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MULTIBUMP SOLUTIONS FOR AN ALMOST PERIODICALLY FORCED SINGULAR HAMILTONIAN SYSTEM

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Abstract

This paper uses variational methods to establish the existence of so-called multibump homoclinic solutions for a family of singular Hamiltonian systems in $\mathbb{R}^2$ which are subjected to almost periodic forcing in time.

Introduction

This paper is a sequel to [1] where the existence of homoclinic solutions was proved for a family of singular Hamiltonian systems which were subjected to almost periodic forcing. More precisely, consider the Hamiltonian system

$$\text{(HS)} \qquad \ddot{q} + a(t)W'(q) = 0$$

where a and W satisfy

(a_1) $a(t)$ is a continuous almost periodic function of t with $a(t) \geq a_0 > 0$ for all $t \in \mathbb{R}$.

(W_1) There is a $\xi \in \mathbb{R}^2 \backslash \{0\}$ such that $W \in C^2(\mathbb{R}^2 \backslash \{\xi\}, \mathbb{R})$.

(W_2) $\lim_{x \to \xi} W(x) = -\infty$.

(W_3) There is a neighborhood $\mathcal{N}$ of ξ and $U \in C^1(\mathcal{N} \backslash \{\xi\}, \mathbb{R})$ such that $|U(x)| \to \infty$ as $x \to \xi$ and

$$|U'(x)|^2 \leq -W(x) \quad \text{for } x \in \mathcal{N} \backslash \{\xi\},$$

(W_4) $W(x) < W(0) = 0$ if $x \neq 0$ and $W''(0)$ is negative definite.

(W_5) There is a constant $W_0 < 0$ such that $\overline{\lim}_{x \to \infty} W(x) \leq W_0$.

Let $E = W^{1,2}(\mathbb{R}, \mathbb{R}^2)$, $\mathcal{L}(q) = \dfrac{1}{2}|\dot{q}(t)|^2 - a(t)W(q(t))$, and define the functional

$$I(q) = \int_{\mathbb{R}} \mathcal{L}(q)\, dt. \tag{0.1}$$

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Introducing the subset of E,

$$\Lambda = \{q \in E | q(t) \neq \xi \quad \text{for all} \quad t \in \mathbb{R}\}, \tag{0.2}$$

it was shown in [1] that $I \in C^1(\Lambda, \mathbb{R})$ and critical points of I in Λ are classical solutions of (HS) which are homoclinic to 0, i.e. $|q(t)|$, $|\dot{q}(t)| \to 0$ as $|t| \to \infty$. Since any $q \in \Lambda$ satisfies $|q(t)| \to 0$ as $|t| \to \infty$, q can be considered to be a closed curve in $\mathbb{R}^2$ which avoids ξ. As such it has an associated Brouwer degree, $d(q)$, which equals its winding number, $WN(q)$ with respect to ξ. Let

$$\Gamma = \{q \in \Lambda | d(q) \neq 0\} = \Gamma^+ \cup \Gamma^-$$

where

$$\Gamma^\pm = \{q \in \Gamma | \pm d(q) > 0\}.$$

The main results in [1] were that I possesses infinitely many critical points in Γ^+ and Γ^- with corresponding critical values near

$$c^\pm = \inf_{\Gamma^\pm} I \tag{0.3}$$

Moreover if $c^\pm$ is attained by I at $Q^\pm \in \Gamma^\pm$ with $Q^\pm$ an isolated critical point of I, then there is an unbounded sequence $(\sigma_m) \subset \mathbb{R}$ such that I has a local minimum near $Q^\pm(t - \sigma_m)$ for large m. The numbers (σ_m) stem from the almost periodicity of the function $a(t)$ which implies there is such a sequence satisfying

$$\|a(\cdot) - a(\cdot + \sigma_m)\|_{L^\infty(\mathbb{R})} \to 0 \tag{0.4}$$

as $m \to \infty$. We do not know whether c^+ (resp. c^-) is attained by $I \equiv I_a$ for the given almost periodic function $a(t)$. However there is always an α in $\mathcal{H}(a)$, the hull of a, i.e. the L^∞ closure of the set of translates of $a(t)$ for which the infimum is achieved by the corresponding I_α.

The main goal of the current paper is to show that when there is an isolated minimizer $Q^\pm \in \Gamma^\pm$ of I with $I(Q^\pm) = c^\pm$, then (HS) possesses so called multibump solutions. To state this a bit more precisely, for $s \in \mathbb{R}$ and $q \in E$, set

$$\tau_s q(t) = q(t - s). \tag{0.5}$$

We will prove that for any $k \in \mathbb{N}$, near $\sum_1^k \tau_{\sigma_{j_i}} Q^\pm$, there is an actual homoclinic solution of (HS) provided that e.g. $0 < \sigma_{j_1} < \cdots < \sigma_{j_k}$ and $\sigma_{j_1}, \sigma_{j_{i+1}} - \sigma_{j_i}$ are sufficiently large. If both Q^+ and Q^- are isolated minimizers, there is a more complicated existence statement. The requirement that Q^+ be isolated is the analogue for the variational approach taken here of the related assumption that one has a transversal intersection of stable and unstable manifolds for a Poincaré map associated with (HS) at a homoclinic point corresponding to Q^+.

An exact formulation of the main existence theorem and its proof will be given in §1. Some extensions and related results will be carried out in §2. Various technical results required for the proofs will be treated in §3.

There have been several recent papers, beginning with Séré [2], which use methods from the calculus of variations to get the existence of multibump homoclinic or heteroclinic solutions of Hamiltonian systems. See e.g. Bessi [3], Bolotin [4], Caldiroli and Montecchiari [5], Coti Zelati and Rabinowitz [6-7], Giannoni and Rabinowitz [8], Montecchiari and Nolasco [9], Rabinowitz [10-11], Séré [12], and Strobel [13]. Aside from [9] these papers deal with periodically forced Hamiltonian systems. Reference [9] treats a perturbation with arbitrary time dependence of a time periodically forced potential which is a superquadratic function of q. See also [11] in this regard. Recently Buffoni and Séré [14] found multibump solutions for an autonomous superquadratic Hamiltonian system. Our work in [1] was motivated in part by [15] and [11] – see also Tanaka [16] - where the existence of basic homoclinic and multibump solutions was studied for (HS) under periodic forcing and weaker conditions than (a_1), (W_1) - (W_5). The second major influence on [1] was the recent work of Serra, Tarallo and Terracini [17] who found a basic homoclinic solution for a family of superquadratic Hamiltonian systems under almost periodic forcing. See also Bertotti and Bolotin [18]. Some results on multibump homoclinics in the setting of [17] have been obtained by Spradlin [19] and as we recently learned by Coti Zelati, Montecchiari and Nolasco [20]. Lastly there has been some recent work in the setting of [15] by Caldiroli and Nolasco [21] who study an autonomous problem and under additional hypotheses on the potential find basic homoclinics which wind k times around the singularity for any $k \in \mathbb{N}$.

§1. Multibump solutions

The existence of multibump solutions of (HS) will be studied in this section. In order to formulate the main result, some preliminaries and notations are needed. Let $B_r(t)$ denote an open ball of radius r about $x \in E$.

As was noted in the Introduction, by (a_1), there is an unbounded sequence $(\sigma_m) \subset \mathbb{R}$ such that

$$\|\tau_{-\sigma_m} a - a\|_{L^\infty} \to 0 \tag{1.1}$$

as $m \to \infty$. Fix $k \in \mathbb{N}$ and let $\sigma_{j_1} < \cdots < \sigma_{j_k}$ with $\sigma_{\sigma_1}, \ldots, \sigma_{j_k} \in (\sigma_m)$. Set $\beta_0 = -\infty$, $\beta_k = \infty$, and for $1 \leq i \leq k-1, \beta_i = \frac{1}{2}(\sigma_{j_i} + \sigma_{j_{i+1}})$. For $x \in E$, set

$$|||x||| = \max_{1 \leq i \leq k} \|x\|_{W^{1,2}[\beta_{i-1},\beta_i]}.$$

Thus $|||x|||$ is an equivalent norm on E. Let $\mathcal{B}_r(x)$ denote the open ball of radius r about $x \in E$ under $|||\cdot|||$. Let

$$\mathcal{K} = \{q \in E\backslash\{0\} | I'(q) = 0\},$$

i.e., $\mathcal{K}$ is the set of nontrivial critical points of I or equivalently solutions of (HS) that are homolinic to 0.

Our main result can now be stated:

Theorem 1.2. *Let $(a_1), (W_1) - (W_5)$ be satisfied. Suppose that $c^{\pm}$ (in (0.3)) is attained at an isolated critical point $Q(\in \Gamma^+)$. Let $k \in \mathbb{N}$ and $\sigma_{j_1} < \cdots < \sigma_{j_k} \in (\sigma_m)$. Set $Q_k = \sum_1^k \tau_{\sigma_{j_i}} Q$. Then there is an $r_0 > 0$ and an $\ell = \ell(r)$ defined for*

$0<r<r_0$ such that whenever $\sigma_{j_1}\ge \ell, \sigma_{j_{i+1}}-\sigma_{j_i}\ge \ell, 1\le i\le k-1$, I possesses a local minimum in $\mathcal{B}_r(Q_k)$.

Remark 1.3. ℓ is independent of k. As will be seen in §2, this leads to the existence of infinite bump solutions of (HS) via a simple limit process.

In order to prove Theorem 1.2, some technical preliminaries are required. They will be stated next and their proofs will be given in §3. The first provides a lower bound for $\|I'\|$ in an annular neighborhood of an isolated minimizer of I.

Proposition 1.4. *Let $(a_1),(W_1)-(W_5)$ be satisfied and suppose $Q\in\Gamma^+$ is an isolated critical point of I with $I(Q)=c^+$. Then there is an $r_1>0$ and $\delta=\delta(r,\underline{r})$ defined for $0<\underline{r}<r\le r_1$ such that $\|I'(x)\|\ge 4\delta$ if $x\in\overline{B}_r(Q)\backslash B_{\underline{r}}(Q)$.*

The next result concerns the existence of a vector field that plays an important role in the proof of Theorem 1.2.

Proposition 1.5. *Under the hypotheses of Proposition 1.4, there is an $r_2>0$, a real valued function $\ell_0(r,\rho)$ defined for $0<\rho<r\le r_2$, and a locally Lipschitz continuous function $\mathcal{V}:\overline{B}_r(Q_k)\to E$ satisfying:*

$$\||\mathcal{V}(x)\||\le 3, \tag{1.6}$$

and

$$I'(x)\mathcal{V}(x)\ge\delta(6r,\frac{\rho}{8})\quad\text{for}\quad x\in\overline{\mathcal{B}}_r(Q_k)\backslash\mathcal{B}_\rho(Q_k) \tag{1.7}$$

provided that $j_1\ge\ell_0$, $\sigma_{j_{i+1}}-\sigma_{j_i}\ge\ell_0$, $1\le 1\le k-1$. Moreover defining

$$\Phi_i(x)\equiv\int_{\beta_{i-1}}^{\beta_i}\mathcal{L}(x)\,dt\qquad 1\le i\le k, \tag{1.8}$$

writing $x=Q_k+z$ where $z\in\mathcal{B}_r(0)$ and setting $z_i=z|_{\beta_{i-1}}^{\beta_i}$, then

$$\Phi_i'(x)\mathcal{V}(x)\ge\delta(6r,\frac{r}{8})\quad\text{if}\quad\frac{r}{4}\le\|z_i\|_{W^{1,2}[\beta_{i-1},\beta_i]}\le r \tag{1.9}$$

and

$$\Phi_i'(x)\mathcal{V}(x)\ge\delta(r,\frac{\rho}{8})\quad\text{if}\quad\rho\le\|z_i\|_{W^{1,2}[\beta_{i-1},\beta_i]}\le r \tag{1.10}$$

The next preliminary yields a decay estimate for solutions of (HS) that are homoclinic to 0:

Proposition 1.11. *Let $P\in E$ be a solution of (HS). Then there are constants $\gamma,A,R>0$ such that*

$$|P(t)|+|P'(t)|+|P''(t)|\le Ae^{-\gamma|t|}\quad\text{for}\quad|t|\ge R. \tag{1.12}$$

The final simple preliminary concerns the existence of minimizers of I in $\mathcal{B}_\alpha(Q_k)$.

Proposition 1.13. *For any $\alpha > 0$, there is a $P = P_\alpha \in \overline{\mathcal{B}}_\alpha(Q_k)$ such that $I(P) = \inf_{\overline{\mathcal{B}}_\alpha(Q_k)} I$.*

Before beginning the formal proof of Theorem 1.2, we briefly indicate the strategy of the argument. After a suitable choice of parameters, $r, \ell(r)$, etc. by Proposition 1.5, $I' \neq 0$ in $\overline{\mathcal{B}}_r(Q_k)\backslash\mathcal{B}_\rho(Q_k)$. If I does not have a local minimum in $\mathcal{B}_\rho(Q_k)$, by Proposition 1.13, $\inf_{\mathcal{B}_\rho(Q_k)} I$ is attained at $z \in \partial\mathcal{B}_\rho(Q_k)$. An ordinary differential equation is introduced using the function $\mathcal{V}$ of Proposition 1.5 and with initial condition $\eta(0) = z$. The solution trajectory $\eta(s)$ lies in $\mathcal{B}_r(Q_k)\backslash\mathcal{B}_\rho(Q_k)$ for all $k > 0$ and analyzing the behavior of η leads to a contradiction.

Proof of Theorem 1.2: The proof involves variants of arguments from [7] and [1].

For convenience, set $c = c^+$. Choose $r_0 = \min(r_1, r_2)$ and let $\delta_1(r) = \delta(r, \frac{r}{4})$ as given by Proposition 1.4 and $\ell_1(r) \geq \ell_0(r, \frac{r}{4})$ as given by Proposition 1.5. Set

$$\epsilon = \epsilon(r) = r\delta_1(r)/48. \tag{1.14}$$

For ℓ_1 sufficiently large,

$$I(Q_k) \leq k(c + \frac{\epsilon}{2}) \tag{1.15}$$

Indeed,

$$I(Q_k) \leq (I(Q_k) - \sum_1^k I(\tau_{\sigma_{j_i}} Q)) + \sum_1^k I(\tau_{\sigma_{j_i}} Q). \tag{1.16}$$

By (1.1),

$$\begin{aligned} I(\tau_{\sigma_{j_i}} Q) &= \int_{\mathbb{R}} (\frac{1}{2}|\tau_{\sigma_{j_i}} \dot{Q}|^2 - a(x)W(\tau_{\sigma_{j_i}} Q))\, dt \\ &= I(Q) + \int_{\mathbb{R}} (a(x) - \tau_{-\sigma_{j_i}} a(t))W(Q)\, dt \\ &\leq c + \frac{\epsilon}{4}, \qquad 1 \leq i \leq k \end{aligned} \tag{1.17}$$

for ℓ_1 sufficiently large. Moreover

$$\begin{aligned} I(Q_k) - \sum_1^k I(\tau_{\sigma_{j_i}} Q) &= 2\int_{\mathbb{R}} \sum_{\substack{i,p \\ i \neq p}} \tau_{\sigma_{j_i}} \dot{Q} \cdot \tau_{\sigma_{j_p}} \dot{Q}\, dt \\ &\quad - \int_{\mathbb{R}} a(x)(W(Q_k) - \sum_1^k W(\tau_{\sigma_{j_i}} Q))\, dt \end{aligned} \tag{1.18}$$

Writing

$$\int_{\mathbb{R}} \tau_{\sigma_{j_i}} \dot{Q} \cdot \tau_{\sigma_{j_p}} \dot{Q}\, dt = \sum_1^k \int_{\beta_{i-1}}^{\beta_i} \tau_{\sigma_{j_i}} \dot{Q} \cdot \tau_{\sigma_{j_p}} \dot{Q}\, dt\,, \tag{1.20}$$

in each interval $[\beta_{i-1}, \beta_i]$, for ℓ_1 large compared to R, at least one factor of the integrand is $\leq e^{-\gamma \ell_1/4}$ via Proposition 1.11. Similarly

$$\begin{aligned} \int_{\mathbb{R}} a(x)(W(Q_k) - \sum_1^k W(\tau_{\sigma_{j_i}} Q))\, dt \\ = \sum_1^k \int_{\beta_{i-1}}^{\beta_i} a(x)(W(Q_k) - \sum_1^k W(\tau_{\sigma_{j_i}} Q))\, dt \end{aligned} \tag{1.21}$$

and on $[\beta_{i-1}, \beta_i]$,

$$|W(Q_k(t)) - W(\tau_{\sigma_{j_i}} Q(t))| \leq M_1 \sum_{p \neq i} |\tau_{\sigma_{k_p}} Q(t)| \tag{1.22}$$

where M_1 depends on L^∞ bounds for $W'(P)$ for P near $\tau_{\sigma_{j_i}} Q$. By (1.22) and Proposition 1.11 again, each integral in (1.21) is exponentially small in ℓ_1. Hence for ℓ_1 sufficiently large, (1.15) holds via (1.16) - (1.22). Note that ℓ_1 is independent of k.

Similar estimates show

$$c - \frac{\epsilon}{4} \leq \Phi_i(Q_k) \leq c + \frac{\epsilon}{4}, \qquad 1 \leq i \leq k \tag{1.23}$$

Next a family of cutoff functions will be introduced. Let $\psi_i(x), \chi_i(x)$ be locally Lipschitz continuous for $x \in \overline{\mathcal{B}}_r(Q_k), 1 \leq i \leq k$ and satisfy

$$\psi_i(x) \begin{cases} = 0 & \text{if } \Phi_i(x) \geq c + 2\epsilon \\ = 1 & \text{if } \Phi_i(x) \leq c + \epsilon \\ \in (0,1) & \text{otherwise.} \end{cases} \tag{1.24}$$

$$\chi_i(x) \begin{cases} = 0 & \text{if } \Psi_i(x) \leq c - 2\epsilon \\ = 1 & \text{if } \Psi_i(x) \geq c - \epsilon \\ \in (0,1) & \text{otherwise.} \end{cases} \tag{1.25}$$

Set

$$\psi(x) = \prod_1^k \psi_i(x); \quad \chi(x) = \prod_1^k \chi_i(x) \tag{1.26}$$

choose $\rho = \rho(r)$ so that

$$0 < \rho < \frac{\epsilon}{24} \tag{1.27}$$

and

$$\sup_{\overline{B}_{4\rho}(Q)} I(x) \leq c + \frac{\epsilon}{8} \tag{1.28}$$

If I has a local minimum in $\mathcal{B}_\rho(Q_k)$, the Theorem is proved. Thus suppose this is not the case. Then by Proposition 1.13, there is a $z \in \partial\mathcal{B}_\rho(Q_k)$ such that

$$I(z) = \inf_{x\in\overline{\mathcal{B}}_\rho(Q_k)} I(x) \tag{1.29}$$

Consider the ordinary differential equations

$$\frac{d\eta}{ds} = -\psi(\eta)\chi(\eta)\mathcal{V}(\eta) \tag{1.30}$$

where $\mathcal{V}$ is given by Proposition 1.5 (with $\rho = \rho(r)$ satisfying (1.27) - (1.28)). Note that by (1.7), $I'(x) \neq 0$ for $x \in \overline{\mathcal{B}}_r(Q_k)\backslash\mathcal{B}_\rho(Q_k)$. As initial conditions for (1.30), take $\eta(0) = z$. Since

$$\Phi_i(z) = \Phi_i(Q_k) + \int_0^1 \Phi_i'(sQ_k + (1-s)z)(Q_k - z)\,ds, \tag{1.31}$$

$\|z - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} \leq \rho$, and Φ_i' is bounded in $\mathcal{B}_\rho(Q_k)$, by making ρ still smaller if necessary, it can be assumed that

$$|\Phi_i(z) - c| \leq \frac{\epsilon}{2} \tag{1.32}$$

Therefore $\psi(z) = \chi(z) = 1$.

The solution of (1.30) certainly exists for small $s > 0$. We claim it exists for all $s > 0$ and lies in $\overline{\mathcal{B}}_r(Q_k)$. Otherwise for some i, some $s_1 < s_2$, s_2 being minimal and all $s \in [s_1, s_2]$,

$$\begin{aligned}\|\eta(s_1) - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} = \frac{r}{2} &\leq \|\eta(s) - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} \\ &\leq \|\eta(s_2) - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} = r.\end{aligned} \tag{1.33}$$

Therefore

$$\begin{aligned}\frac{r}{2} &\leq \|\eta(s_1) - \eta(s_2)\|_{W^{1,2}[\beta_{i-1},\beta_i]} \\ &= \left\|\int_{s_1}^{s_2} \frac{d\eta}{ds}\,ds\right\| \leq \int_{s_1}^{s_2} \psi(\eta(s))\chi(\eta(s))\|\mathcal{V}(\eta(s))\|_{W^{1,2}[\beta_{i-1},\beta_i]}\,ds \\ &\leq 3\int_{s_1}^{s_2} \psi(\eta(s))\chi(\eta(s))\,ds\end{aligned} \tag{1.34}$$

via (1.30) and (1.6). By (1.9)

$$\begin{aligned}\Phi_i(\eta(s_1)) - \Phi_i(\eta(s_2)) &= \int_{s_2}^{s_1} \Phi_i'(\eta(s))\frac{d\eta}{ds}\,ds \\ &= \int_{s_1}^{s_2} \psi(\eta(s))\chi(\eta(s))\Phi_i'(\eta(s))\mathcal{V}(\eta(s))\,ds \\ &\geq \delta_1(r)\int_{s_1}^{s_2} \psi(\eta(s))\chi(\eta(s))\,ds\end{aligned} \tag{1.35}$$

Combining (1.34)-(1.35) yields

$$8\epsilon = \frac{r\delta_1}{6} \le \Phi_i(\eta(s_1)) - \Phi_i(\eta(s_2)). \tag{1.36}$$

Due to the definition of ψ and χ, and the form of (1.30),

$$\Phi_i(\eta(s)) \in (c - 2\epsilon, c + 2\epsilon) \tag{1.37}$$

for all $s \in [0, s_2]$. Hence (1.36) is not possible and as claimed $\eta(s)$ lies in $\mathcal{B}_r(Q_k)$ for all $s > 0$.

Next observe that $\eta(s) \notin \mathcal{B}_\rho(Q_k)$ for all $s > 0$. Indeed

$$\frac{dI}{ds}(\eta(s))|_{s=0} = -I'(z)\mathcal{V}(z) < 0 \tag{1.38}$$

by (1.7) so $I(\eta(s))$ decreases for small s. Thus for such s, $I(\eta(s)) < I(z)$ and $\eta(s) \notin \mathcal{B}_\rho(Q_k)$ by the choice of z. Moreover as long as $\eta(s) \in \overline{\mathcal{B}}_r(Q_k) \backslash \mathcal{B}_\rho(Q_k)$, as in (1.38),

$$\frac{dI}{ds}(\eta(s)) = -\psi(\eta(s))\chi(s))I'(\eta(s))\mathcal{V}(\eta(s)) \le 0 \tag{1.39}$$

so $\eta(s)$ can never return to $\overline{\mathcal{B}}_\rho(Q_k)$.

Suppose for the moment that

$$\psi(\eta(s)) = 1 = \chi(\eta(s)) \tag{1.40}$$

for all $s > 0$. Then by (1.30) and (1.7) again,

$$\begin{aligned} I(\eta(s)) &= I(z) + \int_0^s I'(\eta(s))\mathcal{V}(\eta(s))\,ds \\ &\le I(z) - \delta(6r, \frac{\rho}{8})s \end{aligned} \tag{1.41}$$

In particular for large s,

$$I(\eta(s)) < 0. \tag{1.42}$$

But $I(x) \ge 0$ for all $x \in E$ so (1.42) cannot occur. Consequently I must have a local minimizer in $\mathcal{B}_\rho(Q_k)$ and Theorem 1.2 follows.

It remains to verify (1.40). If it does not hold, there is a smallest $s^* > 0$ beyond which (1.40) is violated. Thus for some i, $|\Phi_i(\eta(s^*)) - c| = \epsilon$. Suppose

$$\Phi_i(\eta(s^*)) = c - \epsilon. \tag{1.43}$$

We will show this leads to the construction of a function $P \in \Gamma^+$ with $I(P) < c = c^+$. Hence (1.43) is not possible and $\chi(\eta(s)) \equiv 1$ for $s > 0$.

To find P, note first that

$$\frac{1}{2}\|\eta(s^*) - Q_k\|_{L^\infty[\beta_{i-1},\beta_i]} \le \|\eta(s^*) - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} \le r \le r_0. \tag{1.44}$$

Hence $\eta(s^*)$ is close to Q_k in $L^\infty[\beta_{i-1},\beta_i]$. By estimates as in (1.16) and (1.22) using Proposition 1.11, Q_k is close to $\tau_{\sigma_{j_i}}Q$ in $L^\infty[\beta_{i-1},\beta_i]$. Hence $WN(\eta(s^*))|_{\beta_{i-1}}^{\beta_i})$ is near $WN(\tau_{\sigma_{j_i}}Q|_{\beta_{i-1}}^{\beta_i}) = WN(Q|_{\beta_{i-1}-\sigma_{j_i}}^{\beta_i-\sigma_{j_i}})$ and for ℓ_1 large, this latter quantity is near $WN(Q)=d(Q)>0$.

As was noted earlier, it can be assumed that ℓ_1 is large compared to R of Proposition 2.11 and in particular Q is exponentially small for $|t|\geq \frac{\ell_1}{4}$. Hence for $t\in[\beta_{i-1},\beta_{i-1}+\frac{\ell_1}{3}]$, Q_k satisfies an estimate of the form

$$\begin{aligned}|Q_k(t)| &= |\sum_1^k \tau_{\sigma_{j_i}}Q(t)| \leq \sum_1^k |Q(t-\sigma_{j_i})| \\ &\leq A\sum_1^k e^{-\gamma\frac{\ell_1}{3}i} \leq 2Ae^{-\gamma\frac{\ell_1}{3}}\end{aligned} \tag{1.45}$$

with a similar estimate for $\dot{Q}_k$. Likewise for $t\in[\beta_i-\frac{\ell_1}{3},\beta_i]$,

$$|\dot{Q}_k(t)|+|Q_k(t)| \leq 2Ae^{-\gamma\frac{\ell_1}{3}}. \tag{1.46}$$

By the proof of Proposition 1.5 - see Proposition 3.17 in §3 - there are subintervals U^-,U^+ of length 3 in $[\beta_{i-1},\beta_{i-1}+\frac{\ell_1}{3}]$, $[\beta_{i-}-\frac{\ell_1}{3},\beta_i]$ in which

$$|((\eta(s^*)-Q_k)(t)| \leq 2\frac{r}{\ell_1^{1/2}}. \tag{1.47}$$

Hence for $t\in U^\pm$,

$$|\eta(s^*)(t)| \leq 2A\ell_1^{-\gamma\frac{\ell}{3}} + 2r\ell_1^{-\frac{1}{2}} \tag{1.48}$$

Suppose $U^-=[\alpha^-,\alpha^-+3]$, $U^+=[\alpha^+,\alpha^++3]$. Define $P(t)$ as follows:

$$P(t) = \begin{cases} 0, & t\in(-\infty,\alpha^-+1]\cup[\alpha^++2,\infty) \\ \eta(s^*)(t) & t\in[\alpha^-+2,\alpha^++1] \\ (t-(\alpha^-+1))\eta(s^*)(t) & t\in(\alpha^-+1,\alpha^-+2) \\ (\alpha^++2-t)\eta(s^*)(t) & t\in(\alpha^++1,\alpha^++2) \end{cases} \tag{1.49}$$

Then by the remarks following (1.44),

$$d(P)=d(Q)>0 \tag{1.50}$$

so $P\in\Gamma^+$. Moreover for ℓ_1 sufficiently large,

$$|\int_{\alpha^-+1}^{\alpha^++2}[\mathcal{L}(P)-\mathcal{L}(\eta(s^*))]\,dt\,| < \frac{\epsilon}{2} \tag{1.51}$$

via (1.48) - (1.49). Hence by (1.49) and (1.51),

$$\begin{aligned} I(P)=\Phi_i(P) &< \int_{\alpha^-+1}^{\alpha^++2}\mathcal{L}(\eta(s^*))\,dt+\frac{\epsilon}{2} \\ &< \Phi_i(\eta(s^*))+\frac{\epsilon}{2} = c-\frac{\epsilon}{2} < c \end{aligned} \tag{1.52}$$

contrary to the definition of c. Thus $\chi(\eta(s)) \equiv 1$.

Remark 1.53. If $i = 1$ or $k = 1$, the above construction simplifies a bit.

It remains to prove that $\psi(\eta(s)) \equiv 1$. Thus suppose that

$$\Phi_i(\eta(s^*)) = c^* + \epsilon. \tag{1.54}$$

If $\rho \le \|\eta(s^*) - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} \le r$, by (1.9) - (1.10),

$$\frac{d\Phi_i(\eta(s^*))}{ds} = -\Phi_i'(\eta(s^*))\mathcal{V}(\eta(s^*)) < 0 \tag{1.55}$$

But then $\Phi_i(\eta(s))$ is decreasing for s near s^*, contrary to the definition of s^*. Consequently

$$\|\eta(s^*) - Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} < \rho. \tag{1.56}$$

We will show (1.56) is incompatible with (1.54).

Define

$$Y(t) = \begin{cases} \tau_{\sigma_{j_i}} Q(t) & t \notin [\beta_{i-1}, \beta_i] \\ (\beta_{i-1} + 1 - t)\tau_{\sigma_{j_i}} Q(t) + (t - \beta_{i-1})\eta(s^*)(t) & t \in [\beta_{i-1}, \beta_{i-1} + 1] \\ \eta(s^*)(t) & t \in [\beta_{i-1} + 1, \beta_i - 1] \\ (\beta_i - t)\eta(s^*)(t) + (t - (\beta_i - 1))\tau_{\sigma_{j_i}} Q(t) & t \in [\beta_i - 1, \beta_i] \end{cases} \tag{1.57}$$

Then a computation shows

$$\|Y - \tau_{\sigma_{j_i}} Q\| \le 3\|\tau_{\sigma_{j_i}} Q - \eta(s^*)\|_{W^{1,2}[\beta_{i-1},\beta_i]}. \tag{1.58}$$

Using (1.57) and (1.58) and Proposition 1.11 shows

$$\begin{aligned} \|Y - \tau_{\sigma_{j_i}} Q\| &\le 3\rho + \|Q_k - \tau_{\sigma_{j_i}} Q\|_{W^{1,2}[\beta_{i-1},\beta_i]} \\ &\le 3\rho + 3Ae^{-\gamma \frac{\ell_1}{3}} < 4\rho \end{aligned} \tag{1.59}$$

for ℓ_1 sufficiently large. Set $Y \equiv \tau_{\sigma_{j_i}} P$. Therefore by (1.59),

$$\|P - Q\| < 4\rho, \tag{1.60}$$

i.e. $P \in B_{4\rho}(Q)$ so by (1.28),

$$I(P) \le c + \frac{\epsilon}{8} \tag{1.61}$$

Consequently

$$\begin{aligned} I(Y) &= I(P) + \int_{\mathbb{R}} (a - \tau_{-\sigma_{j_i}} a) W(P)\, dt \\ &\le I(P) + \frac{\epsilon}{2} \le c + \frac{5\epsilon}{8} \end{aligned} \tag{1.62}$$

if ℓ_1 is sufficiently large. On the other hand,

$$
\begin{aligned}
(1.63)\qquad |\Phi_i(\eta(s^*)) - I(Y)| \le & \int_{\mathbb{R}\setminus[\beta_{i-1},\beta_i]} \mathcal{L}(Y)\,dt \\
& + \left| \int_{\beta_i-1}^{\beta_{i-1}+1} (\mathcal{L}(\eta(s^*)) - \mathcal{L}(Y))\,dt \right| \\
& + \left| \int_{\beta_i-1}^{\beta_i} (\mathcal{L}(\eta(s^*)) - \mathcal{L}(Y))\,dt \right| .
\end{aligned}
$$

By Proposition 1.11,

$$
\int_{\mathbb{R}\setminus[\beta_{i-1},\beta_i]} \mathcal{L}(Y)\,dt \le A_1 e^{\gamma\frac{\ell_1}{3}} \tag{1.64}
$$

where A_1 depends on A and $\|a\|_{L^\infty}$. Using (1.57)

$$
\begin{aligned}
|\int_{\beta_{i-1}}^{\beta_{i-1}+1} [\mathcal{L}(\eta(s^*)) - \mathcal{L}(Y)]\,dt| \le & \|\dot\eta\|_{L^2[\beta_{i-1},\beta_i]} \|\tau_{\sigma_{j_i}} Q - \eta(s^*)\|_{W^{1,2}[\beta_{i-1},\beta_i]} \\
& + \|\tau_{\sigma_{j_i}} Q - \eta(s^*)\|^2_{W^{1,2}[\beta_{i-1},\beta_i]} \\
& + M_2 \|\tau_{\sigma_{j_i}} Q - \eta(s^*)\|_{W^{1,2}[\beta_{i-1},\beta_i]}
\end{aligned} \tag{1.65}
$$

where M_2 depends on $\|a\|_{L^\infty}$ and L^∞ bounds for W' in a neighborhood of Q. Using (1.56) and Proposition 1.11 then gives

$$
|\Phi_i(\eta(s^*)) - I(Y)| \le A_1 e^{-\gamma\frac{\ell_1}{3}} + A_2(\rho + A_e^{-\frac{\gamma\ell_1}{3}}) + (\rho + (M + M_2) A e^{-\gamma\frac{\ell_1}{3}}) \tag{1.66}
$$

where A_2 depends on $\|Q_k\|_{W^{1,2}[\beta_{i-1},\beta_i]} \le 2\|Q\|$. Making ρ possibly still smaller shows

$$
|\Phi_i(\eta(s^*)) - \Phi(Y)| \le \frac{\epsilon}{4} \tag{1.67}
$$

Consequently by (1.67) and (1.62),

$$
\Phi_i(\eta(s^*)) \le \frac{\epsilon}{4} + c + \frac{5\epsilon}{8} = c + \frac{7}{8}\epsilon < c + \epsilon \tag{1.68}
$$

contrary to (1.54). Thus $\psi(\eta(s)) \equiv 1$ and I must have a local minimizer in $\mathcal{B}_\rho(Q_k)$. The proof of Theorem 1.2 is complete with $\ell = \ell_1$.

§2. Related results

This section treats some variants and extensions of Theorem 1.2. In particular, the existence of infinite bump solutions of (HS) will be obtained and the effect of having a pair of isolated minimizers Q^+, Q^- for (0.3) will be studied.

To get infinite bump solutions of (HS), let (σ_m) be as in (0.4) and let (σ_{j_i}) be a subsequence of (σ_m) satisfying $\sigma_{j_1} \ge \ell(r)$, $\sigma_{j_{i+1}} - \sigma_{j_i} \ge \ell(r)$ with $r_0, r, \ell(r)$ as given by Theorem 1.2. Let $\beta_i = \frac{1}{2}(\sigma_{j_i} + \sigma_{j_{i+1}}), i \in \mathbb{N}$ and $\beta_0 = -\infty$. Suppose $Q^+ \in \Gamma^+$ is an isolated critical point of I with $I(Q^+) = c^+$. Then for each $k \in \mathbb{N}$, Theorem 1.2 provides a homoclinic solution P_k of (HS) satisfying

$$\|P_k - \tau_{\sigma_{j_i}} Q^+\|_{W^{1,2}[\beta_{i-1},\beta_i]} \le r, \quad 1 \le i \le k-1 \tag{2.1}$$

and

$$\|P_k - \sigma_{j_{k-1}} Q^+\|_{W^{1,2}[\beta_{k-1},\infty]} \le r. \tag{2.2}$$

By (2.1), the functions (P_k) are bounded in $W^{1,2}_{\mathrm{loc}}$ and therefore in L^∞_{loc}. Since they are solutions of (HS), this yields bounds for (P_k) in C^2_{loc}. Hence along a subsequence, P_k converges to a solution, P of (HS) satisfying

$$\|P - \tau_{\sigma_{j_i}} Q^+\|_{W^{1,2}[\beta_{i-1},\beta_i]} \le r \qquad i \in \mathbb{N} \tag{2.3}$$

Thus P is an infinite bump solution of (HS) with $|P(t)|, |\dot{P}(t)| \to 0$ as $t \to -\infty$. We state this somewhat informally as

Theorem 2.4. *Under the hypotheses of Theorem 1.2, for any subsequence (σ_{j_i}) of (σ_m) satisfying $\sigma_{j_1} \ge \ell(r), \sigma_{j_{i+1}} - \sigma_{j_i} \ge \ell(r)$, there is a solution P of (HS) satisfying (2.3).*

Observe that whenever $Q^- \in \Gamma^-$ is an isolated critical point of I with $I(Q^-) = c^-$, Theorem 1.2 holds with Q^+ replaced by Q^-. Suppose that both Q^+ and Q^- are isolated minimizers of I. Then a stronger version of Theorem 1.2 obtains. Indeed let $Y_i \in \{Q^+, Q^-\}$, $1 \le i \le k$ and set $X_k = \sum_1^k \tau_{\sigma_{j_i}} Y_i$.

Theorem 2.5. Let $(a_1), (W_1) - (W_5)$ be satisfied. Suppose that $I(Q^+) = c^+$, $I(Q^-) = c^-$ with $Q^\pm \in \Gamma^\pm$ and $Q^\pm$ isolated critical points of I. Let $k \in \mathbb{N}$ and $\sigma_{j_i} < \cdots < \sigma_{j_k} \in (\sigma_m)$. Then there is an $r_0 > 0$ and an $\ell = \ell(r)$ defined for $0 < r < r_0$ such that whenever $\sigma_{j_1} \ge \ell$, $\sigma_{j_{i+1}} - \sigma_{j_i} \ge \ell, 1 \le i \le k-1$, and $X_k = \sum_1^k \tau_{\sigma_{j_i}} Y_i$ with $Y_i \in \{Q^+, Q^-\}$, I possesses a local minimizer in $\mathcal{B}_r(X_k)$.

Proof: The proof requires minor modifications from that of Theorem 1.2 and will be sketched. Suppose $Y_i = Q^+$ for m values of i. Then (1.15) becomes

$$I(X_k) \le m(c^+ + \frac{\epsilon}{2}) + (k-m)(c^- + \frac{\epsilon}{2}) \tag{2.6}$$

Similarly (1.23) becomes

$$c^\pm - \frac{\epsilon}{4} \le \Phi_i(X_k) \le c^\pm + \frac{\epsilon}{4} \tag{2.7}$$

when $Y_i = Q^\pm$ with analogous changes in (1.24) - (1.25). We replace (1.28) by the two conditions

$$\sup_{\overline{B}_{4\rho}(Q^\pm)} I(x) \le c^\pm + \frac{\epsilon}{8} \tag{2.8}$$

and c in (1.32) and (1.37) is c^+ or c^- depending on i. The construction of P following (1.43) is modified to yield a P^+ and P^- in Γ^+ or Γ^- with $I(P^\pm) < c^\pm$. Lastly in (1.54) and the construction of Y, $\pm$ cases must be distinguished leading to a contradiction of (2.8).

§3. Some technical results

This section contains the proofs of Proposition 1.4, 1.5, 1.11, and 1.13. With the exception of Proposition 1.5, they are quite straightforward so that result will be proved last.

Proof of Proposition 1.4: Since Q is an isolated critical point of I, it can be assumed that

$$\overline{B}_{r_1}(Q) \cap \mathcal{K} = \{Q\}. \tag{3.1}$$

If Proposition 1.4 is false, there is a sequence $(x_m) \subset \overline{B}_r(Q) \backslash B_{\underline{r}}(Q)$ such that $I'(x_m) \to 0$. For r_1 small, $I(x_m)$ is near c^+. Therefore, along a subsequence, $I(x_m) \to b > 0$. Consequently (x_m) is a Palais-Smale sequence. The behavior of such sequences has been studied in [1]. Let $\mathcal{H}(a)$ denote the closure (in $\|\cdot\|_{L^\infty(\mathbb{R})}$) of the set of uniform limits of translates of a. For $\alpha \in \mathcal{H}(a)$, set

$$I_\alpha(x) = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{x}|^2 - \alpha W(x) \right) dt$$

with associated Hamiltonian system

$$\ddot{x} + \alpha W'(x) = 0.$$

Let

$$\mathcal{K}^* = \{q \in E\backslash\{0\} | I'_\alpha(q) = 0 \quad \text{for some} \quad \alpha \in \mathcal{H}(a)\}.$$

By Proposition 2.7 of [1], if (x_m) is a Palais-Smale sequence for I, there is a $j \in \mathbb{N}, v_1, \cdots, v_j \in \mathcal{K}^*$, and sequences $(k_m^1), \cdots (k_m^j) \subset \mathbb{R}$ such that, along a subsequence, as $m \to \infty$.

$$\left\| x_m - \sum_1^j \tau_{k_m^i} v_i \right\| \to 0 \tag{3.2}$$

and

$$|k_m^i - k_m^p| \to \infty \quad \text{if} \quad i \neq p. \tag{3.3}$$

Since

$$\|x_m - Q\| \le r, \tag{3.4}$$

(3.2) and (3.4) imply

$$\overline{\lim}_{m\to\infty} \left\| Q - \sum_1^j \tau_{k_m^i} v_i \right\| \le r \le r_1 \tag{3.5}$$

It was shown in [1 - Remark 2.6] that there is an $r_2 > 0$ so that $\|v\| \ge r_2$ for all $v \in \mathcal{K}^*$. Hence for e.g. $r_1 < \frac{1}{2}r_2$, (3.5) shows $j = 1$ and (k_m^1) is bounded. Therefore by (3.2), $x_m \to \tau_k v_1 \in \overline{B}_r(Q)\backslash B_{\underline{r}}(Q)$. Moreover $I'(\tau_k v_1) = 0$ so $\tau_k v_1 \in \mathcal{K}$, contrary to (3.1). The Proposition is proved.

Proof of Proposition 1.11: By (W_4), there are constants $a, \beta > 0$ such that $|x| \le \alpha$ implies

$$-x \cdot W'(x) \ge \beta |x|^2 \tag{3.6}$$

Set $y(t) = |P(t)|^2$. By (HS) and (3.6),

$$\begin{aligned} -\ddot{y} &= -2|\dot{P}|^2 - 2P \cdot \ddot{P} = -2|\dot{P}|^2 + 2aP \cdot W'(P) \\ &\le -2|\dot{P}|^2 - 2a\beta |P|^2. \end{aligned} \tag{3.7}$$

Define

$$Ly \equiv -\ddot{y} + 2a_0 \beta y.$$

Then (3.7) and (a_1) show

$$Ly = -|\dot{P}|^2 + 2(a_0 - a)\beta |P|^2 \le 0. \tag{3.8}$$

Let $\epsilon > 0$, $\gamma_0 = \sqrt{2a_0\beta}$ and $A_0 = \alpha \exp \gamma_0 R$. Define

$$z_\epsilon(t) = A_0 e^{-\gamma_0 t} + \epsilon \tag{3.9}$$

Then for any $S > R$,

$$L(z_\epsilon - y) = 2a_0\beta\epsilon - Ly \ge 0, \quad t \in (R, S) \tag{3.10}$$

and

$$z_\epsilon(R) - y(R) = \alpha + \epsilon - y(R) \ge 0 \tag{3.11}$$

$$z_\epsilon(S) - y(S) \ge \epsilon - y(S) \ge 0 \tag{3.12}$$

for R sufficiently large (since $|P(t)| \to 0$ as $|t| \to \infty$). Consequently by the Maximum Principle, $y(t) = |P(t)|^2 \le z_\epsilon(t)$ for all $t \in [R, S]$. Letting first $S \to \infty$ and then $\epsilon \to 0$ shows an estimate of the desired form holds for $|P(t)|$ for $t > R$. Similarly it holds for $t < -R$. By (HS),

$$|\ddot{P}(t)| \le a(t)|W'(P)(t)| \tag{3.13}$$

so (W_4) and the decay estimate for P yield a similar estimate for $\ddot{P}$. Finally standard interpolation inequalities give the decay estimate for $\dot{P}$. The proof is complete.

Proof of Proposition 1.13: Let (q_m) be a minimizing sequence for I in $\overline{\mathcal{B}}_\alpha(Q_k)$. Since (q_m) is bounded, it possesses a subsequence converging weakly in E and strongly in $L^\infty_{\rm loc}$ to $P \in E$. The set $\overline{\mathcal{B}}_\alpha(Q_k)$ is closed and convex. Therefore it is weakly closed and $P \in \overline{\mathcal{B}}_\alpha(Q_k)$. Moreover for any $\ell > 0$,

$$\int_{-\ell}^{\ell} \mathcal{L}(q_m)\,dt \le I(q_m) \tag{3.14}$$

so

$$\int_{-\ell}^{\ell} \mathcal{L}(P)\,dt \le \lim_{m\to\infty} I(q_m). \tag{3.15}$$

Letting $\ell \to \infty$ shows

$$I(P) \le \lim_{m\to 0} I(q_m) \tag{3.16}$$

Consequently $I(P) = \lim_{m\to\infty} I(q_m)$ and the Proposition is proved.

Lastly the proof of Proposition 1.5 will be given. This result is the analogue in the current setting of related results that can be found e.g. in [12] and [7]. The key technical step is its proof is the following:

Proposition 3.17. *Under the hypotheses of Proposition 1.4, there is an $r_2 > 0$, a function $\ell_1(r)$ defined for $0 < r \le r_2$ and a $\varphi_x \in E$ with $\|\varphi_x\| = 1$ defined for $x \in \overline{\mathcal{B}}_r(Q_k)\backslash \mathcal{B}_{\frac{r}{2}}(Q_k)$ such that*

$$I'(x)\varphi_x \ge 2\delta(6r, \frac{r}{8}) \tag{3.18}$$

(δ being as in Proposition 1.4) provided that $j_1 \ge \ell_1, j_{i+1} - j_i \ge \ell_1, 1 \le i \le k-1$.

Proof: If $x \in \overline{\mathcal{B}}_r(Q_k)\backslash \mathcal{B}_{r/2}(Q_k)$, then $x - Q_k \equiv z \in \overline{\mathcal{B}}_r(0)\backslash \mathcal{B}_{r/2}(0)$. Set $z_i = z|_{[\beta_{i-1},\beta_i]}$. Then

$$\|z_i\|_{W^{1,2}[\beta_{i-1},\beta_i]} \le r \qquad 1 \le i \le k \tag{3.19}$$

and for some $p \in [1,k] \cap \mathbb{N}$,

$$\|z_p\|_{W^{1,2}[\beta_{p-1},\beta_p]} \ge \frac{r}{2}. \tag{3.20}$$

Assume for convenience that ℓ_1 is an integer multiple of 12. By (3.19), there is an interval $U_i^+ = [s_i^+, s_i^+ + 3] \subset [\beta_i - \frac{\ell_1}{4}, \beta_i]$ such that

$$\|z_i\|_{W^{1,2}[U_i^+]} \le \sqrt{12} r \ell_1^{-1/2} \tag{3.21}$$

Similarly there is an interval $U_i^- = [s_i^-, s_1^- + 3] \subset [\beta_{i-1}, \beta_{i-1} + \frac{\ell_1}{4}]$ such that

$$\|z_i\|_{W^{1,2}[U_i^-]} \le \sqrt{12} r \ell_1^{-1/2}. \tag{3.22}$$

Set $i = p$ and define a function $z^*(t)$ as follows:
(3.23)

$$z^*(t) = \begin{cases} 0 & t \in \text{center third of } U_{p-1}^+, U_p^\pm, U_{p+1}^- \\ 0 & t \le s_{p-1}^+ + 1 \text{ and } t \ge s_{p+1} + 2 \\ z(t) & t \in [s_{p-1}^+ + 3, s_p^-] \cup [s_p^- + 3, s_p^+] \cup [s_p^+ + 3, s_{p+1}^-] \\ (t - (s_{p-1}^+ + 2))z(t) & t \in [s_{p-1}^+ + 2, s_{p-1}^+ + 3] \\ (s_p^- + 1 - t)z(t) & t \in [s_p^-, s_p^- + 1] \\ (t - (s_p^- + 2))z(t) & t \in [s_p^- + 2, s_p^- + 3] \\ (s_p^+ + 1 - t)z(t) & t \in [s_p^+, s_p^+ + 1] \\ (t - (s_p^+ + 2))z(t) & t \in [s_p^+ + 2, s_p^+ + 3) \\ (s_{p+1}^- + 1 - t)z(t) & t \in [s_{p+1}^-, s_{p+1}^- + 1] \end{cases}$$

(If $p = 1$ we need only deal with U_1^+ and U_2^- while if $p = k$, U_{k-1}^+ and U_k^- suffice). In any of the intervals $U = U_{p-1}^+, U_p^\pm, U_{p+1}^-$,

$$\|z - z^*\|_{W^{1,2}[U]} \le 3\sqrt{12} r \ell_1^{-\frac{1}{2}}. \tag{3.24}$$

Let $\varphi \in E$ with $\|\varphi\| = 1$ and φ having support in $X_p \equiv [s_{p-1}^+, s_{p+1}^- + 3]$. Then

$$\begin{aligned} I'(Q_k + z)\varphi &= I'(\tau_{\sigma_{j_p}} Q + z^*)\varphi \\ &\quad + (I'(\tau_{\sigma_{j_p}} Q + z) - I'(\tau_{\sigma_{j_p}} Q + z^*))\varphi \\ &\quad + (I'(Q_k + z) - I'(\tau_{\sigma_{j_p}} Q + z))\varphi \end{aligned} \tag{3.25}$$

Now on X_p, z and z^* differ only on

$$\hat{U} = U_{p-1}^+ \cup U_p^- \cup U_p^+ \cup U_{p+1}^-.$$

Therefore by (3.24),

$$\begin{aligned} &|(I'(\tau_{\sigma_{j_p}} Q + z) - I'(\tau_{\sigma_{j_p}} Q + z^*))\varphi| \\ &\quad = |\int_{\hat{U}} [(\dot{z} - \dot{z}^*) \cdot \dot{\varphi} - a(t)(W'(\tau_{\sigma_{j_p}} Q + z) - W'(\tau_{\sigma_{j_p}} Q + z^*)) \cdot \varphi]\, dt \\ &\quad \le M_1 \|z - z^*\|_{W^{1,2}[\hat{U}]} \le 12\sqrt{12} r \ell_1^{-\frac{1}{2}} M_1 \equiv M_2 r \ell_1^{-\frac{1}{2}} \end{aligned} \tag{3.26}$$

where M_1 depends on $\|a\|_{L^\infty}$ and L^∞ bounds for the second derivatives of W in a $(2r_1)$ neighborhood of Q. Hence for $\ell_1(r)$ sufficiently large.

$$|(I'(\tau_{\sigma_{j_p}} Q + z) - I'(\tau_{\sigma_{j_p}} Q + z^*))\varphi| \le \frac{1}{4}\delta(6r, \frac{r}{8}). \tag{3.27}$$

The next difference on the right in (3.25) can be estimated as follows:

$$
\begin{aligned}
(3.28)\ &|(I'(Q_k+z)-I'(\tau_{\sigma_{j_p}}Q+z))\varphi| \\
&= |\int_{X_p}[\sum_{i\neq p}\tau_{\sigma_{j_i}}\dot{Q}\cdot\dot{\varphi}-a(t)(W'(Q_k+z)-W'(\tau_{\sigma_{j_p}}Q+z))\cdot\varphi\,dt \\
&\leq (1+M_1)\sum_{i\neq p}\|\tau_{\sigma_{j_i}}Q\|_{W^{1,2}(X_p)}
\end{aligned}
$$

where M_1 is as above. It can be assumed that in Proposition 1.11, $R<\ell_1/4$. Therefore $t\in X_p$ and $i\neq p$, so by Proposition 1.11,

$$
|\tau_{\sigma_{j_i}}\dot{Q}(t)|,|\tau_{\sigma_{j_i}}Q(t)|\leq A_e^{-\gamma|t-\sigma_{j_i}|}. \tag{3.29}
$$

Hence the decay estimate together with the choice of the σ_{j_i}'s: $\sigma_{j_{i+1}}-\sigma_{j_i}\geq\ell_1$ yields

$$
\begin{aligned}
\sum_{i\neq p}\|\tau_{\sigma_{j_i}}Q\|_{W^{1,2}(X_p)} &\leq \frac{A}{\gamma^{1/2}}\sum_{1}^{k}e^{-\frac{\gamma\ell_1 i}{4}} \\
&\leq \frac{2Ae^{-\frac{\gamma\ell_1}{4}}}{\gamma^{1/2}(1-e^{-\frac{\gamma\ell_1}{4}})}\leq\frac{1}{4}\delta(6r,\frac{r}{8})
\end{aligned} \tag{3.30}
$$

for ℓ_1 sufficiently large.

Combining (3.27) and (3.30) gives

$$
I'(Q_k+z)\varphi\geq I'(\tau_{\sigma_{j_p}}Q+z^*)\varphi-\frac{1}{2}\delta(6r,\frac{r}{8}) \tag{3.31}
$$

Now (3.18) can be obtained by making an appropriate choice of φ in (3.31). Since z and z^* differ on X_p only on the region $\hat{U}$ where the difference is small, (3.24), and (3.19) - (3.20), show for ℓ_1 sufficiently large.

$$
2r\geq\|z_p^*\|_{W^{1,2}[\beta_{p-1},\beta_p]}\geq\frac{r}{4}. \tag{3.32}
$$

Set $Y_p=[s_p^-+1,s_p^++2]\subset[\beta_{p-1},\beta_p]$. Two cases will be considered. Suppose that

$$
\|z_p^*\|_{W^{1,2}(Y_p)}\geq\frac{r}{8}. \tag{3.33}
$$

Define

$$
Z_p(t)=\begin{cases} z_p^*(t) & t\in Y_p \\ 0 & t\in R\backslash Y_p\end{cases} \tag{3.34}
$$

Then $Z_p\in E$ and by construction, $Z_p\in\overline{B}_{6r}(0)\backslash B_{\frac{r}{8}}(0)$. Therefore by Proposition 1.4,

$$
\|I'(\tau_{\sigma_{j_p}}Q+Z_p)\|\geq 4\delta(6r,\frac{r}{8}) \tag{3.35}
$$

for r_2 appropriately small. Hence there is a $\varphi \in E$ with $\|\varphi\| = 1$ such that

$$I'(\tau_{\sigma_{j_p}} Q + Z_p)\varphi \geq 3\delta(6r, \frac{r}{8}). \tag{3.36}$$

Moreover since the support of Z_p lies in Y_p and $\tau_{\sigma_{j_p}} Q$ decays exponentially outside of an $\frac{\ell_1}{4}$ neighborhood of σ_{j_p} via Proposition 1.11, it can be assumed that φ has support in Y_p. Therefore since $z^* = z_p^*$ on Y_p,

$$I'(\tau_{\sigma_{j_p}} Q + Z_p)\varphi = I'(\tau_{\sigma_{j_p}} Q + z^*)\varphi \geq 3\delta(6r, \frac{r}{8}) \tag{3.37}$$

and (3.18) obtains for this case with $\varphi_x = \varphi$.

Remark 3.38. For future reference, observe that the above arguments also yield

$$\Phi_p'(x)\varphi_x \geq 2\delta(6r, \frac{r}{8}) \tag{3.39}$$

for this case.

Next suppose that

$$\|z_p^*\|_{W^{1,2}(Y_p)} < \frac{r}{8}. \tag{3.40}$$

Then by (3.32) - (3.34),

$$\|z_p^* - Z_p\|_{W^{1,2}[\beta_{p-1}, \beta_p]} > \frac{r}{8} \tag{3.41}$$

Set

$$\varphi = (z^* - Z_p)\|z^* - Z_p\|^{-1} \equiv (z^* - Z_p)b.$$

Then φ has support in $X_p \backslash Y_p$ and

$$\begin{aligned} I'(\tau_{\sigma_{j_p}} Q + z^*)\varphi &= b \int_{X_p \backslash Y_p} (\tau_{\sigma_{j_p}} \dot{Q} \cdot \dot{z}^* + |\dot{z}^*|^2 - aW'(\tau_{\sigma_{j_p}} Q + z^*) \cdot z^*)\, dt \\ &= b \int_{X_p \backslash Y_p} [|\dot{z}^*|^2 - aW'(z^*) \cdot z^* \\ &\quad + \tau_{\sigma_{j_p}} \dot{Q} \cdot \dot{z}^* - a(W'(\tau_{\sigma_{j_p}} Q + z^*) - W'(z^*)) \cdot z^*]\, dt \end{aligned} \tag{3.42}$$

In the region $X_p \backslash Y_p$, $\tau_{\sigma_{j_p}} Q$ is exponentially small. This yields the estimate

$$\begin{aligned} &|\int_{X_p \backslash Y_p} [\tau_{\sigma_{j_p}} \dot{Q} \cdot \dot{z}^* - a(W'(\tau_{\sigma_{j_p}} Q + z^*) - W'(z^*))\dot{z}^*]\, dt| \\ &\qquad \leq M_3 e^{-\gamma \ell_1/4} \|z^*\|_{W^{1,2}[X_p \backslash Y_p]} \end{aligned} \tag{3.43}$$

where M_3 depends on $A, \gamma, \|a\|_{L^\infty}$, and L^∞ bounds for the second derivatives of W in a neighborhood of 0. Thus by (3.42) - (3.43),

$$\begin{aligned} I'(\tau_{\sigma_{j_p}} Q + z^*)\varphi &\geq b(\min(a_0, 1))\|z^*\|_{W^{1,2}[X_p \backslash Y_p]} \\ &\qquad (\|z^*\|_{W^{1,2}[X_p \backslash Y_p]} - M_3 e^{-\gamma \ell_1/4}). \end{aligned} \tag{3.44}$$

Since

$$\|z^*\|_{W^{1,2}[X_p\backslash Y_p]} \ge \|z^*\|_{W^{1,2}[\beta_{p-1},\beta_p]} = \|z_p^*\|_{W^{1,2}[\beta_{p-1},\beta_p]} \ge \frac{r}{8} \tag{3.45}$$

via (3.41), for ℓ_1 sufficiently large,

$$I'(\tau_{\sigma_{j_p}}Q+z^*)\varphi \ge \frac{b}{16}\min(a_0,1)r^2. \tag{3.46}$$

Finally since $b \ge \frac{1}{6r}$,

$$I'(\tau_{\sigma_{j_p}}Q+z^*)\varphi \ge \frac{1}{96}\min(a_0,1)r. \tag{3.47}$$

It can be assumed that the right hand side of (3.47) is large compared to $\delta(6r,\frac{4}{8})$. Hence we obtain (3.18) for this case.

Remark 3.48. Note that for this case by above estimates and the current choice of φ,

$$\Phi_p'(x)\varphi \ge \Phi_p'(\tau_{\sigma_{j_p}}Q+z)\varphi - \frac{1}{2}\delta(6r,\frac{r}{8}). \tag{3.49}$$

Arguing as in (3.42) - (3.44) gives

$$\begin{aligned}\Phi_p'(\tau_{\sigma_{j_p}}Q+z)\varphi &\ge b\min(a_0,1)\|z_p^*\|_{W^{1,2}[[\beta_{p-1},\beta_p]\backslash Y_p]}\\ &\cdot(\|z_p^*\|_{W^{1,2}[[\beta_{p-1},\beta_p]\backslash Y_p]} - M_3e^{-\gamma\ell_1/4})\end{aligned} \tag{3.50}$$

so by (3.45)

$$\Phi_p'(\tau_{\sigma_{j_p}}Q+z)\varphi \ge \frac{1}{16}\min(a_0,1)r \tag{3.51}$$

as in (3.47). thus (3.18) obtains for this case also. The proof of Proposition 3.17 is complete.

Remark 3.52. Having obtained (3.18) for $x \in \overline{\mathcal{B}}_r(Q_k)\backslash\mathcal{B}_{\frac{r}{2}}(Q_k)$, replacing r by $\frac{r}{2^m}, m = 1,2,\cdots,m_0$ where $\frac{r}{2^{m_0-1}} > \rho \ge \frac{r}{2^{m_0}}$ and appropriately adjusting ℓ_1 yields a φ_x for which

$$I'(x)\varphi_x \ge 2\delta(6r,\frac{\rho}{8}) \tag{3.53}$$

for all $x \in \overline{\mathcal{B}}_r(Q_k)\backslash\mathcal{B}_\rho(Q_k))$.

Remark 3.54. In Case 1 of the proof of Proposition 3.17

$$\|\varphi_x\| = |||\varphi_x||| = 1$$

since the support of φ_x lies in $[\beta_{p-1}, \beta_p]$ while in Case 2, the support of φ_x may extend into the 2 adjacent intervals. Hence $|||\varphi_x||| \le 1$. If there are several values of p for which (3.20) holds, say $p_1, \cdots, p_n$, take $\varphi_x = \varphi_{x_{p_1}} + \cdots + \varphi_{x_{p_n}}$. Then $|||\varphi_x||| \le 3$ due to our observation about the supports of the functions $\varphi_{x_{p_i}}$.

Now finally we have

Proof of Proposition 1.5: A standard construction using convex combinations of the φ_x's and cut-off functions yields $\mathcal{V}(x)$. See e.g. Lemma A.2 of [22] for details. In particular Remark 3.54 gives (1.6), (3.18) and Remark 3.52 prove (1.7), and Remarks 3.38 and 3.48 give (1.9) - (1.10).

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