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LARGE TIME BEHAVIOR OF SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. We study the large time asymptotic behavior of solutions of the doubly degenerate parabolic equation $u_t = \operatorname{div}\left(|u|^{m-1}|\nabla u|^{p-2}\nabla u\right)$ in a cylinder $\Omega \times \mathbb{R}^+$, with initial condition $u(x,0) = u_0(x)$ in Ω and vanishing on the parabolic boundary $\partial\Omega \times \mathbb{R}^+$. Here Ω is a bounded domain in $\mathbb{R}^N$, the exponents m and p satisfy $m + p \geq 3$, $p > 1$, and the initial datum u_0 is in $L^1(\Omega)$.

1. Introduction

The objective of this article is to study the large time asymptotic behavior of weak solutions of nonlinear parabolic equations of the following type

$$u_t = \operatorname{div}\left(|u|^{m-1}|\nabla u|^{p-2}\nabla u\right) \text{ in } \Omega \times \mathbb{R}^+, \tag{1}$$

subject to the boundary condition

$$u(x,t) = 0 \text{ in } \partial\Omega \times \mathbb{R}^+, \tag{2}$$

and satisfying the initial condition

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{3}$$

where $u \in C(\mathbb{R}^+; L^2(\Omega))$ and $u^{\frac{m-1}{p-1}}\nabla u \in L^p_{\mathrm{loc}}(\Omega \times \mathbb{R}^+)$. We always assume that $p > 1$, $m + p \geq 3$, Ω is a bounded domain of $\mathbb{R}^n$ and $u_0 \in L^1(\Omega)$. The notion of weak solution is standard and we refer the reader to the book [DB] for more details. Equations of type (1) are classified as doubly nonlinear [L] or with implicit non linearity [KA]. Two well studied cases, the porous media equation and the p-Laplacian, belong to this larger class. In the last few years several authors have studied these kinds of equations on account of their physical and mathematical interest (see the review paper [KA]). Indeed, it seems interesting to see if and how many of the properties of the solutions of the porous media and the p-Laplacian

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equations are preserved in this more general case. Several papers are devoted to the study of the asymptotic behavior of the solutions of the porous media and the p-Laplacian equations. Among them we quote [AR-PE], [AR-CR-PE], [BE-NA-PE], [BE-PE], [VA2], [KA-VA] and [F-K].

The main point of this paper is to suggest a different approach that gives better results. While in the above references, an elliptic result is used to study the asymptotic behavior, here the basic properties of the evolution equation allow for the study of the asymptotic behavior. Moreover, the elliptic result will follow as a consequence. This approach allows generalizations to a large class of equations and and does not require a non negative datum. Furthermore, we are able to prove our results under weaker regularity assumptions on both the domain Ω and the initial datum u_0. For instance in [AR-PE], $\partial\Omega$ is required to be $C^{2,1/m}$ and u_0 to belong to $C^{1/m}(\bar{\Omega})$.

We denote by γ_i for $i = 1, 2, \dots,$ positive constants depending only on the data N, m, p, the L^1 norm of u_0 and the $C^{1,\alpha}$ norm of $\partial\Omega$. We proceed now to state our results.

Theorem 1.1. *Suppose that $m + p > 3$. There exists a unique non-negative non-trivial solution of the equation*

$$\operatorname{div}(w^{m-1}|Dw|^{p-2}Dw) = \frac{1}{3-m-p}w \tag{4}$$

in Ω, $w \in C^0(\bar{\Omega})$, $w^{\frac{m-1}{p-1}}Dw \in L^p(\Omega)$ and $w(x) = 0$ for $x \in \partial\Omega$. Moreover,

$$|Dw| \le \gamma_1(\operatorname{dist}(x, \partial\Omega))^{\frac{1-m}{m+p-2}} \tag{5}$$

and

$$\gamma_2(\operatorname{dist}(x, \partial\Omega))^{\frac{p-1}{m+p-2}} \le w(x) \le \gamma_3(\operatorname{dist}(x, \partial\Omega))^{\frac{p-1}{m+p-2}}. \tag{6}$$

Theorem 1.2. *Suppose that $m + p > 3$. There exists a unique solution of the evolution equation (1) subject to conditions (2) and (3). Moreover, for all $t > 1$ we have the bound*

$$|u(x, t)| \le \gamma_4 t^{\frac{1}{3-m-p}} w(x). \tag{7}$$

Furthermore, there exists a sequence $t_n \to \infty$ such that

$$\lim_{t_n\to\infty} t_n^{\frac{1}{m+p-3}} u(x, t_n) = z(x)$$

where $z(x) \in C^0(\bar{\Omega})$ is a solution of (4) vanishing on $\partial\Omega$, perhaps of changing sign.

If the initial datum u_0 is non negative (and not identically zero) we can be more precise.

Theorem 1.3. *Under the above assumptions, there exist constants $t_1, t_2 < 1$ depending only on the data, such that for $t > \max\{t_1, t_2\}$ we have*

$$(t - t_1)^{\frac{1}{3-m-p}} w(x) \le u(x, t) \le (t - t_2)^{\frac{1}{3-m-p}} w(x) \tag{8}$$

and

$$|Du(x, t| \le \gamma_5 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}} t^{\frac{1}{3-m-p}}. \tag{9}$$

In order to state the main result for the special case $m + p = 3$ we denote by B_p the best constant of the embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$.

Theorem 1.4. *Suppose that $m+p=3$. There exists a unique solution of the evolution equation (1) satisfying (2) and (3). Moreover, for all $t>1$ we have the bound*

$$|u(x,t)| \le \gamma_6 w(x) e^{(p-1)^{p-1} B_p^{-p} t}, \tag{10}$$

where $w(x)$ is a solution of the equation

$$\operatorname{div}(|w|^{m-1}|Dw|^{p-2}Dw) + (p-1)^{p-1}B_p^{-p} w = 0 \tag{11}$$

in Ω such that $w(x)=0$ in $\partial\Omega$, $w \in C^0(\bar{\Omega})$ and $w^{\frac{p-2}{p-1}} Dw \in L^p(\Omega)$.

In this case too, we obtain a better result assuming the non negativity of the initial datum.

Theorem 1.5. *Assume $u_0 \ge 0$ and not identically zero. Then, for $t>1$ we have*

$$\gamma_7 e^{(p-1)^{p-1}B_p^{-p}t} w(x) \le u(x,t) \le \gamma_8 e^{(p-1)^{p-1}B_p^{-p}t} w(x) \tag{12}$$

and

$$\gamma_9 e^{(p-1)^{p-1}B_p^{-p}t} \operatorname{dist}(x,\partial\Omega)^{p-1} \le u(x,t) \le \gamma_{10} e^{(p-1)^{p-1}B_p^{-p}t} \operatorname{dist}(x,\partial\Omega)^{p-1}. \tag{13}$$

Moreover, we also have

$$|Du(x,t)| \le \gamma_{11} \operatorname{dist}(x,\partial\Omega)^{2-p} e^{(p-1)^{p-1}B_p^{-p}t} \tag{14}$$

Remark 1. The above results hold for more general operators as defined, for example, in [A-DB] and [DB-H] and satisfying more general boundary conditions ([S-V]). For the sake of brevity we choose to analyze only simple operators.

The essential tools used below will be some quantitative L^∞-estimates, Hölder regularity results for bounded solutions (proved in [I], [P-V] and [V]), Harnack inequalities (stated in [V2]) and the introduction of suitable comparison functions.

Following the scheme of ideas in [DB-H] we prove the quantitative L^∞-estimates under more general conditions than needed. More precisely, let μ be a σ-finite Borel measure in $\mathbb{R}^N$ and $r>0$. We write

$$|||\mu|||_r = \sup_{\rho \ge r} \rho^{\frac{-\ell}{m+p-3}} \int_{B_\rho} |d\mu| \,,$$

where $|d\mu|$ is the variation of μ and

$$\ell = N(m+p-3)+p. \tag{15}$$

Theorem 1.6. *Let $m+p>3$. For every σ-finite Borel measure μ in $\mathbb{R}^N$ such that $|||\mu|||_r < +\infty$ for some $r>0$, there exists a weak solution of*

$$u_t = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) \tag{16}$$

in $\mathbb{R}^N \times (0,T(\mu))$ with initial condition

$$u(x,0)=\mu$$

and satisfying $u \in C((0,T(\mu)); L^2_{loc}(\mathbb{R}^N))$ and $u^{\frac{m-1}{p-1}} Du \in L^p_{loc}(\mathbb{R}^N \times (0,T(\mu)))$. Here we have set $T(\mu)=+\infty$ if $\lim_{r\to\infty} |||\mu|||_r = 0$ and, otherwise

$$T(\mu) = c_0 \lim_{r\to\infty} |||\mu|||_r^{-(m+p-3)}$$

where $c_0 = c_0(N,m,p)$.

Moreover, for all $0 < t < T(\mu)$ and $\rho \geq r > 0$ we have

$$|||u(\cdot,t)|||_r \leq \gamma_{12}|||\mu|||_r \tag{17}$$

and

$$\|u(\cdot,t)\|_{\infty,B_\rho} \leq \gamma_{13} t^{\frac{-N}{\ell}} \rho^{\frac{p}{m+p-3}} |||\mu|||_r^{\frac{p}{\ell}}. \tag{18}$$

Furthermore, for every bounded open set $\Omega \subset \mathbb{R}^N$ and for every $\epsilon > 0$ there exist constants $c_1 \equiv c_1(N,m,p,\epsilon,\mathrm{diam}(\Omega))$ and $c_2 \equiv c_2(N,m,p,\epsilon)$ such that

$$\int_0^t \int_\Omega |Du^{\frac{p-1}{m+p-2}}|^q \, dx \, d\tau \leq c_1 |||\mu|||_r^{c_4}, \tag{19}$$

where $q = p - 1 + \frac{1-\epsilon}{Nm+1}$. In particular, if $\epsilon = 1$ we obtain

$$\int_0^t \int_\Omega |Du|^{p-1}|u|^{m-1} \, dx \, d\tau \leq c_2 t^{\frac{1}{\ell}} \rho^{1+\frac{\ell}{m+p-3}} |||\mu|||_r^{1+\frac{m+p-3}{\ell}}, \tag{20}$$

where $c_5 \equiv c_5(N,m,p,\mathrm{diam}(\Omega))$.

As proved in [DB-H] in the case of the p-Laplacian, these estimates are optimal. Finally we also remark that the case $m + p < 3$ studied in [DB-K-V] and [S-V] behaves quite differently from the case considered in this paper since finite extinction time phenomena occur.

2. L^∞-Estimates

To prove Theorem 1.6, we follow the ideas in section 3 of [DB-H] where we refer the reader for more details and remarks. First, note that without loss of generality we can assume $\mu_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Actually, if one proves the quantitave estimates (17)–(20) in such a case, then by approximating μ_0 with regular functions, Theorem 1.6 will follow. As in [DB-H], we will prove the statement via several lemmas.

Lemma 2.1. *Consider the quantity*

$$K(T) = T^{-\frac{N(m+p-3)}{\ell}} \phi^{m+p-3}(T) + T^{-1} \tag{21}$$

where

$$\phi(t) = \sup_{\tau \in (0,t)} \tau^{N/\ell} \sup_{\rho \geq r} \rho^{-\frac{p}{m+p-3}} \|u(\cdot,\tau)\|_{\infty,B_\rho}. \tag{22}$$

Under the assumptions of Theorem 1.6, for each $t > 0$ we have

$$\|u(\cdot,t)\|_{\infty,B_\rho} \leq \gamma_{14}[K(t)]^{\frac{N+p}{\lambda}} \left(\int_{\frac{t}{4}}^t \int_{B_{2\rho}} u^p \, dx \, d\tau \right)^{p/\lambda}, \tag{23}$$

where $\lambda = p^2 + N(m + p - 3)$.

Proof. Fix $T > 0$ and $\rho > 0$ consider sequences $T_n = \frac{T}{2} - \frac{T}{2^{n+1}}$, $\rho_n = \rho + \frac{\rho}{2^{n+1}}$, and $\bar{\rho}_n = \frac{1}{2}(\rho_n + \rho_{n+1})$ for $n = 1, 2, \ldots$ Set $B_n = B_{\rho_n}$, $\bar{B}_n = B_{\bar{\rho}_n}$, $Q_n = B_n \times (T_n, T)$ and $\bar{Q}_n = \bar{B}_n \times (T_{n+1}, T)$ and, let $(x,t) \to \zeta_n(x,t)$ be a smooth cut off function in Q_n satisfying $\zeta_n(x,t) = 1$ for $(x,t) \in \bar{Q}_n$, $|D\zeta_n(x,t)| \leq \frac{2^{n+3}}{\rho}$ and

$$0 \leq \frac{\partial}{\partial t}\zeta_n(x,t) \leq \frac{2^{n+2}}{T}.$$

Finally, for a positive number k to be determined later we will consider the increasing sequences $k_n = k - \frac{k}{2^{n+1}}$ and $\bar{k}_n = \frac{1}{2}(k_n + k_{n+1})$ for $n = 0, 1, 2 \dots$

Assume that u is non negative. Setting $v = u^{\frac{m+p-2}{p-1}}$ we see that v satisfies the equation

$$\frac{\partial}{\partial t} v^{\frac{p-1}{m+p-2}} = \left(\frac{p-1}{m+p-2}\right)^{p-1} \operatorname{div}(|Dv|^{p-2}Dv) \tag{24}$$

If u changes sign we must set $v = |u|^{\frac{m-1}{p-1}} u$ and the proof presented below requires only minor modifications. Multiply (24) by $(v - \bar{k}_n)_+^{q-1}\zeta_n^p$, where $q = \frac{p(p-1)+m-1}{m+p-2}$ and integrate over Q_n. A standard calculation gives

$$\begin{aligned}\sup_{T_n \le t \le T} \int_{\bar{B}_n(t)} G(v(x,t))\,dx + \iint_{\bar{Q}_n} |D(v - \bar{k}_n)^{\frac{p+q-2}{p}}|^p\,dx\,d\tau \\ \le \gamma_{16}\frac{2^{np}}{\rho^p} \iint_{Q_n} (v - \bar{k}_n)^{p+q-2}\,dx\,d\tau + \gamma_{18}\frac{2^n}{T} \iint_{Q_n(t)} G(v(x,t))\,dx\,d\tau\end{aligned} \tag{25}$$

where G is a function defined by

$$\begin{cases} G'(s) &= (s - \bar{k}_n)^{q-1} s^{\frac{p-1}{m+p-2}-1} \\ G(\bar{k}_n) &= 0 \end{cases} \tag{26}$$

We shall use of the following elementary estimates. Suppose that $m \le 1$, then we have

$$\begin{aligned}(s - \bar{k}_n)&^{\frac{p-1}{m+p-2}+q-2} \\ &\le s^{\frac{p-1}{m+p-2}-1}(s - \bar{k}_n)^{q-1} \\ &\le 2^{\frac{p-1}{m+p-2}-1}(s - \bar{k}_n)^{\frac{p-1}{m+p-2}+q-2} + 2^{\frac{p-1}{m+p-2}-1}k^{\frac{p-1}{m+p-2}-1}(s - \bar{k}_n)^{q-1} \\ &\le 2^{\frac{p-1}{m+p-2}-1}(s - \bar{k}_n)^{\frac{p-1}{m+p-2}+q-2} + 2^{\frac{p-1}{m+p-2}-1}4^{n(\frac{p-1}{m+p-2}-1)}(s - k_n)^{\frac{p-1}{m+p-2}+q-2} \\ &\le \gamma_{19}4^{n(\frac{p-1}{m+p-2}-1)}(s - k_n)^{\frac{p-1}{m+p-2}+q-2}\end{aligned} \tag{27}$$

Hence by (25)–(27) and by the definition of $K(T)$ it follows that

$$\sup_{T_{n+1} \le t \le T} \int_{\bar{B}_n(t)} \bar{w}_n^s dx + \iint_{\bar{Q}_n} |D\bar{w}_n|^p\,dx\,d\tau \le \gamma_{20}4^{n(p+1)}K(T) \iint_{Q_n} w_n^s\,dx\,d\tau, \tag{28}$$

where

$$\bar{w}_n = (v - \bar{k}_n)_+^{\frac{p+q-2}{p}}, \quad w_n = (v - k_n)_+^{\frac{p+q-2}{p}}$$

and

$$s = \left[q + \frac{1-m}{m+p-2}\right]\left[\frac{p}{p+q-2}\right].$$

By the Gagliardo-Nirenberg's inequality (see [L-S-U], p. 62) we have

$$\begin{aligned}\iint_{Q_{n+1}} w_{n+1}^{p(1+\frac{s}{N})}\,dx\,d\tau \le \iint_{\bar{Q}_n} |\bar{w}_{n+1}\zeta_n|^{p(1+\frac{s}{N})}\,dx\,d\tau \\ \le \gamma_{21}\left\{\iint_{\bar{Q}_n} |D\bar{w}_n|^p\,dx\,d\tau + \frac{4^{np}}{\rho^p}\iint_{\bar{Q}_n} \bar{w}_n^p\,dx\,d\tau\right\}\left(\sup_{T_{n+1} \le t \le T}\int_{\bar{B}_n(t)} \bar{w}_n^s\,dx\right)^{\frac{p}{N}}\end{aligned} \tag{29}$$

From (28), (29) and the definition of $K(T)$, it follows that

$$\iint_{Q_{n+1}} w_{n+1}^{p(1+\frac{s}{N})}\,dx\,d\tau \le \gamma_{22}\left\{4^{np}K(T)\right\}^{\frac{N+p}{N}}\left(\iint_{Q_n} w_n^s\,dx\,d\tau\right)^{\frac{N+p}{N}} \tag{30}$$

This estimate is the starting point of the iteration process described in Lemma 3.1 of [DB-H]. An application of this result gives

$$(u-k)_+ \equiv 0 \text{ in } Q_\infty \tag{31}$$

Estimate (23) now follows by choosing

$$k=\gamma_{23}\left[K(T)\right]^{\frac{N+p}{\lambda}}\left(\int_{\frac{T}{4}}^{T}\int_{B_{2\rho}} u^p\,dx\,d\tau\right)^{\frac{p}{\lambda}}$$

where $\lambda = p^2+N(m+p-3)$.

Next, consider the case $m>1$. As we did before we need the following elementary estimates for G

$$\begin{aligned}\int_{k_{n+1}}^{v}(s-k_{n+1})^{\frac{p-1}{m+p-2}+q-2}ds &\le \int_{k_{n+1}}^{v}(s-\bar{k}_n)^{\frac{p-1}{m+p-2}+q-2}ds\\ &\le 4^{n(1-\frac{p-1}{m+p-2})}\int_{k_{n+1}}^{v}(s-\bar{k}_n)^{q-1}s^{\frac{p-1}{m+p-2}-1}ds\\ &\le 4^{n(1-\frac{p-1}{m+p-2})}\int_{\bar{k}_n}^{v}(s-\bar{k}_n)^{q-1}s^{\frac{p-1}{m+p-2}-1}ds\\ &\le 4^{n(1-\frac{p-1}{m+p-2})}\int_{\bar{k}_n}^{v}(s-\bar{k}_n)^{\frac{p-1}{m+p-2}+q-2}ds\end{aligned} \tag{32}$$

Hence, as before, by (25), (26), (32) and by definition of $K(T)$, we get

$$\sup_{T_{n+1}\le t\le T}\int_{\bar{B}_n(t)} w_{n+1}^s dx + \iint_{\bar{Q}_n}|Dw_{n+1}|^p\,dx\,d\tau \le \gamma_{25}4^{n(p+1)}K(T)\iint_{Q_n}\bar{w}_n^s\,dx\,d\tau. \tag{33}$$

Once we have (33) we deduce (23) as before. □

For $r>0$ define the function

$$\psi(t)=\sup_{\tau\in(0,t)}|||u(\cdot,\tau)|||_r. \tag{34}$$

Lemma 2.2. *For each $t>0$*

$$\phi(t)\le\gamma_{26}\int_0^t \tau^{-\frac{N}{\ell}(m+p-3)}\phi^{m+p-2}(\tau)\,d\tau+\gamma_{26}\psi(t)^{\frac{p}{k}}, \tag{35}$$

where we have set $\ell = N(m+p-3)+p$.

Proof. Multiply (33) by $\rho^{-\frac{p}{m+p-3}}\tau^{\frac{N}{\ell}}$ to obtain

$$\begin{aligned}\tau^{\frac{N}{\ell}}\frac{|||u(\cdot,\tau)|||_{\infty,B_{2\rho}}}{\rho^{\frac{p}{m+p-3}}} &\le \gamma_{26}\phi(t)^{\frac{N+p}{\lambda}(m+p-3)}t^{\frac{p}{\lambda}\frac{(3-m)N}{\ell}}\left(\int_{\frac{t}{4}}^{t}\int_{B_{2\rho}}\rho^{\frac{-\lambda}{m+p-3}}u^p\,dx\,d\tau\right)^{\frac{p}{\lambda}}\\ &\quad+\gamma_{26}t^{(\frac{N(p-1)}{\ell}-1)\frac{p}{\lambda}}\left(\int_{\frac{t}{4}}^{t}\int_{B_{2\rho}}\rho^{\frac{-\lambda}{m+p-3}}u^p\,dx\,d\tau\right)^{\frac{p}{\lambda}}\\ &\equiv H^{(1)}+H^{(2)}.\end{aligned}$$

We proceed to estimate $H^{(1)}$ and $H^{(2)}$ separately.

$$
\begin{aligned}
H^{(1)} &\le \gamma_{27}\phi(t)^{1+\frac{p}{\lambda}(m-3)}\left(\int_{\frac{t}{4}}^{t}\tau^{-\frac{N(m+p-3)}{\ell}}(2\rho)^{\frac{-p^2}{m+p-3}}\left(\tau^{\frac{N}{\ell}}|||u(\cdot,\tau)|||_{\infty,B_{2\rho}}\right)^{p}d\tau\right)^{\frac{p}{\lambda}}\\
&\le \gamma_{28}\phi(t)^{1-\frac{p}{\lambda}}\left(\int_0^t\tau^{-\frac{N(m+p-3)}{\ell}}\phi^{m+p-2}(\tau)\,d\tau\right)^{\frac{p}{\lambda}}\\
&\le \frac14\phi(t)+\gamma_{29}\int_0^t\tau^{-\frac{N(m+p-3)}{\ell}}\phi^{m+p-2}(\tau)\,d\tau.
\end{aligned}
$$

$$
\begin{aligned}
H^{(2)} &\le \gamma_{30}\left\{\frac1t\int_{\frac{t}{4}}^{t}\tau^{\frac{N(p-1)}{\ell}}\left(\frac{|||u(\cdot,\tau)|||_{\infty,B_{2\rho}}}{(2\rho)^{\frac{p}{m+p-3}}}\right)^{p-1}(2\rho)^{\frac{-\ell}{m+p-3}}\int_{B_{2\rho}}u(x,\tau)\,dx\,d\tau\right\}^{\frac{p}{\lambda}}\\
&\le \gamma_{31}\phi(t)^{\frac{p(p-1)}{\lambda}}\left(\frac1t\int_0^t|||u(\cdot,\tau)|||_r d\tau\right)^{\frac{p}{\lambda}}\\
&\le \frac14\phi(t)+\gamma_{32}\psi(t)^{\frac{p}{\ell}}.
\end{aligned}
$$

□

Lemma 2.3. *Let $\rho\ge r>0$ and let $x\to\zeta(x)$ be a piecewise smooth cut-off function in $B_{2\rho}$ such that $\zeta=1$ on B_ρ and $|D\zeta|\le\frac1\rho$. For each $t>0$ we have*

$$
\begin{aligned}
\int_0^t\int_{B_{2\rho}}|Du|^{p-1}|u|^{m-1}\zeta^{p-1}dx\,d\tau &\le \gamma_{33}\rho^{1+\frac{\ell}{m+p-3}}\left(\int_0^t\left[\tau^{\frac{p+1}{\ell}-1}\phi(\tau)^{\frac{(m+p-3)(p+1)}{p}}\psi(\tau)\right.\right.\\
&\left.\left.+\tau^{\frac1\ell-1}\phi(\tau)^{\frac{m+p-3}{p}}\psi(\tau)\right]d\tau\right)^{\frac{p-1}{p}}\left(\int_0^t\tau^{\frac1\ell-1}\phi(\tau)^{\frac{m+p-3}{p}}\psi(\tau)\,d\tau\right)^{\frac1p}. \qquad (36)
\end{aligned}
$$

Proof. Let $v=u^{\frac{m+p-2}{p-1}}$. By Hölder's inequality we have

$$
\begin{aligned}
\int_0^t\int_{B_{2\rho}}|Dv|^{p-1}\zeta^{p-1}\,dx\,d\tau &\le \gamma_{34}\left(\int_0^t\int_{B_{2\rho}}\tau^{\frac1p}|Dv|^p v^{(\frac{p-1}{p}\frac{m+p-3}{m+p-2})-1}\zeta^p\,dx\,d\tau\right)^{\frac{p-1}{p}}\\
&\times\left(\int_0^t\int_{B_{2\rho}}\tau^{(\frac1p-1)}v^{\frac{p-1}{m+p-2}\frac{m+2p-3}{p}}\,dx\,d\tau\right)^{\frac1p}\\
&\equiv J_1(t)^{\frac{p-1}{p}}J_2(t)^{\frac1p}.
\end{aligned}
$$

The lemma will follow by estimating $J_1(t)$ and $J_2(t)$ separately. Multiply (24) by $\tau^{\frac1p}v^{(p-1)(\frac{m+p-3}{m+p-2}\frac1p)}\zeta^{p-1}$ and integrate by parts to get

$$
\begin{aligned}
J_1 &= \int_0^t\int_{B_{2\rho}}\tau^{\frac1p}v^{\left(\frac{p-1}{m+p-2}\frac{m+p-3}{p}-1\right)}|Dv|^p\zeta^p\,dx\,d\tau\\
&\le \gamma_{35}\rho^{-p}\int_0^t\int_{B_{2\rho}}\tau^{\frac1p}v^{(p-1)(\frac{m+p-3}{m+p-2}\frac1p+1)}\,dx\,d\tau+J_2\\
&= L_1+J_2
\end{aligned}
$$

We estimate L_1 as follows

$$\begin{aligned} L_1 \le \gamma_{36}\rho^{1+\frac{\ell}{m+p-3}} \int_0^t \tau^{\frac{p+1}{\ell}-1} \left(\tau^{\frac{N}{\ell}} (2\rho)^{-\frac{p}{m+p-3}} |||u(\cdot,\tau)|||_{\infty,B_{2\rho}}\right)^{\frac{(m+p-3)(p+1)}{p}} \\ \times \left(\rho^{-\frac{\ell}{m+p-3}} \int_{B_{2\rho}} u(x,\tau)\,dx\right) d\tau \\ \le \gamma_{37}\rho^{1+\frac{\ell}{m+p-3}} \int_0^t \tau^{(\frac{p+1}{\ell}-1)} \phi(\tau)^{\frac{(m+p-3)(p+1)}{p}} \psi(\tau)\,d\tau. \end{aligned}$$

and J_2 by

$$\begin{aligned} J_2 \le \gamma_{38}\rho^{1+\frac{\ell}{m+p-3}} \int_0^t \tau^{\frac{1}{\ell}-1} \left(\tau^{\frac{N}{\ell}} \rho^{-\frac{p}{m+p-3}} |||u(\cdot,\tau)|||_{\infty,B_{2\rho}}\right)^{\frac{m+p-3}{p}} \\ \times \left(\rho^{-\frac{\ell}{m+p-3}} \int_{B_{2\rho}} u(x,\tau)\,dx\right) d\tau \\ \le \gamma_{39}\rho^{1+\frac{\ell}{m+p-3}} \int_0^t \tau^{\frac{1}{\ell}-1} \phi(\tau)^{\frac{m+p-3}{p}} \psi(\tau)\,d\tau. \end{aligned}$$

Multiply (16) by ζ^p and integrate in $[0,t]$ to get

$$\int_{B_\rho} u(x,t)\,dx \le \int_{B_{2\rho}} u_0(x)\,dx + \gamma_{40}\rho^{-1} \int_0^t \int_{B_{2\rho}} |Du|^{p-1}|u|^{m-1}\zeta^{p-1}\,dx\,d\tau.$$

Hence, by the previous lemma we have

$$\begin{aligned} \psi(t) \le \gamma_{41}|||u_0|||_r + \gamma_{41} \bigg(\int_0^t \tau^{\frac{p+1}{\ell}-1} \phi(\tau)^{\frac{(m+p-3)(p+1)}{p}} \psi(\tau)\,d\tau \\ + \int_0^t \tau^{\frac{1}{\ell}-1} \phi(\tau)^{\frac{m+p-3}{p}} \psi(\tau)\,d\tau\bigg). \end{aligned} \tag{37}$$

Via an algebraic lemma (see Lemma 3.5 of [DB-H]), (17) and (18) come from (35) and (37). Moreover (17), (18) and (36) imply (20). It remains only to check (19) to finish the proof of Theorem 1.6. We proceed as follows

$$\begin{aligned} \int_{\frac{t}{2}}^t \int_{B_\rho} |Dv|^q\,dx\,d\tau &\le \int_{\frac{t}{2}}^t \int_{B_\rho} t^{-\beta} v^{-\alpha} |Dv|^q t^\beta v^\alpha\,dx\,d\tau \\ &\le \left(\int_{\frac{t}{2}}^t \int_{B_\rho} t^{\beta\frac{p}{q}} v^{-\alpha\frac{p}{q}} |Dv|^p\,dx\,d\tau\right)^{\frac{q}{p}} \left(\int_{\frac{t}{2}}^t \int_{B_\rho} t^{-\beta\frac{p}{p-q}} v^{\frac{\alpha p}{p-q}}\,dx\,d\tau\right)^{1-\frac{q}{p}} \\ &\equiv (I^{(1)})^{\frac{q}{p}} (I^{(2)})^{1-\frac{q}{p}} \end{aligned}$$

where $q = p - 1 + \frac{1-\varepsilon}{Nm+1}$.

At this point (20) follows from the previous inequality by choosing $\alpha p = p - q$ and arguing as in [DB-H] pages 204–205. □

Remark 2. If we consider the Cauchy problem (1) instead of (16) we get for each $t > s > 0$

$$\|u(\cdot,t)\|_{\infty,\Omega} \le \gamma_{42}(t-s)^{-\frac{N}{\ell}} \left(\int_\Omega u(\cdot,s)\,dx\right)^{\frac{p}{\ell}}. \tag{38}$$

A similar result holds in the case $m+p=3$; that is, for each $t>0$

$$\|u\|_{L^\infty(\Omega\times(t+1,t+2))} \le \gamma_{43}\left(\iint_{\Omega\times(t,t+3)} u^{\frac{p}{p-1}}\,dx\,d\tau\right)^{\frac{p-1}{p}}. \tag{39}$$

One could prove (35) by repeating an argument analogous to the previous one, but we prefer to show another method. As remarked by Trudinger [TR], all the classical estimates for parabolic equations hold in such a particular case. Hence, by considering cylinders $B(R)\times[0,-R^p]$ and repeating the classical argument of the L^∞-estimates (see, for instance, [L-S-U]) one gets (39).

Before concluding this section, let us state a straightforward consequence of the previous estimates (see [DB-H]):

Proposition 2.1. *Let $u\ge 0$ be a weak solution of (1). Then for each $R>0$, and for each $\epsilon\in(0,1]$ we have*

$$\fint_{B_{(1+\epsilon)R}} u(x,t)\,dx \ge$$

$$\fint_{B_R} u(x,\tau)\,dx\left\{(1+\epsilon)^{-N}-\frac{\gamma_{44}}{\epsilon}\left(\left[\fint_{B_r} u(x,\tau)\,dx\right]^{m+p-3} R^{-p}(t-\tau)\right)^{\frac{1}{\ell}}\right\} \tag{40}$$

for all

$$0<\tau<t\le\gamma_{45}R^p\left(\fint_{B_R} u(x,\tau)\,dx\right)^{m+p-3}.$$

This proposition is usually referred as a lemma on "how fast the material can escape a given ball" ([A-C]) and it is useful in studying the initial trace. More precisely one can prove (see [DB-H]):

Theorem 2.1. *Let u be a non–negative solution of (1) in $\mathbb{R}^N\times[0,T]$ for some $0<T\le\infty$. Then, there exists a unique σ-finite Borel measure μ on $\mathbb{R}^N$ such that*

$$\lim_{t\to 0^+}\int_{\mathbb{R}^N} u(x,t)\phi(x)dx = \int_{\mathbb{R}} \phi\,d\mu$$

for all ϕ continuous and compactly supported in $\mathbb{R}^N$. Moreover for each $R>0$, and for each $0<t\le T$

$$R^{-N}\int_{B_R} d\mu \le \gamma_{46}\left\{\left(\frac{R^p}{t}\right)^{\frac{1}{m+p-3}} + \left(\frac{t}{R^p}\right)^{\frac{N}{p}}[u(0,t)]^{\frac{k}{p}}\right\}.$$

3. L^∞-Estimates in the Large

In the sequel, we need sharp estimates for the function $\mathcal{E}(t)$ defined as the Sobolev ratio

$$\mathcal{E}(t) = \frac{\int_\Omega |Dv|^p(x,t)\,dx}{\left(\int_\Omega v(x,t)^{\frac{m+2p-3}{m+p-2}}\,dx\right)^{\frac{m+p-2}{m+2p-3}p}} \tag{41}$$

where $v=u^{\frac{m+p-2}{p-1}}$ is a solution of the equation

$$\frac{\partial}{\partial t}\left(v^{\frac{p-1}{m+p-2}}\right) = \left(\frac{p-1}{m+p-2}\right)^{p-1}\operatorname{div}(|Dv|^{p-2}Dv) \tag{42}$$

in $\Omega \times (0,\infty)$ subject to the boundary condition

$$v(x,t) = 0 \text{ in } \partial\Omega \times \mathbb{R}^+$$

and to the initial condition

$$v(0,x) = u_0(x)^{\frac{m+p-2}{p-1}}.$$

In [B-H] (see also [S-V]) the following basic fact is proved.

Theorem 3.1. *The function $\mathcal{E}(t)$ is non increasing in time.*

We deduce from it the following L^∞-bound:

Proposition 3.1. *Assume $m+p>3$ and u_0 not identically equal to 0. Then, for each $t>0$,*

$$\gamma_{47} t^{\frac{1}{3-m-p}} \le \|u(x,t)\|_{\infty,\Omega} \le \gamma_{48} t^{\frac{1}{3-m-p}}, \tag{43}$$

where $\gamma_{48} \ge \gamma_{47} > 0$.

Proof. Without any loss of generality, we may assume $u_0 \in L^\infty(\Omega)$. Multiplying (42) by v and integrating in Ω, we have

$$\frac{p-1}{m+2p-3}\frac{d}{dt}\int_\Omega v^{\frac{m+2p-3}{m+p-2}} + \left(\frac{p-1}{m+p-2}\right)^{p-1}\int_\Omega |Dv|^p = 0. \tag{44}$$

Let $B_{m,p}$ the best Sobolev constant such that for each $w \in W_0^{1,p}(\Omega)$

$$\left(\int_\Omega w^{\frac{m+2p-3}{m+p-2}}\,dx\right)^{\frac{m+p-2}{m+2p-3}} \le B_{p,m}\left(\int_\Omega |Dw|^p\,dx\right)^{\frac{1}{p}}. \tag{45}$$

Set $z(t) = \int_\Omega v^{\frac{m+2p-3}{m+p-2}}\,dx$. From (44) and (45) we get

$$\frac{p-1}{m+2p-3}\frac{d}{dt}z + \left(\frac{p-1}{m+p-2}\right)^{p-1} B_{p,m}^{-p} z^{\frac{m+p-2}{m+2p-3}p} \le 0.$$

Solve the differential inequality and put $\alpha = \frac{m+p-2}{m+2p-3}p$ to get

$$z(t) \le (\alpha-1)\left(\left((p-1)^{p-2}\frac{m+2p-3}{(m+2p-2)^{p-1}}\right)B_{p,m}^{-p}t + \left(\frac{z(0)}{\alpha-1}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}.$$

Hence, we obtain

$$\left(\int_\Omega u^{\frac{m+2p-3}{p-1}}(\cdot,t)\,dx\right)^{\frac{p-1}{m+2p-3}} \le \gamma_{49} t^{\frac{1}{3-m-p}}. \tag{46}$$

On applying (38) we get

$$\begin{aligned}
\|u(\cdot,t)\|_{\infty,\Omega} &\le \gamma_{50} 2^{\frac{N}{\ell}} t^{-\frac{N}{\ell}} \left(\int_\Omega u(\cdot,\frac{t}{2})\,dx\right)^{\frac{p}{\ell}} \\
&\le \gamma_{51} 2^{\frac{N}{\ell}} t^{-\frac{N}{\ell}} |\Omega|^{\frac{m+p-2}{m+p-3}\frac{p}{\ell}} \gamma_{49}^{\frac{p}{\ell}} t^{\frac{1}{3-m-p}\frac{p}{\ell}} \\
&\le \gamma_{52} t^{\frac{1}{3-m-p}}.
\end{aligned}$$

From Theorem 3.1 and (44) we have

$$\frac{p-1}{m+2p-3}\frac{d}{dt}z + (\mathcal{E}(0))^{-1} z^{\frac{m+p-2}{m+2p-3}p} \ge 0.$$

Solve the differential inequality to obtain

$$\gamma_{53} t^{\frac{1}{3-m-p}} \le \left(\int_\Omega u^{\frac{m+2p-3}{p-1}}(\cdot,t)dx \right)^{\frac{p-1}{m+2p-3}}.$$

The statement is now proved because

$$\left(\int_\Omega u^{\frac{m+2p-3}{p-1}}(\cdot,t)dx \right)^{\frac{p-1}{m+2p-3}} \le |\Omega|^{\frac{p-1}{m+2p-3}} \|u(x,t)\|_{\infty,\Omega}.$$

□

The case $m+p=3$ follows along the same lines.

Proposition 3.2. *Suppose that $m+p=3$ and assume u_0 not identically equal to 0, then for $t>1$ we have*

$$\gamma_{54} e^{-\alpha_1 t} \le \|u(\cdot,t)\|_{\infty,\Omega} \le \gamma_{55} e^{-\alpha_2 t}, \tag{47}$$

where $\alpha_1 = (\mathcal{E}(0))^{-1}(p-1)^{p-1}$ and $\alpha_2 = -(p-1)^{p-1} B_p^{-p}$.

Proof. Reasoning as above, we get

$$\frac{p-1}{p}\frac{d}{dt} z + (p-1)^{p-1} B_p^{-p} z \le 0.$$

Hence,

$$z(t) \le e^{-\frac{p}{p-1}(p-1)^{p-1} B_p^{-p} t} z(0)$$

and

$$\|u(\cdot,t)\|_{\infty,\Omega} \le \gamma_{56} z(0)^{\frac{p-1}{p}} e^{-(p-1)^{p-1} B_p^{-p} t}.$$

The lower bound is obtained analogously. □

4. Behavior near the boundary

It is of interest here to prove estimates from above and below near $\partial\Omega$. The argument we follow is the same of [DB-K-V], with the only difference that we need to use the super and subsolutions introduced in [S-V] instead of the ones of [DB-K-V]. For this reason, we state only the main results, leaving the easy proofs to the reader. As all the constants are stable when $m+p \to 3$, we consider only the case $m+p>3$.

4.1. Estimates from above near $\partial\Omega$.

Theorem 4.1. *Let u be a bounded solution $|u| \le M$ of*

$$u_t - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = 0 \text{ in } \Omega \times (s,t) \tag{48}$$

satisfying $u \in C(s,t;L^2(\Omega))$, $u^{\frac{m-1}{p-1}} Du \in L^p(s,t,\Omega)$ for some $s,t \in \mathbb{R}^+$ and some $M>0$.

Then, for all $(x,t) \in \Omega \times (s,t)$ we have

$$|u(x,t)|^{\frac{m+p-2}{p-1}} \le \gamma_{58} M^{\frac{m+p-2}{p-1}} \left(\frac{e^\lambda}{e^\lambda - 1} \right)^{\frac{3-m-p+mp}{pm}} \operatorname{dist}(x,\partial\Omega), \tag{49}$$

where $\lambda = \min\left\{1, \frac{t-s}{M^{3-m-p}}\right\}$ and γ_{44} depends only upon N, m, p and $\|\partial\Omega\|_{C^{1,\alpha}}$.

Corollary 4.1. *For every $\eta > 0$, there exists a constant γ_{59} (depending only upon N, m, p and $\|\partial\Omega\|_{C^{1,\alpha}}$) such that for all $t - s \geq \eta M^{3-m-p}$ we have*

$$|u(x,t)| \leq \gamma_{60} M \left(\operatorname{dist}(x,\partial\Omega)\right)^{\frac{p-1}{m+p-2}}. \tag{50}$$

Remark 3. Estimates (50), (43) and (47) imply that for each $t > 1$, if $m + p > 3$

$$|u(x,t)| \leq \gamma_{61} \operatorname{dist}(x,\partial\Omega)^{\frac{p-1}{m+p-2}} t^{\frac{1}{3-m-p}}, \tag{51}$$

and if $m + p = 3$

$$|u(x,t)| \leq \gamma_{62} \operatorname{dist}(x,\partial\Omega)^{p-1} e^{-B_{p,m}^{-p} \frac{(p-1)^p}{p} t}. \tag{52}$$

4.2. **Estimates from below near $\partial\Omega$.** Suppose now that u is a non negative bounded solution $u \leq M$ of (48). For $r > 0$ let $\Omega_r = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq r\}$, $\Omega_{r,t} = \Omega_r \times [s,t]$ and $\mu(r) = \inf\{u(x,\tau) \colon (x,\tau) \in \Omega_{r,\tau}\}$. For $0 < s < t$ let

$$r(M,s,t) = r_0 \min \left\{ 1; \left(\frac{t-s}{M^{3-m-p}} \right)^{\frac{1}{p}} \right\}, \tag{53}$$

where $r_0 \leq 1$ is a positive constant depending only upon N, m, p and $\|\partial\Omega\|_{C^{1,\alpha}}$.

Theorem 4.2. *For each $0 < s < t$ and for each $x \in \Omega$ such that $\operatorname{dist}(x,\partial\Omega) \leq r(M,s,t)$ we have*

$$u^{\frac{m+p-2}{p-1}} \geq \gamma_{63} (\mu(r(M,s,t)))^{\frac{m+p-2}{p-1}} \left(\frac{M^{3-m-p}}{t-s} \right)^{\frac{1}{p}} \operatorname{dist}(x,\partial\Omega), \tag{54}$$

where γ_{63} depends only upon N, m, p and $\|\partial\Omega\|_{C^{1,\alpha}}$.

Corollary 4.2. *For every $\eta > 0$, there exist r_0 and γ_{64} (depending only upon N, p, m, $|\Omega|$, η and $\|\partial\Omega\|_{C^{1,\alpha}}$) such that*

$$u(x,t) \geq \gamma_{64} \mu(r_0) \left(\operatorname{dist}(x,\partial\Omega)\right)^{\frac{p-1}{m+p-2}}, \tag{55}$$

where $x \in \Omega$, $0 < s < t$ and $(t - s) \leq \eta M^{3-p-m}$.

5. The case $m + p > 3$

Denote by $x_0(t)$ a point in Ω where the maximum of $|u|$ is attained at time t. Let

$$\tilde{u}_s(x,t) = \frac{u(x,(t+s)u(x_0(s),s)^{-(m+p-3)})}{u(x_0(s),s)} \tag{56}$$

The function $\tilde{u}_s$ satisfies the equation

$$\frac{\partial}{\partial t}\tilde{u}_s = \operatorname{div}(|\tilde{u}_s|^{m-1}|D\tilde{u}_s|^{p-2}D\tilde{u}_s) \text{ in } \Omega \times [-1,1] \tag{57}$$

and it vanishes in $\partial\Omega \times [-1,1]$. By (43) we get that for each $s \geq 1$, $\tilde{u}_s$ is uniformly bounded in $\Omega \times [-1,1]$. Hence, by the regularity results of [P-V] and [V], we have that there is $\alpha > 0$ such that for all $s > 1$

$$\sup_{0 \leq t \leq 1} \|\tilde{u}_s(x,t)\|_{C^\alpha(\bar{\Omega})} \leq \gamma_{65} \tag{58}$$

This estimate implies

$$\sup_{s \geq 1} \|s^{\frac{1}{m+p-3}} u(x,s)\|_{C^\alpha(\bar{\Omega})} \leq \gamma_{66}. \tag{59}$$

On the other hand, since $\mathcal{E}(t)$ is decreasing, reasoning as in [B-H] (see also [S-V]) we have that there is a sequence of times $s_n \to \infty$ such that $u(x,s_n)s_n^{\frac{1}{m+p-3}} \rightharpoonup w$, where w solves (4). Therefore, by Minty's lemma, $u(x,s_n)s_n^{\frac{1}{m+p-3}} \to w$. Moreover by (43), $w \equiv 0$ if and only if $u_0(x) \equiv 0$.

If $u_0(x)$ is assumed to be non negative we can be more precise. Let us recall the Harnack inequality stated in [V2].

Proposition 5.1. *Let $u \geq 0$ be a local weak solution of the equation*

$$u_t = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du)$$

in some cylindrical domain $\Omega_T = \Omega \times [0,T]$.

Let $(x_0,t_0) \in \Omega_T$, let $B_\rho(x_0)$ be the ball of radius ρ centered at x_0 and assume $u(x_0,t_0) > 0$. Then, there exist two constants $c_i = c_i(N,m,p)$, $i = 0,1$, such that

$$u(x_0,t_0) \leq c_0 \inf_{x\in B_\rho(x_0)} u(x, t_0 + \frac{c_1\rho^p}{u^{m+p-3}(x_0,t_0)}) \tag{60}$$

provided the box

$$Q_0 = B_{2\rho}(x_0) \times \left\{ t_0 - c_1\frac{\rho^p}{(u(x_0,t_0))^{m+p-3}}, t_0 + c_1\frac{\rho^p}{(u(x_0,t_0))^{m+p-3}} \right\}$$

is all contained in Ω_T.

Consider now the function $\tilde{u}_s$ defined in (56). First, let us estimate the point at which the maximum is attained. By (43) it follows that $u(x_0(s),s) \geq \gamma_{67} t^{-\frac{1}{m+p-3}}$.

On the other hand, by (50) we obtain

$$u(x_0(s),s) \leq \gamma_{68} \operatorname{dist}(x_0(s),\partial\Omega)^{\frac{p-1}{m+p-2}} t^{-\frac{1}{m+p-3}}.$$

Therefore,

$$\operatorname{dist}(x_0(s),\partial\Omega) \geq \left(\frac{\gamma_{67}}{\gamma_{68}}\right)^{\frac{m+p-2}{p-1}} = \sigma. \tag{61}$$

Let $s \geq 1$ and without loss of generality assume $x_0(s) = 0$. Apply (56) at (0,0) and choose $\rho = \frac{\sigma}{2}$ to get

$$\inf_{x\in B(\frac{\sigma}{2})} \tilde{u}_s(x,\underline{t}) \geq \bar{c}_0,$$

where $\underline{t} = c_1(\frac{\sigma}{2})^p$.

We may now repeat this process starting from each point $(x,t) \in \{|x| < \frac{\sigma}{2}\} \times \{\underline{t}\}$ and continue in this fashion.

Let r_0 be the number determined in Corollary 4.2 and let

$$\tilde{\Omega}_{r_0,\tau} = \{x \in \Omega \text{ such that } \operatorname{dist}(x,\partial\Omega) \geq r_0\} \times [\tau, \tau+1].$$

The arguments indicated above prove that there are two constants τ and γ_{69} that can be determined apriori only in terms of N, m, p, $\mathcal{E}(0)$, $\|\partial\Omega\|_{C^{1,\alpha}}$ and r_0 such that

$$\inf_{(x,t)\in\tilde{\Omega}_{r_0,\tau}} \tilde{u}_s(x,t) \geq \gamma_{69} > 0.$$

To summarize, we have determined a constant t_2 such that for each $t \geq t_2$

$$\inf_{(x,t)\in\Omega_{r_0,t}} u(x,s) \geq \gamma_{70} t^{-\frac{1}{m+p-3}}, \tag{62}$$

where

$$\Omega_{r_0,t} = \{x \in \Omega \text{ such that } \operatorname{dist}(x, \partial\Omega) \geq r_0\} \times [t, 2t].$$

Therefore, from (55) and (62) we have that for each $t \geq 2t_2$

$$u(x,t) \geq \gamma_{71} t^{-\frac{1}{m+p-3}} \left(\operatorname{dist}(x, \partial\Omega)\right)^{\frac{p-1}{m+p-2}}. \tag{63}$$

Remark 4. Note that inequality (63) implies that the support of $u(x,t)$ is $\bar{\Omega}$ for each $t \geq 2t_2$.

In order to get a stronger regularity result, let $(x_0, \bar{t}_0) \in \Omega \times \mathbb{R}^+$ and assume $\bar{t}_0 > 2t_2$. Denote by σ the distance between x_0 and $\partial\Omega$ and let $R = \min(\sigma, 1)$. Consider the change of variables

$$x \to \frac{2(x - x_0)}{R}, \qquad t \to \frac{2^p(t + \bar{t}_0) u(x_0, \bar{t}_0)^{-(m+p-3)}}{R^p}, \qquad v \to \frac{u}{u(x_0, \bar{t}_0)}.$$

The function v satisfies the equation

$$v_t = \operatorname{div}(|v|^{m-1} |Dv|^{p-2} Dv) \text{ in } B(1) \times [-1, 1].$$

Moreover, we have $0 < \gamma_{72} \leq v < \gamma_{73}$ in $B(1) \times [-1, 1]$, and as above we conclude that v is uniformly α-Hölder continuous. Hence, reasoning as in [S-V], we get that there is a constant $\beta > 0$ such that

$$v \in C^{1,\beta} \text{ and } |Dv(0,0)| \leq \gamma_{74}. \tag{64}$$

This inequality implies (9) in a straightforward way. Finally, reasoning as above, we obtain that there is a sequence $t_n \to \infty$ such that $t_n^{\frac{1}{m+p-3}} u(x, t_n) \longrightarrow w$, where w is a solution (4). Hence (5) and (6) are direct consequences of (9), (63) and (50). Moreover (8) holds because it follows from (5) and (6). Therefore, we obtain

$$(t - t_1)^{-\frac{1}{m+p-3}} w(x) \leq u(x,t) \leq (t - t_2)^{-\frac{1}{m+p-3}} w(x)$$

and

$$\|u(x,t) t^{-\frac{1}{m+p-3}} - w(x)\|_{\infty,\Omega} \leq \gamma_{75} t^{-\frac{1}{m+p-3}} \operatorname{dist}(x, \partial\Omega)^{\frac{p-1}{m+p-2}}. \tag{65}$$

Note that (65) implies the uniqueness of a non negative solution of (4).

Indeed, if there are two solutions w and z of (4) then, by (65),

$$\|z(x) - w(x)\|_{\infty,\Omega} \leq \gamma_{75} t^{-\frac{1}{m+p-3}} \operatorname{dist}(x, \partial\Omega) \text{ for all } t > 2t_2.$$

Therefore we must have $z(x) \equiv w(x)$.

6. The case $m + p = 3$

This case is analogous to the previous one. The only difference is that we cannot deduce the existence of a solution of (11) because (47) is not as good as (43). The existence of a minimizer of the functional

$$\left(\int_\Omega |Du|^p \right)^{\frac{1}{p}}$$

on the manifold

$$\left(\int_\Omega u^p \right)^{\frac{1}{p}} = \text{ constant}$$

is well known. Therefore, we can assume the existence of a positive function w that satisfies (11).

Considering now a solution to the evolution problem

$$u_t = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) \text{ in } \Omega \times \mathbb{R}^+$$

satisfying the homogeneous boundary condition

$$u(x,t) = 0 \text{ for } x \in \partial\Omega$$

and the initial condition

$$u(x,0) = u_0(x) \text{ for } x \in \Omega.$$

Reasoning as in the previous section we have that u satisfies (5) and (6). If we assume that u_0 is non negative, arguing as in the previous section we get that for each $1 \le t \le 2$

$$\gamma_{76} \operatorname{dist}(x,\partial\Omega)^{p-1} \le u(x,t) \le \gamma_{77} \operatorname{dist}(x,\partial\Omega)^{p-1}. \tag{66}$$

(Note that in this case the estimate deteriorates as $t \longrightarrow +\infty$). Estimate (12) follows now by applying the maximum principle.

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