Electronic Journal of Differential Equations, Vol. **1996**(1996) No. 02, pp. 1–7.
ISSN 1072-6691. URL: http://ejde.math.swt.edu (147.26.103.110)
telnet (login: ejde), ftp, and gopher access: ejde.math.swt.edu or ejde.math.unt.edu

A LOWER BOUND FOR THE GRADIENT OF ∞-HARMONIC FUNCTIONS

Edi Rosset

Abstract

We establish a lower bound for the gradient of the solution to ∞-Laplace equation in a strongly star-shaped annulus with capacity type boundary conditions. The proof involves properties of the radial derivative of the solution, so that starshapedness of level sets easily follows.

§1. Introduction

In this paper we deal with solutions to the ∞-Laplace equation

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0, \tag{1_∞}$$

in a domain Ω of $\mathbb{R}^n$.

Equation (1_∞) was first considered by G. Aronsson ([Ar1], [Ar2]) and naturally arises as the Euler equation of minimal Lipschitz extensions. It is a highly degenerate elliptic equation which is formally the limit, as $p \to \infty$, of the p-Laplace equation

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) = 0. \tag{1_p}$$

This limit process has been recently made rigorous by R. Jensen in [J], where he establishes the fundamental result that any Dirichlet problem for equation (1_∞) has a unique viscosity solution $u \in W^{1,\infty}(\Omega) \cap C^0(\bar{\Omega})$ which is the limit, as $p \to \infty$, of the unique solution $u_p \in W^{1,p}(\Omega)$ to equation (1_p) satisfying the same Dirichlet data (which exists and is unique by standard variational arguments), in the sense that $u_p \to u$ uniformly in $\bar{\Omega}$ and weakly in $W^{1,q}(\Omega)$ for any q such that $q < \infty$. For a discussion of the related concepts of absolutely minimizing Lipschitz extension, variational solution and viscosity solution to equation (1_∞), we also refer to [B-D-M].

Concerning the critical points of ∞-harmonic functions, Aronsson proved that any non-constant C^2 solution to (1_∞) in the plane has non-vanishing gradient ([Ar2]), this result has been recently extended to C^4 solutions in higher dimensions by L. C. Evans ([E]). On the other hand, Aronsson gave examples of C^1 non-constant (viscosity) solutions to (1_∞) having an interior critical point ([Ar3]).

1991 *Subject Classification:* 35J70, 35B05, 35B50, 31C45.
Key words and phrases: ∞-harmonic functions, p-harmonic functions.

Submitted: November 20, 1995. Published February 6, 1996.
Partially supported by MURST.

We consider a strongly star-shaped annulus Ω, that is $\Omega = \Omega_2 \setminus \bar{\Omega}_1$, where Ω_1 and Ω_2, $\Omega_1 \subset\subset \Omega_2$, are bounded domains with C^2-smooth boundaries $\partial\Omega_1$ and $\partial\Omega_2$ satisfying the following strong starshapedness condition with respect to the origin

$$\exists\, \alpha_0 > 0 \text{ such that } \mathbf{n}(x) \cdot x \geq \alpha_0 |x|, \quad \forall x \in \partial\Omega_i,\ i = 1, 2, \tag{S}$$

where $\mathbf{n}(x)$ is the outer unit normal to Ω_i in $x \in \partial\Omega_i$, i=1,2. We establish a lower bound for the gradient of the solution u to equation (1_∞) with capacity type boundary conditions

$$\begin{cases} u = 0 & \text{on } \partial\Omega_1 \\ u = 1 & \text{on } \partial\Omega_2 \end{cases} \tag{BC}$$

Let us recall that solutions to equation (1_p) satisfying boundary conditions (BC) were studied by J. A. Pfaltzgraff in [P], proving starshapedness of the level sets, and also by J. L. Lewis in [L], who proved, having convexity results as objective, a lower bound for the gradient for any p, $1 < p < \infty$.

In this note we show that such a lower bound is uniform with respect to p in a neighbourhood of $+\infty$ and furthermore we prove that it extends to the solution to equation (1_∞) satisfying boundary conditions (BC).

We note that in the two dimensional case one can prove a lower bound for $|Du_p|$ in all of Ω uniformly as $p \to \infty$, without any starshapedness assumption, since $\log |Du_p|$ satisfies a uniformly elliptic equation, due to a result of G. Alessandrini ([Al]), but unfortunately the type of convergence we have for the u_p's, as described previously, does not ensure the convergence of $\inf |Du_p|$ as $p \to \infty$.

On the other hand the starshapedness assumption becomes necessary when $n \geq 3$, in fact by an example of A. W. J. Stoddart ([S]) with $p = 2$ there exist simply connected domains Ω_1, Ω_2 for which the solution to problem (1_p)-(BC) has the gradient vanishing at at least one interior point. By standard arguments it can be seen that the example by Stoddart can also be generalized to the case $p \neq 2$ for any $p \in (1, \infty)$.

Notation. We shall denote the Euclidean scalar product of two elements u, $v \in \mathbb{R}^n$ by $u \cdot v$.

§2. Statement and proof of main result

Let us briefly sketch the steps of the proof. Let u, u_p denote the solutions to the Dirichlet problems (1_∞)-(BC) and (1_p)-(BC) respectively.

We establish a lower bound for $|Du_p|$ on $\partial\Omega$, uniformly in p, for large p (see Lemma 2). By the starshapedness assumption, this implies a lower bound on $\partial\Omega$ for the radial derivative $v_p = Du_p \cdot x$, uniformly in p. In Lemma 1 we prove that v_p satisfies in the distributional sense a degenerate elliptic equation, see also the Remark following Lemma 1 and, constructing an appropriate test function, we deduce a lower bound for v_p in all of Ω. Passing to the limit as $p \to \infty$, a lower bound for the radial derivative $v = Du \cdot x$ follows. Finally, starshapedness of level sets is a consequence of the positivity of the radial derivative.

Lemma 1. *Let u_p be a weak solution to (1_p) in a domain Ω, where $p > 2$. Let*

$$v_p = Du_p \cdot x = \sum_{i=1}^{n} x_i u_{p,x_i},$$

$$A^p = (A^p_{ij})_{i,j=1,\dots,n},$$

$$(A^p_{ij}) = |Du_p|^{p-2}\delta_{ij} + (p-2)|Du_p|^{p-4} u_{p,x_i} u_{p,x_j}.$$

Then v_p satisfies the following identity

$$\int_\Omega A^p Dv_p \cdot D\phi = 0, \qquad \forall \phi \in C_0^\infty(\Omega). \tag{2}$$

Proof. For simplicity of notations we shall drop the index p from A^p, u_p, v_p.

Let us recall that, by a well known result of K. Uhlenbeck ([U]), weak solutions to (1_p) belong to $C^{1,\alpha}_{loc}(\Omega)$ and satisfy $|Du|^{\frac{p-2}{2}} u_{x_i x_j} \in L^2_{loc}(\Omega)$, so that $ADv \in L^2_{loc}(\Omega)$. Let $\phi \in C_0^\infty(\Omega)$. Then, by equation (1_p),

$$\int_\Omega ADv \cdot D\phi = \int_\Omega |Du|^{p-2} x_i Du_{x_i} \cdot D\phi + \int_\Omega |Du|^{p-2} Du \cdot D\phi +$$

$$+ \int_\Omega (p-2)|Du|^{p-4} x_i u_{x_j} u_{x_i x_j} Du \cdot D\phi + \int_\Omega (p-2)|Du|^{p-4}|Du|^2 Du \cdot D\phi =$$

$$= \int_\Omega x_i \left(|Du|^{p-2} Du\right)_{x_i} \cdot D\phi = -\int_\Omega |Du|^{p-2} Du \cdot \left(\sum_i (x_i D\phi)_{x_i}\right)$$

$$= -\int_\Omega |Du|^{p-2} Du \cdot Dh = 0,$$

where $h = \sum_{i=1}^n x_i \phi_{x_i} + (n-1)\phi$. □

Remark. Let us point out that (2) means that v_p is a distributional solution to the following degenerate elliptic equation

$$\operatorname{div}(|Du_p|^{p-2} Dv_p + (p-2)|Du_p|^{p-4} Du_p \cdot Dv_p Du_p) = 0.$$

The degeneracy of this equation prevents to replace C_0^∞ with $W_0^{1,2}$ test functions and therefore a maximum principle is not straightforward. The suitable maximum principle will be derived in the proof of Theorem 1, under hypotheses which ensure that a Hopf-type Lemma holds.

Lemma 2 (A uniform Hopf-type Lemma). *Let $\Omega = \Omega_2 \setminus \bar{\Omega}_1$, where Ω_1 and Ω_2, $\Omega_1 \subset\subset \Omega_2$ are two simply connected bounded domains in $\mathbb{R}^n$ with C^2 boundaries $\partial\Omega_1$, $\partial\Omega_2$. Let u_p be the solution to (1_p)-(BC). Then there exist a constant $\tilde{C} > 0$, $\tilde{C}$ independent of p, and $\bar{p} > n$ such that*

$$|Du_p(x)| \geq \tilde{C} \qquad \forall x \in \partial\Omega, \quad \forall p \geq \bar{p}. \tag{3}$$

Proof. There exists $R>0$ such that for any $x_0\in\partial\Omega$ there exists a ball of center $y\in\Omega$ and radius R such that $B_R(y)\subset\Omega$ and $x_0\in\partial B_R(y)$. Let, for instance, $x_0\in\partial\Omega_1$ and y as above. Let us define, for $p>n$, $w(x)=K\left(R^{\frac{p-n}{p-1}}-r^{\frac{p-n}{p-1}}\right)$, where $r=|x-y|$, $\frac{R}{2}<r<R$ and K is a positive constant to be chosen later. Let $A=\{x\in\Omega\mid\frac{R}{2}<|x-y|<R\}$, $\Omega_d=\{x\in\Omega\mid d(x,\partial\Omega)>d\}$, for $d>0$. Let us consider the unique viscosity solution u to the Dirichlet problem (1_∞)-(BC). Then the comparison principle ([J]) and Harnack inequality ([L-M]) or viscosity solutions to equation (1_∞) imply that $0<u<1$ in Ω. Let $\gamma=\max_{\bar\Omega_{\frac{R}{2}}\setminus\Omega_{\frac{3}{2}R}}u>0$. For sufficiently large p, $u_p\geq\frac{\gamma}{2}$ in $\partial B_{\frac{R}{2}}$ so that, choosing $K=\frac{\gamma}{2}\left(R^{\frac{p-n}{p-1}}-\left(\frac{R}{2}\right)^{\frac{p-n}{p-1}}\right)^{-1}$, we have $w\leq u_p$ on $\partial B_{\frac{R}{2}}$. Moreover, $w\equiv0\leq u_p$ on ∂B_R and $\Delta_pw=0$ in A. The comparison principle for solutions to equation (1_p) implies that $w\leq u_p$ in A. Since $w(x_0)=u_p(x_0)=0$, we have

$$\frac{\partial u_p}{\partial\mathbf{n}}(x_0)\geq-w_r(x_0)=\frac{\gamma}{2}\left(\frac{p-n}{p-1}\right)R^{\frac{1-n}{p-1}}\left(R^{\frac{p-n}{p-1}}-\frac{R}{2}^{\frac{p-n}{p-1}}\right)^{-1}\to\frac{\gamma}{R},$$

as $p\to\infty$, where $\mathbf{n}$ is the outer unit normal to Ω_1, and (3) follows.

Theorem 1. *Let Ω be as in the hypotheses of Lemma 2 and moreover let us assume that the starshapedness condition (S) holds. Let u be the solution to the Dirichlet problem (1_∞)-(BC) extended by 0 to all of $\bar\Omega_2$. Then there exist a constant $\delta>0$, δ independent of p, and $\bar p>n$ such that*

$$\frac{\partial u_p}{\partial r}(x)\geq\delta\qquad\forall x\in\Omega,\quad\forall p\geq\bar p,\tag{4}$$

$$\frac{\partial u}{\partial r}(x)\geq\delta\qquad\text{a.e. } x\in\Omega,\tag{5}$$

where u_p is the solution to (1_p)-(BC). Moreover the set $\Omega_k=\{x\in\Omega_2\mid u(x)<k\}$ is star-shaped w. r. t. the origin, for any $k\in(0,1)$.

Proof. Let us consider the unique solution $u_p\in W^{1,p}(\Omega)$ to the Dirichlet problem (1_p)-(BC) and notice that u_p solves a uniformly elliptic quasilinear equation in a neighbourhood of $\partial\Omega$ in $\bar\Omega$, due to the lower bound for $|Du_p|$. Therefore, by standard estimates (see [L-U]), $|Du_p|^{(p-2)/2}u_{p,x_ix_j}\in L^2(\Omega)$ so that $ADv_p\in L^2(\Omega)$ and (2) extends to test functions in $W_0^{1,2}(\Omega)$. From (3) and (S) it follows that

$$v_p\geq C:=\tilde C\alpha_0r_1\qquad\text{on }\partial\Omega,\quad\forall p\geq\bar p,\tag{6}$$

where $r_1=d(0,\partial\Omega_1)$. Using the test function $\phi=(|v_p|^{\frac{p-2}{2}}v_p-C^{\frac{p}{2}})^-\in W_0^{1,2}(\Omega)$ in (2), we get

$$(p-2)\int_{v_p<C}|Du_p|^{p-4}|v_p|^{\frac{p-2}{2}}(Dv_p\cdot Du_p)^2+\int_{v_p<C}|Du_p|^{p-2}|v_p|^{\frac{p-2}{2}}|Dv_p|^2=0.\tag{7}$$

Let $W^p=\{x\in\Omega\mid v_p<C\}=U^p\cup V^p$, where $U^p=\{x\in\Omega\mid v_p<C,\ v_p\neq0\}$, $V^p=\{x\in\Omega\mid v_p<C,\ v_p=0\}$. Let U_j^p, $j\in J\subset\mathbb{N}$, be the open connected

components of U^p. Since $Du_p \neq 0$ in U^p, we have $u_p \in C^2(U^p)$, and therefore from (7) it follows that $v_p \equiv c_j^p$ in U_j^p. From $\partial U_j^p \subset \{x \in \Omega \mid v_p = C\} \cup \{x \in \Omega \mid v_p = 0\}$ and the definition of U^p it follows that $U^p = \emptyset$. Therefore $v_p \equiv 0$ in W^p. Since Ω is connected and (6) holds it follows that $W^p = \emptyset$, that is $v_p \geq C$ in Ω for $p \geq \bar{p}$.

Now let $C' < C$. If there exists a set S of positive measure such that $v := Du \cdot x \leq C'$ in S, since $v_p \to v$ weakly in L^q for any $q < \infty$, we get the contradiction

$$C\mu(S) \leq \int_\Omega \chi_S v_p \to \int_\Omega \chi_S v \leq C'\mu(S).$$

Therefore $v > C'$ almost everywhere, for any $C' < C$. We have $v \geq C$ almost everywhere and (4) and (5) follow with $\delta = \frac{C}{r_2}$, where $r_2 = \sup_{x \in \Omega_2} |x|$.

In order to prove that Ω_k is star-shaped w. r. t. the origin, let us assume by contradiction that there exist points $x \in \Omega_k$, $\lambda x \notin \Omega_k$, with $\lambda \in (0,1)$ and let $\beta = u(\lambda x) - u(x) > 0$. There exists $p \geq \bar{p}$ such that $u_p(\lambda x) - u_p(x) \geq \frac{\beta}{2} > 0$, but this contradicts (4). □

References

[Al] G. Alessandrini, Isoperimetric inequalities for the length of level lines of solutions of quasilinear capacity problems in the plane, *J. Appl. Math. Phys. (ZAMP)* **40**, (1989), 920–924.

[Ar1] G. Aronsson, Extension of functions satisfying Lipschitz conditions, *Ark. Mat.* **6**, (1967), 551–561.

[Ar2] G. Aronsson, On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, *Ark. Mat.* **7**, (1968), 395–425.

[Ar3] G. Aronsson, On certain singular solutions of the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, *Manuscripta Math.* **47**, (1984), 133–151.

[B-D-M] T. Bhattacharya, E. DiBenedetto, J. Manfredi, Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems, *Rend. Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale Nonlinear PDE's*, (1989), 15–68.

[E] L. C. Evans, Estimates for smooth absolutely minimizing Lipschitz extensions, *Electron. J. Differential Equations* **1993**, (1993), No. 3, 1–9.

[J] R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, *Arch. Rational Mech. Anal.* **123**, (1993), 51–74.

[L] J. L. Lewis, Capacitary functions in convex rings, *Arch. Rational Mech. Anal.* **66**, (1977), 201–224.

[L-M] P. Lindqvist, J. J. Manfredi, The Harnack inequality for ∞-harmonic functions, *Electron. J. Differential Equations* **1995**, (1995), No. 4, 1–5.

[L-U] O. A. Ladyzhenskaya, N. N. Ural'tseva, Linear and quasilinear elliptic equations, Academic Press, New York, 1968.

[P] J. A. Pfaltzgraff, Radial symmetrization and capacities in space, *Duke Math. J.* **34**, (1967), 747–756.

[S] A. W. J. Stoddart, The shape of level surfaces of harmonic functions in three dimensions, *Michigan Math. J.* **11**, (1964), 225–229.

[U] K. Uhlenbeck, Regularity for a class of nonlinear elliptic systems, *Acta Math.* **138**, (1977), 219–240.

Edi Rosset
Dipartimento di Scienze Matematiche
Università di Trieste
Piazzale Europa 1
34100 Trieste, Italy

E-mail: rossedi@univ.trieste.it