Electronic Journal of Differential Equations
Vol. **1995**(1995), No. 02, pp. 1-16. Published February 24, 1995.
ISSN 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

STRONG SOLUTIONS OF QUASILINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH SINGULAR KERNELS IN SEVERAL SPACE DIMENSIONS

Hans Engler

Abstract. For quasilinear integro-differential equations of the form $u_t - a * A(u) = f$, where a is a scalar singular integral kernel that behaves like $t^{-\alpha}$, $\frac{1}{2} \le \alpha < 1$ and A is a second order quasilinear elliptic operator in divergence form, solutions are found for which $A(u)$ is integrable over space and time.

1. Introduction

The purpose of this note is a study of quasilinear integro-differential equations of the form

$$\frac{\partial}{\partial t}u(x,t) - \int_0^t a(t-s)\,\mathrm{div}\,\tilde{g}(\nabla u(x,s))\,ds = f(x,t) \tag{1.1}$$

in cylindrical domains $\Omega \times [0,T] \subset \mathbb{R}^{N+1}$, together with zero boundary data for u on $\partial\Omega \times [0,T]$ and initial data on $\Omega \times \{0\}$. Here $a(\cdot)$ is a scalar integral kernel that behaves like $t^{-\alpha}$, $0 < \alpha < 1$ for t near zero. The equation occurs in the description of simple shear motions ($N = 1$) and of antiplane shear ($N = 2$) of certain idealized viscoelastic materials ([14]), and "power law" kernels $a(t) = t^{-\alpha}$ have been proposed elsewhere for such models ([13]). We are interested in large time solutions without restrictions on the size of the data.

Recently, for $\Omega = [0,1] \subset \mathbb{R}$, "strong solutions" with u_{xx}^2 integrable over $\Omega \times [0,T]$ have been found under two different sets of assumptions, namely when $\alpha \ge \frac{1}{2}$ and $0 < c \le \tilde{g}'(r) \le c' < \infty$ ([9]) and when the Laplace transform $\hat{a}$ maps the right half plane into a sector in the right half plane, under a restriction for $\frac{c}{c'}$ that depends on the angle of aperture of this sector ([8]). Also, weaker solutions with only u_x^2 integrable can be found without restrictions for α, c, c' ([11]), and smooth solutions (as smooth as the data permit) can be found if $\frac{2}{3} < \alpha < 1$ and only $0 < \tilde{g}'(r) \le C < \infty$ ([3]). In other work, large time solutions for small smooth data have been constructed, weak solutions for more regular kernels have been found, and the break-down of smooth solutions for regular kernels has been established; see the references in [8] and [9].

1991 Mathematics Subject Classifications: 45K05.
Key words and phrases: Integro-differential equation, strong solution, singular kernel, quasilinear

Submitted: December 15, 1994.
Supported by the National Science Foundation under grant # DMS - 9003543

We want to extend the results of [9] in several ways, namely by treating also higher space dimensions and by allowing also nonlinear terms $\tilde{g}$ that grow polynomially in their arguments, corresponding to "shear-thickening" behavior for viscoelastic materials. The function $\tilde{g} : \mathbb{R}^N \to \mathbb{R}^N$ will be assumed to be "isotropic", i.e. to have the form $\tilde{g}(\xi) = g(|\xi|^2)\xi$. We also use a slightly different class of integral kernels and modified assumptions for the data. The main restriction, namely $\frac{1}{2} \leq \alpha < 1$, however is not weakened in the present paper. We also give linear examples that show that more must be assumed for the data of the problem if strong solutions are to be found for smaller values of α.

Throughout, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 - smooth boundary $\partial\Omega$, with exterior unit normal vector field $\nu : \partial\Omega \to S^{N-1}$. Subscripts $,i$ denote partial differentiation with respect to x_i, $1 \leq i \leq N$, and a subscript t is used to denote partial differentiation with respect to t. We shall use the common symbols ∇ for the gradient of a scalar function and for the derivative matrix of a vector field and the symbol div for the divergence of a vector field: If $\phi = (\phi_i)_{1\leq i\leq M}$ and $\psi = (\psi_i)_{1\leq i\leq N}$ are vector fields, then

$$\nabla\phi = (\phi_{i,j})_{\substack{1\leq i\leq M \\ 1\leq j\leq N}}\,, \quad \operatorname{div}\psi = \sum_i \psi_{i,i}\,.$$

Also, $\Delta u = \operatorname{dvi}\nabla u$ is the Laplacian of a scalar function u, and $\nabla^2 u$ is its Hessian matrix.

The common notation for Sobolev spaces will be employed, and we write H^k for $W^{k,2}$ and H_0^k for $W_0^{k,2}$. We also use spaces of vector-valued $H^s([0,T],V)$, for non-integer $s = k + \gamma > 0$, $0 < \gamma < 1$, with a separable Hilbert space V. Such spaces can be defined as complex interpolation spaces between $H^k([0,T],V)$ and $H^{k+1}([0,T],V)$ and can alternatively be characterized by the conditions that $v \in H^s([0,T],V)$ if and only if $v^{(k)} \in L^2(0,T;V)$ and

$$\int_0^T \int_0^T \frac{\|v^{(k)}(t) - v^{(k)}(\tau)\|_V^2}{|t-\tau|^{1+2\gamma}}\, d\tau\, dt < \infty\,,$$

see [1]. The norms of $L^2(\Omega,\mathbb{R})$ and $L^2(\Omega,\mathbb{R}^N)$ are denoted by $\|\cdot\|$, and the scalar products on these spaces are denoted by $\langle\cdot,\cdot\rangle$. Other norms on function spaces are identified by subscripts. Euclidean norms of vectors and matrices are denoted by $|\cdot|$.

The convolution of two functions u, v that are supported on the positive half axis is written as $u * v(t) = \int_0^t u(t-s)v(s)\,ds$. The letter C is used for constants whose values may be different in each occurrence and which can in principle be estimated explicitly in terms of known quantities.

2. Main Result

Consider the quasilinear integro-differential equation

$$u_t(x,t) - \int_0^t a(t-s)\operatorname{div}\left(g(|\nabla u(x,s)|^2)\nabla u(x,s)\right)\, ds = f(x,t) \tag{2.1}$$

in $\Omega \times [0,T]$, with initial and boundary data

$$u(\cdot,0) = u_0, \quad u\Big|_{\partial\Omega\times[0,T]} = 0\,. \tag{2.2}$$

The kernel $a : [0,T] \to \mathbb{R}$ is assumed to be of the form

$$a(t) = a_\alpha(t) + c * a_\alpha(t) \tag{2.3}$$

with

$$a_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad c \in W^{1,1}([0,T];\mathbb{R}) \tag{2.4}$$

and

$$\frac{1}{2} < \alpha < 1\,. \tag{2.5}$$

The case $\alpha = \frac{1}{2}$ will be discussed in section 3. We assume that the function $g : [0,\infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$g(r) + (2+\epsilon) r g'(r) \geq \epsilon > 0, \quad r g'(r) \leq \beta g(r) \tag{2.6}$$

for some ϵ, β, for all $r \geq 0$. Note that (2.6) implies that

$$\epsilon \leq g(r) \leq g(1) r^\beta \tag{2.7}$$

for all $r \geq 1$. We define

$$G(r) = \frac{1}{2}\int_0^r g(s)\,ds\,; \tag{2.8}$$

then the function $r \to G(r^2)$ is strictly convex on $\mathbb{R}$ by (2.6). For the data, we assume that

$$\begin{gathered}
\|f(\cdot,0)\| + \int_0^T \|f_t(\cdot,t)\|\,dt < \infty \\
\int_\Omega G(|\nabla u_0(x)|^2) < \infty \\
\int_\Omega |\nabla(g(|\nabla u_0|^2)\nabla u_0)|^2 < \infty\,.
\end{gathered} \tag{2.9}$$

The main result of this note is the following.

Theorem 1. *Under these assumptions, there exists a function u in*

$$W^{1,\infty}([0,T], L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^{\frac{1+\alpha}{2}}([0,T], H^1_0(\Omega))$$

that satisfies (2.1) in the sense of distributions and for which $\lim_{t\to 0} u(\cdot,t) = u_0$ in $H^1(\Omega)$. Moreover, for some $q > 1$, $\nabla\left(g(|\nabla u|^2)\nabla u\right) \in L^q(\Omega \times [0,T])$, and thus (2.1) holds in this space.

Proof. We construct Galerkin approximations, establish suitable *a priori* estimates, and pass to the limit. This will yield a solution in the desired solution class. Thus let

$$\{\varphi_i \,\big|\, i \geq 1\} \subset L^2(\Omega)$$

be a complete orthonormal system in $L^2(\Omega)$ of eigenfunctions of the negative Laplacian with zero boundary data. These functions are known to be in $W^{2,p}(\Omega)\cap C^\infty(\Omega)$ for all $p<\infty$. Let $V_n = span(\varphi_1,\dots,\varphi_n)$ for $n=1,\,2,\,3,\dots$. For technical reasons, we shall construct solutions on $\Omega\times[0,T+1]$. Thus let $T'=T+1$; we can extend f and a such that all their properties still hold on the larger interval. Let f^n be a sequence in $C^1([0,T'],L^2(\Omega))$ that converges to f in $W^{1,1}[0,T'],L^2(\Omega))$, and let $u^n\in C^1([0,t_n),V_n)$ be the unique maximal solution of the system

$$\langle u^n_t, v\rangle + \langle a * g(|\nabla u^n|^2)\nabla u^n, \nabla v\rangle = \langle f^n, v\rangle \tag{2.10}$$

for all $v\in V_n$, with initial data defined by

$$\langle u^n(0)-u_0, v\rangle = 0$$

for all $v\in V_n$. It follows from standard results on functional differential equations ([10]) that u^n exists and is unique for each n; possibly, $t_n<T'$, in which case

$$\|u^n(t)\| \to \infty \quad \text{as} \quad t\to t_n\,.$$

It is also easy to see (by differentiating the equation) that in fact $u^n\in C^2((0,t_n),V_n)$.

A Priori Estimate I. Let $d\in W^{1,1}([0,T'],\mathbb{R})$ be the resolvent kernel of c ([7]), i.e. the function satisfying

$$c+d+c*d=0\,. \tag{2.11}$$

We use the notation

$$g^n(t) = g(|\nabla u^n(\cdot,t)|^2)\nabla u^n(\cdot,t)\,,\quad g^n(0)=g^n_0\,. \tag{2.12}$$

Then (2.10) is equivalent to

$$\langle u^n_t + d*u^n_t, v\rangle + \langle a_\alpha * g^n, \nabla v\rangle = \langle f^n + d*f^n, v\rangle \tag{2.13}$$

for all $v\in V_n$. We differentiate with respect to t; after some manipulations, the result is the equation

$$\begin{aligned}&\langle u^n_{tt} + d(0)u^n_t + d'*u^n_t, v\rangle + \langle a_\alpha(g^n-g^n_0), \nabla v\rangle\\ &+\langle \int_0^t (-a'_\alpha(t-s))(g^n(t)-g^n(s))\,ds, \nabla v\rangle = \langle F^n(\cdot,t), v\rangle\,,\end{aligned} \tag{2.14}$$

where

$$F^n(\cdot,t) = f^n_t(\cdot,t) + d(0)f^n(\cdot,t) + d'*f^n(\cdot,t) - a_\alpha(t)\mathrm{dvi}\, g^n_0\,.$$

Note that the integral involving a'_α is defined for all t, since $|g^n(t)-g^n(s)| = O(|t-s|)$, due to the differentiability of u^n. We now use the t-dependent test function $v=u^n_t(\cdot,t)\in V_n$. The result is the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^n_t(\cdot,t)\|^2 + d(0)\|u^n_t(\cdot,t)\|^2 + \langle d'*u^n_t(\cdot,t),\, u^n_t(\cdot,t)\rangle\\ &+a_\alpha(t)\langle g^n(t)-g^n_0, \nabla u^n_t(\cdot,t)\rangle + \int_0^t (-a'_\alpha(t-s))\langle g^n(t)-g^n(s), \nabla u^n_t(\cdot,t)\rangle\,ds\\ &= \langle F^n(\cdot,t),\, u^n_t(\cdot,t)\rangle\,.\end{aligned} \tag{2.15}$$

Now

$$\langle g^n(t) - g^n(s), \nabla u_t^n(\cdot,t)\rangle = \frac{d}{dt}\Phi^n(t,s) \tag{2.16}$$

where

$$\begin{aligned} \Phi^n(t,s) &= \int_\Omega H(\nabla u^n(\cdot,t), \nabla u^n(\cdot,s)), \\ H(\xi,\zeta) &= G(|\xi|^2) - G(|\zeta|^2) - (\xi-\zeta)^T g(|\zeta|^2)\zeta. \end{aligned} \tag{2.17}$$

Thus

$$\begin{aligned} &\int_0^t (-a_\alpha'(t-s))\langle g^n(t) - g^n(s), \nabla u_t^n(\cdot,t)\rangle\, ds \\ &= \frac{d}{dt}\int_0^t (-a_\alpha'(t-s))\Phi^n(t,s)\, ds + \int_0^t a_\alpha''(t-s)\Phi^n(t,s)\, ds \end{aligned} \tag{2.18}$$

and

$$a_\alpha(t)\langle g^n(t) - g_0^n, \nabla u_t^n(\cdot,t)\rangle = \frac{d}{dt}\left(a_\alpha(t)\Phi^n(t,0)\right) - a_\alpha'(t)\Phi^n(t,0). \tag{2.19}$$

Note that $|\Phi^n(t,s)| = O(|t-s|^2)$; thus the integral involving $a_\alpha''(t) = \frac{t^{-\alpha-2}}{\Gamma(-\alpha-1)}$ converges. We insert these formulae into (2.15) and integrate with respect to t. After a trivial estimate, the result is

$$\begin{aligned} &\frac{1}{2}\|u_t^n(\cdot,t)\|^2 - \frac{1}{2}\|u_t^n(\cdot,0)\|^2 + \int_0^t (-a_\alpha'(t-s))\Phi^n(t,s)\, ds + a_\alpha(t)\Phi^n(t,0) \\ &\quad + \int_0^t \big(\int_0^s a_\alpha''(s-\tau)\Phi^n(s,\tau)\, d\tau - a_\alpha'(s)\Phi^n(s,0)\big)\, ds \\ &\le \int_0^t \big((\|F^n(\cdot,s)\| + \|d' * u_t^n(\cdot,s)\| + |d(0)|\|u_t^n(\cdot,s)\|)\|u_t^n(\cdot,s)\|\big)\, ds. \end{aligned} \tag{2.20}$$

Next we note that for any $\xi,\zeta \in \mathbb{R}^n$

$$H(\xi,\zeta) = \int_0^1 (1-s)(\xi-\zeta)^T K(\zeta + s(\xi-\zeta))(\xi-\zeta)\, ds$$

where

$$K(\lambda) = g(|\lambda|^2)\mathbf{I} + 2g'(|\lambda|^2)\lambda\lambda^T$$

is the derivative matrix of the function $\lambda \to g(|\lambda|^2)\lambda$. By assumption (2.6),

$$\mu^T K(\lambda)\mu \ge \epsilon|\mu|^2$$

for all $\mu \in \mathbb{R}^n$, and thus for all ξ,ζ

$$H(\xi,\zeta) \ge \frac{\epsilon}{2}|\xi-\zeta|^2. \tag{2.21}$$

In particular,

$$\Phi^n(t,s) \ge \frac{\epsilon}{2}\|\nabla u^n(\cdot,t) - \nabla u^n(\cdot,s)\|^2 \ge 0 \tag{2.22}$$

for all $0 \le s,t \le T'$. Dropping now first all terms involving Φ^n on the left hand side of (2.20) and applying Gronwall's inequality shows that

$$\sup_{0\le t\le t_n} \|u^n_t(\cdot,t)\| \le C \tag{2.23}$$

for some constant C that does not depend on n. Consequently, $t_n = T'$ for all n. Next we return to (2.20) and infer that also

$$\begin{gathered}\sup_{0\le t\le T'} \left(\int_0^t (-a'_\alpha(t-s))\Phi^n(t,s)\,ds + a_\alpha(t)\Phi^n(t,0)\right) \le C\,,\\ \int_0^{T'} \left(\int_0^s a''_\alpha(s-\tau)\Phi^n(s,\tau)\,d\tau + (-a'_\alpha(s))\Phi^n(s,0)\right)\,ds \le C\end{gathered} \tag{2.24}$$

and therefore in particular

$$\int_0^{T'}\int_0^s (s-\tau)^{-\alpha-2}\|\nabla u^n(\cdot,s)-\nabla u^n(\cdot,\tau)\|^2\,d\tau\,ds + \sup_{0\le t\le T'}\|\nabla u^n(\cdot,t)\| \le C\,. \tag{2.25}$$

The first estimate in (2.25) is equivalent to an *a priori* estimate in $H^{\frac{1+\alpha}{2}}([0,T'],H^1(\Omega))$ for the approximate solutions u^n. Finally, from (2.25), (2.24), and (2.9) we deduce that

$$\sup_{0\le t\le T'}\int_\Omega G(|\nabla u^n(\cdot,t)|^2) \le C\,. \tag{2.26}$$

A Priori Estimate II. As in (2.11), let d be the resolvent kernel of c, and set

$$b = b_\alpha + d * b_\alpha \quad \text{with} \quad b_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\,.$$

Then $a_\alpha * b_\alpha(t) = 1$ and therefore also $a * b(t) = 1$ for all $t > 0$. We form the convolution of (2.10) with b and differentiate with respect to t. The result is the identity

$$\frac{d}{dt}\langle b * u^n_t(\cdot,t), v\rangle + \langle g^n(t),\nabla v\rangle = \langle b(t)f^n(\cdot,0) + b * f^n_t(\cdot,t), v\rangle \tag{2.27}$$

for all $v \in V_n$. We now use the test function $v = (t-T')\Delta u^n(\cdot,t) \in V_n$ (recall that Δ maps V_n into itself) and integrate from 0 to T'. After some integrations by parts, we obtain

$$\begin{aligned}\int_0^{T'} \langle b * \nabla u^n_t(\cdot,t), (t-T')\nabla u^n_t(\cdot,t) + \nabla u^n(\cdot,t)\rangle\,dt + \int_0^{T'} (t-T')\langle g^n(t),\nabla\Delta u^n(\cdot,t)\rangle\,dt\\ = \int_0^{T'} \langle b(t)f^n(\cdot,0) + b * f^n_t(\cdot,t), (t-T')\Delta u^n(\cdot,t)\rangle\,dt\,.\end{aligned} \tag{2.28}$$

The absolute value of the integral on the right hand side of this identity can be estimated from above by

$$
\begin{aligned}
&\int_0^{T'} \left(|b(t)|\|f^n(\cdot,0)\| + \|b * f_t^n(\cdot,t)\|\right)(T'-t)\|\Delta u^n(\cdot,t)\|\,dt \qquad (2.29)\\
&\le C(\epsilon,T')\|b\|_{L^2(0,T')}\left(\|f^n(\cdot,0)\|^2 + \|f_t^n\|^2_{L^1(0,T';L^2)}\right) + \epsilon\int_0^{T'}(T'-t)\|\Delta u^n(\cdot,t)\|^2\,dt
\end{aligned}
$$

for any $\epsilon > 0$, since $b \in L^2(0,T')$, due to the restriction $\alpha > \frac{1}{2}$. To estimate the first integral on the left hand side of (2.28) in absolute value, we define

$$
e_\alpha(t) = \frac{t^{\alpha/2-1}}{\Gamma(\frac{\alpha}{2})}
$$

such that $e_\alpha * e_\alpha = b_\alpha$ and thus $b = e_\alpha * e_\alpha + d * e_\alpha * e_\alpha$. Then the integral can be rewritten as

$$
\cdots = \int_0^{T'} \langle (e_\alpha + d*e_\alpha)*\nabla u^n(\cdot,t), \int_t^{T'} e_\alpha(\tau-t)\left((\tau-T')\nabla u^n(\cdot,\tau) + \nabla u^n(\cdot,\tau)\right)\rangle d\tau\,dt\,.
$$

Now

$$
(e_\alpha + d*e_\alpha)*\nabla u_t^n(\cdot,t) = \frac{d}{dt}(e_\alpha + d*e_\alpha)*w^n(\cdot,t) \qquad (2.30)
$$

with $w^n(\cdot,t) = \nabla u^n(\cdot,t) - \nabla u^n(\cdot,0)$, and since $\nabla u^n \in H^{\frac{1+\alpha}{2}}([0,T'],L^2(\Omega))$ due to (2.25), with an estimate in this space that does not depend on n, it follows that $(e_\alpha + d*e_\alpha)*w^n(\cdot,t)$ is in $H^{\alpha+\frac{1}{2}}([0,T'],L^2(\Omega))$. Its derivative is therefore in $H^{\alpha-\frac{1}{2}}([0,T'],L^2(\Omega))$, with an estimate that does not depend on n ([6]). Similarly,

$$
\begin{aligned}
&\int_t^{T'} e_\alpha(\tau-t)\left((\tau-T')\nabla u_t^n(\cdot,\tau) + \nabla u^n(\cdot,\tau)\right)\,d\tau\\
&= \frac{d}{dt}\left(\int_0^{T'-t} e_\alpha(\sigma)((t-T'+\sigma)\nabla u^n(\cdot,t+\sigma))\,d\sigma\right) \qquad (2.31)\\
&= \frac{d}{dt}\left((t-T')\int_0^{T'-t} e_\alpha(\sigma)\nabla u^n(\cdot,t+\sigma)\,d\sigma\right) + \frac{d}{dt}\int_0^{T'-t}\sigma e_\alpha(\sigma)\nabla u^n(\cdot,t+\sigma)\,d\sigma\,.
\end{aligned}
$$

As above, both integrals are in $H^{\alpha+\frac{1}{2}}([0,T'],L^2(\Omega))$ as function of t, and the entire expression is therefore bounded *a priori* in $H^{\alpha-\frac{1}{2}}([0,T'],L^2(\Omega))$. The first integral on the left hand side of (2.28) can now be bounded independently of n.

Finally consider the second integral on the left hand side of (2.28). Fix t and set $w = u^n(\cdot,t)$, then we want to estimate

$$
\int_\Omega g(|\nabla w|^2)\nabla w\cdot\nabla(\Delta w) = \int_\Omega \sum_{i,j} g(|\nabla w|^2) w_{,i} w_{,i,j,j}\,.
$$

The following arguments are taken from [2], where more details can be found. After an integration by parts, the integral equals

$$-\int_\Omega \sum_{i,j} w_{,i,j}(g(|\nabla w|^2)w_{,i})_{,j} + \int_{\partial\Omega} \sum_{i,j} g(|\nabla w|^2)w_{,i}w_{,i,j}\nu_j \,. \qquad (2.32)$$

Since the function w vanishes on the hypersurface $\partial\Omega$, its gradient is perpendicular to it and therefore parallel to the unit normal ν, i.e. $\nabla w = \nu\,\partial_\nu w$, where $\partial_\nu w = \nabla w \cdot \nu$. Also, at any point of this hypersurface, $\Delta w = \Delta_{\partial\Omega} w + (N-1)\mathbf{H}\partial_\nu w + \partial_\nu^2 w$, where $\Delta_{\partial\Omega}$ is the induced Laplacian, $\mathbf{H}$ is the mean curvature of $\partial\Omega$ with respect to ν and $\partial_\nu^2 u = \nu^T \nabla^2 w \nu$ ([15]). Since w vanishes on $\partial\Omega$, this means that $\Delta w = (N-1)\mathbf{H}\partial_\nu w + \partial_\nu^2 w$. Since also $\Delta w \in V_n$ vanishes on $\partial\Omega$, we obtain at each boundary point

$$\sum_{i,j} w_{,i}w_{,i,j}\nu_j = \partial_\nu w \partial_\nu^2 w = -(N-1)\mathbf{H}|\partial_\nu w|^2 = -(N-1)\mathbf{H}|\nabla w|^2 \,.$$

Therefore the boundary integral reduces to a term containing only first derivatives that can be estimated from above by means of a trace theorem:

$$\begin{aligned}\int_{\partial\Omega} \sum_{i,j} g(|\nabla w|^2)w_{,i,j}w_{,i}\nu_j &= -\int_{\partial\Omega} (N-1)\mathbf{H}g(|\nabla w|^2)|\nabla w|^2 \\ &\le \epsilon \|\nabla \left(\sqrt{g(|\nabla w|^2)}\nabla w\right)\|^2 + C\|g(|\nabla w|^2)|\nabla w|^2\|_{L^1} \\ &\le \epsilon \|\nabla \left(\sqrt{g(|\nabla w|^2)}\nabla w\right)\|^2 + C_1(1 + \int_\Omega G(|\nabla w|^2))\end{aligned} \qquad (2.33)$$

for any $\epsilon > 0$ with suitable $C, C_1 > 0$. Here Lemma 2 was used to arrive at the last line.

The integral over Ω in (2.32) is

$$\begin{aligned}&-\int_\Omega \sum_{i,j} w_{,i,j}(g(|\nabla w|^2)w_{,i})_{,j} \\ &= -\int_\Omega \sum_{i,j} \left(g(|\nabla w|^2)w_{,i,j}w_{,i,j} + 2\sum_k g'(|\nabla w|^2)w_{,i,j}w_{,i}w_{,j,k}w_{,k} \right) \\ &= -\int_\Omega \left(g(r^2)D^2 + 2g'(r^2)d^2 \right)\end{aligned}$$

with the abbreviations $r^2 = |\nabla w|^2$, $D^2 = |\nabla^2 w|^2 = \sum_{i,j} w_{,i,j}w_{,i,j}$, $d^2 = |\nabla^2 w \nabla w|^2 = \sum_{i,j,k} w_{,i,j}w_{,i}w_{,j,k}w_{,k}$. On the other hand,

$$|\nabla \left(\sqrt{g(|\nabla w|^2)}\nabla w\right)|^2 = g(r^2)D^2 + 2g'(r^2)d^2 + \frac{(g'(r^2))^2 r^2 d^2}{g(r^2)} \,.$$

Using (2.6), this term can be estimated from above by a constant multiple of $g(r^2)D^2 + 2g'(r^2)d^2$, and therefore

$$-\int_\Omega \sum_{i,j}(g(|\nabla w|^2)w_{,i})_{,j}w_{,i,j} \le -\delta\|\nabla\left(\sqrt{g(|\nabla w|^2)}\nabla w\right)\|_{L^2}^2$$

with some fixed $\delta > 0$. Combining this last estimate and (2.33) with $\epsilon = \delta/2$, one finally obtains

$$\int_\Omega g(|\nabla w|^2)\nabla w \cdot \nabla(\Delta w) \le -\epsilon\int_\Omega |\nabla\left(\sqrt{g(|\nabla w|^2)}\nabla w\right)|^2 + C(1+\int_\Omega G(|\nabla w|^2))\,, \tag{2.34}$$

for some ϵ, $C > 0$. Using estimate (2.26), rearranging terms in (2.28), and combining all estimates, we thus deduce that

$$\int_0^{T'}(T'-t)\|\nabla\left(\sqrt{g(|\nabla u^n(\cdot,t)|^2)}\nabla u^n(\cdot,t)\right)\|^2\,dt \le C \tag{2.35}$$

independent of n. In particular,

$$\int_0^T \|\nabla^2 u^n(\cdot,t)\|^2\,dt \le C\,. \tag{2.36}$$

Passage to the Limit. Estimates (2.23) and (2.36) imply that the set of all ∇u^n is in a relatively compact subset of $L^2(0,T;L^2(\Omega))$ ([12]). We can therefore extract a subsequence $u^{M(n)}$ such that

$$\begin{aligned}
\nabla u^{M(n)} &\to \nabla u \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \\
&\qquad\quad\ \text{and pointwise a.e. on } \Omega\times(0,T) \\
\nabla^2 u^{M(n)} &\to \nabla^2 u \quad \text{weakly in } L^2(0,T;L^2(\Omega)) \\
u_t^{M(n)} &\to u_t \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega))
\end{aligned}$$

Also, $u \in H^{\frac{1+\alpha}{2}}([0,T],H^1(\Omega))$ by (2.25). Estimate (2.26) and Lemma 2 below imply that the $g^n = g(|\nabla u^n|^2)\nabla u^n$ are in a bounded subset of $L^\infty(0,T;L^p(\Omega))$ for some $p > 1$. Therefore the subsequence can be chosen such that also

$$g(|\nabla u^{M(n)}|^2)\nabla u^{M(n)} \to \xi \quad \text{weakly-* in } L^\infty(0,T;L^p(\Omega))$$

for some $\xi \in L^\infty(0,T;L^p(\Omega))$. Since also $g(|\nabla u^{M(n)}|^2)\nabla u^{M(n)} \to g(|\nabla u|^2)\nabla u$ pointwise a.e., Lemma 3 implies that $\xi = g(|\nabla u|^2)\nabla u$. Passing now to the limit in equation (2.10), we see that

$$\langle u_t, v\rangle + \langle a * g(|\nabla u|^2)\nabla u, \nabla v\rangle = \langle f, v\rangle$$

for all $v \in \cup_n V_n$ and therefore by density for all $v \in C_0^\infty(\Omega)$. Therefore, u is a distributional solution of (1.1) on $\Omega\times[0,T]$.

Additional Regularity. For any differentiable function $h : [0,\infty) \to \mathbb{R}$ and any sufficiently smooth function $w : \bar{\Omega} \to \mathbb{R}$, we compute

$$\nabla\left(h(|\nabla w|^2)\nabla w\right) = \left(h(|\nabla w|^2)w_{,i,j} + 2h'(|\nabla w|^2)\sum_k w_{,k,j}w_{,i}\right)_{i,j}$$

Using again the notation $r^2 = |\nabla w|^2$, $D^2 = |\nabla^2 w|^2$, $d^2 = |\nabla^2 w \nabla w|^2$ and setting $g = h$ we obtain after estimating

$$|\nabla\left(g(|\nabla w^2)\nabla w\right)| \le Cg(r^2)D\,. \tag{2.37}$$

With $h = \sqrt{g}$, we obtain at points where $g'(|\nabla w|^2) < 0$

$$\begin{aligned}|\nabla\left(\sqrt{g(|\nabla u|^2)}\nabla u\right)|^2 &= g(|r^2)D^2 + 2g'(r^2)d^2 + \frac{(g'(r^2))^2}{g(r^2)}r^2d^2 \\ &\ge g(r^2)D^2 + 2g'(r^2)d^2 \\ &\ge g(r^2)D^2 + 2g'(r^2)D^2r^2 \\ &\ge \frac{\epsilon}{2+\epsilon}g(r^2)D^2 \end{aligned} \tag{2.38}$$

from (2.6). The same estimate holds trivially at points where $g'(|\nabla w|^2) \ge 0$. Now let $1 < q < 2$, to be fixed later. Then by (2.37), (2.38), and Hölder's inequality

$$\begin{aligned}\|\nabla\left(g(|\nabla w|^2)\nabla w\right)\|_{L^q}^q &\le C\int_\Omega \left(g(r^2)D\right)^q \\ &= C\int_\Omega g(r^2)^{q/2}D^q g(r^2)^{q/2} \\ &\le C\left(\int_\Omega g(r^2)D^2\right)^{q/2}\left(\int_\Omega g(r^2)^{q/(2-q)}\right)^{1-q/2} \\ &\le C\left(\int_\Omega |\nabla\left(\sqrt{g(|\nabla w|^2)}\nabla w\right)|^2\right)^{q/2}\left(\int_\Omega g(r^2)^{q/(2-q)}\right)^{1-q/2}. \end{aligned} \tag{2.39}$$

If $q = 1 + \frac{\delta}{1+2\delta} \in (1,2)$ with δ as in Lemma 2, we can estimate further

$$\int_\Omega g(r^2)^{q/(2-q)} \le C\left(1 + \int_\Omega G(r^2)\right).$$

We use these estimates for $w = u(\cdot,t)$ for any fixed t and infer that

$$\|\nabla\left(g(|\nabla u(\cdot,t)|^2)\nabla u(\cdot,t)\right)\|_{L^q}^2 \le C\|\nabla\left(\sqrt{g(|\nabla u(\cdot,t)|^2)}\nabla u(\cdot,t)\right)\|^2$$

The right hand side is integrable over $[0,T]$, and therefore $\nabla\left(g(|\nabla u|^2)\nabla u\right) \in L^2(0,T;L^q(\Omega))$ with this q.

Lemma 2. *If $g : [0, \infty) \to \mathbb{R}$ satisfies (2.6), then there exist $\delta, C > 0$ such that for all $r \geq 0$*

$$\begin{aligned} |g(r^2)r|^{1+\delta} &\leq C(1 + G(r^2))\,, \\ g(r^2)r^2 &\leq C(1 + G(r^2))\,. \end{aligned}$$

Proof. A constant C can be found such that both inequalities are true for all $r \leq 1$. For $r \geq 1$, we have with β as in (2.6) and $\delta = \dfrac{1}{2\beta + 1}$

$$\begin{aligned} \frac{d}{dr}\left(g(r^2)r\right)^{1+\delta} &= (1+\delta)g^{\delta}(r^2)r^{\delta}(g(r^2) + 2r^2 g'(r^2)) \\ &\leq Cg^{1+\delta}(r^2)r^{\delta} \\ &\leq Cg(r^2)r^{2\delta\beta+\delta} \\ &\leq Cg(r^2)r = C\frac{d}{dr}G(r^2)\,. \end{aligned}$$

An integration implies the first estimate. The second estimate follows from (2.6) in the form

$$g'(r^2)r^3 + g(r^2)r \leq (\beta + 1)g(r^2)r$$

by integrating.

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be measurable and bounded and let z_n be a sequence in $L^1(\Omega)$ satisfying*

$$\begin{aligned} z_n(x) &\to \xi(x) \quad \textit{for a.e. } x \in \Omega \\ z_n &\to \zeta \quad \textit{weakly in } L^1(\Omega). \end{aligned}$$

Then $\zeta = \xi$ almost everywhere.

Proof. The function ξ must be finite almost everywhere, otherwise the z_n cannot converge weakly. Let $\epsilon > 0$. By Egorov's Theorem, there exists a subset $\Omega_\epsilon \subset \Omega$ such that $z_n \to \xi$ uniformly on Ω_ϵ, $|\xi| \leq \epsilon^{-1}$ on Ω_ϵ, and $meas(\Omega - \Omega_\epsilon) < \epsilon$. Set $\varphi(x) = sign(\xi(x) - \zeta(x)) \cdot I_{\Omega_\epsilon}(x)$, then

$$\int_{\Omega_\epsilon} |\xi - \zeta| = \int_{\Omega_\epsilon} (\xi - z_n)\varphi_M - \int_{\Omega_\epsilon} (\zeta - z_n)\varphi_M\,.$$

As $n \to \infty$, both terms go to zero by assumption and because of the Dominated Convergence Theorem. Therefore $\xi = \zeta$ a.e. on Ω_ϵ. As $\epsilon \to 0$, the conclusion follows.

3. Extensions and Counterexamples

The question arises whether a result like Theorem 1 is still true if $0 < \alpha \leq \frac{1}{2}$. The short answer is that the theorem is no longer correct for this range of α's. Below, we give some counterexamples to show this. However, these counterexamples are essentially related to regularity properties for linear equations. A suitable modification can still be expected to hold for $0 < \alpha \leq \frac{1}{2}$. For $\alpha = \frac{1}{2}$, this is the content of the next corollary. For nonlinear problems with $0 < \alpha < \frac{1}{2}$, the question of existence of solutions u for which $u(\cdot, t) \in H^2(\Omega)$ for $t > 0$ remains open, except for the result in [8].

Corollary 4. *Let $\alpha = \frac{1}{2}$, let the assumptions of Theorem 1 hold, and assume that for some $r > 1$*

$$\int_0^T \|f_t(\cdot,t)\|^r \, dt < \infty \,. \tag{3.1}$$

a) If $f(\cdot,0) \in H_0^1(\Omega)$, then the statement of Theorem 1 remains true.
b) If $f(\cdot,0) \in L^2(\Omega)$, then a solution u of (2.1) exists for which

$$\int_0^T t\|u(\cdot,t)\|_{H^2}^2 \, dt < \infty \tag{3.2}$$

and $u \in W^{1,\infty}([0,T], L^2(\Omega)) \cap H^{\frac{3}{4}}([0,T], H_0^1(\Omega))$.

Outline of Proof. We use the same approximation as in the proof of Theorem 1 and derive estimates (2.23) – (2.26) in the same way. In case a), we can assume that the $f^n(\cdot,0)$ are all in $H_0^1(\Omega)$. We can then carry out *a priori estimate II* by integrating by parts in the right hand side of (2.28), obtaining

$$\int_0^{T'} \big(\langle -b(t)\nabla f^n(\cdot,0), (t-T')\nabla u^n(\cdot,t)\rangle + \langle b * f_t^n(\cdot,t), (t-T')\Delta u^n(\cdot,t)\rangle\big) \, dt \,.$$

Since $b \in L^q(0,T')$ for all $q < 2$ and due to (2.25), this can be estimated by

$$C(b, \|\nabla f^n(\cdot,0)\|) + C(\epsilon, T')\|b\|_{L^q(0,T')}\|f_t^n\|^2_{L^r(0,T';L^2)} + \epsilon \int_0^{T'} (T'-t)\|\Delta u^n(\cdot,t)\|^2 \, dt$$

with $q = \dfrac{2r}{3r-2}$. No change is required for the remaining arguments, and the conclusion follows as before.

In case b), we proceed as before and obtain approximating solutions that satisfy (2.23) – (2.26). We then use the test function $v = t(t-T')\Delta u^n(\cdot,t)$ in the arguments that follow (2.27). Instead of (2.29), we then obtain an estimate with the right hand side

$$C_0(\epsilon, T', b)\|f^n(\cdot,0)\|^2 + C_1(\epsilon, T')\|b\|_{L^q(0,T')}\|f_t^n\|^2_{L^r(0,T';L^2)}$$
$$+\epsilon \int_0^{T'} t(T'-t)\|\Delta u^n(\cdot,t)\|^2 \, dt \,.$$

The other terms in the identity that is analogous to (2.28) can be treated as before. The additional factor t in the test function v leads to some longer version of (2.31); we leave the details to the interested reader. In the passage to the limit, we now only assert that

$$\nabla^2 u^{M(n)} \to \nabla^2 u \quad \text{weakly in} \quad L^2(\delta, T; L^2(\Omega))$$

for all $\delta > 0$. This is sufficient to conclude the proof.

To construct counterexamples that show that the conditions in these results are sharp, several facts about scalar integro-differential equations will be used.

Lemma 5. *For $0 < \alpha < 1$ fixed and for arbitrary $\lambda > 0$, let u_λ, $v_\lambda : [0,1] \to \mathbb{R}$ be the solutions of*

$$\begin{aligned} u_\lambda'(t) + \lambda a_\alpha * u_\lambda &= 0 \quad (0 \le t \le 1), \quad u_\lambda(0) = 1 \\ v_\lambda'(t) + \lambda a_\alpha * v_\lambda &= 1 \quad (0 \le t \le 1), \quad v_\lambda(0) = 0 \end{aligned}$$

where a_α is as in (2.4). Then

$$\sup_\lambda \int_0^1 \lambda^2 t^\gamma u_\lambda^2(t)\, dt < \infty \quad \textit{iff} \quad 2\alpha + \gamma > 3 \tag{3.3}$$

$$\sup_\lambda \int_0^1 \lambda^2 t^\gamma v_\lambda^2(t)\, dt < \infty \quad \textit{iff} \quad 2\alpha + \gamma > 1\,. \tag{3.4}$$

Proof. It is known that

$$u_\lambda(t) = E_{2-\alpha}(-\lambda t^{2-\alpha})\,, \quad v_\lambda(t) = \int_0^t E_{2-\alpha}(-\lambda s^{2-\alpha})\, ds$$

where

$$E_\sigma(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\sigma k)}$$

is the Mittag-Leffler function ([4], [5]). Therefore the estimates (3.3) and (3.4) follow immediately for, say, $0 \le \lambda \le K$, where K is any fixed finite value, and we only have to prove them for large values of λ. If $0 < \sigma < 2$, then

$$E_\sigma(-z) = \frac{z^{-1}}{\Gamma(1-\sigma)} + O(z^{-2})$$

as $z \to \infty$ ([4]). Therefore there exist constants c_1, c_2, c_3, $c_4 > 0$ such that

$$\begin{aligned} |u_\lambda(t)| &\le c_1 & (\lambda t^{2-\alpha} \le c_2) \\ c_3 \lambda^{-1} t^{\alpha-2} \le |u_\lambda(t)| &\le c_4 \lambda^{-1} t^{\alpha-2} & (\lambda t^{2-\alpha} \ge c_2)\,. \end{aligned} \tag{3.5}$$

The equation for u_λ shows that the Laplace transform of u_λ is given by

$$\hat{u_\lambda}(s) = \frac{1}{s + \lambda s^{\alpha-1}}$$

which implies $\int_0^\infty u_\lambda(t)\, dt = 0$. Therefore, with the same constants $c_1, \ldots, c_4$ we can also assume that

$$\begin{aligned} |v_\lambda(t)| &\le c_1 & (\lambda t^{2-\alpha} \le c_2) \\ c_3 \lambda^{-1} t^{\alpha-1} \le |v_\lambda(t)| &\le c_4 \lambda^{-1} t^{\alpha-1} & (\lambda t^{2-\alpha} \ge c_2)\,. \end{aligned} \tag{3.6}$$

Suppose now that $2\alpha+\gamma>3$. Then

$$\begin{aligned}\int_0^1 \lambda^2 t^\gamma u_\lambda^2(t)\,dt &= \int_0^{(c_2/\lambda)^{1/(2-\alpha)}} \lambda^2 t^\gamma u_\lambda^2(t)\,dt + \int_{(c_2/\lambda)^{1/(2-\alpha)}}^1 \lambda^2 t^\gamma u_\lambda^2(t)\,dt \\ &\le \int_0^{(c_2/\lambda)^{1/(2-\alpha)}} \lambda^2 c_1^2 t^\gamma dt + \int_0^1 t^\gamma c_4^2 t^{2\alpha-4}\,dt \\ &\le C\lambda^{2-\frac{\gamma+1}{2-\alpha}} + C \\ &\le C < \infty\end{aligned}$$

independently of λ. The same argument, using (3.6) instead of (3.5), shows that

$$\sup_\lambda \int_0^1 \lambda^2 t^\gamma v_\lambda^2(t)\,dt < \infty$$

if $2\alpha+\gamma>1$. Suppose next that $2\alpha+\gamma\le 3$. Then

$$\int_0^1 \lambda^2 t^\gamma u_\lambda^2(t)\,dt \ge \int_{(c_2/\lambda)^{1/(2-\alpha)}}^1 \lambda^2 t^\gamma u_\lambda^2(t)\,dt \ge \int_{(c_2/\lambda)^{1/(2-\alpha)}}^1 c_3^2 t^{\gamma+2\alpha-4}\,dt$$

and the last integral does not remain bounded as $\lambda\to\infty$. Thus the supremum in (3.3) is infinite if $2\alpha+\gamma\le 3$. Using (3.6), it follows that the supremum in (3.4) is infinite if $2\alpha+\gamma\le 1$.

It is also possible to show that

$$\sup_\lambda \int_0^1 \lambda^{2\delta} t^\gamma u_\lambda^2(t)\,dt = \infty$$

if $\delta>1$ and that this supremum is finite for $\delta<1$ iff in addition $2\delta(2-\alpha)\le\gamma+1$. Similar statements hold for the supremum that can be formed with the v_λ. Note that the supremum is finite if $2\delta(2-\alpha)=\gamma+1$ for $\delta<1$, but infinite in the case $\delta=1$.

Counterexamples. Consider the problems

$$u_t(x,t) - \int_0^t a_\alpha(t-s)\Delta u(x,s)\,ds = F(x,t) \quad (0<t\le 1,\ x\in\Omega) \tag{3.7}$$

with

$$F(x,t) = f_0(x) + \int_0^t f_1(x,s)\,ds$$

and initial data $u(x,0)=0$ in Ω. As before, let $(\varphi_i)_i$ be the set of eigenfunctions of $-\Delta$ with zero Dirichlet boundary conditions, with eigenvalues $0<\lambda_1<\lambda_2\le\lambda_3\le\dots$.

Suppose first that $f_0=\sum_i c_i\varphi_i\in L^2(\Omega)$ and $f_1=0$. The solution of (3.7) then is given by

$$u(\cdot,t) = \sum_{i=1}^\infty c_i v_{\lambda_i}(t)\varphi_i$$

and thus

$$\begin{aligned}\|u(\cdot,t)\|^2_{H^2} &\geq C\|\Delta u(\cdot,t)\|^2 \\ &= C\sum_{i=1}^{\infty} c_i^2\lambda_i^2|v_{\lambda_i}(t)|^2 \,. \end{aligned} \tag{3.8}$$

If $\alpha \leq \frac{1}{2}$, then there exist c_i such that $f_0 \in L^2(\Omega)$ but $\int_0^1 \|u(\cdot,t)\|^2_{H^2}dt = \infty$, according to Lemma 5. However, if $\gamma > 1-2\alpha \geq 0$, then $\int_0^1 t^\gamma\|u(\cdot,t)\|^2_{H^2}dt < \infty$ for any such f_0.

Suppose next that $f_0 = 0$ and $f_1(\cdot,t) = h_i(t)\varphi_i$. Then $u(\cdot,t) = \int_0^t h_i(t-s)v_{\lambda_i}(s)\,ds\cdot\varphi_i$. Pick $h_i(t) = k_ih(k_it)$, where $h \geq 0$ is smooth, supported on $[0,1]$, $\int_0^1 h(t)\,dt = 1$, and $k_i > 1$ is so large that

$$\int_0^1 |v_{\lambda_i} * h_i(t)|^2\,dt \geq \frac{1}{2}\int_0^1 |v_{\lambda_i}(t)|^2\,dt \,.$$

It follows that

$$\|u(\cdot,t)\|^2_{H^2} \geq C\|\Delta u(\cdot,t)\|^2 = C\lambda_i^2\int_0^1 |v_{\lambda_i} * h_i(t)|^2\,dt \geq C\lambda_i^2\int_0^1 |v_{\lambda_i}(t)|^2\,dt = M_i \,. \tag{3.9}$$

If $\alpha \leq \frac{1}{2}$, then $\limsup M_i = \infty$. On the other hand,

$$\int_0^1 \|f_1(\cdot,t)\|\,dt = \int_0^1 h_i(t)\,dt = 1 \,.$$

Choose c_i such that $\sum_{i=1}^\infty |c_i| < \infty$ and $\sum_{i=1}^\infty c_i^2M_i = \infty$ and set $f_1(\cdot,t) = \sum_{i=1}^\infty c_ih_i(t)\varphi_i$, then

$$\int_0^1 \|f_1(\cdot,t)\|\,dt < \infty \quad \text{and} \quad \int_0^T \|u(\cdot,t)\|^2_{H^2}\,dt = \infty \,. \tag{3.10}$$

These arguments show that Theorem 1 does not extend to the case $\alpha \leq \frac{1}{2}$ without strengthening some assumptions or weakening some conclusions, as in Corollary 4. Similar constructions can be used to show that the assumption $u_0 \in H^2(\Omega)$ can only be weakened slightly if solutions are still to belong to $L^2(0,T;H^2(\Omega))$.

Acknowledgment. The author would like to thank the referee for several useful comments.

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Department of Mathematics
Georgetown University
Washington, D.C. 20057
E-mail address: engler@guvax.acc.georgetown.edu