Electronic Journal of Differential Equations, Vol. **1997**(1997), No. 03, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

Positive solutions and nonlinear multipoint conjugate eigenvalue problems *

Paul W. Eloe & Johnny Henderson

Abstract

Values of λ are determined for which there exist solutions in a cone of the n^{th} order nonlinear differential equation, $u^{(n)} = \lambda a(t)f(u)$, $0 < t < 1$, satisfying the multipoint boundary conditions, $u^{(j)}(a_i) = 0$, $0 \le j \le n_i - 1$, $1 \le i \le k$, where $0 = a_1 < a_2 < \cdots < a_k = 1$, and $\sum_{i=1}^{k} n_i = n$, where a and f are nonnegative valued, and where both $\lim_{|x|\to 0^+} f(x)/|x|$ and $\lim_{|x|\to\infty} f(x)/|x|$ exist.

1 Introduction

Let $n \ge 2$ and $2 \le k \le n$ be integers, and let $0 = a_1 < a_2 < \cdots < a_k = 1$ be fixed. Also, let $n_1, \ldots, n_k$ be positive integers such that $\sum\limits_{i=1}^{k} n_i = n$.

We are concerned with determining eigenvalues, λ, for which there exist solutions, that are positive with respect to a cone, of the nonlinear multipoint conjugate boundary value problem,

$$u^{(n)} = \lambda a(t)f(u), \quad 0 < t < 1, \tag{1.1}$$

$$u^{(j)}(a_i) = 0, \quad 0 \le j \le n_i - 1, \quad 1 \le i \le k, \tag{1.2}$$

where

(A) $f : \mathbb{R} \to [0, \infty)$ is continuous,

(B) $a : [0, 1] \to [0, \infty)$ is continuous and does not vanish identically on any subinterval, and

(C) $f_0 = \lim\limits_{|x|\to 0^+} \frac{f(x)}{|x|}$ and $f_\infty = \lim\limits_{|x|\to\infty} \frac{f(x)}{|x|}$ exist.

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* *1991 Mathematics Subject Classifications:* 34B10, 34B15.
Key words and phrases: multipoint, nonlinear eigenvalue problem, cone.

Submitted: December 17, 1996. Published January 22, 1997.

This work constitutes a complete generalization, in the conjugate problem setting, of the paper by Henderson and Wang [16] which was devoted to the eigenvalue problem (1.1), (1.2) for the case $n = 2$ and $k = 2$. While the paper [16] arose from a cornerstone paper by Erbe and Wang [12], which was devoted to $n = 2$ and $k = 2$ for the cases when f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$) and when f is sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$), the development since has been rapid. For example, Eloe and Henderson [6] gave a most general extension of [16] for (1.1), (1.2) in the case of arbitrary n and $k = 2$. Other partial extensions have been given for higher order boundary value problems, as well as results for multiple solutions, in both the continuous and discrete settings; see for example [1, 2, 4, 8, 9, 11, 15, 17, 21, 22, 23]. Foundational work for this paper is the recent study by Eloe and McKelvey [10] of (1.1), (1.2), for arbitrary n, $k = 3$ and $n_1 = n_3 = 1$.

For the case of $n = 2$ and $k = 2$, (1.1), (1.2) describes many phenomena in the applied mathematical sciences such as, to name a few, nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and chemical concentrations in biological problems where only positive solutions are meaningful; see, for example [13, 14, 20, 27]. Higher order boundary value problems for ordinary differential equations arise naturally in technical applications. Frequently, these occur in the form of a multipoint boundary value problem for an n^{th} order ordinary differential equation, such as an n-point boundary value problem model of a dynamical system with n degrees of freedom in which n states are observed at n times; see Meyer [24]. It is noted in [24] that, strictly speaking, boundary value problems for higher order ordinary differential equations are a particular class of interface problems. One example in which this is exhibited is given by Keener [18] in determining the speed of a flagellate protozoan in a viscous fluid. Another particular case of a boundary value problem for a higher order ordinary differential equation arising as an interface problem is given by Wayner, *et al.* [28] in dealing with a study of perfectly wetting liquids.

We now observe that, for $n = 2$, positive solutions of (1.1), (1.2) are concave. This concavity was exploited in [12, 16] and in many of the extensions cited above in defining a cone on which a positive operator was defined. A fixed point theorem due to Krasnosel'skii [19] was then applied to yield positive solutions for certain intervals of eigenvalues. In defining an appropriate cone, inequalities that provide lower bounds for positive functions as a function of the supremum norm have been applied. The inequality to which we refer may be stated as follows:

If $y \in C^{(2)}[0,1]$ is such that $y(t) \geq 0$, $0 \leq t \leq 1$, and $y''(t) \leq 0$, $0 \leq t \leq 1$, then

$$y(t) \geq \frac{1}{4} \max_{0 \leq s \leq 1} |y(s)|, \quad \frac{1}{4} \leq t \leq \frac{3}{4}. \tag{1.3}$$

Inequality (1.3) was recently generalized by Eloe and Henderson [5] in the following sense:

Let $n \geq 2$ and $2 \leq \ell \leq n-1$. If $y \in C^{(n)}[0,1]$ is such that

$$\begin{aligned} (-1)^{n-\ell} y^{(n)}(t) \geq 0, &\quad 0 \leq t \leq 1, \\ y^{(j)}(0) = 0, &\quad 0 \leq j \leq \ell - 1, \\ y^{(j)}(1) = 0, &\quad 0 \leq j \leq n - \ell - 1, \end{aligned}$$

then

$$y(t) \geq \frac{1}{4^m}\|y\|, \quad \frac{1}{4} \leq t \leq \frac{3}{4}, \tag{1.4}$$

where $\|y\| = \max_{0 \leq s \leq 1} |y(s)|$ and $m = \max\{\ell, n-\ell\}$.

An inequality analogous to (1.4) for a Green's function was also given in [6].

In a later paper, Eloe and Henderson [7] obtained a further generalization of (1.4) for solutions of differential inequalities satisfying the multipoint conjugate boundary conditions (1.2). In that same paper [7], an analogous inequality was also derived for a Green's function associated with $y^{(n)} = 0$ and (1.2). It is that generalization of (1.4) as it applies to solutions of (1.1), (1.2) which eventually leads to the main results of this paper.

In Section 2, we state the generalization of (1.4) as it applies to solutions of (1.1), (1.2). We also state the analogous inequality for a Green's function that will be used in defining a positive operator on a cone. The Krasnosel'skii fixed point theorem is also stated in that section. Then, in Section 3, we give an appropriate Banach space and construct a cone on which we apply the fixed point theorem to our positive operator, thus yielding solutions of (1.1), (1.2), for open intervals of eigenvalues.

2 Preliminaries

In this section, we state the Krasnosel'skii fixed pointed theorem to which we referred in the introduction. Prior to this, we will state the generalization of (1.4) as given in [7]. For notational purposes, set $\alpha_i = \sum_{j=i+1}^{k} n_j$, $1 \leq i \leq k-1$, let $S_i \subset (a_i, a_{i+1})$, $1 \leq i \leq k-1$, be defined by

$$S_i = [(3a_i + a_{i+1})/4, (a_i + 3a_{i+1})/4],$$

let

$$a = \min_{1 \leq i \leq k-1} \{a_{i+1} - a_i\},$$

and let

$$m = \max\{n - n_1, n - n_k\}.$$

Theorem 2.1 *Assume $y \in C^{(n)}[0,1]$ is such that $y^{(n)}(t) \geq 0$, $0 \leq t \leq 1$, and y satisfies the multipoint boundary conditions (1.2). Then, for each $1 \leq i \leq k-1$,*

$$(-1)^{\alpha_i} y(t) \geq \|y\| (\frac{a}{4})^m, \quad t \in S_i, \tag{2.1}$$

where $\|y\| = \max_{0\le t\le 1} |y(t)|$.

The Krasnosel'skii fixed point theorem will be applied to a completely continuous integral operator whose kernel, $G(t,s)$, is the Green's function for

$$y^{(n)} = 0, \quad 0 \le t \le 1, \tag{2.2}$$

satisfying (1.2). It is well-known [3] that

$$(-1)^{\alpha_i} G(t,s) > 0 \text{ on } (a_i, a_{i+1}) \times (0,1), \quad 1 \le i \le k-1. \tag{2.3}$$

For the remainder of the paper, for $0 < s < 1$, let $\tau(s) \in (0,1)$ be defined by

$$|G(\tau(s), s)| = \sup_{0\le t\le 1} |G(t,s)|, \tag{2.4}$$

so that, for each $1 \le i \le k-1$,

$$(-1)^{\alpha_i} G(t,s) \le |G(\tau(s), s)| \text{ on } [a_i, a_{i+1}] \times [0,1]. \tag{2.5}$$

Then in analogy to (2.1), Eloe and Henderson [7] proved the following inequality for $G(t,s)$.

Theorem 2.2 *Let $G(t,s)$ denote the Green's function for (2.2), (1.2). Then, for $0 < s < 1$ and $1 \le i \le k-1$,*

$$(-1)^{\alpha_i} G(t,s) \ge (\frac{a}{4})^m |G(\tau(s), s)|, \quad t \in S_i. \tag{2.6}$$

We mention that inequality (2.6) is closely related to inequalities derived for $G(t,s)$ by Pokornyi [25, 26]. Inequalities (2.5) and (2.6) are of fundamental importance in defining positive operators to which we will apply the following fixed point theorem [19].

Theorem 2.3 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3 Solutions in a Cone

In this section, we apply Theorem 2.3 to the eigenvalue problem (1.1), (1.2). The keys to satisfying the hypotheses of the theorem are in selecting a suitable cone and in inequalities (2.5) and (2.6). As is standard, $u \in C[0,1]$ is a solution of (1.1), (1.2) if, and only if,

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds, 0 \leq t \leq 1,$$

where $G(t,s)$ is the Green's function for (2.2), (1.2).

We let $\mathcal{B} = C[0,1]$, and for $y \in \mathcal{B}$, define $\|y\| = \sup_{0\leq t\leq 1}|y(t)|$. Then $(\mathcal{B}, \|\cdot\|)$ is a Banach space. The cone, $\mathcal{P}$, in which we shall exhibit solutions is defined by

$$\begin{aligned}\mathcal{P} = \{x \in \mathcal{B} \mid \text{ for } 1 \leq i \leq k-1, (-1)^{\alpha_i}x(t) \geq 0 \text{ on } [a_i, a_{i+1}],\\ \text{and } \min_{t\in S_i}(-1)^{\alpha_i}x(t) \geq (\tfrac{a}{4})^m\|x\|\}.\end{aligned}$$

Theorem 3.1 *Assume that conditions (A), (B) and (C) are satisfied. Then, for each λ satisfying,*

$$\frac{4^m}{a^m \sum_{i=1}^{k-1}\int_{S_i}|G(\frac{1}{2},s)|a(s)dsf_\infty} < \lambda < \frac{1}{\int_0^1|G(\tau(s),s)|a(s)dsf_0}, \tag{3.1}$$

there is at least one solution of (1.1), (1.2) belonging to $\mathcal{P}$.

Proof We remark that a special case in the arguments result when $f_\infty = \infty$. However, the modifications required for that case, in the following proof, are straightforward, and so we omit those details.

Let λ be given as in (3.1), and let $\epsilon > 0$ be such that

$$\frac{4^m}{a^m \sum_{i=1}^{k-1}\int_{S_i}|G(\frac{1}{2},s)|a(s)ds(f_\infty-\epsilon)} \leq \lambda \leq \frac{1}{\int_0^1|G(\tau(s),s)|a(s)ds(f_0+\epsilon)}.$$

We seek a fixed point of the integral operator $T : \mathcal{P} \to \mathcal{B}$ defined by

$$Tu(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds, \quad u \in \mathcal{P}. \tag{3.2}$$

First, let $u \in \mathcal{P}$ and let $t \in [0,1]$. Then, for some $1 \leq i \leq k-1$, we have $t \in [a_i, a_{i+1}]$, and by (2.3) and (2.5),

$$\begin{aligned}0 \leq (-1)^{\alpha_i}Tu(t) &= \lambda \int_0^1 (-1)^{\alpha_i}G(t,s)a(s)f(u(s))ds\\ &\leq \lambda \int_0^1 |G(\tau(s),s)|a(s)f(u(s))ds,\end{aligned}$$

so that

$$\|Tu\| \leq \lambda \int_0^1 |G(\tau(s),s)|a(s)f(u(s))ds. \tag{3.3}$$

Moreover, for $u \in \mathcal{P}$ and $t \in S_i$, $1 \leq i \leq k-1$, we have from (2.6) and (3.3),

$$\begin{aligned}
\min_{t\in S_i}(-1)^{\alpha_i}Tu(t) &= \min_{t\in S_i}\lambda\int_0^1(-1)^{\alpha_i}G(t,s)a(s)f(u(s))ds \\
&\geq (\frac{a}{4})^m\lambda\int_0^1|G(\tau(s),s)|a(s)f(u(s))ds \\
&\geq (\frac{a}{4})^m\|Tu\|.
\end{aligned}$$

As a consequence $T : \mathcal{P} \to \mathcal{P}$. The standard arguments can also be used to verify that T is completely continuous.

We begin with f_0. There exists an $H_1 > 0$ such that $f(x) \leq (f_0 + \epsilon)|x|$, for $0 < |x| < H_1$. So, if we choose $u \in \mathcal{P}$ with $\|u\| = H_1$, then from (2.5)

$$\begin{aligned}
|Tu(t)| &\leq \lambda\int_0^1|G(\tau(s),s)|a(s)f(u(s))ds \\
&\leq \lambda\int_0^1|G(\tau(s),s)|a(s)(f_0+\epsilon)|u(s)|ds \\
&\leq \lambda\int_0^1|G(\tau(s),s)|a(s)ds(f_0+\epsilon)\|u\| \\
&\leq \|u\|, 0 \leq t \leq 1.
\end{aligned}$$

So, $\|Tu\| \leq \|u\|$. We set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\}.$$

Then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.4}$$

Next, we consider f_∞. There exists an $\bar{H}_2 > 0$ such that $f(x) \geq (f_\infty - \epsilon)|x|$, for all $|x| \geq \bar{H}_2$. Let $H_2 = \max\{2H_1, (\frac{4}{a})^m\bar{H}_2\}$, and define

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\}.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_2$. Then, for each $1 \leq i \leq k-1$, $\min_{t\in S_i}(-1)^{\alpha_i}u(t) \geq (\frac{a}{4})^m\|u\| \geq \bar{H}_2$. Moreover, there exists $1 \leq i_0 \leq k-1$ such that $\frac{1}{2} \in [a_{i_0}, a_{i_0+1}]$. Then, by (2.3),

$$\begin{aligned}
(-1)^{\alpha_{i_0}} Tu(\frac{1}{2}) &= \lambda \int_0^1 (-1)^{\alpha_{i_0}} G(\frac{1}{2}, s)a(s)f(u(s))ds \\
&= \lambda \int_0^1 |G(\frac{1}{2}, s)|a(s)f(u(s))ds \\
&\geq \lambda \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)|a(s)f(u(s))ds \\
&\geq \lambda \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)|a(s)(f_\infty - \epsilon)|u(s)|ds \\
&\geq \lambda (\frac{a}{4})^m \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)|a(s)ds(f_\infty - \epsilon)\|u\| \\
&\geq \|u\|.
\end{aligned}$$

Thus, $\|Tu\| \geq \|u\|$. Hence,

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.5}$$

We apply part (i) of Theorem 2.3 in obtaining a fixed point, u, of T that belongs to $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$. The fixed point, u, is a desired solution of (1.1), (1.2), for the given λ. The proof is complete. □

Remark 3.1 *It follows from Theorem 3.1, if f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$), then (1.1), (1.2) has a solution, for each $0 < \lambda < \infty$.*

Theorem 3.2 *Assume that conditions (A), (B) and (C) are satisfied . Then, for each λ satisfying*

$$\frac{4^m}{a^m \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)|a(s)ds f_0} < \lambda < \frac{1}{\int_0^1 |G(\tau(s), s)|a(s)ds f_\infty}, \tag{3.6}$$

there is at least one solution of (1.1), (1.2) belonging to $\mathcal{P}$.

Proof Let λ be as in (3.6), and choose $\epsilon > 0$ such that

$$\frac{4^m}{a^m \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)|a(s)ds(f_0 - \epsilon)} \leq \lambda \leq \frac{1}{\int_0^1 |G(\tau(s), s)|a(s)ds(f_\infty + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2).

Beginning with f_0, there exists an $H_1 > 0$ such that $f(x) \geq (f_0 - \epsilon)|x|$, for $0 < |x| \leq H_1$. Choose $u \in \mathcal{P}$ with $\|u\| = H_1$. As in Theorem 3.1, there exists $1 \leq i_0 \leq k-1$ such that $\frac{1}{2} \in [a_{i_0}, a_{i_0+1}]$. Then

$$\begin{aligned}
(-1)^{\alpha_{i_0}} Tu(\frac{1}{2}) &= \lambda \int_0^1 (-1)^{\alpha_{i_0}} G(\frac{1}{2}, s) a(s) f(u(s)) ds \\
&= \lambda \int_0^1 |G(\frac{1}{2}, s)| a(s) f(u(s)) ds \\
&\geq \lambda \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)| a(s) f(u(s)) ds \\
&\geq \lambda \sum_{k=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)| a(s) (f_0 - \epsilon) |u(s)| ds \\
&\geq \lambda (\frac{a}{4})^m \sum_{i=1}^{k-1} \int_{S_i} |G(\frac{1}{2}, s)| a(s) ds (f_0 - \epsilon) \|u\| \\
&\geq \|u\|.
\end{aligned}$$

Therefore, if we let

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.7}$$

We now consider f_∞. There exists an $\bar{H}_2 > 0$ such that $f(x) \leq (f_\infty + \epsilon)|x|$, for all $|x| \geq \bar{H}_2$. There are the two cases, (a) f is bounded, or (b) f is unbounded.

For (a), suppose $N > 0$ is such that $f(x) \leq N$, for all $x \in \mathbb{R}$. Let $H_2 = \max\{2H_1,$
$N\lambda \int_0^1 |G(\tau(s), s)| a(s) ds\}$. Then, for $u \in \mathcal{P}$ with $\|u\| = H_2$,

$$\begin{aligned}
|Tu(t)| &\leq \lambda \int_0^1 |G(t, s)| a(s) f(u(s)) ds \\
&\leq \lambda N \int_0^1 |G(\tau(s), s)| a(s) ds \\
&\leq \|u\|, \quad 0 \leq t \leq 1.
\end{aligned}$$

Thus, $\|Tu\| \leq \|u\|$. So, if

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.8}$$

For case (b), let $H_2 > \max\{2H_1, \bar{H}_2\}$ be such that $f(x) \leq f(H_2)$, for $0 < |x| \leq H_2$. Let $u \in \mathcal{P}$ with $\|u\| = H_2$, and choose $t \in [0,1]$. Then, for some $1 \leq i \leq k-1$, $t \in [a_i, a_{i+1}]$, and by (2.5),

$$\begin{aligned}
(-1)^{\alpha_i} Tu(t) &= \lambda \int_0^1 (-1)^{\alpha_i} G(t,s) a(s) f(u(s)) ds \\
&= \lambda \int_0^1 |G(t,s)| a(s) f(u(s)) ds \\
&\leq \lambda \int_0^1 |G(\tau(s),s)| a(s) f(H_2) ds \\
&\leq \lambda \int_0^1 |G(\tau(s),s)| a(s) ds (f_\infty + \epsilon) H_2 \\
&= \lambda \int_0^1 |G(\tau(s),s)| a(s) ds (f_\infty + \epsilon) \|u\| \\
&\leq \|u\|,
\end{aligned}$$

so that $\|Tu\| \leq \|u\|$. For this case, if we let

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_2\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2.$$

Thus, regardless of the cases, an application of part (ii) of Theorem 2.3 yields a fixed point of T which belongs to $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$. This fixed point is a solution of (1.1), (1.2) corresponding to the given λ. The proof is complete. □

Remark 3.2 *We observe that, if f is sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$), then Theorem 3.2 yields a solution of (1.1), (1.2), for all* $0 < \lambda < \infty$.

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Paul W. Eloe
Department of Mathematics
University of Dayton
Dayton, Ohio 45469-2316 USA
E-mail address: eloe@saber.udayton.edu

Johnny Henderson
Department of Mathematics
Auburn University
Auburn, AL 36849 USA
E-mail address: hendej2@mail.auburn.edu