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Behaviour near the boundary for solutions of elasticity systems *

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Abstract

In this article we study the behaviour near the boundary for weak solutions of the system

$$u'' - \mu\Delta u - (\lambda+\mu)\nabla(\alpha(x)\,\operatorname{div} u) = h\,,$$

with $u(x,t)=0$ on the boundary of a domain $\Omega \in \mathbf{R}^n$, and $u(x,0)=u^0$, $u'(x,0)=u^1$ in Ω. We show that the Sobolev norm of the solution in an ε-neighbourhood of the boundary can be estimated independently of ε.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with a C^3-boundary Γ, and let $\nu(x)$ be the unit exterior normal of Γ at a point x. For $T>0$, we denote by Q the finite cylinder $\Omega\times]0,T[$, and by Σ its lateral boundary $\Gamma\times]0,T[$. For an open subset Γ_0 of Γ, Σ_0 denotes $\Gamma_0\times]0,T[$. For each $\varepsilon>0$, ω_ε denotes the ε-neighbourhood of Γ_0 in Ω defined by

$$\omega_\varepsilon = \bigcup_{x\in\Gamma_0} B(x,\varepsilon)\cap\Omega\,,$$

where $B(x,\varepsilon)$ is the open ball with center x and radius ε. Functional spaces and their inner products are denoted as follows

$$H=[L^2(\Omega)]^n, \qquad (u,v)_H=\sum_{i=1}^n (u_i,v_i)_{L^2(\Omega)} \quad \forall u,v\in H$$

$$V=[H_0^1(\Omega)]^n, \qquad ((u,v))_V=\sum_{i=1}^n (\nabla u_i,\nabla v_i)_{L^2(\Omega)} \quad \forall u,v\in V\,.$$

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This article is concerned with the behaviour near the boundary for weak solutions of the system

$$\begin{aligned} &u'' - \mu\Delta u - (\lambda+\mu)\nabla(\alpha(x)\operatorname{div} u) = h \quad \text{in} \quad Q \\ &u = 0 \quad \text{on} \quad \Sigma \\ &u(0) = u^0, \quad u'(0) = u^1 \quad \text{in} \quad \Omega\,, \end{aligned} \tag{1.1}$$

with

$$\{u^0, u^1, h\} \in V \times H \times L^1(0,T;H)\,, \tag{1.2}$$

where $\lambda, \mu > 0$ are the Lamé constants, and $\alpha \in C^1(\overline{\Omega})$ is a real function such that for all x in $\overline{\Omega}$ and some α_0, $\alpha(x) \geq \alpha_0 > 0$.

Recall that if u is the unique solution of the above system, for the energy function

$$E(t) = \frac{1}{2}|u'(t)|_H^2 + \frac{\mu}{2}||u(t)||_V^2 + \frac{(\lambda+\mu)}{2}|\alpha^{\frac{1}{2}}(x)\operatorname{div} u(t)|^2_{L^2(\Omega)}$$

there exists a positive constant c such that

$$\sup_{t\in[0,T]} E(t) \leq cE_0\,,$$

where

$$E_0 = ||u_0||_V^2 + |u^1|_H^2 + ||h||^2_{L^1(0,T;H)}. \tag{1.3}$$

We shall prove that the H norm of the solution u of (1.1) in an ε-neighbourhood of the boundary can be estimated independently of ε. It will be done by studying first the behaviour of ∇u in the same neighbourhood. We use the method developed by J.P. Puel and C. Fabre [7] for solutions of the wave equation, which is an extension of the multipliers method introduced by F. Rellich for elliptic equations and used by C. Morawetz in hyperbolic equations. Their method was also applied to Schrodinger equations, and to equations for vibrating beams (cf. C. Fabre [4] and C. Fabre - J. P. Puel [6]).

The interest in the above result lies in the fact that this estimate, combined with other results, allow us to obtain the boundary control as the limit of internal controls. Then, provided that the internal controls exist we obtain the boundary control passing to the limit as $\varepsilon \to 0$. This is a way of getting boundary control when we only have the guarantee that the system is internal controllable.

To use this kind of argument is a powerful tool since we act where it is convenient and then pass to the limit. In this direction we can cite the work by M. E. Bradley and M. A. Horn [1] where they control the Von Kármán system $w'' - \gamma^2\Delta w' + \Delta^2 w + b(x)w' = [w, \chi(w)]$, by showing that the boundary control which stabilizes its solution when $\gamma \neq 0$ also stabilizes the solution to the system obtained in the limit as $\gamma \to 0$. Here γ is a parameter which is proportional to the thickness of the Von Kármán plate.

2 Geometrical Properties of Ω

Using the fact that Ω is a bounded domain of $\mathbf{R}^n$ with a C^3-boundary Γ, we obtain the following lemma.

Lemma 2.1 *There exist open sets $U_1, \dots U_m$ and a positive constant ε_0 which satisfy*

- *For each ε in $]0, \varepsilon_0[$, $\overline{\omega_\varepsilon} \subset \bigcup_{i=1}^m U_i$, where $\overline{\omega_\varepsilon}$ denotes the closure of ω_ε in $\mathbf{R}^n$.*
- *For each $x \in \omega_\varepsilon \cap U_i$, there exists a unique $(y, z) \in (\Gamma \cap U_i) \times]0, \varepsilon[$ such that $x = y - z\nu(y)$; $i = 1, ..., m$.*
- *There are functions $F_i^{-1} : x \to (w, z)$, C^2-diffeomorphisms defined from $\omega_\varepsilon \cap U_i$ onto their image; $i = 1, \dots, m$.*

Proof. Due to the regularity of Γ, which is the C^3-boundary of Ω, by M. Do Carmo [2] (section 2.7, proposition 1) for each point $p \in \Gamma$ and $X_p : U \to \Gamma$, a C^3-parametrization of a neighbourhood of p, there exist a neighbourhood $W_p \subset X_p(U)$ of p in Γ and $\varepsilon_p > 0$ such that the segments of the normal lines through points $q \in W_p$, with center at q and length $2\varepsilon_p$, are disjoint; that is, W_p has a tubular neighbourhood.

Considering $0 < 2r_p < \varepsilon_p$ where $\overline{B_{2r_p}(p)} \subset B_p$, $W_p = B_p \cap \Gamma$ and B_p is an open set of $\mathbf{R}^n$ we obtain:

$$\overline{\Omega} = \Omega \cup \Gamma \subset \Omega \cup \left(\bigcup_{p \in \Gamma} B_{r_p}(p) \right).$$

Using the compactness of $\overline{\Omega}$, there exist $p_1, \dots, p_m$ such that

$$\Gamma \subset \bigcup_{i=1}^{m} B_{r_{p_i}}(p_i).$$

Defining

$$U_i = B_{2r_{p_i}}(p_i); \quad i = 1, \dots, m \quad \text{and} \quad \varepsilon_0 = \frac{1}{2} \min \{r_{p_1}, \dots, r_{p_m}\},$$

we conclude that for every $\varepsilon \in]0, \varepsilon_0]$, $\overline{\omega_\varepsilon} \subset \bigcup_{i=1}^m U_i$.

On the other hand, we observe that for each $p_i \in \Gamma$ the related C^3- parametrization X_i implies that the function

$$F_i : X_i^{-1}(\Gamma \cap U_i) \times]0, \varepsilon[\to \omega_\varepsilon \cap U_i$$
$$(w, z) \mapsto F_i(w, z) = X_i(w) - zN_i(w),$$

where $\varepsilon \in]0, \varepsilon_0[$ and $N_i(w)$ is the normal vector at point $X_i(w)$, is a C^2-diffeomorphism.

Moreover, according to the choice of the open sets U_i we conclude that for every $x \in \omega_\varepsilon \cap U_i$ where $\varepsilon \in]0, \varepsilon_0]$ there exist a unique normal projection y of x on Γ and unique $z \in]0, \varepsilon[$ such that $x = y - z\nu(y)$. $\square$

Remark. It follows from the above lemma that if $x \in \omega_\varepsilon \cap U_i$ for sufficiently small ε, the normal projection of x onto Γ is uniquely defined. Furthermore,

$$p(x) = X_i(\omega) = F_i(\omega, 0)\,,$$

where $F_i(w, z) = X_i(w) - zN_i(w)$.

Lemma 2.2 *Let* $\det JF_i(w, z)$ *denote the determinant of the Jacobian matrix of* F_i *at* (w, z)*. Then*

(i) there exist positive constants ε_0, m, M*, such that* $\forall (w, z) \in X_i^{-1}(\Gamma \cap U_i) \times [0, \varepsilon_0]$,
$$m \leq |\det JF_i(w, z)| \leq M$$

(ii) $|\det JF_i(w, 0)| = 1$, $\forall w \in X_i^{-1}(\Gamma \cap U_i)$

(iii) the function $(w, z) \to |\det JF_i(w, z)|$ *is a* C^1*-function.*

(iv) for function v *on* $\omega_\varepsilon \cap U_i$*, define* $\hat{v}(w, z) = (v \circ F_i)(w, z)$*. If* $v \in H^1(\omega_\varepsilon \cap U_i)$ *then*
$$\frac{\partial}{\partial z}\hat{v}(w, z) = -\nabla v(F_i(w, z)) \cdot \nu(F_i(w, 0)\,. \tag{2.1}$$

Proof. Since $F_i(w, z) = X_i(w) - zN_i(w)$ it follows that $|\det JF_i(\omega, 0)| = ||N_i(\omega)||$ and consequently $|\det JF_i(\omega, 0)| = 1$, $\forall \omega \in X_i^{-1}(\Gamma \cap U_i)$. Besides, taking into account the regularity of the boundary of Ω, we get that $\det JF_i(\omega, z)$ is a C^1-function. For a $\varepsilon_i > 0$ small enough we obtain that $\det JF_i(\omega, z)$ does not change its sign on $]0, \varepsilon_i[$, which allow us to conclude (iii). Combining the two results obtained in (ii) and (iii) we get (i). Finally, from the regularity of the functions $F_i(w, z)$, observing that $\frac{\partial}{\partial z}F_{i_k}(w, z) = -N_{i_k}(w)$ and from the identity

$$\frac{\partial}{\partial z}(v \circ F_i)(w, z) = \sum_{k=1}^{n} \frac{\partial v}{\partial x_k}(F_i(w, z))\frac{\partial}{\partial z}F_{i_k}(w, z)\,,$$

which holds for regular functions v, we obtain (iv). $\square$

3 Fundamental Identity

In this section we prove an identity that is essential in the proof of our main result.

Lemma 3.1 *If $u = (u_1, ..., u_n)$ is the solution of (1.1)-(1.2), for every vector valued function $g \in W^{2,\infty}(\Omega, \mathbf{R}^n)$ collinear to the normal vector on Γ, we have*

$$\begin{aligned}
& 2\mu \sum_{i,j,k=1}^{n} \int_Q \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial g_k}{\partial x_j}\, dx\, dt + \mu \sum_{i=1}^{n} \int_Q (\nabla u_i \cdot \nabla(\operatorname{div} g)) u_i\, dx\, dt \\
& + 2(\lambda+\mu) \sum_{i,j=1}^{n} \int_Q \alpha(x) \operatorname{div} u \frac{\partial u_i}{\partial x_j} \frac{\partial g_j}{\partial x_i}\, dx\, dt \\
& + (\lambda+\mu) \int_Q \alpha(x) \operatorname{div} u (u \cdot \nabla(\operatorname{div} g))\, dx\, dt \qquad (3.1) \\
= \; & 2 \sum_{i=1}^{n} \int_Q h_i (\nabla u_i \cdot g)\, dx\, dt \; + \sum_{i=1}^{n} \int_Q h_i u_i divg\, dx\, dt \\
& + \mu \sum_{i=1}^{n} \int_\Sigma |\frac{\partial u_i}{\partial \nu}|^2 g \cdot \nu d\Gamma dt \; + \; (\lambda+\mu) \int_\Sigma \alpha(x)(divu)^2 g \cdot \nu d\Gamma\, dt \\
& + (\lambda+\mu) \int_Q \nabla \alpha \cdot g (\operatorname{div} u)^2\, dx\, dt \; - \; 2 \sum_{i=1}^{n} \int_\Omega u_i'(T)(\nabla u_i(T) \cdot g)\, dx \\
& + 2 \sum_{i=1}^{n} \int_\Omega u_i'(0)(\nabla u_i(0) \cdot g)\, dx - \sum_{i=1}^{n} \int_\Omega u_i'(T) u_i(T) \operatorname{div} g\, dx \\
& + \sum_{i=1}^{n} \int_\Omega u_i'(0) u_i(0) \operatorname{div} g\, dx\, .
\end{aligned}$$

Proof. For initial data $\{u^0, u^1, h\} \in \left[H_0^1(\Omega) \times H^2(\Omega)\right]^n \times V \times L^1(0,T;V)$ let

$$2\nabla u \cdot g = (2\nabla u_1 \cdot g, \ldots, 2\nabla u_n \cdot g)\ ,$$

where

$$2\nabla u_i \cdot g = 2 \sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k} g_k\, .$$

Multiplying (1.1) by $2\nabla u \cdot g$ we obtain

$$\begin{aligned}
& \sum_{i=1}^{n} \int_Q u_i''(2\nabla u_i \cdot g)\, dx\, dt - \mu \sum_{i=1}^{n} \int_Q \Delta u_i (2\nabla u_i \cdot g)\, dx\, dt \\
& -(\lambda+\mu) \sum_{i=1}^{n} \int_Q \frac{\partial}{\partial x_i} \left(\alpha(x) \operatorname{div} u\right) \left(2\nabla u_i \cdot g\right)\, dx\, dt \qquad (3.2)
\end{aligned}$$

$$= \sum_{i=1}^{n} \int_Q h_i \left(2\nabla u_i \cdot g\right) \, dx \, dt \,.$$

Integrating by parts with respect to t, by Gauss Theorem,

$$\begin{aligned} & 2\sum_{i=1}^{n} \int_Q u_i'' \left(\nabla u_i \cdot g\right) \, dx \, dt \\ = \;& 2\sum_{i=1}^{n} \int_\Omega u_i'(T) \left(\nabla u_i(T) \cdot g\right) \, dx - 2\sum_{i=1}^{n} \int_\Omega u_i'(0) \left(\nabla u_i(0) \cdot g\right) dx \\ & + \sum_{i=1}^{n} \int_Q \left(u_i'\right)^2 \operatorname{div} g \, dx \, dt \,. \end{aligned} \tag{3.3}$$

Using the fact that

$$\frac{\partial u_i}{\partial x_k} = \nu_k \frac{\partial u_i}{\partial \nu} \quad \text{on} \quad \Gamma \,, \tag{3.4}$$

by Green and Gauss Theorems we have

$$\begin{aligned} & -\mu \sum_{i=1}^{n} \int_Q \Delta u_i (2\nabla u_i \cdot g) \, dx \, dt \\ = \;& -\mu \sum_{i=1}^{n} \int_Q \left|\nabla u_i\right|^2 \operatorname{div} g \, dx \, dt + 2\mu \sum_{i,j,k=1}^{n} \int_Q \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial g_k}{\partial x_j} \, dx \, dt \\ & -\mu \sum_{i=1}^{n} \int_\Sigma \left(\frac{\partial u_i}{\partial \nu}\right)^2 g \cdot \nu \, d\Gamma dt \,. \end{aligned} \tag{3.5}$$

Finally, by (3.4) and by Gauss theorem we have

$$\begin{aligned} & -(\lambda+\mu) \sum_{i=1}^{n} \int_Q \frac{\partial}{\partial x_i} \left(\alpha(x) \operatorname{div} u\right) \left(2\nabla u_i \cdot g\right) \, dx \, dt \\ = \;& 2(\lambda+\mu) \sum_{k=1}^{n} \int_Q \alpha(x) \operatorname{div} u \frac{\partial}{\partial x_k} (\operatorname{div} u) g_k \, dx \, dt \\ & +2(\lambda+\mu) \sum_{i,k=1}^{n} \int_Q \alpha(x) \operatorname{div} u \frac{\partial u_i}{\partial x_k} \frac{\partial g_k}{\partial x_i} \, dx \, dt \,. \\ & -2(\lambda+\mu) \int_\Sigma \alpha(x) \left(\operatorname{div} u\right)^2 g \cdot \nu \, d\Gamma \, dt \,. \\ = \;& -(\lambda+\mu) \int_Q \left(\operatorname{div} u\right)^2 \left(\nabla \alpha(x) \cdot g\right) dx \, dt - (\lambda+\mu) \int_Q \left(\operatorname{div} u\right)^2 \alpha(x) \operatorname{div} g \, dx \, dt \\ & +(\lambda+\mu) \int_\Sigma \alpha(x) \left(\operatorname{div} u\right)^2 g \cdot \nu \, d\Gamma dt \,. \end{aligned} \tag{3.6}$$

$$+2(\lambda+\mu)\sum_{i,k=1}^{n}\int_Q \alpha(x)\operatorname{div}u\frac{\partial u_i}{\partial x_k}\frac{\partial g_k}{\partial x_i}\,dx\,dt\,.$$
$$-2(\lambda+\mu)\int_\Sigma \alpha(x)\left(\operatorname{div}u\right)^2 g\cdot\nu\,d\Gamma dt\,.$$

Replacing (3.3), (3.5) and (3.6) in (3.2) it follows that

$$\begin{aligned}
&2\sum_{i=1}^{n}\int_\Omega u_i'(T)\left(\nabla u_i(T)\cdot g\right)\,dx-2\sum_{i=1}^{n}\int_\Omega u_i'(0)\left(\nabla u_i(0)\cdot g\right)dx \qquad (3.7)\\
&+\sum_{i=1}^{n}\int_Q \left(u_i'\right)^2\operatorname{div}g\,dx\,dt-\mu\sum_{i=1}^{n}\int_Q\left|\nabla u_i\right|^2\operatorname{div}g\,dx\,dt\\
&+2\mu\sum_{i,j,k=1}^{n}\int_Q\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\frac{\partial g_k}{\partial x_j}\,dx\,dt-\mu\sum_{i=1}^{n}\int_\Sigma\left(\frac{\partial u_i}{\partial \nu}\right)^2 g\cdot\nu\,d\Gamma dt\\
&-(\lambda+\mu)\int_Q\left(\operatorname{div}u\right)^2\left(\nabla\alpha(x)\cdot g\right)dx\,dt-(\lambda+\mu)\int_Q\left(\operatorname{div}u\right)^2\alpha(x)\operatorname{div}g\,dx\,dt\\
&+2(\lambda+\mu)\sum_{i,k=1}^{n}\int_Q\alpha(x)\operatorname{div}u\frac{\partial u_i}{\partial x_k}\frac{\partial g_k}{\partial x_i}\,dxdt-(\lambda+\mu)\int_\Sigma\alpha(x)\left(\operatorname{div}u\right)^2 g\cdot\nu\,d\Gamma dt\\
=\;&\sum_{i=1}^{n}\int_Q h_i\left(2\nabla u_i\cdot g\right)\,dx\,dt\,.
\end{aligned}$$

Let $u\operatorname{div}g=(u_1\operatorname{div}g,\dots,u_n\operatorname{div}g)$, where $u_i\operatorname{div}g=\sum_{j=1}^{n}u_i\frac{\partial g_j}{\partial x_j}$. Now, multiplying (1.1) by $u\operatorname{div}g$ we get

$$\begin{aligned}
&\sum_{i=1}^{n}\int_Q u_i''u_i\operatorname{div}g\,dx\,dt-\mu\sum_{i=1}^{n}\int_Q\Delta u_i u_i\operatorname{div}g\,dx\,dt\\
&-(\lambda+\mu)\sum_{i=1}^{n}\int_Q\frac{\partial}{\partial x_i}\left(\alpha(x)\operatorname{div}u\right)u_i\operatorname{div}g\,dx\,dt \qquad (3.8)\\
=\;&\sum_{i=1}^{n}\int_Q h_iu_i\operatorname{div}g\,dx\,dt\,.
\end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int_Q u_i''u_i\operatorname{div}g\,dx\,dt \qquad (3.9)\\
=\;&\sum_{i=1}^{n}\int_\Omega u_i'(T)u_i(T)\operatorname{div}g\,dx-\sum_{i=1}^{n}\int_\Omega u_i'(0)u_i(0)\operatorname{div}g\,dx\,.\\
&-\sum_{i=1}^{n}\int_Q\left|u_i'\right|^2\operatorname{div}g\,dx\,dt\,.
\end{aligned}$$

and by Green's Theorem

$$\begin{aligned} & -\mu\sum_{i=1}^n\int_Q \Delta u_i u_i \operatorname{div} g\,dx\,dt \qquad (3.10)\\ = &\quad \mu\sum_{i=1}^n\int_Q |\nabla u_i|^2 \operatorname{div} g\,dx\,dt+\mu\sum_{i=1}^n\int_Q \nabla u_i\cdot\nabla(\operatorname{div} g)\,u_i\,dx\,dt. \end{aligned}$$

Since $u=0$ on Γ, by Gauss Theorem we have

$$\begin{aligned} & -(\lambda+\mu)\sum_{i=1}^n\int_Q \frac{\partial}{\partial x_i}(\alpha(x)\operatorname{div} u)\,u_i \operatorname{div} g\,dx\,dt \qquad (3.11)\\ = &\quad (\lambda+\mu)\int_Q \alpha(x)(\operatorname{div} u)^2 \operatorname{div} g\,dxdt\\ &\quad +(\lambda+\mu)\int_Q \alpha(x)(\operatorname{div} u)u\cdot\nabla(\operatorname{div} g)\,dx\,dt\,. \end{aligned}$$

Then, from (3.8)-(3.11), it follows that

$$\begin{aligned} &\sum_{i=1}^n\int_\Omega u_i'(T)u_i(T)\,div\,g\,dx-\sum_{i=1}^n\int_\Omega u_i'(0)u_i(0)\operatorname{div} g\,dx\\ &-\sum_{i=1}^n\int_Q |u_i'|^2\operatorname{div} g\,dx\,dt+\mu\sum_{i=1}^n\int_Q|\nabla u_i|^2\operatorname{div} g\,dx\,dt \qquad (3.12)\\ &+\mu\sum_{i=1}^n\int_Q\nabla u_i\cdot\nabla(\operatorname{div} g)\,u_i\,dx\,dt+(\lambda+\mu)\int_Q\alpha(x)(\operatorname{div} u)^2\operatorname{div} g\,dx\,dt\\ &+(\lambda+\mu)\int_Q\alpha(x)(\operatorname{div} u)u\cdot\nabla(\operatorname{div} g)\,dx\,dt\\ &=\sum_{i=1}^n\int_Q h_iu_i\operatorname{div} g\,dx\,dt \end{aligned}$$

Adding (3.7) and (3.12), and using that $\left[H_0^1(\Omega)\cap H^2(\Omega)\right]^n\times V\times L^1(0,T;V)$ is dense in $V\times H\times L^1(0,T;V)$, we obtain (3.1). □

Lemma 3.2 *Let u be the solution of (1.1) with initial data satisfying (1.2), and let $v_r=\theta_r u$, where $\{\theta_r\}_{1\le r\le m}$ is a C^∞ partition of the unity relative to the open sets $U_1,\dots,U_m$. Then $v_r=(v_{r_1},\dots,v_{r_n})$ is the solution to*

$$\begin{aligned} & v_r''-\mu\Delta v_r-(\lambda+\mu)\nabla(\alpha(x)\operatorname{div} v_r)=h_r \quad \textit{in } Q\\ & v_r=0 \quad \textit{on} \quad \Sigma \qquad (3.13)\\ & v_r(0)=\theta_r u^0,\ v_r'(0)=\theta_r u^1 \quad \textit{on} \quad \Omega\,, \end{aligned}$$

where

$h_r = \theta_r h - 2\mu\nabla\theta_r \cdot \nabla u - \mu u\Delta\theta_r - (\lambda+\mu)\alpha(x) div u \nabla\theta_r - (\lambda+\mu)\nabla(\alpha(x)u\cdot\nabla\theta_r)$

and $supp\, v_r \subset U_r \times [0,T]$. *Furthermore, if* ε_0 *is the minimum between the two epsilons found in Lemmas 2.1 and 2.2, the function*

$$G(\varepsilon) = \begin{cases} \frac{1}{\varepsilon}\int_0^T \int_{\omega_\varepsilon \cap U_r} \sum_{i=1}^n |\nabla v_{r_i}(x,t)\cdot \nu(p(x))|^2\, dx\, dt, & \varepsilon \in]0,\varepsilon_0] \\ \int_0^T \int_{\Gamma\cap U_r} \sum_{i=1}^n \left|\frac{\partial v_{r_i}}{\partial \nu}\right|^2 d\Gamma\, dt & \varepsilon = 0\,, \end{cases}$$

where $F_r(W_r \times]0,\varepsilon[) = \omega_\varepsilon \cap U_r$, *is continuous on* $[0,\varepsilon_0]$.

Proof. The continuity of G on $[0,\varepsilon_0]$ follows from Lemma 2.2. □

Lemma 3.3 *Let* δ *be a positive number. Then there exists a real number* $\gamma \in]0,\delta[$, *and there exist positive decreasing functions* $\rho_\varepsilon \in W^{2,\infty}(0,\varepsilon)$, *where* $\varepsilon = \delta+\gamma$, *such that*

$$\rho_\varepsilon(\varepsilon) = 0 \quad \rho'_\varepsilon(\varepsilon) = 0 \quad \textit{and} \quad \rho'_\varepsilon = -\frac{1}{\delta}\,, \quad \textit{in} \quad [0,\delta]\,. \tag{3.14}$$

Furthermore,

$$||\rho_\varepsilon||_{L^\infty(0,\varepsilon)} \le C_1,\ \ ||\rho'_\varepsilon||_{L^\infty(0,\varepsilon)} \le \frac{C_2}{\varepsilon},\ \ ||\rho''_\varepsilon||_{L^\infty(0,\varepsilon)} \le \frac{C_3}{\varepsilon^2} \tag{3.15}$$

for positive constants C_1, C_2, C_3, *and*

$$\frac{\gamma}{2}\int_0^\varepsilon |\rho''_\varepsilon(z)|^2 z^2 dz = \frac{9}{16}\,. \tag{3.16}$$

Proof. Let γ be the positive value when solving for x in

$$1 + \frac{x}{\delta} + \frac{1}{3}(\frac{x}{\delta})^2 = \frac{9}{8}\,. \tag{3.17}$$

Then $\gamma = \left(\sqrt{\frac{7}{6}} - 1\right)\frac{3}{2}\delta$ which belongs to the interval $]0,\delta[$. Put $\varepsilon = \gamma+\delta$, and define $\rho_\varepsilon : [0,\varepsilon] \to R$, by

$$\rho_\varepsilon(z) = \begin{cases} 1 - \frac{\gamma}{2\delta} - \frac{z}{\delta}\,, & z \in [0,\delta] \\ \frac{1}{2\gamma\delta}(\delta+\gamma-z)^2\,, & z \in [\delta,\delta+\gamma]\,. \end{cases}$$

Then $\rho_\varepsilon \in W^{2,\infty}(0,\varepsilon)$ and satisfies (3.14),(3.15). To show that ρ_ε satisfies (3.16), we use (3.17) as follows

$$\begin{aligned} \frac{\gamma}{2}\int_0^\varepsilon |\rho''_\varepsilon(z)|^2 z^2\, dz &= \frac{\gamma}{2}\int_\gamma^{\delta+\gamma} \frac{1}{\gamma^2\delta^2} z^2\, dz \\ &= \frac{1}{2}\left(1 + \frac{\gamma}{\delta} + \frac{\gamma^2}{3\delta^2}\right) \\ &= \frac{9}{16}\,. \end{aligned}$$

4 Behaviour of the Solution u in $\omega_\varepsilon\times]0,T[$

Our goal in this section, which contains the main result of this work, is to study the behaviour of the solution u of the elasticity system given in (1.1) in $\omega_\varepsilon\times]0,T[$.

Theorem 4.1 *There exist positive constants C and ε_0 such that every solution u of (1.1) where (1.2) holds with $\alpha_1 < \mu/(3n(\lambda+\mu))$ satisfies*

$$\frac{1}{\varepsilon}\sum_{i=1}^{n}\int_0^T\int_{\omega_\varepsilon}|\nabla u_i(x,t)\cdot\nu(p(x))|^2\,dx\,dt \le cE_0 \quad \forall\varepsilon\in]0,\varepsilon_0]\,.$$

Furthermore, C and ε_0 depend only on the positive number T, the function $\alpha(x)$, the geometry of Ω, and the Lamé constants.

Proof. Without loss of generality assume that $\Gamma_0=\Gamma$. Let u be the solution of (1.1) and define $v_r=\theta_r u$ where $\{\theta_r\}_{1\le r\le m}$ is a C^∞ partition of the unity relative to the open sets $U_1,\dots,U_m$ given in lemma 2.1. According to Lemma 3.2, v_r is the solution of (3.13) and the function G is continuous on $[0,\varepsilon_0]$. Let $\delta_0\in[0,\varepsilon_0]$ be a value such that

$$G(\delta_0)=\max_{\varepsilon\in[0,\varepsilon_0]}G(\varepsilon)\,.$$

If $\delta_0=0$, we obtain

$$\max_{\varepsilon\in[0,\varepsilon_0]}G(\varepsilon)=\sum_{i=1}^{n}\int_0^T\int_{\Gamma\cap U_r}\left|\frac{\partial v_{r_i}}{\partial\nu}\right|^2 d\Gamma\,dt\,.$$

But, considering the trace theory developed by M. Milla Miranda [9], we get

$$\frac{\partial v_{r_i}}{\partial\nu}=\theta|_\Gamma\frac{\partial u_i}{\partial\nu}$$

and consequently,

$$\max_{\varepsilon\in[0,\varepsilon_0]}G(\varepsilon)\le c^*\sum_{i=1}^{n}\int_0^T\int_\Gamma\left|\frac{\partial u_i}{\partial\nu}\right|^2 d\Gamma\,dt\,.$$

On the other hand, proceeding as in J. L. Lions [8] (chapter IV - pages 224, 225), that is, using the multiplier method, we obtain that $\frac{\partial u_i}{\partial\nu}\in L^2(\Sigma)$, $i=1,2,...,n$ and

$$\sum_{i=1}^{n}\int_0^T\int_\Sigma\left|\frac{\partial u_i}{\partial\nu}\right|^2 d\Gamma dt\le cE_0\,.$$

Then, we conclude that

$$\max_{\varepsilon\in[0,\varepsilon_0]}G(\varepsilon)\le cE_0\,.$$

If $\delta_0 \in]\frac{\varepsilon_0}{2}, \varepsilon_0[$, we have

$$\max_{\varepsilon\in[0,\varepsilon_0]} G(\varepsilon) \leq \frac{2}{\varepsilon_0}\sum_{i=1}^{n}\int_0^T\int_{\omega_{\delta_0}\cap U_r} |\nabla v_{r_i}(x,t)|^2\,dx\,dt \leq cE_0$$

where c, in both cases, is the desired constant. We can then restrict ourselves to the case $0 < \delta_0 < \frac{\varepsilon_0}{2}$.

As $\delta_0 > 0$, according to lemma 3.3, there exists a real number $\gamma \in]0, \delta_0[$. Put $\varepsilon = \gamma + \delta_0$, then there exists a decreasing function $\rho_\varepsilon \in W^{2,\infty}(0,\varepsilon)$ such that

$$\rho_\varepsilon(\varepsilon) = 0, \quad \rho_\varepsilon'(\varepsilon) = 0, \quad \text{and} \quad \rho_\varepsilon' = -\frac{1}{\delta_0} \ \ in\ [0,\delta_0] \tag{4.1}$$

and (3.15) and (3.16) are satisfied. Let us consider now the following vector valued function

$$g_\varepsilon(x) = \rho_\varepsilon(z)\nu(y), \quad x\in\omega_\varepsilon\cap U_r, \quad \text{where} \quad x = y - z\nu(y). \tag{4.2}$$

Noting that $\frac{\partial z}{\partial x_k} = -\nu_k$ we have

$$\frac{\partial g_{\varepsilon_j}}{\partial x_k}(x) = -\rho_\varepsilon'(z)\nu_k(p(x))\nu_j(p(x)) + \rho_\varepsilon(z)\frac{\partial \nu_j}{\partial x_k}(p(x)), \quad x\in\omega_\varepsilon\,.$$

So, we obtain

$$\operatorname{div} g_\varepsilon(x) = -\rho_\varepsilon'(z) + \rho_\varepsilon(z)\operatorname{div}\nu(p(x)), \quad x\in\omega_\varepsilon$$

and

$$\nabla(\operatorname{div} g_\varepsilon)(x) = \rho_\varepsilon''(z)\nu(p(x)) - \rho_\varepsilon'(z)\operatorname{div}(\nu(p(x)) + \rho_\varepsilon(z)\nabla(\operatorname{div}\nu(p(x)))\,.$$

Next, we use identity (3.1) taking as function g the family of functions g_ε defined in (4.2), replacing u by v_r the solution of (3.13) and observing that $\operatorname{supp} v_r \subset U_r\times[0,T]$.

From the above equalities, Lemma 2.2, and taking into account that $F_r(W_r\times]0,\varepsilon[) = \omega_\varepsilon\times U_r$, we get by lemma 3.1 the following expression

$$\begin{aligned}
&-2\mu\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^\varepsilon \rho_\varepsilon'(z)\left|\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\right|^2|\det JF_r(w,z)|dz\,dw\,dt \qquad (4.3)\\
&-\mu\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^\varepsilon \rho_\varepsilon''(z)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\hat v_{r_i}(w,z,t)|\det JF_r(w,z)|dz\,dw\,dt\\
&+(\lambda+\mu)\sum_{i=1}^{n}\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho_\varepsilon''(z)\operatorname{div} v_r(v_r\cdot\nu(p(x)))\,dx\,dt\\
&= R_1 + R_2\,.
\end{aligned}$$

where R_1 is given by

$$
\begin{aligned}
&-2\mu\sum_{i,j,k=1}^{n}\int_0^T\int_{\omega_\varepsilon\cap U_r}\rho_\varepsilon(z)\frac{\partial v_{r_i}}{\partial x_j}\frac{\partial v_{r_i}}{\partial x_k}\frac{\partial \nu_j}{\partial x_k}(p(x))\,dx\,dt\\
&-\mu\sum_{i=1}^{n}\int_0^T\int_{\omega_\varepsilon\cap U_r}\rho_\varepsilon(z)(\nabla v_{r_i}\cdot\nabla(\operatorname{div}\nu(p(x))))v_{r_i}\,dx\,dt\\
&-2(\lambda+\mu)\sum_{i,j=1}^{n}\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho_\varepsilon(z)\operatorname{div}v_r\frac{\partial v_{r_i}}{\partial x_j}\frac{\partial \nu_j}{\partial x_i}(p(x))\,dx\,dt\\
&-(\lambda+\mu)\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho_\varepsilon(z)\operatorname{div}v_r(v_r\cdot\nabla(\operatorname{div}\nu(p(x))))\,dx\,dt\\
&-2\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^\varepsilon\rho_\varepsilon(z)\hat h_{r_i}(w,z,t)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\hat v_{r_i}(w,z,t)|\det JF_r(w,z)|dz\,dw\,dt\\
&+\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^\varepsilon\rho_\varepsilon(z)\operatorname{div}\nu(F_r(w,0))\hat h_{r_i}(w,z,t)|\det JF_r(w,z)|dz\,dw\,dt\\
&+\int_\Sigma\{\mu\sum_{i=1}^{n}|\frac{\partial v_{r_i}}{\partial\nu}|^2+(\lambda+\mu)\alpha(x)|\operatorname{div}v_r|^2\}\rho_\varepsilon(0)d\Gamma\,dt\\
&+(\lambda+\mu)\int_0^T\int_{\omega_{\varepsilon\cap U_r}}\rho_\varepsilon(z)(\nabla\alpha(x)\cdot\nu(p(x)))|\operatorname{div}v_r|^2\,dx\,dt\\
&+2\sum_{i=1}^{n}\int_{W_r}\int_0^\varepsilon\rho_\varepsilon(z)\hat v_{r_i}(w,z,T)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,T)|\det JF_r(w,z)|dz\,dw\\
&-2\sum_{i=1}^{n}\int_{W_r}\int_0^\varepsilon\rho_\varepsilon(z)\hat v_{r_i}(w,z,0)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,0)|\det JF_r(w,z)|dz\,dw\\
&-\sum_{i=1}^{n}\int_{\omega_{\varepsilon\cap U_r}}\rho_\varepsilon(z)\operatorname{div}\nu(p(x))v'_{r_i}(T)v_{r_i}(T)\,dx\\
&+\sum_{i=1}^{n}\int_{\omega_{\varepsilon\cap U_r}}\rho_\varepsilon(z)\operatorname{div}\nu(p(x))v'_{r_i}(0)v_{r_i}(0)\,dx
\end{aligned}
$$

and R_2 is given by

$$
\begin{aligned}
&-\mu\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^\varepsilon\rho'_\varepsilon(z)\operatorname{div}\nu(F_r(w,0))\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\hat v_{r_i}(w,z,t,)\times\\
&\qquad|\det JF_r(w,z)|dz\,dw\,dt\\
&-2(\lambda+\mu)\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^\varepsilon\alpha(F_r(w,z))\rho'_\varepsilon(z)\operatorname{div}v_r(F_r(w,z),t)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\times\\
&\qquad\nu_i(F_r(w,0))|\det JF_r(w,z)|dz\,dw\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+(\lambda+\mu)\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho'_\varepsilon(z)\operatorname{div}\nu(p(x))\operatorname{div}v_r(v_r\cdot\nu(p(x)))\,dx\,dt\\
&-\sum_{i=1}^n\int_0^T\int_{W_r}\int_0^\varepsilon\rho'_\varepsilon(z)\hat h_{r_i}(w,z,t)\hat v_{r_i}(w,z,t)|\det JF_r(w,z)|dz\,dw\,dt\\
&+\sum_{i=1}^n\int_{\omega_\varepsilon\cap U_r}\rho'_\varepsilon(z)v'_{r_i}(T)v_{r_i}(T)\,dx\;-\;\sum_{i=1}^n\int_{\omega_\varepsilon\cap U_r}\rho'_\varepsilon(z)v'_{r_i}(0)v_{r_i}(0)\,dx\,.
\end{aligned}
$$

Note that the expression for R_1 collects the terms which involves $\rho_\varepsilon(z)$ and the expression R_2 the terms related to $\rho'_\varepsilon(z)$.

Taking into account the regularity of the functions α, ρ_ε and of the normal vector ν, and considering that

$$
\frac{\partial v_{r_i}}{\partial\nu}=\theta_r|_\Gamma\frac{\partial u_i}{\partial\nu}\quad\text{and}\quad\operatorname{div}v_r|_\Gamma=\theta_r|_\Gamma\operatorname{div}u|_\Gamma \tag{4.4}
$$

we have a positive constant C_1 such that $|R_1|\le C_1E_0$.

On the other hand, observing that:

$$
\frac{\partial v_{r_i}}{\partial x_j}(x)=-\frac{\partial\hat v_{r_i}}{\partial z}(w,z)\nu_j(F_r(w,0))+D_w\hat v_{r_i}D_{x_j}w
$$

where

$$
D_w\hat v_{r_i}D_{x_j}w=\sum_{k=1}^{n-1}\frac{\partial\hat v_{r_i}}{\partial w_k}(w,z)\frac{\partial w_k}{\partial x_j},
$$

we can write

$$
\operatorname{div}v_r(x)=\sum_{j=1}^n\{-\frac{\partial\hat v_{r_j}}{\partial z}(w,z)\nu_j(F_r(w,0))+D_w\hat v_{r_j}D_{x_j}w\}. \tag{4.5}
$$

Using the Hölder inequality, we have

$$
|\hat v_{r_i}(w,z,t)|^2\le z\int_0^z\left|\frac{\partial\hat v_{r_i}}{\partial z}(w,s,t)\right|^2ds.
$$

From this inequality, and using in (4.5) an analogous argument to the one used by C. Fabre and J. P. Puel in [6] (page 196) we conclude that there exists $C_2>0$ such that

$$
|R_2|\le C_2E_0. \tag{4.6}
$$

From (4.3), (4.4) and (4.6) we find a positive constant C_3 independent of ε, δ_0 and u for which,

$$
-2\mu\sum_{i=1}^n\int_0^T\int_{W_r}\int_0^\varepsilon\rho'_\varepsilon(z)\left|\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\right|^2|\det JF_r(w,z)|dz\,dw\,dt \tag{4.7}
$$

$$\begin{aligned}\leq\ & \mu\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^{\varepsilon}\rho''_{\varepsilon}(z)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\hat v_{r_i}(w,z,t)|\det JF_r(w,z)|\,dz\,dw\,dt\\ & -(\lambda+\mu)\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho''_{\varepsilon}(z)\operatorname{div}v_r(v_r\cdot\nu(p(x)))\,dx\,dt\ +\ C_3E_0\,.\end{aligned}$$

But using (4.5) we have

$$\begin{aligned}-(\lambda+\mu)&\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho''_{\varepsilon}(z)\operatorname{div}v_r(v_r\cdot\nu(p(x)))\,dx\,dt \qquad (4.8)\\ =\ & (\lambda+\mu)\sum_{i,j=1}^{n}\int_0^T\int_{W_r}\int_0^{\varepsilon}\alpha(F_r(w,z))\rho''_{\varepsilon}(z)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\nu_i(F_r(w,0))\hat v_{r_j}(w,z,t)\times\\ &\qquad\qquad \nu_j(F_r(w,0))|\det JF_r(w,z)|\,dz\,dw\,dt\\ & -(\lambda+\mu)\int_0^T\int_{W_r}\int_0^{\varepsilon}\alpha(F_r(w,z))\rho''_{\varepsilon}(z)\sum_{i=1}^{n}D_w\hat v_{r_i}D_{x_i}w\times\\ &\qquad\qquad (\hat v_r(w,z,t)\cdot\nu(F_r(w,0)))|\det JF_{\rm r}(w,z)|\,dz\,dw\,dt\,.\end{aligned}$$

Using the Cauchy-Schwarz inequality and (4.1) we prove that

$$\begin{aligned}&(\lambda+\mu)\sum_{i,j=1}^{n}\int_0^T\int_{W_r}\int_0^{\varepsilon}\alpha(F_r(w,z))\rho''_{\varepsilon}(z)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\nu_i(F_r(w,0))\times\\ &\qquad\qquad \hat v_{r_j}(w,z,t)\nu_j(F_r(w,0))|\det JF_r(w,z)|\,dz\,dw\,dt \qquad (4.9)\\ \leq\ & (\lambda+\mu)\alpha_1 n\{\frac{1}{\gamma}\int_0^T\int_{W_r}\int_{\delta_0}^{\varepsilon}\sum_{i=1}^{n}\left|\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\right|^2|\det JF_r(w,z)|\,dz\,dw\,dt\}^{1/2}\times\\ &\qquad \{\gamma\int_0^T\int_{W_r}\int_{\delta_0}^{\varepsilon}\sum_{i=1}^{n}|\rho''_{\varepsilon}(z)|^2|\hat v_{r_i}(w,z,t)|^2|\det JF_r(w,z)|\,dz\,dw\,dt\}^{1/2}\\ \leq\ & (\lambda+\mu)\alpha_1 nG(\delta_0)^{1/2}(\frac{9}{4}G(\delta_0)+C_4E_0)^{1/2}\,,\end{aligned}$$

where the last inequality comes from (3.16) and the identity

$$\begin{aligned}&\hat v_{r_j}(w,z,t)|\det JF_r(w,z)|^{1/2}\\ =\ & \int_0^z\frac{\partial\hat v_{r_j}}{\partial z}(w,s,t)|\det JF_r(w,z)|^{1/2}ds+\int_0^z\hat v_{r_j}(w,s,t)\frac{\partial}{\partial z}(|\det JF_r(w,s)|^{1/2})ds\,.\end{aligned}$$

Now, using an analogous argument to the one used in C. Fabre and J.P. Puel [6](page 197) we get

$$\begin{aligned}&(\lambda+\mu)|\textstyle\sum_{i=1}^{n}\int_0^T\int_{W_r}\int_0^{\varepsilon}\alpha(F_r(w,z))\rho''_{\varepsilon}(z)[D_w\hat v_{r_i}D_{x_i}w]\times\\ &(\hat v_r(w,z,t)\cdot\nu(F_r(w,0)))|\det JF_r(w,z)|\,dz\,dw\,dt\,|\leq C_5E_0\,. \qquad (4.10)\end{aligned}$$

So, by (4.8)-(4.10) we have

$$
\begin{aligned}
&-(\lambda+\mu)\int_0^T\int_{\omega_\varepsilon\cap U_r}\alpha(x)\rho''_\varepsilon(z)\,\mathrm{div}\, v_r(v_r\cdot\nu(p(x)))\,dx\,dt \\
&\le\quad \frac{3}{2}(\lambda+\mu)\alpha_1 nG(\delta_0)+C_6(\lambda+\mu)\alpha_1 nG(\delta_0)^{\frac12}\,E_0^{\frac12}+C_7E_0\,.
\end{aligned}
\tag{4.11}
$$

Proceeding in the same way we did to obtain (4.9) we get

$$
\begin{aligned}
&\mu\sum_{i=1}^n\int_0^T\int_{W_r}\int_0^\varepsilon\rho''_\varepsilon(z)\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\hat v_{r_i}(w,z,t)|\det JF_r(w,z)|\,dz\,dw\,dt \\
&\le\frac{3}{2}\mu G(\delta_0)+C_8\mu G(\delta_0)^{1/2}E_0^{1/2}\,.
\end{aligned}
\tag{4.12}
$$

Finally, using the fact that $\rho'_\varepsilon(z)\le 0$ we have

$$
\begin{aligned}
&-2\mu\sum_{i=1}^n\int_0^T\int_{W_r}\int_0^\varepsilon\rho'_\varepsilon(z)\left|\frac{\partial\hat v_{r_i}}{\partial z}(w,z,t)\right|^2|\det JF_r(w,z)|\,dz\,dw\,dt \\
&\ge 2\mu G(\delta_0)\,.
\end{aligned}
\tag{4.13}
$$

Replacing the inequalities (4.11)-(4.13) in (4.7) we obtain

$$
2\mu G(\delta_0)\left[\frac14-\frac34 n\frac{(\lambda+\mu)}{\mu}\alpha_1\right]\le C_9G(\delta_0)^{1/2}\,E_0^{1/2}+C_{10}E_0\,.
$$

Since $\alpha_1<\frac{\mu}{3n(\lambda+\mu)}$, there exists $\zeta_0>0$ such that

$$
0<\zeta_0<\frac14-\frac34 n\frac{(\lambda+\mu)}{\mu}\alpha_1
$$

and consequently there exists $C>0$, independent of ε, u and δ_0 such that

$$
\max_{\varepsilon\in[0,\varepsilon_0]}G(\varepsilon)\le CE_0\,.
$$

So, in any case, there exists $C>0$ independent of ε, u and δ_0 such that

$$
\begin{aligned}
&\frac{1}{\varepsilon}\sum_{i=1}^n\int_0^T\int_{\omega_\varepsilon\cap U_r}|\nabla v_{r_i}(x,t)\cdot\nu(p(x))|^2\,dx\,dt \\
&\qquad\le\quad C\{||u_0||_V^2+|u_1|_H^2+||h||^2_{L^1(0,T,H)}\},\quad\forall\varepsilon\in]0,\varepsilon_0]\,.
\end{aligned}
$$

But since $u=\sum_{r=1}^m v_r$, Theorem 4.1 is proved. □

From the above theorem we obtain the following result.

Theorem 4.2 *There exist positive constants C and ε_0 such that every solution u of (1.1) where (1.2) holds with $\alpha_1 \leq \frac{\mu}{3n(\lambda+\mu)}$ satisfies*

$$\frac{1}{\varepsilon^3}\sum_{i=1}^{n}\int_0^T\int_{\omega_\varepsilon}|u_i(x,t)|^2\,dx\,dt \leq CE_0\,,\forall\varepsilon\in]0,\varepsilon_0]\,.$$

Furthermore, C and ε_0 depend only on the real positive number T, the function $\alpha(x)$, the geometry of Ω, and the Lamé constants.

Proof. Let $u=(u_1,\dots,u_n)$ be the solution of (1.1) with (1.2) and let ε_0 be the minimum between the two epsilons found in Lemmas 2.1 and 2.2. Considering $\{\theta_r\}_{1\leq r\leq m}$ a C^∞ partition of the unity relative to the open sets $U_1,\dots,U_m$ given in Lemma 2.1 we obtain $u=\sum_{r=1}^m u\theta_r$. Then, for every $\varepsilon\in]0,\varepsilon_0]$ we obtain

$$\begin{aligned}
\int_0^T\int_{\omega_\varepsilon}|u_i(x,t)|^2\,dx\,dt &= \int_0^T\int_{\omega_\varepsilon}|\sum_{r=1}^m u_i\theta_r|^2\,dx\,dt\\
&\leq C_{11}\sum_{r=1}^m\int_0^T\int_{\omega_\varepsilon\cap U_r}|u_i|^2|\theta_r|^2\,dx\,dt\\
&\leq C_{11}\sum_{r=1}^m\int_0^T\int_{\omega_\varepsilon\cap U_r}|u_i|^2\,dx\,dt\\
&= C_{11}\sum_{r=1}^m\int_0^T\int_{W_r}\int_0^\varepsilon|\hat{u}_i(w,z,t)|^2|\det JF_r(w,z)|\,dz\,dw\,dt\\
&\leq C_{12}\sum_{r=1}^m\int_0^T\int_{W_r}\int_0^\varepsilon|\hat{u}_i(w,z,t)|^2\,dz\,dw\,dt\,, \qquad (4.14)
\end{aligned}$$

where $\hat{u}_i(w,z,t)=u_i(F_r(w,z),t)$. Since

$$\hat{u}_i(w,z,t)=\int_0^z\frac{\partial\hat{u}_i}{\partial s}(w,s,t)\,ds\,,$$

it follows that

$$|\hat{u}_i(w,z,t)|^2\leq z\int_0^z\left|\frac{\partial\hat{u}_i}{\partial s}(w,s,t)\right|^2\,ds\,.$$

Consequently from (4.14) we have

$$\begin{aligned}
&\int_0^T\int_{\omega_\varepsilon}|u_i(x,t)|^2\,dx\,dt\\
&\leq C_{12}\sum_{r=1}^m\int_0^T\int_{W_r}\int_0^\varepsilon(z\int_0^z\left|\frac{\partial\hat{u}_i}{\partial s}(w,s,t)\right|^2\,ds)dz\,dw\,dt
\end{aligned}$$

$$\begin{aligned}
&\leq C_{12}\sum_{r=1}^{m}\int_0^T\int_{W_r}\int_0^{\varepsilon} z\int_0^{\varepsilon}\left|\frac{\partial\hat{u}_i}{\partial s}(w,s,t)\right|^2 ds\,dz\,dw\,dt\\
&= C_{12}(\sum_{r=1}^{m}\int_0^T\int_{W_r}\int_0^{\varepsilon}\left|\frac{\partial\hat{u}_i}{\partial s}(w,s,t)\right|^2 ds\,dw\,dt)(\int_0^{\varepsilon} z\,dz)\\
&= C_{13}\varepsilon^2\sum_{r=1}^{m}\int_0^T\int_{W_r}\int_0^{\varepsilon}|\nabla u_i(F_r(w,s),t)\cdot\nu(F_r(w,0))|^2\,ds\,dw\,dt
\end{aligned}$$

where the last equality comes from (2.1). Therefore,

$$\begin{aligned}
\int_0^T\int_{\omega_\varepsilon}|u_i(x,t)|^2\,dx\,dt &\leq C_{14}\varepsilon^2\sum_{r=1}^{m}\int_0^T\int_{\omega_\varepsilon\cap U_r}|\nabla u_i(x,t)\cdot\nu(p(x))|^2\,dx\,dt\\
&\leq C_{14}\varepsilon^2\int_0^T\int_{\omega_\varepsilon}|\nabla u_i(x,t)\cdot\nu(p(x))|^2\,dx\,dt\,.
\end{aligned}$$

Then, we obtain the desired result from Theorem 4.1, in view of

$$\frac{1}{\varepsilon^3}\int_0^T\int_{\omega_\varepsilon}|u_i(x,t)|^2\,dx\,dt \leq C_{14}\frac{1}{\varepsilon}\int_0^T\int_{\omega_\varepsilon}|\nabla u_i(x,t)\cdot\nu(p(x))|^2\,dx\,dt.$$

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