \operatorname{supp}\phi$, then $K \subset \Omega$ is compact.

$$\begin{aligned}
|\langle v_{\overline{z}}, \phi\rangle_{H^1}| &= \Big|\int_\Omega \nabla v_{\overline{z}}(z)\nabla\phi(z)dz + \int_\Omega v_{\overline{z}}(z)\phi(z)dz\Big| \\
&= \Big|\int_K \nabla\left[\psi_R(z)\bar{u}(z-\overline{z})\right]\nabla\phi(z)dz + \int_K \psi_R(z)\bar{u}(z-\overline{z})\phi(z)dz\Big| \\
&\le \|\nabla\left[\psi_R(z)\bar{u}(z-\overline{z})\right]\|_{L^2(K)}\|\nabla\phi\|_{L^2(K)} \\
&\quad + \|\psi_R(z)\bar{u}(z-\overline{z})\|_{L^2(K)}\|\phi\|_{L^2(K)} \\
&= o(1) \quad \text{as } |\overline{z}| \to \infty.
\end{aligned}$$

By part (i), there is a $c > 0$ such that $\|v_{\overline{z}}\|_{H^1} \le c$. For $\varepsilon > 0$ and $\varphi \in H_0^1(\Omega)$, there exist $\phi \in C_c^1(\Omega)$ and $l_0 > 0$ such that

$$\|\varphi - \phi\|_{H^1} < \varepsilon/2c$$
$$|\langle v_{\overline{z}}, \phi\rangle_{H^1}| < \varepsilon/2 \quad \text{for } |\overline{z}| \ge l_0.$$

Thus,

$$\begin{aligned}\langle v_{\overline{z}}, \varphi\rangle_{H^1} &= \langle v_{\overline{z}}, \varphi - \phi\rangle_{H^1} + \langle v_{\overline{z}}, \phi\rangle_{H^1}\\ &\le \|v_{\overline{z}}\|_{H^1}\|\varphi - \phi\|_{H^1} + \langle v_{\overline{z}}, \phi\rangle_{H^1}\\ &< c\|\varphi - \phi\|_{H^1} + \frac{\varepsilon}{2}\\ &< \varepsilon \text{ for } |\overline{z}| \ge l_0.\end{aligned}$$

Therefore, $v_{\overline{z}} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $|\overline{z}| \to \infty$. □

Lemma 13.19. *Let D be a domain in $\mathbb{R}^N$. If $f : D \to \mathbb{R}$ satisfies*

$$\int_D |f(z)e^{\sigma|z|}|dz < \infty \quad \textit{for some } \sigma > 0,$$

then

$$\Big(\int_D f(z)e^{-\sigma|z-\overline{z}|}dz\Big)e^{\sigma|\overline{z}|} = \int_D f(z)e^{\sigma\frac{\langle z,\overline{z}\rangle}{|\overline{z}|}}dz + o(1) \quad \textit{as } |\overline{z}| \to \infty.$$

Proof. Since $\sigma|\overline{z}| \le \sigma|z| + \sigma|z - \overline{z}|$, we have

$$|f(z)e^{-\sigma|z-\overline{z}|}e^{\sigma|\overline{z}|}| \le |f(z)e^{\sigma|z|}|.$$

Since $-\sigma|z - \overline{z}| + \sigma|\overline{z}| = \sigma\frac{\langle z,\overline{z}\rangle}{|\overline{z}|} + o(1)$ as $|\overline{z}| \to \infty$, then the lemma follows from Theorem 2.20. □

Since Ω here is nonachieved, we need a delicate result.

Lemma 13.20. *$l_0 > 0$ exists such that for $|\overline{z}| \ge l_0$*

$$\sup_{t\ge 0} J_h(u_0 + tv_{\overline{z}}) < J_h(u_0) + \alpha(\Omega),$$

where u_0 is the local minimum in Theorem 13.12.

Proof. Since J_h is continuous in $H_0^1(\Omega)$ and $\{v_{\overline{z}}\}$ is bounded in $H_0^1(\Omega)$, there is $t_0 > 0$ such that for $0 \le t < t_0$ and each $v_{\overline{z}} \in H_0^1(\Omega)$

$$J_h(u_0 + tv_{\overline{z}}) < J_h(u_0) + \alpha(\Omega).$$

Thus, we only need to show that there exists $l_0 > 0$ such that for $|\overline{z}| \ge l_0$

$$\sup_{t\ge t_0} J_h(u_0 + tv_{\overline{z}}) < J_h(u_0) + \alpha(\Omega).$$

First, we observe that if $a \ge 0$ and $b \ge 0$, then there is $c = c(p) > 0$ independent of a and b such that

$$(a + b)^p \ge a^p + b^p + p(a^{p-1}b + ab^{p-1}) - ca^{p/2}b^{p/2}.$$

Hence, we get

$$\begin{aligned}&\int_\Omega (u_0 + tv_{\overline{z}})^p dz\\ &\ge \int_\Omega (u_0^p + (tv_{\overline{z}})^p + ptu_0^{p-1}v_{\overline{z}} + pu_0(tv_{\overline{z}})^{p-1})dz - c\int_\Omega u_0^{p/2}(tv_{\overline{z}})^{p/2}dz.\end{aligned}$$

By Lemma 4.2 and Theorem 4.3, there exists a $t^0(v_{\overline{z}}) > 0$ such that $t^0(v_{\overline{z}})v_{\overline{z}} \in \mathbf{M}(\Omega)$ and

$$\max t \geq 0 J(tv_{\overline{z}}) = J(t^0(v_{\overline{z}})v_{\overline{z}}) = \alpha(\Omega) + o(1) \quad \text{as } |\overline{z}| \to \infty. \tag{13.20}$$

By (13.20) and Lemma 13.18, we deduce for $t \geq t_0$

$$\begin{aligned} J_h(u_0 + tv_{\overline{z}}) &= \frac{1}{2}\int_\Omega \big[|\nabla(u_0 + tv_{\overline{z}})|^2 + (u_0 + tv_{\overline{z}})^2\big]dz \\ &\quad - \frac{1}{p}\int_\Omega (u_0 + tv_{\overline{z}})^p dz - \int_\Omega h(u_0 + tv_{\overline{z}})dz \\ &\leq J_h(u_0) + \alpha(\Omega) - t^{\frac{p}{2}}\big(t_0^{\frac{p-2}{2}}\int_\Omega v_{\overline{z}}^{p-1}u_0 dz - \frac{c}{p}\int_\Omega u_0^{p/2}v_{\overline{z}}^{\frac{p}{2}}dz\big) + o(1). \end{aligned}$$

Let $0 < \delta' < \min\{\varepsilon, \frac{p-2}{p+2}\}$, then $\frac{(1-\delta')p}{2} - (1+\delta') > 0$. We choose

$$0 < \delta < \delta' < (p-2) + (p-1)\delta.$$

By Lemma 13.17, 13.18 and u_0 is a positive solution of the Equation (1.2), we have

$$\begin{aligned} \int_\Omega v_{\overline{z}}^{p-1}u_0 dz &= \int_\Omega (\nabla v_{\overline{z}}\nabla u_0 + v_{\overline{z}}u_0)dz + o(1) \\ &= \int_\Omega (v_{\overline{z}}u_0^{p-1} + hv_{\overline{z}})dz + o(1) \\ &= \int_\Omega v_{\overline{z}}u_0^{p-1}dz + o(1) \\ &\geq c_1\int_{\{|z|\geq R+1\}} e^{(-(1+\delta')|z-\overline{z}|)}e^{(p-1)(-(1+\delta)|z|)}dz + o(1), \end{aligned}$$

and

$$\int_\Omega u_0^{p/2}v_{\overline{z}}^{p/2}dz \leq c_2\int_{\{|z|\geq R\}} e^{(-\frac{(1-\delta')p}{2}|z-\overline{z}|)}e^{(-\frac{(1-\delta)p}{2}|z|)}dz,$$

where $c_1 = c_1(\delta,\delta')$ and $c_2 = c_2(\delta,\delta')$. Since

$$\int_{\{|z|\geq R+1\}} e^{(p-1)(-(1+\delta)|z|)}e^{(1+\delta')|z|}dz < \infty$$

and

$$\int_{\{|z|\geq R\}} e^{(-\frac{(1-\delta)p}{2}|z|)}e^{\frac{(1-\delta')p}{2}|z|}dz < \infty,$$

by Lemma 13.19, we deduce that as $|\overline{z}| \to \infty$

$$\begin{aligned} \int_\Omega v_{\overline{z}}^{p-1}u_0 dz &\geq c_1\int_{\{|z|\geq R+1\}} e^{(-(1+\delta')|z-\overline{z}|)}e^{(p-1)(-(1+\delta)|z|)}dz + o(1) \\ &= c_1\big\{\int_{\{|z|\geq R+1\}} e^{(p-1)(-(1+\delta)|z|)}e^{(1+\delta')\frac{\langle z,\overline{z}\rangle}{|\overline{z}|}}dz + o(1)\big\}e^{-(1+\delta')|\overline{z}|} \end{aligned}$$

and

$$\begin{aligned} \int_\Omega u_0^{p/2}v_{\overline{z}}^{p/2}dz &\leq c_2\int_{\{|z|\geq R\}} e^{(-\frac{(1-\delta')p}{2}|z-\overline{z}|)}e^{(-\frac{(1-\delta)p}{2}|z|)}dz \\ &= c_2\big\{\int_{\{|z|\geq R\}} e^{(-\frac{(1-\delta)p}{2}|z|)}e^{\frac{(1-\delta')p}{2}\frac{\langle z,\overline{z}\rangle}{|\overline{z}|}}dz + o(1)\big\}e^{-\frac{(1-\delta')p}{2}|\overline{z}|}. \end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\int_\Omega v_{\overline{z}}^{p-1}u_0dz}{\int_\Omega u_0^{\frac{p}{2}}v_{\overline{z}}^{p/2}dz}\\&\geq \frac{c_1\{\int_{\{|z|\geq R+1\}}e^{(p-1)(-(1+\delta)|z|)}e^{(1+\delta')\frac{\langle z,\overline{z}\rangle}{|\overline{z}|}}dz+o(1)\}}{c_2\{\int_{\{|z|\geq R\}}e^{(-\frac{(1-\delta)p}{2}|z|)}e^{\frac{(1-\delta')p}{2}\frac{\langle z,\overline{z}\rangle}{|\overline{z}|}}dz+o(1)\}}e^{(\frac{(1-\delta')p}{2}-(1+\delta'))|\overline{z}|}\end{aligned}$$

which approaches ∞ as $|\overline{z}|\to\infty$. Then $l_0>0$ exists such that for $|\overline{z}|\geq l_0$

$$t_0^{\frac{p-2}{2}}\int_\Omega v_{\overline{z}}^{p-1}u_0dz-\frac{c}{p}\int_\Omega u_0^{p/2}v_{\overline{z}}^{p/2}dz>0.$$

Hence, we have that for $|\overline{z}|\geq l_0$,

$$\sup_{t\geq t_0}J_h(u_0+tv_{\overline{z}})<J_h(u_0)+\alpha(\Omega).$$

□

Take the sequence $\{t_n\}\subset\mathbb{R}$ such that $|t_n|\nearrow+\infty$ as $n\to+\infty$. For $n\in\mathbb{N}$, we define

$$v_n(z)=\psi_R(z)\bar{u}(z-t_n\overline{z}),$$

where $\overline{z}$ is a unit vector in $\mathbb{R}^N$. Clearly, $v_n\in H_0^1(\Omega)$.

Remark 13.21. There exists $n_0>0$ such that for $n\geq n_0$

$$\sup_{t\geq 0}J_h(u_0+tv_n)<J_h(u_0)+\alpha(\Omega)\quad\text{uniformly in }\overline{z},$$

where u_0 is the local minimum in Theorem 13.12.

We use the notation: For $c\in\mathbb{R}$,

$$[J_h\leq c]=\{u\in\mathbf{M}_h^-:J_h(u)\leq c\}.$$

In this section, we show for a sufficiently small $\sigma>0$

$$\operatorname{cat}([J_h\leq\alpha_h(\Omega)+\alpha(\mathbb{R}^N)-\sigma])\geq 2. \tag{13.21}$$

To prove (13.21), we need some preliminaries. Recall the definition of Lusternik-Schnirelman category.

Definition 13.22. (i) For a topological space X, we say a non-empty, closed subset $A\subset X$ is contractible to a point in X if and only if there exists a continuous mapping

$$\eta:[0,1]\times A\to X$$

such that for some $x_0\in X$

$$\begin{aligned}\eta(0,x)&=x\quad\text{for all }x\in A,\\ \eta(1,x)&=x_0\quad\text{for all }x\in A.\end{aligned}$$

(ii) We define

$$\begin{aligned}\operatorname{cat}(X)=\min\big\{&k\in\mathbb{N}:\text{there exist closed subsets }A_1,\dots,A_k\subset X\text{ such that}\\&A_j\text{ is contractible to a point in }X\text{ for all }j\text{ and }\cup_{j=1}^kA_j=X\}.\end{aligned}$$

When we do not have finitely many closed subsets $A_1, \dots, A_k \subset X$ such that A_j is contractible to a point in X for all j and $\bigcup_{j=1}^{k} A_j = X$, we say $\operatorname{cat}(X) = \infty$.

For fundamental properties of Lusternik-Schnirelman category, we refer to Ambrosetti [2] and Schwartz [66]. Here we use the following property:

Theorem 13.23. *Suppose that X is a Hilbert manifold and $\Psi \in C^1(X, \mathbb{R})$. Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$,*
(i) $\Psi(x)$ satisfies the (PS)$_c$-condition for $c \le c_0$;
(ii) $\operatorname{cat}(\{x \in X : \Psi(x) \le c_0\}) \ge k$.
Then $\Psi(x)$ has at least k critical points in $\{x \in X; \Psi(x) \le c_0\}$.

Theorem 13.24. *Let $N \ge 1$, $S^{N-1} = \{x \in \mathbb{R}^N; |x| = 1\}$, and let X be a topological space. Suppose that there are two continuous maps*

$$F : S^{N-1} \to X, \quad G : X \to S^{N-1}$$

such that $G \circ F$ is homotopic to the identity map of S^{N-1}, that is, a continuous map $\zeta : [0,1] \times S^{N-1} \to S^{N-1}$ exists such that

$$\zeta(0,x) = (G \circ F)(x) \quad \text{for each } x \in S^{N-1},$$
$$\zeta(1,x) = x \quad \text{for each } x \in S^{N-1}.$$

Then $\operatorname{cat}(X) \ge 2$.

Proof. We argue indirectly and suppose that $\operatorname{cat}(X) = 1$, that is, that X is contractible to a point in itself. Thus, a continuous map $\eta : [0,1] \times X \to X$ exists such that for some $x_0 \in X$

$$\eta(0,x) = x \quad \text{for all } x \in X,$$
$$\eta(1,x) = x_0 \quad \text{for all } x \in X.$$

Consider a homotopy $\beta : [0,1] \times S^{N-1} \to S^{N-1}$ defined by

$$\beta(s,x) = G(\eta(s, F(x))).$$

Then

$$\beta(0,x) = (G \circ F)(x) \quad \text{for all } x \in X,$$
$$\beta(1,x) = G(x_0) \quad \text{for all } x \in X.$$

Thus $G \circ F$ is homotopic to a constant map. However,by assumption, $G \circ F$ is homotopic to the identity. Thus S^{N-1} is contractible to a point in S^{N-1}, which is a contradiction. Therefore $\operatorname{cat}(X) \ge 2$. □

Let

$$A_1 = \{u \in H_0^1(\Omega) \backslash \{0\} : \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) > 1\} \cup \{0\}$$
$$A_2 = \{u \in H_0^1(\Omega) \backslash \{0\} : \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) < 1\}.$$

Lemma 13.25. *We have the following results:*
(i) $H_0^1(\Omega) \backslash \mathbf{M}_h^- = A_1 \cup A_2$;
(ii) $\mathbf{M}_h^+ \subset A_1$;
(iii) $t_0 > 1$ and $n_1 \ge n_0$ exist such that $u_0 + t_0 v_n \in A_2$ for each $n \ge n_1$, where n_0 is defined as in Remark 13.21;
(iv) a sequence $\{s_n\} \subset (0,1)$ exists such that $u_0 + s_n t_0 v_n \in \mathbf{M}_h^-$ for each $n \ge n_1$.

Proof. (i) By Lemma 13.4 (iii), we have

$$\mathbf{M}_h^- = \{u \in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}}) = 1\}.$$

Then $H_0^1(\Omega)\backslash\mathbf{M}_h^- = A_1 \cup A_2$.
(ii) For each $u \in \mathbf{M}_h^+$, we have

$$1 < t_{\max}(u) < t^-(u).$$

Since $t^-(u) = \frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$, then $\mathbf{M}_h^+ \subset A_1$. In particular, $u_0 \in A_1$.
(iii) There is a constant $c > 0$ such that $0 < t^-(\frac{u_0+tv_n}{\|u_0+tv_n\|_{H^1}}) < c$ for each $t \geq 0$ and each $n \in \mathbb{N}$. Otherwise, a sequence $\{t_n\}$ and a subsequence $\{v_n\}$ exist such that $t^-(\frac{u_0+t_nv_n}{\|u_0+t_nv_n\|_{H^1}}) \to \infty$ as $n \to \infty$. Let $w_n = \frac{u_0+t_nv_n}{\|u_0+t_nv_n\|_{H^1}}$. We claim that $b(w_n)$ is bounded below away from zero.
Case (a) : $t_n = c_0 + o(1)$ as $n \to \infty$, where $c_0 > 0$. By Lemma 13.18, we have

$$a(v_n) = b(v_n) + o(1) = \frac{2p}{p-2}\alpha(\Omega) + o(1).$$

Thus,

$$\begin{aligned} b(w_n) &= \frac{1}{\|\frac{u_0}{t_n} + v_n\|_{H^1}^p} \int_\Omega (\frac{u_0}{t_n} + v_n)^p \\ &\geq \frac{b(v_n)}{2^{p-1}\big(\|\frac{u_0}{t_n}\|_{H^1}^p + \|v_n\|_{H^1}^p\big)} \\ &= \frac{\frac{2p}{p-2}\alpha(\Omega)}{2^{p-1}\big(\frac{\|u_0\|_{H^1}^p}{c_0^p} + \big(\frac{2p}{p-2}\alpha(\Omega)\big)^{p/2}\big)} + o(1). \end{aligned}$$

Case (b) : $t_n \to \infty$ as $n \to \infty$. The proof is similar to Case (a).
Case (c) : $t_n = o(1)$ as $n \to \infty$. By Lemma 13.18, we have

$$\|u_0 + t_nv_n\|_{H^1}^2 = \|u_0\|_{H^1}^2 + t_n^2\|v_n\|_{H^1}^2 + 2t_n\langle v_n, u_0\rangle_{H^1} = \|u_0\|_{H^1}^2 + o(1).$$

Thus,

$$b(w_n) \geq \frac{1}{\|u_0 + t_nv_n\|_{H^1}^p} \int_\Omega u_0^p = \frac{1}{\|u_0\|_{H^1}^p} \int_\Omega u_0^p + o(1).$$

From Case (a), (b) and (c), $b(w_n)$ is bounded below away from zero.

Since $t^-(w_n)w_n \in \mathbf{M}_h^- \subset \mathbf{M}_h$, we have

$$J_h(t^-(w_n)w_n) = \frac{1}{2}[t^-(w_n)]^2 - \frac{1}{p}[t^-(w_n)]^p b(w_n) - t^-(w_n)\int_\Omega hw_n$$

which approaches $-\infty$ as $n \to \infty$. However, J_h is bounded below on $\mathbf{M}_h$, which is a contradiction. Let

$$t_0 = (\frac{p-2}{2p\alpha(\Omega)}|c^2 - a(u_0)|)^{1/2} + 1,$$

then

$$\begin{aligned} \|u_0 + t_0v_n\|_{H^1}^2 &= a(u_0) + t_0^2(\frac{2p}{p-2})\alpha(\Omega) + o(1) \\ &> c^2 + o(1) \geq \Big[t^-(\frac{u_0 + t_0v_n}{\|u_0 + t_0v_n\|_{H^1}})\Big]^2 + o(1). \end{aligned}$$

Thus, there is an $n_1 \geq n_0$, where n_0 is defined as in Remark 13.21, such that, or $n \geq n_1$,

$$\frac{1}{\|u_0 + t_0 v_n\|_{H^1}} t^-(\frac{u_0 + t_0 v_n}{\|u_0 + t_0 v_n\|_{H^1}}) < 1,$$

or $u_0 + t_0 v_n \in A_2$.

(iv) Define a path $\gamma_n(s) = u_0 + s t_0 v_n$ for $s \in [0,1]$ and each $n \geq n_1$where $t_0 > 1$, then

$$\gamma_n(0) = u_0 \in A_1,\ \gamma_n(1) = u_0 + t_0 v_n \in A_2.$$

Since $\frac{1}{\|u\|_{H^1}} t^-(\frac{u}{\|u\|_{H^1}})$ is a continuous function for nonzero u and $\gamma_n([0,1])$ is connected, a sequence $\{s_n\} \subset (0,1)$ exists such that $u_0 + s_n t_0 v_n \in \mathbf{M}_h^-$. □

Define a map $F_n : S^{N-1} \to H_0^1(\Omega)$ by, for $\overline{z} \in S^{N-1}$,

$$F_n(\overline{z})(z) = u_0(z) + s_n t_0 v_n(z) \quad \text{for } n \geq n_1,$$

where $v_n(z) = \psi_R(z)\overline{u}(z - t_n \overline{z})$ and n_1 is defined as in Lemma 13.25. Then we have the following proposition.

Proposition 13.26. *A sequence $\{\sigma_n\} \subset \mathbb{R}^+$ exists such that*

$$F_n(S^{N-1}) \subset \left[J_h \leq \alpha_h(\Omega) + \alpha(\mathbb{R}^N) - \sigma_n\right].$$

Proof. By Lemma 13.25 (iv) and Remark 13.21, we have that for each $n \geq n_1$ $u_0 + s_n t_0 v_n \in \mathbf{M}_h^-$ and $J_h(u_0 + s_n t_0 v_n) \leq \alpha_h(\Omega) + \alpha(\mathbb{R}^N) - \sigma_n$, the conclusion holds. □

For $c > 0$, we define

$$b_c(u) = \int_\Omega c|u|^p;$$
$$I_c(u) = \frac{1}{2}a(u) - \frac{1}{p}b_c(u);$$
$$\mathbf{M}_{I_c} = \{u \in H_0^1(\Omega)\backslash\{0\} : \langle I_c'(u), u\rangle = 0\}.$$

Recall that a unique $t^- = t^-(u) > 0$ and $t^1 = t^1(u) > 0$ exist such that $t^- u \in \mathbf{M}_h^-$ and $t^1 u \in \mathbf{M}(\Omega)$.

Lemma 13.27. *For $u \in \Sigma = \{u \in H_0^1(\Omega) \mid \|u\|_{H^1} = 1\}$, we have the following results:*

(i) a unique $t^c(u) > 0$ exists such that $t^c(u)u \in \mathbf{M}_{I_c}$ and

$$\max_{t \geq 0} I_c(tu) = I_c(t^c(u)u) = (\frac{1}{2} - \frac{1}{p}) b_c(u)^{-\frac{2}{p-2}};$$

(ii) for $0 < \mu < 1$, $d_1(\mu) > 0$ exists such that for $\|h\|_{L^2} < d_1(\mu)$

$$J_h(t^- u) \geq (1-\mu)^{\frac{p}{p-2}} J(t^0 u) - \frac{1}{2\mu}\|h\|_{L^2}^2.$$

Proof. (i) For each $u \in \Sigma$, let $f(t) = I_c(tu) = \frac{1}{2}t^2 - \frac{1}{p}t^p b_c(u)$, then $f(t) \to -\infty$ as $t \to \infty$, $f'(t) = t - t^{p-1}b_c(u)$ and $f''(t) = 1 - (p-1)t^{p-2}b_c(u)$. Let

$$t^c(u) = \big[\frac{1}{b_c(u)}\big]^{\frac{1}{p-2}} > 0.$$

Then $f'(t^c(u)) = 0$, $t^c(u)u \in \mathbf{M}_{I_c}$ and

$$(t^c(u))^2 f''(t^c(u)) = a(t^c(u)u) - (p-1)b_c(t^c(u)u) \\ = (2-p)(t^c(u))^2 a(u) < 0.$$

Thus, a unique $t^c(u) > 0$ exists such that $t^c(u)u \in \mathbf{M}_{I_c}$ and

$$\max_{t\ge 0} I_c(tu) = I_c(t^c(u)u) = (\frac{1}{2} - \frac{1}{p}) b_c(u)^{-\frac{2}{p-2}}.$$

(ii) Let $c = \frac{1}{1-\mu}$, $t^c = t^{\frac{1}{1-\mu}} > 0$ and $t^1 = t^1(u) > 0$ such that $t^c u \in M_{I_c}$ and $t^1 u \in \mathbf{M}(\Omega)$. For $\mu \in (0,1)$, we have

$$|\int_\Omega h t^c u dz| \le \|t^c u\|_{H^1} \|h\|_{L^2} \le \frac{\mu}{2}\|t^c u\|^2_{H^1} + \frac{1}{2\mu}\|h\|^2_{L^2}.$$

Then by part (i),

$$\begin{aligned}
\sup_{t\ge 0} J_h(tu) &\ge J_h(t^c u) \ge \frac{1-\mu}{2}\|t^c u\|^2_{H^1} - \frac{1}{p} b(t^c u) - \frac{1}{2\mu}\|h\|^2_{L^2} \\
&= (1-\mu)\Big[\frac{1}{2}\|t^c u\|^2_{H^1} - \frac{1}{(1-\mu)p}\int_\Omega |t^c u|^p\Big] - \frac{1}{2\mu}\|h\|^2_{L^2} \\
&= (1-\mu) I_{\frac{1}{1-\mu}}(t^c u) - \frac{1}{2\mu}\|h\|^2_{L^2} \\
&= (1-\mu)^{\frac{p}{p-2}} (\frac{1}{2} - \frac{1}{p}) b(u)^{-\frac{2}{p-2}} - \frac{1}{2\mu}\|h\|^2_{L^2} \\
&= (1-\mu)^{\frac{p}{p-2}} J(t^1 u) - \frac{1}{2\mu}\|h\|^2_{L^2} \\
&\ge (1-\mu)^{\frac{p}{p-2}} \alpha(\Omega) - \frac{1}{2\mu}\|h\|^2_{L^2}.
\end{aligned}$$

For $\mu \in (0,1)$, there exists $d_1(\mu) > 0$ such that for $\|h\|_{L^2} < d_1(\mu)$

$$\sup_{t\ge 0} J_h(tu) > 0.$$

By Lemma 13.4, there exists $t^- = t^-(u) > 0$ such that $t^- u \in \mathbf{M}_h^-$ and

$$\sup_{t\ge 0} J_h(tu) = J_h(t^- u).$$

Thus, for $\|h\|_{L^2} < d_1(\mu)$,

$$J_h(t^- u) \ge (1-\mu)^{\frac{p}{p-2}} J(t^1 u) - \frac{1}{2\mu}\|h\|^2_{L^2}.$$

□

Lemma 13.28. *A $\delta_0 > 0$ exists such that if $u \in \mathbf{M}(\Omega)$ and $J(u) \le \alpha(\mathbb{R}^N) + \delta_0$, then*

$$\int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u|^2 + u^2) dz \ne 0.$$

Proof. If not, a sequence $\{u_n\} \subset \mathbf{M}(\Omega)$ exists such that $J(u_n) = \alpha(\mathbb{R}^N) + o(1)$ and

$$\int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u_n|^2 + u_n^2) dz = 0.$$

By Theorem 4.4, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbb{R}^N)}$-sequence in $H_0^1(\Omega)$ for J. By Theorem 4.18, $\inf_{v\in\mathbf{M}(\Omega)} J(v) = \alpha(\Omega) = \alpha(\mathbb{R}^N)$ is not achieved. Let $\overline{u}$ be the unique positive solution of Equation (1.1) in $\mathbb{R}^N$. It follows from Theorem 3.1 that a sequence $\{z_n\}$ exists in $\mathbb{R}^N$ such that $|z_n| \to \infty$ as $n \to \infty$ and

$$u_n(z) = \overline{u}(z - z_n) + o(1) \quad \text{strongly in } H^1(\mathbb{R}^N).$$

Assume $\frac{z_n}{|z_n|} \to z_0$ as $n \to \infty$, where z_0 is a unit vector in $\mathbb{R}^N$. Then by Theorem 2.20, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u_n|^2 + u_n^2)dz \\ &= \int_{\mathbb{R}^N} \frac{z+z_n}{|z+z_n|}(|\nabla \overline{u}|^2 + \overline{u}^2)dz + o(1) \\ &= (\frac{2p}{p-2})z_0\alpha(\mathbb{R}^N) + o(1), \end{aligned}$$

which is a contradiction. □

Lemma 13.29. $d_0 > 0$ *exists such that for* $\|h\|_{L^2} < d_0$, *we have*

$$\int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u|^2 + u^2)dz \neq 0,$$

for $u \in \big[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\big]$.

Proof. For $u \in \big[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\big]$, then $u/\|u\|_{H^1} \in \Sigma$. There exists a $t^0 > 0$ such that $t^0 u/\|u\|_{H^1} \in \mathbf{M}$. By Lemma 13.27 (ii), we have for any $\mu \in (0,1)$ and $\|h\|_{L^2} < d_1(\mu)$

$$J(\frac{t^0 u}{\|u\|_{H^1}}) \leq (1-\mu)^{-\frac{p}{p-2}}\big(J_h(u) + \frac{1}{2\mu}\|h\|_{L^2}^2\big). \tag{13.22}$$

Since $\alpha_h(\Omega) < 0$, we have $\big[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\big] \subset \big[J_h < \alpha(\mathbb{R}^N)\big]$. Thus by (13.22), we have, for $u \in \big[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\big]$,

$$J(\frac{t^0 u}{\|u\|_{H^1}}) \leq (1-\mu)^{-\frac{p}{p-2}}\big(\alpha(\mathbb{R}^N) + \frac{1}{2\mu}\|h\|_{L^2}^2\big).$$

Take $\mu \in (0,1)$ such that $d_1(\mu) > d_0 > 0$ and $\delta_0 > 0$ exist such that for $\|h\|_{L^2} < d_0$

$$J(\frac{t^0 u}{\|u\|_{H^1}}) \leq \alpha(\mathbb{R}^N) + \delta_0. \tag{13.23}$$

Since $t^0 u/\|u\|_{H^1} \in \mathbf{M}$, by Lemma 13.28 and (13.23)

$$\int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla(\frac{t^0 u}{\|u\|_{H^1}})|^2 + (\frac{t^0 u}{\|u\|_{H^1}})^2)dz \neq 0,$$

or,

$$\int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u|^2 + (u)^2)dz \neq 0.$$

□

By Lemma 13.29

$$\int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u|^2 + |u|^2)dz \neq 0$$

for all $u \in \left[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\right]$. We define

$$G : \left[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\right] \to S^{N-1}$$

by

$$G(u) = \int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u|^2 + |u|^2)dz \Big/ \Big| \int_{\mathbb{R}^N} \frac{z}{|z|}(|\nabla u|^2 + |u|^2)dz \Big|.$$

Proposition 13.30. *For $n \geq n_1$ and $\|h\|_{L^2} < d_0$, the map*

$$G \circ F_n : S^{N-1} \to S^{N-1}$$

is homotopic to the identity.

Proof. We define $\zeta_n(\theta, \overline{z}) : [0,1] \times S^{N-1} \to S^{N-1}$ by

$$\zeta_n(\theta, \overline{z}) = \begin{cases} G((1-2\theta)F_n(\overline{z}) + 2\theta\overline{u}(z - t_n\overline{z})) & \text{for } \theta \in [0, 1/2); \\ G\big(\overline{u}\big(z - \frac{t_n}{2(1-\theta)}\overline{z}\big)\big) & \text{for } \theta \in [1/2, 1); \\ \overline{z} & \text{for } \theta = 1. \end{cases}$$

Since $h \in L^2(\Omega) \cap L^{\frac{N}{2}}(\Omega) \cap L^s(\Omega)$ for some $s > N$, then by Theorem 8.11 we have $u_0 \in C^{1,\theta}(\overline{\Omega})$. First, we claim that $\lim_{\theta \to 1^-} \zeta_n(\theta, \overline{z}) = \overline{z}$ and $\lim_{\theta \to \frac{1}{2}^-} \zeta_n(\theta, \overline{z}) = G(\overline{u}(z - t_n\overline{z}))$.
(*a*) $\lim\limits_{\theta \to 1^-} \zeta_n(\theta, \overline{z}) = \overline{z}$: since

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{z}{|z|}\Big(\big|\nabla\overline{u}\big(z - \frac{t_n}{2(1-\theta)}\overline{z}\big)\big|^2 + \overline{u}\big(z - \frac{t_n}{2(1-\theta)}\overline{z}\big)^2\Big)dz \\ &= \int_{\mathbb{R}^N} \frac{z + \frac{t_n}{2(1-\theta)}\overline{z}}{|z + \frac{t_n}{2(1-\theta)}\overline{z}|}(|\nabla\overline{u}(z)|^2 + \overline{u}(z)^2)dz \\ &= (\frac{2p}{p-2})\alpha(\mathbb{R}^N)\overline{z} + o(1) \quad \text{as } \theta \to 1^-, \end{aligned}$$

then $\lim\limits_{\theta \to 1^-} \zeta_n(\theta, \overline{z}) = \overline{z}$.
(*b*) $\lim\limits_{\theta \to \frac{1}{2}^-} \zeta_n(\theta, \overline{z}) = G(\overline{u}(z - t_n\overline{z}))$: since $\overline{u}$ and $u_0 \in C^{1,\theta}(\overline{\Omega})$, then

$$\|(1-2\theta)F_n(\overline{z}) + 2\theta\overline{u}(z - t_n\overline{z})\|_{H^1} = \|\overline{u}(z - t_n\overline{z})\| + o(1) \quad \text{as } \theta \to \frac{1}{2}^-.$$

By the continuity of G, we obtain $\lim\limits_{\theta \to \frac{1}{2}^-} \zeta_n(\theta, \overline{z}) = G(\overline{u}(z - t_n\overline{z}))$. Thus, $\zeta_n(\theta, \overline{z}) \in C([0,1] \times S^{N-1}, S^{N-1})$ and

$$\zeta_n(0, \overline{z}) = G(F_n(\overline{z})) \quad \text{for all } \overline{z} \in S^{N-1},$$
$$\zeta_n(1, \overline{z}) = \overline{z} \quad \text{for all } \overline{z} \in S^{N-1},$$

provided $n \geq n_1$ and $\|h\|_{L^2} < d_0$. This completes the proof. □

Thus we have the following theorem.

Theorem 13.31. *$J_h(u)$ has at least two critical points in*

$$\left[J_h < \alpha_h(\Omega) + \alpha(\mathbb{R}^N)\right].$$

Proof. Applying Theorem 13.24 and Proposition 13.30, we have for sufficiently large $n \geq n_1$ and $\|h\|_{L^2} < d_0$,

$$\operatorname{cat}\big(\big[J_h \leq \alpha_h(\Omega) + \alpha(\mathbb{R}^N) - \sigma_n\big]\big) \geq 2.$$

By Proposition 13.13 and Theorem 13.23, the theorem holds. □

By Theorem 13.11, there is a nontrivial solution in $\mathbf{M}_h^+$ and by Theorem 13.31, there are two nontrivial solutions in $\mathbf{M}_h^-$. Therefore, we have

Theorem 13.32. *Let the domain Ω to be $\mathbf{A}^r$, $\mathbb{R}^N$, $\mathbf{A}^r \backslash \overline{D}$, or $\mathbb{R}^N \backslash \overline{D}$, where D is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$ and assume that $h \in L^{\frac{N}{2}}(\Omega) \cap L^s(\Omega) \cap L^2(\Omega)$ for some $s > N$, $0 < \|h\|_{L^2} < d(p,\alpha)$, and $h \gneqq 0$*

$$0 \leq h(z) \leq c \exp(-(1+\varepsilon)|z|)$$

for any $z \in \Omega$ and for some positive constants c, ε, where $d(p,\alpha)$ is defined as in (13.1). Then there are three positive solutions of equation (1.2).

Bibliographical notes: The results of this section are from Adachi-Tanaka [1].

13.2. **Symmetry Breaking in a Bounded Symmetry Domain.** The main purpose of this sectionis to present the breaking of symmetry by a perturbation of the finite strip $\mathbf{A}^r_{-t,t}$. Let $0 < r_1 < r$ and consider the finite strip with a hole,

$$\Omega_t = \mathbf{A}^r_{-t,t} \setminus \overline{B^N((x,0);r_1)}.$$

We prove that $t_0 > 0$ exists such that for $t \geq t_0$, Equation (1.1) on Ω_t has three positive solutions, one of which is y-symmetric while the other two are nonaxially symmetric.

13.2.1. *Existence of Three Solutions.*

Example 13.33 (y-symmetric large domain)**.** (i) For $0 < r_1 < r$ and x in $B^{N-1}(0; r + r_1)$, consider the infinite strip with holes

$$\Omega_l = \mathbf{A}^r \setminus \big[\overline{B^N((x,l);r_1)} \cup \overline{B^N((x,-l);r_1)}\big] \quad \text{for some } l > 0.$$

Then Ω_l is a y-symmetric large domain in $\mathbf{A}^r$;
(ii) Let $0 < r_1 < r$ and $0 < y_{n+1} = ny_n$ for $n = 1, 2, \dots$. Consider the infinite strip with infinite holes

$$D = \mathbf{A}^r \setminus \big\{ \bigcup_{n=1}^{\infty} \big[\overline{B^N((0,y_n);r_1)} \cup \overline{B^N((0,-y_n);r_1)}\big] \big\}.$$

Then D is a y-symmetric large domain in $\mathbf{A}^r$.

Proposition 13.34. $\alpha_s(\mathbf{A}^r_{-t,t}) = \alpha(\mathbf{A}^r_{-t,t})$ *and* $\alpha_s(\mathbf{A}^r) = \alpha(\mathbf{A}^r)$.

Proof. By Gidas-Ni-Nirenberg [34] and Chen-Chen-Wang [23], every positive solution of (1.1) in a finite strip $\mathbf{A}^r_{-t,t}$ and in an infinite strip $\mathbf{A}^r$is y-symmetric. □

The following symmetric results are required to assert our main result.

Theorem 13.35. (i) *Suppose that Ω is a proper y-symmetric large domain in $\mathbf{A}^r$ and J does not satisfy the $(PS)_{\alpha_s(\Omega)}$-condition in $H_s^1(\Omega)$. Then $\alpha_s(\Omega) \geq 2\alpha(\mathbf{A}^r)$;*
(ii) *If Ω is a proper y-symmetric large domain in $\mathbf{A}^r$, then $\alpha(\mathbf{A}^r) < \alpha_s(\Omega)$.*

Proof. (*i*) Suppose that J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}$-condition in $H_s^1(\Omega)$. By Lemma 2.41, a subsequence $\{u_n\}$ exists such that $J(\xi_n u_n) = \alpha_s(\Omega) + o(1)$ and $J'(\xi_n u_n) = o(1)$ in $H_s^{-1}(\Omega)$, where ξ_n is defined as in (2.1). Let $w_n = \xi_n u_n$. Then by Lemma 2.9, we obtain

$$\begin{aligned} J(w_n) &= \alpha_s(\Omega) + o(1), \\ J'(w_n) &= o(1) \quad \text{in } H^{-1}(\Omega). \end{aligned} \tag{13.24}$$

Since $\Omega \subsetneqq \mathbf{A}^r$, $K > 0$ exists such that $w_n = 0$ in $\overline{Q_K}$, and two disjoint large domains Ω^1 and Ω^2 in $\mathbf{A}^r$ exist such that

$$(x, y) \in \Omega^2 \quad \text{if and only if} \quad (x, -y) \in \Omega^1,$$
$$\Omega \backslash \overline{Q_K} = \Omega^1 \cup \Omega^2 \quad \text{where } Q_K = \Omega \cap B^N(0; K).$$

Let

$$w_n^i(x) = \begin{cases} w_n(x) & \text{for } x \in \Omega^i, \\ 0 & \text{for } x \notin \Omega^i, \end{cases}$$

for $i = 1, 2$. Then $w_n^i \in H_0^1(\Omega^i)$, $w_n^1(x, y) = w_n^2(x, -y)$, $w_n = w_n^1 + w_n^2$, and $J(w_n^1) = J(w_n^2)$. Moreover, we have

$$\alpha_s(\Omega) + o(1) = J(w_n) = J(w_n^1) + J(w_n^2) = 2J(w_n^i) \quad \text{for } i = 1, 2,$$

or

$$J(w_n^i) = \frac{1}{2}\alpha_s(\Omega) + o(1) \quad \text{for } i = 1, 2.$$

By (13.24), we have $J'(w_n^i) = o(1)$ in $H^{-1}(\Omega^i)$ for $i = 1, 2$. Therefore, $\frac{1}{2}\alpha_s(\Omega)$ is a positive (PS)-value in $H_0^1(\Omega^i)$ for J. By Lemma 4.12, Theorem 4.13, and Definition 4.14,

$$\frac{1}{2}\alpha_s(\Omega) \geq \alpha_{\mathbf{M}}(\Omega^i) = \alpha(\Omega^i).$$

Since Ω^i is a large domain in $\mathbf{A}^r$, by Lemma 4.18 (*ii*), we have

$$\alpha(\Omega^i) = \alpha(\mathbf{A}^r).$$

Thus, $\alpha_s(\Omega) \geq 2\alpha(\mathbf{A}^r)$.
(*ii*) Clearly, we have $\alpha(\mathbf{A}^r) \leq \alpha_s(\Omega)$. Assume that $\alpha(\mathbf{A}^r) = \alpha_s(\Omega)$, then by Theorem 5.7, J does not satisfy the $(\mathrm{PS})_{\alpha_s(\Omega)}$-condition in $H_s^1(\Omega)$ for J. By Lemma 13.35, $2\alpha(\mathbf{A}^r) \leq \alpha_s(\Omega) = \alpha(\mathbf{A}^r)$, which is a contradiction. □

Theorem 13.36. *Let $0 < r_1 < r$, and for each $t > 0$*

$$\Theta_t = \mathbf{A}^r \setminus \left[\overline{B^N((x, t + r_1); r_1)} \cup \overline{B^N((x, -(t + r_1)); r_1)}\right].$$

Then $t_0 > 0$ exists such that $\alpha_s(\Theta_t) < 2\alpha(\mathbf{A}^r)$ for all $t \geq t_0$. In particular, there is a y-symmetric positive ground state solution of Equation (1.1) *in Ω_t.*

Proof. By Lien-Tzeng-Wang [47], $\alpha(\mathbf{A}^r_{-t,t})$ is strictly decreasing as t is strictly increasing and

$$\alpha(\mathbf{A}^r_{-t,t}) \searrow \alpha(\mathbf{A}^r) \quad \text{as } t \to \infty.$$

Thus, there is a $t_0 > 0$ such that $\alpha(\mathbf{A}^r_{-t,t}) < 2\alpha(\mathbf{A}^r)$ for each $t \geq t_0$. By Proposition 13.34, $\alpha(\mathbf{A}^r_{-t,t}) = \alpha_s(\mathbf{A}^r_{-t,t})$ for each t. Thus, $\alpha_s(\mathbf{A}^r_{-t,t}) < 2\alpha(\mathbf{A}^r)$ for each $t \geq t_0$. Since $\Theta_t \supset \mathbf{A}^r_{-t,t}$ for each $t \geq t_0$. Therefore, we have $\alpha_s(\Theta_t) \leq \alpha_s(\mathbf{A}^r_{-t,t})$ for each $t \geq t_0$. We then conclude that

$$\alpha_s(\Theta_t) \leq \alpha_s(\mathbf{A}^r_{-t,t}) < 2\alpha(\mathbf{A}^r) \quad \text{for each } t \geq t_0.$$

Since Θ_t is a y-symmetric large domain in $\mathbf{A}^r$, by Theorem 13.35, J satisfies the $(\mathrm{PS})_{\alpha_s(\Omega)}$-condition in $H_s^1(\Omega)$, or telse here is a y-symmetric positive ground state solution of Equation (1.1) in Θ_t for each $t \geq t_0$. □

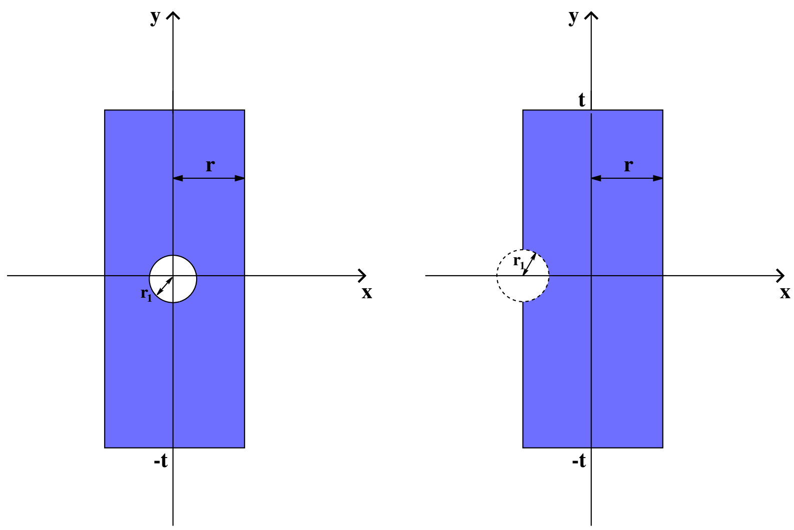

FIGURE 12. the finite strip with a hole

Let $0 < r_1 < r$ and consider the finite strip with a hole

$$\Omega_t = \mathbf{A}^r_{-t,t} \backslash \overline{B^N((x,0);r_1)}.$$

Then we have the following assertion.

Theorem 13.37. *$t_0 > 0$ exists such that for $t \geq t_0$, Equation (1.1) on Ω_t has three positive solutions of which one is y-symmetric and the other two are nonaxially symmetric.*

Proof. Let $\Omega = \mathbf{A}^r \setminus \overline{B^N((x,0);r_1)}$. Then Ω is a y-symmetric large domain in $\mathbf{A}^r$. By Theorem 13.35, we have $\alpha(\mathbf{A}^r) < \alpha_s(\Omega)$. By Lien-Tzeng-Wang [47], (1.1) admits a ground state solution in $\mathbf{A}^r_{0,t}$ and in $\mathbf{A}^r$, and $\alpha(\mathbf{A}^r_{0,t})$ is strictly decreasing as t is strictly increasing and

$$\alpha(\mathbf{A}^r_{0,t}) \searrow \alpha(\mathbf{A}^r) \quad \text{as } t \to \infty.$$

Take $t_1 > 0$ such that for $t \geq t_1$,

$$\alpha(\mathbf{A}^r) < \alpha(\mathbf{A}^r_{0,t}) < \alpha_s(\Omega). \tag{13.25}$$

Note that $\mathbf{A}^r_{r_1,t_1+r_1} \subsetneqq \Omega_t \subsetneqq \mathbf{A}^r$ for $t \geq t_0 = t_1 + r_1$. By Theorem 5.7, we conclude that

$$\alpha(\mathbf{A}^r) < \alpha(\Omega_t) < \alpha(\mathbf{A}^r_{r_1,t_1+r_1}). \tag{13.26}$$

By Lien-Tzeng-Wang [47], if Ω is a domain of $\mathbb{R}^N$, then $\alpha(\Omega)$ is invariant by rigid motions. Thus,

$$\alpha(\mathbf{A}^r_{r_1,t_1+r_1}) = \alpha(\mathbf{A}^r_{0,t_1}). \tag{13.27}$$

Therefore, by (13.25)-(13.27)

$$\alpha(\mathbf{A}^r) < \alpha(\Omega_t) < \alpha(\mathbf{A}^r_{0,t_1}) < \alpha_s(\Omega). \tag{13.28}$$

Since $\Omega_t \subset \Omega$, we have

$$\alpha_s(\Omega) \le \alpha_s(\Omega_t). \tag{13.29}$$

By (13.28) and (13.29), we obtain

$$\alpha(\Omega_t) < \alpha_s(\Omega_t). \tag{13.30}$$

By Theorem 12.3, there are a $y-$symmetry solution u_1 and a solution u_2 of Equation (1.1) in domain Ω_t such that

$$J(u_1) = \alpha_s(\Omega_t),$$
$$J(u_2) = \alpha(\Omega_t).$$

By Theorem 5.7, we may take u_1 and u_2 to be positive. Let

$$u_3(x, y) = u_2(x, -y).$$

Then u_3 is the third solution. By (13.30), u_1, u_2 and u_3 are different. Moreover, u_1 is a y-symmetric solution while both u_2 and u_3 are nonaxially symmetric solutions of Equation (1.1) in domain Ω_t. □

Bibliographical notes: The results of this section are from Wang-Wu [74].

13.3. **Multiple Solutions in Domains with Two Bumps.** That the existence of solutions of (1.1) is affected by the shape of the domain Ω has been the focus of a great deal of research in recent years . By the Rellich compactness theorem, it is easy to obtain a solution of (1.1) in a bounded domain. For general unbounded domains Ω, because of the lack of compactness, the existence of solutions of Equation (1.1) is an important open question. Recently, there has been some progress in determining the existence and multiplicity of solutions as follows: Bahri-Lions [13], Coti Zelati [28], Chabrowski [20], Chen-Lee-Wang [24], Chen-Wang[26] , Chen-Lin-Wang [25], Lien-Tzeng-Wang [47], del Pino-Felmer [30], [31], and Wang [71] used the (PS)$-$theory to treat the existence of solutions of (1.1). Byeon [16], Chen-Ni-Zhou [22], Dancer [29], and Wang-Wu [74] asserted the existence of three positive solutions of semilinear elliptic equations in a dumbbell domain. Jimbo [43] and [44] asserted the existence of solutions depending on the width of the corridor of the dumbbell.

In this section we assert that there is a $R_0 > 0$ such that for $R > R_0$ Equation (1.1) on the two bumps domain D_R has three positive solutions in which one is y-symmetric and other two are nonaxially symmetric. (see Theorem 13.41). Since finite dumbbell is a two bumps domain, the results of Byeon [16], Chen-Ni-Zhou [22], and Dancer [29] are the consequences of our Theorem 13.41.

13.3.1. *Existence of Three Solutions.* We have the following results.

Theorem 13.38. (*i*) *The bounded domains in $\mathbb{R}^N$ are the achieved domains in $\mathbb{R}^N$;*
(*ii*) *The C^1 quasi-bounded domains are the achieved domains in $\mathbb{R}^N$;*
(*iii*) *$\mathbb{R}^N$ is an achieved domain in $\mathbb{R}^N$;*
(*iv*) *The periodic domains in $\mathbb{R}^N$ are the achieved domains. In particular, the infinite strip $\mathbf{A}^r$ is an achieved domain in $\mathbb{R}^N$;*
(*v*) *$s_0 > 0$ exists such that $\mathbf{F}^r_s$ is an achieved domain in $\mathbb{R}^N$ if $s > s_0$.*

Proof. (*i*) By Theorem 12.3.
(*ii*) By Theorem 5.4.
(*iii*) and (*iv*) follows from Lien-Tzeng-Wang [47].
(*v*) By Theorem 12.7. □

Throughout this section, let Θ be a proper achieved y-symmetric domain in $\mathbb{R}^N$ bounded in the $x-$direction, such as bounded domains and the infinite strip $\mathbf{A}^r$. For $R > 0$, let Ω_R^1 and Ω_R^2 be two bounded domains in $\mathbb{R}^N$ such that

$$R = \operatorname{dist}\{0, \Omega_R^1\},$$
$$\Omega_R^2 = \{(x, y) : (x, -y) \in \Omega_R^1\}.$$

and the y-symmetric domain

$$D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2.$$

We call D_R a two-bumps domain. Here are some examples of two-bumps domains.

Example 13.39. (*i*) For $t > R > r > 0$. The bounded dumbbell domain D_R is a two-bumps domain, where

$$D_R = B^N((0, -t), r) \cup \mathbf{A}^r_{-t,t} \cup B^N((0, t), r);$$

(*ii*) For $t > R > r > 0$. The unbounded dumbbell domain D_R is a two-bumps domain, where

$$D_R = B^N((0, -t), r) \cup \mathbf{A}^r \cup B^N((0, t), r);$$

(*iii*) For $t > R > r > 0$. The curved dumbbell domain $D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2$ is a two bumps domain, where Ω_R^1 and Ω_R^2 are two bounded domains in $\mathbb{R}^N$ such that

$$R = \operatorname{dist}\{0, \Omega_R^1\},$$
$$\Omega_R^2 = \{(x, y) : (x, -y) \in \Omega_R^1\}.$$

and Θ is a curved bounded y-symmetric domain in $\mathbb{R}^N$.

Then we have the following assertion.

Theorem 13.40. *Let D_R be a two-bumps domain, where $D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2$, and let Θ be a proper achieved y-symmetric domain in $\mathbb{R}^N$ bounded in the $x-$direction. Then we have, for all $R > 0$,*
(i) $\alpha(\Theta) \geq \alpha(D_R) > \alpha(\mathbb{R}^N)$;
(ii) J satisfies the $(PS)_{\alpha_X(D_R)}$-condition in $X(D_R)$.

Proof. By Theorem 5.7 and Theorem 12.3, it suffices to assume that Θ is unbounded.
(*i*) Since $\Theta \subset D_R \subsetneqq \mathbb{R}^N$, we have $\alpha(\Theta) \geq \alpha(D_R) \geq \alpha(\mathbb{R}^N)$. Suppose that $\alpha(D_R) = \alpha(\mathbb{R}^N)$, by Theorem 5.7, J does not satisfy the $(\mathrm{PS})_{\alpha(D_R)}$-condition. By Theorem 5.11, a sequence $\{u_n\}$ in $H_0^1(D_R)$ exists such that $\{u_n\}$ and $\{\xi_n u_n\}$ are the $(\mathrm{PS})_{\alpha(D_R)}$-sequences for J, where ξ_n is defined as in (2.1). Let $w_n = \xi_n u_n$. Then

$$J(w_n) = \alpha(D_R) + o(1),$$
$$J'(w_n) = o(1) \quad \text{in } H^{-1}(D_R).$$

Since $D_R = \Omega_R^1 \cup \Theta \cup \Omega_R^2$ is a y-symmetric domain in $\mathbb{R}^N$ separated by a bounded domain, $n_0 > 0$ exists such that for $n \geq n_0$, $w_n \in H_0^1(\Theta)$, $J(w_n) = \alpha(D_R) + o(1)$, and $a(w_n) = b(w_n) + o(1)$. By Theorem 4.3, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such

that $\{s_n w_n\}$ is in $\mathbf{M}(\Omega)$ and $\{s_n w_n\}$ is a $(\mathrm{PS})_{\alpha(D_R)}$-sequence in $X(\Omega)$ for J. Thus $\alpha(\Theta) \le \alpha(D_R)$. We then conclude that $\alpha(\Theta) = \alpha(D_R) = \alpha(\mathbb{R}^N)$. However, since Θ is a proper achieved y-symmetric domain in $\mathbb{R}^N$, by Theorem 5.7, $\alpha(\Theta) > \alpha(\mathbb{R}^N)$. This is a contradiction. Thus, $\alpha(\Theta) \ge \alpha(D_R) > \alpha(\mathbb{R}^N)$ for all $R > 0$.
(*ii*) It suffices to prove the case $X(D_R) = H_0^1(D_R)$. Since Ω_R^1, Θ, and Ω_R^2 are achieved, by Theorem 5.7,

$$\alpha(D_R) < \min\{\alpha(\Omega_R^1), \alpha(\Theta), \alpha(\Omega_R^2)\}.$$

By Theorem 5.12, J satisfies the $(\mathrm{PS})_{\alpha(D_R)}$-condition in $H_0^1(D_R)$. □

We apply Theorems 6.7 and 13.40 to prove the following result.

Theorem 13.41. *There is an $R_0 > 0$ such that for $R > R_0$ Equation* (1.1) *on D_R has three positive solutions, of which one is y-symmetric and other two are nonaxially symmetric.*

Proof. Take $\rho > 0$ such that $\Omega = (\mathbb{R}^N \setminus \overline{\mathbb{R}^N_{-\rho,\rho}}) \cup \Theta$ is connected. Then Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by a bounded domain. By Theorem 6.7, we have $\alpha(\mathbb{R}^N) = \alpha(\Omega) < \alpha_s(\Omega)$. By Lien-Tzeng-Wang [47], $\alpha(B^N(0, R))$ is strictly decreasing as R is strictly increasing and

$$\alpha(B^N(0, R)) \searrow \alpha(\mathbb{R}^N) \quad \text{as } R \to \infty.$$

Take $R_0 > 0$, such that for $R > R_0$

$$\alpha(\mathbb{R}^N) < \alpha(B^N(0, R)) < \alpha_s(\Omega). \tag{13.31}$$

Since $B^N((x_R, y_R), R) \subsetneqq D_R \subsetneqq \mathbb{R}^N$, by Theorem 5.7 and Theorem 13.40, we conclude that

$$\alpha(\mathbb{R}^N) < \alpha(D_R) < \alpha(B^N((x_R, y_R), R)) = \alpha(B^N(0, R)). \tag{13.32}$$

Therefore, by (13.31) and (13.32) and $D_R \subset \Omega$, we have

$$\alpha(D_R) < \alpha(B^N(0, R)) < \alpha_s(\Omega) \le \alpha_s(D_R). \tag{13.33}$$

Thus,

$$\alpha(D_R) < \alpha_s(D_R). \tag{13.34}$$

By Theorem 12.3, there are a $y-$symmetry positive solution u_1 and a positive solution u_2 of Equation (1.1) in domain D_R for $R > R_0$ such that

$$J(u_1) = \alpha_s(D_R),$$
$$J(u_2) = \alpha(D_R).$$

Let $u_3(x, y) = u_2(x, -y)$, then u_3 is the third positive solution. By (13.34), u_1, u_2, and u_3 are different. Moreover, u_1 is a y-symmetric positive solution while both u_2, and u_3 are nonaxially symmetric positive solutions of (1.1) in domain D_R. □

Bibliographical notes: The results of this section are from Wang-Wu [74].

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Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan
E-mail address: hwang@mail.math.nthu.edu.tw