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REGULARITY FOR NON-UNIFORMLY ELLIPTIC SYSTEMS AND APPLICATION TO SOME VARIATIONAL INTEGRALS

Francesco Leonetti
and
Chiara Musciano

Abstract. This paper deals with higher integrability for minimizers of some variational integrals whose Euler equation is elliptic but not uniformly elliptic. This setting is also referred to as elliptic equations with p, q-growth conditions, following Marcellini. Higher integrability of minimizers implies the existence of second derivatives. This improves on a result by Acerbi and Fusco concerning the estimate of the (possibly) singular set of minimizers.

0. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, u be a (possibly) vector-valued function, $u : \Omega \to \mathbb{R}^N, N \geq 1$, F be a continuous function, $F : \mathbb{R}^{nN} \to \mathbb{R}$; we consider the integral

$$I(u) = \int_\Omega F(Du(x))\, dx\,, \tag{0.1}$$

where

$$|F(\xi)| \leq c\,(1 + |\xi|^p)\,, \tag{0.2}$$

$u \in W^{1,p}(\Omega), 2 \leq p$. Regularity of minimizers has been widely studied when

$$\hat{m}\,(1 + |\xi|^{p-2})|\lambda|^2 \leq DDF(\xi)\lambda\lambda\,, \qquad 0 < \hat{m}\,, \tag{0.3}$$

$$|DDF(\xi)| \leq c\,(1 + |\xi|^{p-2}), \tag{0.4}$$

see [24], [14], [16], [17] (and [10], [12], [18], [11], [20], where (0.3) is weakened in order to consider quasi-convex integrals but (0.4) is still present). We refer to (0.3), (0.4) as *uniform ellipticity condition*. When dealing with

$$\hat{J}(u) = \int_\Omega \{a|Du|^2 + a|Du|^p + \sqrt{1 + (\det Du)^2}\}\, dx\,. \tag{0.5}$$

where $2 \leq n \leq p < 2n$, $u : \mathbb{R}^n \to \mathbb{R}^n$, $a > 0$, it turns out that (0.4) does not hold true any longer; conversely, the following growth condition applies:

$$|DDF(\xi)| \leq c\,(1 + |\xi|^{2n-2})\,. \tag{0.6}$$

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Moreover, if a is large enough [13], namely $a \geq a(n) = 2n^4[(n-2)!]$, then (0.3) is still true: we are lead to consider integrals (0.1) verifying (0.2), (0.3) and

$$|DDF(\xi)| \leq c\,(1 + |\xi|^{q-2})\,, \tag{0.7}$$

for some $q > p$. We refer to (0.3), (0.7) as *nonuniform ellipticity condition* [13], *nonstandard growth condition* [22], or *p,q-growth condition* [23]. In this paper we prove higher integrability and differentiability for minimizers of integrals verifying the nonuniform ellipticity (0.3), (0.7). Our results apply to the model integral (0.5) in this way: assume that $u : \mathbb{R}^n \to \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ is bounded and open, $2 \leq n \leq 2n-2 < p < 2n$; if u minimizes $\hat{J}$ and $a \geq a(n)$, then

$$DDu \text{ and } D(|Du|^{(p-2)/2}Du) \in L^2_{\mathrm{loc}}(\Omega). \tag{0.8}$$

We can also apply a partial regularity theorem contained in [1], see also [15], in order to get

$$Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_0), \qquad \forall \alpha \in (0,1)\,, \tag{0.9}$$

for some open $\Omega_0 \subset \Omega$, with

$$|\Omega \setminus \Omega_0| = 0\,, \tag{0.10}$$

where $|E|$ is the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. Now we are able to improve on (0.10), because of our result (0.8):

$$\mathcal{H}^{n-2+\epsilon}(\Omega \setminus \Omega_0) = 0 \qquad \forall \epsilon > 0\,, \tag{0.11}$$

where $\mathcal{H}^{n-2+\epsilon}$ is the $(n-2+\epsilon)$-dimensional Hausdorff measure.

1. Notation and main results

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, u be a (possibly) vector-valued function, $u : \Omega \to \mathbb{R}^N, N \geq 1$, F be a function $F : \mathbb{R}^{nN} \to \mathbb{R}$. We consider the integral

$$I(u) = \int_\Omega F(Du(x))\,dx\,, \tag{1.1}$$

where

$$F \in C^1(\mathbb{R}^{nN}) \tag{1.2}$$

and, for some positive constants c, p, m,

$$|F(\xi)| \leq c\,(1 + |\xi|^p)\,, \tag{1.3}$$

$$|DF(\xi)| \leq c\,(1 + |\xi|^{p-1})\,, \tag{1.4}$$

$$m\,(|\xi| + |\hat{\xi}|)^{(p-2)}|\xi - \hat{\xi}|^2 \leq (DF(\xi) - DF(\hat{\xi}))(\xi - \hat{\xi})\,, \tag{1.5}$$

for every ξ, $\hat{\xi} \in \mathbb{R}^{nN}$. About p, we assume that

$$2 \leq p\,. \tag{1.6}$$

We say that u minimizes the integral (1.1) if $u : \Omega \to \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ and

$$I(u) \leq I(u + \phi)\,. \tag{1.7}$$

for every $\phi : \Omega \to \mathbb{R}^N$ with $\phi \in W^{1,p}_0(\Omega)$. We will prove the following higher integrability result for Du:

Theorem 1. *Let $u \in W^{1,p}(\Omega)$ minimize the integral (1.1) and F satisfy (1.2–1.6); then*

$$Du \in L^{\sigma}_{\text{loc}}(\Omega), \qquad \forall \sigma < p\frac{n}{n-1}\,. \tag{1.8}$$

The higher integrability result (1.8) for Du allows us to get existence of second weak derivatives under additional conditions on F. Now we assume that

$$F \in C^2(\mathbb{R}^{nN}) \tag{1.9}$$

and, for some constants $c, p, q, \hat{m}, \mu$,

$$|F(\xi)| \le c\,(1 + |\xi|^p)\,, \tag{1.10}$$

$$|DF(\xi)| \le c\,(1 + |\xi|^{p-1})\,, \tag{1.11}$$

$$\hat{m}\,(\mu + |\xi|^{p-2})|\lambda|^2 \le DDF(\xi)\lambda\lambda\,, \qquad 0 < \hat{m}\,, \quad 0 \le \mu\,, \tag{1.12}$$

$$|DDF(\xi)| \le c\,(1 + |\xi|^{q-2}), \tag{1.13}$$

$$2 \le p < q < p\frac{n}{n-1}, \tag{1.14}$$

for every $\xi, \lambda \in \mathbb{R}^{nN}$. Let us remark that (1.12) implies (1.5): compare with Corollary 2.8 in the next section 2.

Theorem 2. *Let $u \in W^{1,p}(\Omega)$ minimize the integral (1.1) and F satisfy (1.9–1.14); then*

$$D(|Du|^{(p-2)/2}Du) \in L^2_{\text{loc}}(\Omega). \tag{1.15}$$

Moreover, if (1.12) holds true with $0 < \mu$, then

$$DDu \in L^2_{\text{loc}}(\Omega). \tag{1.16}$$

In this setting we can apply the partial regularity result contained in [1], see also [15], in order to get

$$Du \in C^{0,\alpha}_{\text{loc}}(\Omega_0), \qquad \forall \alpha \in (0,1), \tag{1.17}$$

for some open $\Omega_0 \subset \Omega$, with

$$|\Omega \setminus \Omega_0| = 0. \tag{1.18}$$

Now Theorem 2 allows us to improve on the estimate (1.18) of the (possibly) singular set. This is achieved in the following:

Theorem 3. *Let $u \in W^{1,p}(\Omega)$ minimize the integral (1.1) and F satisfy (1.9–1.14); moreover, we assume that (1.12) holds true with $0 < \mu$: then, there exists an open set Ω_0, $\Omega_0 \subset \Omega$, such that*

$$Du \in C^{0,\alpha}_{\text{loc}}(\Omega_0), \qquad \forall \alpha \in (0,1) \tag{1.19}$$

and

$$\mathcal{H}^{n-2+\epsilon}(\Omega \setminus \Omega_0) = 0, \qquad \forall \epsilon > 0, \tag{1.20}$$

where $\mathcal{H}^{n-2+\epsilon}$ is the $(n-2+\epsilon)$-dimensional Hausdorff measure.

A model functional for the previous theorems is

$$J(u)=\int_\Omega \{a|Du|^2+a|Du|^p+h(\det Du)\}\,dx, \tag{1.21}$$

where $u:\mathbb{R}^n\to\mathbb{R}^n$, $h:\mathbb{R}\to\mathbb{R}$, $h\in C^2(\mathbb{R})$, and for some positive constants c_1,d,

$$1\le d<2, \tag{1.22}$$

$$|h(t)|\le c_1\,(1+|t|)^d, \tag{1.23}$$

$$|h'(t)|\le c_1\,(1+|t|)^{d-1}, \tag{1.24}$$

$$0\le h''(t)\le c_1, \tag{1.25}$$

for every $t\in\mathbb{R}$. Under these assumptions if a is large enough, see [13],

$$a\ge a(n,d,c_1)=c_1n^4[(n-2)!]\{1+[n!]^{d-1}\} \tag{1.26}$$

and

$$2\le n\le 2n-2<p<2n, \qquad nd\le p, \tag{1.27}$$

then (1.9), ... , (1.14) hold true with $q=2n$ and $\mu=1$ in (1.12). For example, we can take $h(t)=\sqrt{1+t^2}$, $d=1$, $c_1=1$; the resulting functional is

$$\hat{J}(u)=\int_\Omega \{a|Du|^2+a|Du|^p+\sqrt{1+(\det Du)^2}\}\,dx. \tag{1.28}$$

In order to deal with weak solutions of non-uniformly elliptic systems which are not Euler equations of variational integrals, we find out that Theorem 1 remains true; with regard to Theorem 2, we need a more restrictive range of q. More precisely, we consider $A:\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ and the system of partial differential equations

$$\operatorname{div}\big(A(Du(x))\big)=0, \tag{1.29}$$

where

$$A\in C^0(\mathbb{R}^{nN}), \tag{1.30}$$

and for some positive constants c,p,m,

$$|A(\xi)|\le c\,(1+|\xi|^{p-1}), \tag{1.31}$$

$$m\,(|\xi|+|\hat\xi|)^{(p-2)}|\xi-\hat\xi|^2\le (A(\xi)-A(\hat\xi))(\xi-\hat\xi), \tag{1.32}$$

for every ξ, $\hat\xi\in\mathbb{R}^{nN}$. About p, we keep on assuming

$$2\le p. \tag{1.33}$$

We say that u is a weak solution of (1.29) if $u:\Omega\to\mathbb{R}^N$, $u\in W^{1,p}(\Omega)$ and

$$\int_\Omega A(Du(x))\,D\phi(x)\,dx=0, \tag{1.34}$$

for every $\phi:\Omega\to\mathbb{R}^N$ with $\phi\in W_0^{1,p}(\Omega)$. We have the following higher integrability result for Du:

Theorem 4. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.29) and A satisfy (1.30–1.33); then*

$$Du \in L^{\sigma}_{\mathrm{loc}}(\Omega), \qquad \forall \sigma < p\frac{n}{n-1}\,. \tag{1.35}$$

As in the case of minimizers, the higher integrability of Du allows us to get higher differentiability; let us remark that, when dealing with elliptic systems that are not the Euler equation of some variational integral, we do not know any longer that the bilinear form $(\lambda, \xi) \to DA\,\lambda\,\xi$ is symmetric: this lack of information is responsible for the more restrictive range of q in the following (1.40). Now we assume that

$$A \in C^1(\mathbb{R}^{nN}), \tag{1.36}$$

and, for some constants $c, p, q, \hat{m}, \mu$,

$$|A(\xi)| \le c\,(1 + |\xi|^{p-1}), \tag{1.37}$$

$$\hat{m}\,(\mu + |\xi|^{p-2})|\lambda|^2 \le DA(\xi)\lambda\lambda, \qquad 0 < \hat{m}, \quad 0 \le \mu, \tag{1.38}$$

$$|DA(\xi)| \le c\,(\mu + |\xi|^{q-2}), \tag{1.39}$$

$$2 \le p < q < p\frac{2n-1}{2n-2}\,, \tag{1.40}$$

for every $\xi, \lambda \in \mathbb{R}^{nN}$. Let us remark that (1.38) implies (1.32).

Theorem 5. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of (1.29) and A satisfy (1.36–1.40); then*

$$D(|Du|^{(p-2)/2}Du) \in L^2_{\mathrm{loc}}(\Omega). \tag{1.41}$$

Moreover, if (1.38), (1.39) hold true with $0 < \mu$, then

$$DDu \in L^2_{\mathrm{loc}}(\Omega). \tag{1.42}$$

Remark. The most important result of this paper is Theorem 1: in our framework, the main step towards regularity is the improvement from $Du \in L^p$ to $Du \in L^\sigma$, $\sigma < pn/(n-1)$, which is contained in Theorem 1. This higher integrability result is achieved by a careful use of difference quotient technique: when dealing with $DF(Du(x + he_s)) - DF(Du(x))$, where $h \in \mathbb{R}$ and e_s is the unit vector in the x_s direction, we do not use any second derivatives of F but we employ growth condition (1.4) for DF: $|DF(\xi)| \le c\,(1 + |\xi|^{p-1})$; this allows us to gain only a fractional derivative of $|Du|^{(p-2)/2}Du$ but it is enough in order to improve on the integrability of Du: see (3.5), (3.6) with the discussion between (4.1) and (4.5). This proof collects some ideas found in [7], [6], [9], [27], [25].

Remark. Regularity for *scalar* minimizers of variational integrals and *scalar* weak solutions to elliptic equations with p, q-growth condition (0.3), (0.7) can be found in [22], [23].

2. Preliminaries

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h} f(x) = f(x + he_s) - f(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \ldots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R. We will often suppress x_0 whenever there is no danger of confusion. We now state several lemmas that are crucial to our work. In the following $f : \Omega \to \mathbb{R}^k$, $k \geq 1$; B_ρ, B_R, $B_{2\rho}$ and B_{2R} are concentric balls.

Lemma 2.1. *If $0 < \rho < R$, $|h| < R - \rho$, $1 \leq t < \infty$, $s \in \{1, \ldots, n\}$, f, $D_s f \in L^t(B_R)$, then*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^t dx \leq |h|^t \int_{B_R} |D_s f(x)|^t dx.$$

(See [**14**,p. 45], [**5**,p. 28]*.)*

Lemma 2.2. *Let $f \in L^t(B_{2\rho})$, $1 < t < \infty$, $s \in \{1, \ldots, n\}$; if there exists a positive constant C such that*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^t dx \leq C|h|^t,$$

for every h with $|h| < \rho$, then there exists $D_s f \in L^t(B_\rho)$. (See [**14**,p. 45], [**5**,p. 26]*.)*

Lemma 2.3. *If $f \in L^2(B_{3\rho})$ and for some $d \in (0, 1)$ and $C > 0$*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} f(x)|^2 dx \leq C|h|^{2d},$$

for every h with $|h| < \rho$, then $f \in L^r(B_{\rho/4})$ for every $r < 2n/(n - 2d)$.

Proof. The previous inequality tells us that $f \in W^{b,2}(B_{\rho/2})$ for every $b < d$, so we can apply the embedding theorem for fractional Sobolev spaces. [3,chapter VII].

Lemma 2.4. *For every t with $1 \leq t < \infty$, for every $f \in L^t(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$ we have*

$$\int_{B_R} |f(x + he_s)|^t dx \leq \int_{B_{2R}} |f(x)|^t dx.$$

Lemma 2.5. *For every $p \geq 2$*

$$\left| \tau_{s,h} \left(|f(x)|^{(p-2)/2} f(x) \right) \right|^2 \leq k^3 \left(\frac{p}{2} \right)^2 \int_0^1 |f(x) + t\, \tau_{s,h} f(x)|^{p-2} |\tau_{s,h} f(x)|^2 dt$$

for every $f \in L^p(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$, for every $x \in B_R$.

Lemma 2.6. *For every $\gamma > -1$, for every $k \in \mathbb{N}$ there exist positive constants c_2, c_3 such that*

$$c_2(|v|^2 + |w|^2)^{\gamma/2} \leq \int_0^1 |v + tw|^\gamma dt \leq c_3(|v|^2 + |w|^2)^{\gamma/2} \tag{2.1}$$

for every $v, w \in \mathbb{R}^k$. (See [**2**]*.)*

Lemma 2.6 allows us to easily get the following Corollaries.

Corollary 2.7. *For every $p \geq 2$, for every $k \in \mathbb{N}$ there exists a positive constant c_4 such that*

$$c_4 \int_0^1 |\lambda + t(\xi - \lambda)|^{p-2} dt \leq (|\lambda| + |\xi|)^{p-2} \tag{2.2}$$

for every $\lambda, \xi \in \mathbb{R}^k$.

Corollary 2.8. *Let F be a function $F : \mathbb{R}^{nN} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{nN})$ and $p \geq 2$; if there exists $\hat{m} > 0$ such that*

$$\hat{m}\, |\xi|^{p-2} |\lambda|^2 \leq DDF(\xi)\lambda\lambda,$$

for every $\xi, \lambda \in \mathbb{R}^{nN}$, then there exists $m > 0$ such that

$$m\, (|\xi| + |\hat{\xi}|)^{(p-2)} |\xi - \hat{\xi}|^2 \leq (DF(\xi) - DF(\hat{\xi}))(\xi - \hat{\xi}),$$

for every ξ, $\hat{\xi} \in \mathbb{R}^{nN}$.

Corollary 2.9. *For every $p \geq 2$, for every $k \in \mathbb{N}$ there exists a positive constant $\hat{c}$ such that*

$$|\lambda - \xi|^p \leq \hat{c} \left| |\lambda|^{\frac{p-2}{2}} \lambda - |\xi|^{\frac{p-2}{2}} \xi \right|^2 \tag{2.3}$$

for every $\lambda, \xi \in \mathbb{R}^k$.

3. Proof of Theorem 1

Since u minimizes the integral (1.1) with growth conditions as in (1.3), (1.4), u solves the Euler equation,

$$\int_\Omega DF(Du(x)) D\phi(x)\, dx = 0, \tag{3.1}$$

for all functions $\phi : \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$. Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$ and let B_ρ and B_R be concentric balls, $0 < \rho < R$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a “cut off” function in $C_0^\infty(B_R)$ with $\eta \equiv 1$ on B_ρ, $0 \leq \eta \leq 1$. Fix $s \in \{1, \ldots, n\}$, take $0 < |h| < R$. Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ in (3.1) we get, as usual

$$(I) = \int_{B_R} \eta^2 \tau_{s,h} \left(DF(Du)\right) \tau_{s,h} Du\, dx = -\int_{B_R} \tau_{s,h} \left(DF(Du)\right) 2\eta D\eta\, \tau_{s,h} u\, dx = (II) \tag{3.2}$$

We apply (1.5) so that

$$m \int_{B_R} (|Du(x+he_s)| + |Du(x)|)^{p-2} |\tau_{s,h} Du(x)|^2 \eta^2(x)\, dx \le (I). \tag{3.3}$$

Now we use Lemma 2.5 and Corollary 2.7 in order to get, for some positive constant c_5, independent of h,

$$\begin{aligned} c_5 \int_{B_R} \left| \tau_{s,h} \left(|Du(x)|^{(p-2)/2} Du(x) \right) \right|^2 \eta^2(x)\, dx \\ \le m \int_{B_R} (|Du(x+he_s)| + |Du(x)|)^{p-2} |\tau_{s,h} Du(x)|^2 \eta^2(x)\, dx. \end{aligned} \tag{3.4}$$

In order to estimate (II), we first use the growth condition (1.4):

$$\begin{aligned} |\tau_{s,h}\left(DF(Du(x))\right)| &= |DF(Du(x+he_s)) - DF(Du(x))| \\ &\le |DF(Du(x+he_s))| + |DF(Du(x))| \\ &\le c\left(1 + |Du(x+he_s)|^{p-1}\right) + c\left(1 + |Du(x)|^{p-1}\right). \end{aligned} \tag{3.5}$$

We apply inequality (3.5) and the properties of the "cut off" function η, then Hölder inequality, finally Lemma 2.1 and 2.4:

$$\begin{aligned} (II) &\le c_6 \int_{B_R} (1 + |Du(x+he_s)|^{p-1} + |Du(x)|^{p-1}) |\tau_{s,h} u(x)|\, dx \\ &\le c_7 \left(\int_{B_R} (1 + |Du(x+he_s)|^p + |Du(x)|^p)\, dx \right)^{(p-1)/p} \left(\int_{B_R} |\tau_{s,h} u(x)|^p dx \right)^{1/p} \\ &\le c_8 \left(\int_{B_{2R}} (1 + |Du(x)|^p)\, dx \right)^{(p-1)/p} \left(\int_{B_{2R}} |D_s u(x)|^p dx \right)^{1/p} |h| \le c_9 |h|, \end{aligned} \tag{3.6}$$

for some positive constants c_6, c_7, c_8, c_9 independent of h. Collecting the estimates for (I) and (II) yields, for some positive constant c_{10}, independent of h,

$$\int_{B_R} \left| \tau_{s,h} \left(|Du(x)|^{(p-2)/2} Du(x) \right) \right|^2 \eta^2(x)\, dx \le c_{10} |h|, \tag{3.7}$$

for every $s = 1, \dots, n$, for every h with $|h| < R$. Since $\eta = 1$ on B_ρ, inequality (3.7) allows us to apply Lemma 2.3 in order to get

$$|Du|^{(p-2)/2} Du \in L^r(B_{\rho/4}), \qquad \forall r < 2n/(n-1).$$

We remark that $|\,|Du|^{(p-2)/2} Du| = |Du|^{p/2}$, thus (1.8) is completely proven. $\square$

4. Proof of Theorem 2

We start as in the proof of Theorem 1 and we arrive at (3.2); now F has second derivatives, thus

$$\begin{aligned}\tau_{s,h}\left(DF(Du(x))\right) &= DF(Du(x+he_s)) - DF(Du(x)) \\ &= \int_0^1 \frac{d}{dt}\big(DF(Du(x)+t\tau_{s,h}Du(x))\big)\,dt \\ &= \int_0^1 DDF(Du(x)+t\tau_{s,h}Du(x))\tau_{s,h}Du(x)\,dt\,.\end{aligned} \tag{4.1}$$

Let us remark that second derivatives of F verify (1.13) with $p < q$: (4.1) is not very useful when carrying on the standard difference quotient technique if we only know that $Du \in L^p$. But we have already proven, in Theorem 1, that $Du \in L^\sigma$, for every $\sigma < pn/(n-1)$. Since we assumed (1.14), then $Du \in L^q$ and we can go on using (4.1) in (3.2):

$$\int_{B_R}\int_0^1 DDF(Du+t\tau_{s,h}Du)\,\eta\,\tau_{s,h}Du\,\eta\,\tau_{s,h}Du\,dt\,dx = (I)$$

$$= (II) = \int_{B_R}\int_0^1 -2DDF(Du+t\tau_{s,h}Du)\,\eta\,\tau_{s,h}Du\,D\eta\,\tau_{s,h}u\,dt\,dx. \tag{4.2}$$

Since F is C^2, the bilinear form $(\lambda,\xi) \to DDF(Du+t\tau_{s,h}Du)\,\lambda\,\xi$ is symmetric; moreover, it is positive because of (1.12). Therefore we can use Cauchy-Schwartz inequality in order to get

$$\begin{aligned}(II) \le& \frac{1}{2}\int_{B_R}\int_0^1 DDF(Du+t\tau_{s,h}Du)\,\eta\,\tau_{s,h}Du\,\eta\,\tau_{s,h}Du\,dt\,dx \\ &+ 2\int_{B_R}\int_0^1 DDF(Du+t\tau_{s,h}Du)\,D\eta\,\tau_{s,h}u\,D\eta\,\tau_{s,h}u\,dt\,dx \\ =& \frac{1}{2}(I) + 2(III).\end{aligned} \tag{4.3}$$

As we have already pointed out, in Theorem 1 we have proven the higher integrability (1.8), so that, with the aid of (1.14), we can get

$$Du \in L^q_{\mathrm{loc}}(\Omega). \tag{4.4}$$

Growth condition (1.13) and higher integrability (4.4) make the two integrals in (4.3) finite, so we can subtract $\frac{1}{2}(I)$ from both sides of (4.2) in order to get

$$\frac{1}{2}(I) \le 2(III). \tag{4.5}$$

Let us estimate (III). First we use the properties of the "cut-off" function η with the growth condition (1.13), then Hölder inequality, finally Lemma 2.1 and 2.4 (Lemma 2.1 is avaliable with $t = q$ because of (4.4)):

$$\begin{aligned}(III) \leq & c_{11} \int\limits_{B_R} (1 + |Du(x)| + |Du(x + he_s)|)^{q-2} |\tau_{s,h} u|^2 \, dx \\ \leq & c_{12} \left(\int\limits_{B_R} (1 + |Du(x)|^q + |Du(x + he_s)|^q) \, dx \right)^{\frac{q-2}{q}} \left(\int\limits_{B_R} |\tau_{s,h} u|^q \, dx \right)^{\frac{2}{q}} \\ \leq & c_{13} \int\limits_{B_{2R}} (1 + |Du|^q) \, dx \, |h|^2 = c_{14} |h|^2,\end{aligned} \tag{4.6}$$

for some positive constants $c_{11}, c_{12}, c_{13}, c_{14}$ independent of h. Now we estimate (I) from below: using (1.12) we have, for some positive constant c_{15} independent of h,

$$\mu \, c_{15} \int\limits_{B_R} |\tau_{s,h} Du|^2 \, \eta^2 \, dx + c_{15} \int\limits_{B_R} \left| \tau_{s,h} \left(|Du|^{(p-2)/2} \, Du \right) \right|^2 \eta^2 \, dx \leq (I). \tag{4.7}$$

Collecting the previous inequalities yields, for some positive constant c_{16} independent of h,

$$\mu \int\limits_{B_R} |\tau_{s,h} Du|^2 \, \eta^2 \, dx + \int\limits_{B_R} \left| \tau_{s,h} \left(|Du|^{(p-2)/2} \, Du \right) \right|^2 \eta^2 \, dx \leq c_{16} |h|^2, \tag{4.8}$$

for every $s = 1, \dots, n$, for every h with $|h| < R$. Since $\eta = 1$ on B_ρ, inequality (4.8) allows us to apply Lemma 2.2 with $f = |Du|^{(p-2)/2} \, Du$ (and $f = Du$ provided $\mu > 0$), thus giving (1.15) (and (1.16), provided $\mu > 0$). This ends the proof. $\square$

5. Proof of Theorem 3. We can use the partial regularity result contained in [1], see [15] too, in order to get

$$Du \in C^{0,\alpha}_{\text{loc}}(\Omega_0), \qquad \forall \alpha \in (0,1),$$

for the open set Ω_0 defined as follows

$$\Omega_0 = \left\{ x \in \Omega : \lim_{r \to 0} (Du)_{B(x,r)} \in \mathbb{R}^{nN}, \quad \lim_{r \to 0} r^{-n} \int\limits_{B(x,r)} |Du(y) - (Du)_{B(x,r)}|^p dy = 0 \right\}.$$

where

$$(g)_{B(x,r)} = |B(x,r)|^{-1} \int\limits_{B(x,r)} g(y) \, dy.$$

So, for the singular set, we have

$$\Omega \setminus \Omega_0 \subset S_1 \cup S_2,$$

where

$$S_1 = \left\{ x \in \Omega : \qquad \nexists \lim_{r \to 0} (Du)_{B(x,r)} \qquad \text{or} \qquad \lim_{r\to 0} |(Du)_{B(x,r)}| = \infty \right\},$$

$$S_2 = \left\{ x \in \Omega : \limsup_{r\to 0} r^{-n} \int\limits_{B(x,r)} |Du(y) - (Du)_{B(x,r)}|^p dy > 0 \right\}.$$

Let us take $\xi \in \mathbb{R}^{nN}$ such that $|\xi|^{\frac{p-2}{2}}\xi = \left(|Du|^{\frac{p-2}{2}} Du\right)_{B(x,r)}$; then

$$r^{-n} \int\limits_{B(x,r)} |Du(y) - (Du)_{B(x,r)}|^p dy \le 2^p r^{-n} \int\limits_{B(x,r)} |Du(y) - \xi|^p dy = (V);$$

we can use Corollary 2.9 with $\lambda = Du(y)$ and, if we keep in mind the particular choice of ξ and Poincarè inequality, we get

$$\begin{aligned}
(V) \le& \hat{c}\, 2^p\, r^{-n} \int\limits_{B(x,r)} \left| |Du(y)|^{\frac{p-2}{2}} Du(y) - |\xi|^{\frac{p-2}{2}}\xi \right|^2 dy \\
=& \hat{c}\, 2^p\, r^{-n} \int\limits_{B(x,r)} \left| |Du(y)|^{\frac{p-2}{2}} Du(y) - (|Du|^{\frac{p-2}{2}} Du)_{B(x,r)} \right|^2 dy \\
\le& \tilde{c}\, \hat{c}\, 2^p\, r^{2-n} \int\limits_{B(x,r)} \left| D\left(|Du(y)|^{\frac{p-2}{2}} Du(y)\right)\right|^2 dy\,.
\end{aligned}$$

Thus

$$S_2 \subset \left\{ x \in \Omega : \quad \limsup_{r\to 0} r^{2-n} \int\limits_{B(x,r)} \left| D\left(|Du(y)|^{\frac{p-2}{2}} Du(y)\right)\right|^2 dy > 0 \right\}.$$

Since we have proven that

$$DDu \text{ and } D(|Du|^{(p-2)/2} Du) \in L^2_{\text{loc}}(\Omega),$$

we can use standard technique [19], [14], in order to get (1.20). This ends the proof. □

6. Proof of Theorem 4 and 5

Theorem 4 is proven just in the same way as Theorem 1, so we skip it and we go to Theorem 5. Arguing as in Theorem 2 we get

$$\begin{aligned}
&\int\limits_{B_R} \int\limits_0^1 DA(Du + t\tau_{s,h} Du)\, \eta\, \tau_{s,h} Du\, \eta\, \tau_{s,h} Du\, dt\, dx = (I) \\
&\qquad = (II) = \int\limits_{B_R} \int\limits_0^1 -2DA(Du + t\tau_{s,h}Du)\, \eta\, \tau_{s,h} Du\, D\eta\, \tau_{s,h} u\, dt\, dx. \qquad (6.1)
\end{aligned}$$

Since the bilinear form $(\lambda, \xi) \to DA\,\lambda\,\xi$ is no longer symmetric, we cannot use Cauchy-Schwartz inequality as we did in (4.3). Let us remark that $q < p(2n-1)/(2n-2) < pn/(n-1)$, so we can use the higher integrability result proven in Theorem 4:

$$Du \in L^{\sigma}_{\mathrm{loc}}(\Omega), \qquad \forall \sigma < p\frac{n}{n-1}. \tag{1.35}$$

We apply the nonuniform ellipticity conditions (1.38) and (1.39), then we use (1.35) with $\sigma = q$:

$$0 \le \hat{m} \int\limits_{B_R} \int\limits_0^1 (\mu + |Du + t\tau_{s,h}Du|^{p-2})|\tau_{s,h}Du|^2\,\eta^2\,dt\,dx = \hat{m}\,(IV) \le (I) < \infty. \tag{6.2}$$

Let us estimate (II). First of all we use the growth condition (1.39):

$$\begin{aligned} &|2DA(Du + t\tau_{s,h}Du)\,\eta\,\tau_{s,h}Du\,D\eta\,\tau_{s,h}u| \\ &\qquad \le c_{17}\,(\mu + |Du + t\tau_{s,h}Du|^{q-2})|\eta\,\tau_{s,h}Du|\,|\tau_{s,h}u| \\ &\qquad \le \epsilon\,(\mu + |Du + t\tau_{s,h}Du|^{p-2})|\eta\,\tau_{s,h}Du|^2 \\ &\qquad\quad + \frac{c_{17}^2}{\epsilon}(\mu + |Du + t\tau_{s,h}Du|^{2q-p-2})|\tau_{s,h}u|^2, \qquad \forall \epsilon > 0, \end{aligned} \tag{6.3}$$

for some positive constant c_{17} independent of h and ϵ, so that

$$|(II)| \le \epsilon(IV) + \frac{c_{17}^2}{\epsilon} \int\limits_{B_R} \int\limits_0^1 (\mu + |Du + t\tau_{s,h}Du|^{2q-p-2})|\tau_{s,h}u|^2\,dt\,dx. \tag{6.4}$$

Because of (1.40), $p < q < 2q - p < pn/(n-1)$, so (1.35) allows us to use Lemma 2.1 and 2.4 with $t = 2q - p$ and $f = Du$:

$$\begin{aligned} &\int\limits_{B_R} \int\limits_0^1 (\mu + |Du + t\tau_{s,h}Du|^{2q-p-2})|\tau_{s,h}u|^2\,dt\,dx \\ &\qquad \le c_{18} \left\{ \int\limits_{B_R} \left(\mu^{\frac{2q-p}{2q-p-2}} + |Du(x)|^{2q-p} + |Du(x + he_s)|^{2q-p} \right) dx \right\}^{\frac{2q-p-2}{2q-p}} \\ &\qquad\quad \times \left\{ \int\limits_{B_R} |\tau_{s,h}u|^{2q-p}\,dx \right\}^{\frac{2}{2q-p}} \\ &\qquad \le c_{19} \int\limits_{B_{2R}} \left(\mu^{\frac{2q-p}{2q-p-2}} + |Du(x)|^{2q-p} \right) dx\,|h|^2 \\ &\qquad \le c_{20}\,|h|^2, \end{aligned} \tag{6.5}$$

for some positive constants c_{18}, c_{19}, c_{20} independent of h and ϵ. Inequalities (6.4) and (6.5) give

$$(II) \le \epsilon(IV) + \frac{c_{21}}{\epsilon}\,|h|^2, \tag{6.6}$$

for some positive constant c_{21} independent of h and ϵ. Now we use (6.1), (6.2) and (6.6):

$$\hat{m}\,(IV) \le (I) = (II) \le \epsilon(IV) + \frac{c_{21}}{\epsilon}\,|h|^2; \tag{6.7}$$

we select $\epsilon = \frac{1}{2}\hat{m}$ in (6.7); since $(IV) < \infty$, we can subtract $\epsilon(IV)$ from both sides of (6.7) thus giving

$$\frac{\hat{m}}{2}(IV) \le \frac{2c_{21}}{\hat{m}}|h|^2. \tag{6.8}$$

This last inequality and Lemma 2.5 end the proof. □

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Francesco Leonetti
Dipartimento di Matematica
Università di L'Aquila
Piazzale Aldo Moro 5, 00185 Roma, Italy
E-mail: leonetti@vxscaq.aquila.infn.it

Chiara Musciano
Istituto Nazionale di Alta Matematica Francesco Severi
Città Universitaria
Piazzale Aldo Moro 5, 00185 Roma, Italy