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A Dirichlet problem in the strip *

Eugenio Montefusco

Abstract

In this paper we investigate a Dirichlet problem in a strip and, using the sliding method, we prove monotonicity for positive and bounded solutions. We obtain uniqueness of the solution and show that this solution is a function of only one variable. From these qualitative properties we deduce existence of a classical solution for this problem.

1 Introduction

In 1979 B. Gidas, W. M. Ni and L. Nirenberg studied the problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } B(0,r) \\ u \equiv 0 & \text{on } \partial B(0,r) \end{cases} \tag{1}$$

where f is a locally Lipschitz function. In [GNN] they showed that the solution of (1) is a radial function, therefore this solution reflects the symmetry of the domain. The proof of this result is based on the moving plane method and the maximum Principle.

In the last years the interest in qualitative properties of solutions of nonlinear elliptic equations has increased. H. Berestycki and L. Nirenberg [BN2] have simplified the moving plane method and proved the symmetry of solutions of elliptic equations in nonsmooth domains. In the same paper H. Berestycki and L. Nirenberg have also simplified the sliding method, which is a technique for proving monotonicity of solutions of nonlinear elliptic equations.

At the same time some mathematicians are interested in qualitative properties of solutions of elliptic equations in unbounded domains. H. Berestycki and L. Nirenberg have studied the flame propagation in cylindrical domains [BN1]. C. Li investigated elliptic equations in various unbounded domains [L]. H. Berestycki, M. Grossi and F. Pacella showed (using the moving plane method) that an equation with critical growth does not admit a solution in the half space [BGP].

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In 1993, H. Berestycki, L. A. Caffarelli and L. Nirenberg considered a Dirichlet problem in the half space, they showed that the solution is a function of only one variable (under suitable hypotheses) using the sliding method [BCN].

With the same technique we want to prove a similar result in the strip. In fact, using the sliding method, we show that problem (4) has a unique classical solution depending on one variable only. As a matter of fact, the problem is reduced to an ODE.

This paper is organized as follows. In section 2, we study the qualitative properties of the solution to (4). In section 3 we show some simple corollaries to the qualitative study.

In this paper we use frequently the following two theorems.

Theorem 1.1 *Let Ω be an arbitrary bounded domain of $\mathbb{R}^N$ which is convex in the x_1-direction. Let $u \in W^{2,N}_{loc}(\Omega) \cap C\left(\overline{\Omega}\right)$ be a solution of:*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega\,, \\ u \equiv \varphi & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

The function f is assumed to be Lipschitz continuous. Here we assume that for any three points $x' = (x'_1, y)$, $x = (x, y)$, $x'' = (x''_1, y)$ lying on a segment parallel to the x_1-axis, $x'_1 < x_1 < x''_1$, with $x', x'' \in \partial\Omega$, the following holds

$$\begin{array}{ll} \varphi(x') < u(x) < \varphi(x'') & \text{if } x \in \Omega, \\ \varphi(x') \le \varphi(x) \le \varphi(x'') & \text{if } x \in \partial\Omega. \end{array} \tag{3}$$

Then u is monotone with respect to x_1 in Ω:

$$u(x_1 + \tau, y) > u(x_1, y) \quad \text{for } (x_1, y)\,,\ (x_1 + \tau, y) \in \Omega \text{ and } \tau > 0.$$

Furthermore, if f is differentiable, then $u_{x_1} > 0$ in Ω. Finally, u is the unique solution of (2) in $W^{2,N}_{\mathrm{loc}}(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfying (3).

Proof. See Theorem 1.4 of [BN2].

Theorem 1.2 *Let Ω be a bounded domain and suppose $u_1 \in H^1(\Omega)$ is a subsolution while $u_2 \in H^1(\Omega)$ is a supersolution to problem (2), let be $f \in C(\mathbb{R})$ and assume that with constants $c_1, c_2 \in \mathbb{R}$ there holds $-\infty < c_1 \le u_1 \le u_2 \le c_2 < -\infty$, almost everywhere in Ω. Then there exists a weak solution $u \in H^1(\Omega)$ of (2), satisfying the condition $u_1 \le u \le u_2$ almost everywhere in Ω.*

Proof. See Theorem 2.4 in chapter I of [S].

2 Statements and proofs

Theorem 2.1 *Let u be a classical solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } S \\ 0 < u < M & \text{in } S \\ u \equiv 0 & \text{on } \{x_1 = 0\} \\ u \equiv M & \text{on } \{x_1 = h\} \end{cases} \tag{4}$$

where $S := \{x \in \mathbb{R}^N \text{ such that } 0 < x_1 < h\}$. Let f be a locally Lipschitz function with $f(0) \geq 0$ and $f(M) \leq 0$. Then $u \equiv u(x_1)$ and $u_{x_1} > 0$ in S. Moreover, u is the unique solution of (4).

The proof of Theorem 2.1 relies on the following propositions:

Proposition 2.2 *There exists $w(t)$, a solution of*

$$\begin{cases} w''(t) + f(w(t)) = 0 & \text{in } (0,h) \\ 0 < w < M & \text{in } (0,h) \\ w(0) = 0 & \\ w(h) = M & \end{cases} \tag{5}$$

such that $u(x) \leq w(x_1)$ in S.

Proposition 2.3 *There exists $\rho(t)$, a solution of (5), such that $u(x) \geq \rho(x_1)$ in S.*

Proof of Theorem 2.1 By Theorem 1.1 problem (5) has a unique solution such that $0 < w < M$; indeed, we have that $\rho(t) \equiv w(t)$ in $(0, h)$.

Then Theorem 2.1 is proved, since we have that $\rho(x_1) \leq u(x) \leq w(x_1) \equiv \rho(x_1)$, so u is a function of only x_1; furthermore , Theorem 1.1 shows that $u_{x_1} > 0$ in S and u is the unique solution of (4).

Now we must prove Propositions 2.2 and 2.3, which are more difficult than Theorem 2.1.

Proof of Proposition 2.2

Since the function u is a classical solution of (4), by the Schauder estimates we can say that $|\nabla u| \leq K$ in the strip, with K depending only on $\max_{[0,h]} f(s)$, (see Theorem 8.33 in [GT]).

For $\varepsilon \in (0, \min[1/K, h/M])$, consider the function on $\mathbb{R}^+$:

$$\sigma_\varepsilon(t) = \begin{cases} \frac{t}{\varepsilon} & \text{in } [0, \varepsilon M] \\ M & \text{in } [\varepsilon M, h]. \end{cases}$$

Set $\Omega_R := \left\{ x \in S : \left(x_2^2 + \ldots + x_N^2\right)^{1/2} < R \right\}$, the cylinder in S of radius R.

We note that there is a unique function w_ε such that:

$$\begin{cases} -\Delta w_\varepsilon = f(w_\varepsilon) & \text{in } \Omega_R \\ u < w_\varepsilon < M & \text{in } \Omega_R \\ w_\varepsilon \equiv \sigma_\varepsilon(x_1) & \text{on } \partial\Omega_R\,. \end{cases}$$

In fact, u is a subsolution of this problem (on the boundary u takes values less than w because u cannot increases faster, since $|\nabla u| \le K$), and the constant function M is a supersolution. By the theorem of sub and supersolution (see Theorem 1.2), there exists a solution.

This solution is smooth by the classical regularity results (recall that f is locally Lipschitz, and we can apply Lemma B.3 of [S]); therefore, by Theorem 1.1 the function w_ε is unique and strictly increasing in x_1 (note that $u < w_\varepsilon < M$ holds by the strong maximum Principle).

If we slide u vertically (i.e. without changing x_1), we find that similarly to the proof of the first Lemma 6 in [BCN]:

$$u(x_1, x' + a) < w_\varepsilon \quad \text{in } \Omega_R,\ \forall a \in \mathbb{R}^{N-1}.$$

Now let $\varepsilon \to 0$ through a sequence. The sequence w_ε is bounded in $C^2(\Omega_R)$ by Theorem 8.33 in [GT], indeed it is compact in $C^1(A)$, where A is a compact subset of $\Omega_R \setminus \{(0, x') : |x'| = R\}$.

Then w_ε converges to a function w_R satisfying:

$$\begin{cases} -\Delta w_R = f(w_R) & \text{in } \Omega_R \\ w(0, x') \equiv 0 & \text{for } |x'| < R \\ w \equiv M & \text{on } \partial\Omega_R \setminus \{x_1 = 0\} \\ w \text{ is increasing in } x_1 & \\ u(x_1, x' + a) < w_R < M & \text{in } \Omega_R,\ \forall a \in \mathbb{R}^{N-1}. \end{cases} \tag{6}$$

Now we want to prove that w_R is the unique solution of (6). In order to prove the uniqueness we cannot use Theorem 1.1, since w_R is not continuous on $\overline{\Omega}_R$,but in [BCN] (section 2, p. 34-35) this fact is proved, using the sliding method and a little bit of care.

Now we can consider $R' > R$. For $a \in \mathbb{R}^{N-1}$, such that $|a| < R' - R$, and for $\delta \in (0, h)$, we slide Ω_R so that its center is at $\left(a, \delta - \frac{h}{2}\right)$. For δ small, by continuity, the translated w_R is greater than $w_{R'}$ in the overlapped region with $\Omega_{R'}$. Moving the displaced Ω_R up, and using the sliding method as above, we conclude that

$$w_R(x) > w_{R'}(x_1, x' + a) \quad \forall a \in \mathbb{R}^{N-1}, \text{ with } |a| < R' - R. \tag{7}$$

It follows that w_R is a decreasing sequence in R. We let $R \to \infty$, through a sequence, and we find that the sequence converges to a function w which satisfies problem (4).

From (7) we find that

$$w(x) \geq w(x_1, x' + a) \quad \forall a \in \mathbb{R}^{N-1}\,.$$

Then it follows that w is independent of x', since a is an arbitrary vector in $\mathbb{R}^{N-1}$. We also have

$$u(x) \leq w(x_1) \leq M,$$

(it follows from the formulation of problem (6) and letting $R \to \infty$).

Then the function w satisfies problem (5) and is greater than u

Proof of Proposition 2.3

We wish to prove Proposition 2.3 following the proof of Proposition 2.2; this is possible because we are in the strip and we can build a suitable boundary condition for the problem in the cylinder. In [BCN] this is not possible, since the only condition at the boundary that we can consider is the constant 0, and with this condition we cannot apply the Maximum Principle.

For $\varepsilon \in (0, \min[1/K, h/M])$, define the function

$$\gamma(t) = \begin{cases} 0 & \text{in } [0, h - \varepsilon M] \\ (t-h)/\varepsilon + M & \text{in } [h - \varepsilon M, h]. \end{cases}$$

As above, there exists a unique function ρ_ε such that:

$$\begin{cases} -\Delta\rho_\varepsilon = f(\rho_\varepsilon) & \text{in } \Omega_R \\ 0 < \rho_\varepsilon < u & \text{in } \Omega_R \\ \rho_\varepsilon \equiv \gamma(x_1) & \text{on } \partial\Omega_R\,. \end{cases}$$

In this case, u is a supersolution of the problem (on the boundary u takes values greater than ρ_ε), and the constant 0 is a subsolution (as $f(0) \geq 0$). By the theorem of sub and supersolution, there is a solution of the equation.

By regularity theory, the function ρ_ε is smooth; indeed, by the sliding method (Theorem 1.14), ρ_ε is unique and strictly increasing in x_1 (note that $0 < \rho_\varepsilon < u$ is true by the strong maximum Principle). As for the function w_ε we find, by the sliding method, that

$$0 < \rho_\varepsilon < u(x_1, x' + a) \quad \text{in } \Omega_R,\ \forall a \in \mathbb{R}^{N-1}.$$

Now let $\varepsilon \to 0$ through a sequence. The sequence ρ_ε is bounded in $C^2(\Omega_R)$ by the Schauder theory, then by the Ascoli-Arzelà Theorem, it is compact in $C^1(A)$, where A is a compact subset of $\Omega_R \setminus \{(h, x') : |x'| = R\}$. Then ρ_ε converges to a function ρ_R satisfying:

$$\begin{cases} -\Delta\rho_R = f(\rho_R) & \text{in } \Omega_R \\ \rho_R \equiv 0 & \text{on } \partial\Omega_R \setminus \{x_1 = h\} \\ \rho_R(h, x') & \text{for } |x'| < R \\ \rho_R \text{ is increasing in } x_1 & \\ 0 < \rho_R < u(x_1, x' + a) & \text{in } \Omega_R,\ \forall a \in \mathbb{R}^{N-1}. \end{cases} \tag{8}$$

To show that the function ρ_R is the unique solution of (8), we proceed as in [BCN]. Let ζ be another solution of (8). Let Σ_δ denote the intersection $\Omega_R \cap \{\Omega_R - (h-\delta)e_1\}$, and let ρ' be the shifted ρ_R. Then the function ρ' satisfies:

$$-\Delta\rho' = f(\rho') \quad \text{in } \Sigma_\delta$$

and

$$\lim\inf_{x\to\partial\Sigma_\delta} (\rho' - \zeta) \geq 0.$$

Since the function f is locally Lipschitz, by the mean value Theorem, $\rho_\delta := (\rho' - \zeta)$ satisfies:

$$-\Delta\rho_\delta = c(x)\rho_\delta \quad \text{in } \Sigma_\delta\,, \tag{9}$$

where $|c(x)|$ is a function bounded by the Lipschitz constant of f.

For δ small we may apply the maximum Principle in narrow domains (see [GNN] Corollary p. 213), as Σ_δ is narrow in the x_1 direction, and conclude that $\rho_\delta > 0$ (using also the strong maximum Principle).

We want to slide the translated Ω_R by increasing δ, and use the maximum Principle to show that $\rho_\delta > 0$ in Σ_δ for every positive $\delta < h$. Suppose that $\rho_\delta > 0$ in Σ_δ for a maximal open interval $(0,\mu)$, with $\mu \leq \delta$.

We want to show that $\mu = h$ by contradiction. Assume that $\mu < h$.

By continuity, $\rho_\mu \geq 0$ in Σ_μ and satisfies (9), since $\rho_\mu > 0$ on $\{x_1 = 0\}$ we have that $\rho_\mu \not\equiv 0$; therefore, by the maximum Principle, we can say that $\rho_\mu > 0$ in Σ_μ.

We choose a small positive real number α such that $\alpha < \min[(h-\mu), R]$, and consider the subset $A := \{(x,x') \in \Omega_R : x_1 < (\mu - \alpha)\,,\, |x'| < (R-\alpha)\}$.

As $\rho_\mu > 0$ in Σ_μ, there exists some constant $\varepsilon > 0$ such that $\rho_\mu \geq \varepsilon$ in $\overline{A}$. Thus for $\mu' > \mu$, with $(\mu' - \mu)$ sufficiently small, we obtain that $\rho_{\mu'} > 0$ in $\overline{A}$.

To conclude that $\rho_{\mu'} > 0$ in $\Sigma_{\mu'}$ we use the maximum Principle again. In $\Sigma_{\mu'} \setminus \overline{A}$ the function $\rho_{\mu'}$ verifies $-\Delta\rho_{\mu'} = c(x)\rho_{\mu'}$, and since $\rho_{\mu'} > 0$ in $\overline{A}$, we also have $\lim\inf_{x\to\partial(\Sigma_{\mu'}\setminus\overline{A})} \rho_{\mu'}(x) \geq 0$. Since $\left(\Sigma_{\mu'} \setminus \overline{A}\right)$ has small measure for $(\mu' - \mu)$ small, the maximum principle holds in $\left(\Sigma_{\mu'} \setminus \overline{A}\right)$ (see Proposition 1.1 in [BN2]), and we conclude that $\rho_{\mu'} > 0$ in $\left(\Sigma_{\mu'} \setminus \overline{A}\right)$, and hence in all of $\Sigma_{\mu'}$. This is impossible for the maximality of $(0,\mu)$, therefore we have proved that $\rho_\delta > 0$ in Σ_δ, $\forall \delta < h$.

Now let $\delta \to h$; by continuity, it follows that $\rho_R \geq \zeta$ in Ω_R. We may interchange the roles of ρ_R and ζ in the proof, and state that $\rho_R \equiv \zeta$.

Now we can consider $R' > R$. For $a \in \mathbb{R}^{N-1}$ such that $|a| < \left(R' - R\right)$, and for $\delta \in (0,h)$, we slide Ω_R so that its center is at $\left(a, \delta - \frac{h}{2}\right)$. For δ small, by continuity, the translated ρ_R is less than $\rho_{R'}$ in the overlapped region with $\Omega_{R'}$. Moving the displaced Ω_R, and using the sliding method, we conclude that

$$\rho_{R'}(x) > \rho_R(x_1, x' + a) \quad \forall a \in \mathbb{R}^{N-1}, \text{ with } |a| < R' - R. \tag{10}$$

It follows that ρ_R is an increasing sequence in R. We let $R \to \infty$, through a sequence, and find that the sequence converges to a function ρ, which satisfies equation (5). From (10) we find that

$$\rho(x) \geq \rho(x_1, x' + a) \quad \forall a \in \mathbb{R}^{N-1},$$

then it follows that ρ is independent of x', since a is arbitrary. Furthermore, we also have that

$$0 \leq \rho(x_1) \leq u(x),$$

(it follows from the formulation of problem (5) and letting $R \to \infty$). This concludes the proof of Proposition 2.3.

3 Remarks and corollaries

Proposition 3.1 *Theorem 2.1 is true even if the function f depends on x_1 provided the function $f(t,s)$ is increasing in the variable t.*

Proof Indeed if $f(s,u)$ is increasing in the first variable, we obtain

$$\begin{aligned} -\Delta\rho_\delta &= f(x_1 + (\delta - h), \rho) - f(x_1, \zeta) \\ &\leq f(x_1, \rho) - f(x_1, \zeta) \\ &= c(x)(\rho - \zeta) \ . \end{aligned}$$

This inequality is sufficient for the proofs of the above Propositions. In fact the maximum Principle holds with inequality, and we don't need the equality.

Corollary 3.2 *There exists a unique classical solution to problem (4) which satisfies the claim of Theorem 2.1, supposing that f is a locally Lipschitz function such that $f(0) \geq 0$ and $f(M) \leq 0$.*

Proof We have proved that every solution of the problem is a function depending only on x_1, thus problem (4) has a solution if and only if problem (5) has a solution. But the one-dimensional problem admits a weak solution u by the Theorem of sub and supersolution (see theorem 1.2), since the constants 0 and M are functions of $H^1(0,h)$ and they are sub and supersolution of (5), respectively.

Since f is a locally Lipshitz function this weak solution u is a classical solution (see appendix B in [S]) and is unique by the Theorem 1.1. The Corollary is proved.

This Corollary is a simple generalization of Theorem 1.2; in fact our domain is not bounded, but the domain S has a particular geometry and this is crucial for the proof of Theorem 2.3 and Corollary 3.2.

Corollary 3.3 *Consider the problem:*

$$\begin{cases} -\Delta u = f(u) & \text{in } (0,h)\times\Omega \\ 0 < u < M & \text{in } (0,h)\times\Omega \\ u \equiv 0 & \text{on } \{0\}\times\Omega \\ u \equiv M & \text{on } \{h\}\times\Omega \\ u_\nu \equiv 0 & \text{on } (0,h)\times\partial\Omega\ , \end{cases} \tag{11}$$

where Ω is a Domain in $\mathbb{R}^{N-1}$ and u_ν is the exterior normal derivative. Let f be a locally Lipschitz function with $f(0) \geq 0$ and $f(M) \leq 0$.

Then there exists at least a classical solution of (11) depending only on the variable x_1 and strictly increasing in the x_1 direction.

Proof. Under these hypotheses we know that if there is a unique classical solution to this problem with $\Omega = \mathbb{R}^{N-1}$, this solution satisfies problem (11). In fact the function u is a function of x_1 only, thus the derivative u_ν is zero on $(0,h)\times\partial\Omega$ since the normal vector ν is orthogonal to the x_1-direction on this part of the boundary. Then the solution of (4) is a solution of (11), and this concludes the proof of the statement.

Remark. In Corollary 3.3 we don't require all of the hypotheses on the domain $\Omega \subset \mathbb{R}^{N-1}$. In fact, this domain can be unbounded, and the argument still holds.

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Eugenio Montefusco
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa
E-mail address: montefus@cibs.sns.it