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A NOTE ON VERY WEAK P–HARMONIC MAPPINGS

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Abstract. We prove a new *a priori* estimate for very weak p–harmonic mappings when p is close to two. This sheds some light on a conjecture posed by Iwaniec and Sbordone.

1. Introduction

Let Ω be a bounded regular domain (Definition 2.1 in [IS]) in $\mathbf{R}^n$ and $1 < p < \infty$. We are interested in the p–harmonic system

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.1}$$

under minimal assumptions on u. System (1.1) is understood in the weak sense: a weak p–harmonic mapping is defined as $u \in W^{1,p}_{\text{loc}}(\Omega, \mathbf{R}^m)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = 0 \tag{1.2}$$

for every $\phi \in C^\infty_0(\Omega, \mathbf{R}^m)$. However, equation (1.2) makes sense when $u \in W^{1,r}_{\text{loc}}(\Omega, \mathbf{R}^m)$ with $r \geq \max\{1, p-1\}$. Such solutions are called *very weak p–harmonic mappings*.

Iwaniec and Sbordone showed in [IS] that there exist $r_1 = r_1(p, m, \Omega)$ and $r_2 = r_2(p, m, \Omega)$ satisfying

$$1 < r_1 < p < r_2 < \infty \tag{1.3}$$

such that every very weak p–harmonic mapping $u \in W^{1,r_1}_{\text{loc}}(\Omega, \mathbf{R}^m)$ belongs to $W^{1,r_2}_{\text{loc}}(\Omega, \mathbf{R}^m)$. They conjectured (Conjecture 1 in [IS]) that every $r_1 > \max\{1, p-1\}$ would do for the p–harmonic system, but their estimate for r_1 is very close to p.

The objective of our note is to study the case when p is close to two. We show that r_1 in (1.3) can be chosen arbitrarily close to one if p is close to two. This does not solve the conjecture by Iwaniec and Sbordone, but we are able to obtain estimates when r_1 is arbitrarily close to $\max\{1, p-1\}$.

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Theorem 1.4. *Let $0 < \eta < 1$. For every exponent $r_1 \geq 1 + \eta$ there are $\delta = \delta(\eta, m, \Omega) > 0$ and $r_2 = r_2(p, m, \Omega) > p$ such that every very weak p–harmonic mapping $u \in W^{1,r_1}_{\mathrm{loc}}(\Omega, \mathbf{R}^m)$ belongs to $W^{1,r_2}_{\mathrm{loc}}(\Omega, \mathbf{R}^m)$ provided $|p-2| < \delta$.*

Our method is a sharpening of [IS]. If u belongs to the Sobolev space with the natural exponent p, then we may use the classical method of reverse Hölder inequalities, see [ME]. However, if u is assumed to belong to a Sobolev space below the natural exponent, then we cannot choose ηu, where η is a cut-off function, as a test function in (1.2) and hence it is difficult to obtain any *a priori* estimates. Iwaniec and Sbordone used the Hodge decomposition ingeniously to build a test function for equation (1.2). Another argument has been given by Lewis in [L]. The main new feature in our proof is that we use the Hodge decomposition twice and also employ the fact that the matrix field involved is divergence free. This trick works only for p–harmonic operator. As Serrin's example in [S] shows, our result is not true for $\mathcal{A}$–harmonic operators studied by [IS] and [L].

2. Main results

For the convenience of the reader we recall the formulation of the Hodge decomposition (Theorem 3 in [IS]).

Theorem 2.1. *Let Ω be a regular domain in $\mathbf{R}^n$ and $w \in W^{1,r}_0(\Omega, \mathbf{R}^m)$ with $r > 1$, and let $-1 < \varepsilon < r - 1$. Then there exist $\phi \in W^{1,r/(1+\varepsilon)}_0(\Omega, \mathbf{R}^m)$ and a divergence free matrix field $H \in L^{r/(1+\varepsilon)}(\Omega, \mathbf{R}^{m\times n})$ such that*

$$|\nabla w|^{\varepsilon} \nabla w = \nabla \phi + H, \tag{2.2}$$

and

$$\|H\|_{r/(1+\varepsilon),\Omega} \leq C_r(\Omega, m)|\varepsilon| \, \|\nabla w\|^{1+\varepsilon}_{r,\Omega}. \tag{2.3}$$

An examination of [IS] reveals that all their results are based on Theorem 5.1 in [IS]. Instead of rewriting all results in [IS], we only prove a sharpening of Theorem 5.1 in [IS] when p is close to two.

We consider the very weak solutions $w \in W^{1,r}_0(\Omega, \mathbf{R}^m)$ with $r > \max\{1, p-1\}$ of the non homogeneous system

$$\operatorname{div}\left(|g + \nabla w|^{p-2}(g + \nabla w)\right) = \operatorname{div} h \tag{2.4}$$

where $g \in L^r(\Omega, \mathbf{R}^{m\times n})$ and $h \in L^{r/(p-1)}(\Omega, \mathbf{R}^{m\times n})$ are matrix fields. Equation (2.4) is understood in the weak sense, that is,

$$\int_\Omega |g + \nabla w|^{p-2}(g + \nabla w) \cdot \nabla \phi \, dx = \int_\Omega h \cdot \nabla \phi \, dx \tag{2.5}$$

for every $\phi \in W^{1,r/(r-p+1)}_0(\Omega, \mathbf{R}^m)$.

Theorem 2.6. *Let $0 < \eta < 1$. Suppose that $w \in W^{1,r}_0(\Omega, \mathbf{R}^m)$ with $r \geq 1 + \eta$ satisfies (2.3). Then there is $\delta = \delta(\eta, m, \Omega) > 0$ such that if $|p-2| < \delta$, then*

$$\int_\Omega |\nabla w|^r \, dx \leq C(\eta, p, m, \Omega) \int_\Omega \left(|g|^r + |h|^{r/(p-1)}\right) dx. \tag{2.7}$$

Proof. Using Theorem 2.1 with $\varepsilon = r-p$ we obtain functions $\phi_1 \in W_0^{1,r/(r-p+1)}(\Omega, \mathbf{R}^m)$ and $H_1 \in L^{r/(r-p+1)}(\Omega, \mathbf{R}^{m\times n})$ such that

$$|\nabla w|^{r-p}\nabla w = \nabla\phi_1 + H_1, \tag{2.8}$$

$$\int_\Omega H_1 \cdot \nabla\varphi\, dx = 0, \qquad \text{for every } \varphi \in W_0^{1,r/(p-1)}(\Omega, \mathbf{R}^m), \tag{2.9}$$

and

$$\|H_1\|_{r/(r-p+1),\Omega} \le c_1|r-p|\, \|\nabla w\|_{r,\Omega}^{r-p+1}, \qquad c_1 = C_r(\Omega, m). \tag{2.10}$$

In particular, we have

$$\|\nabla\phi_1\|_{r/(r-p+1),\Omega} \le (c_1+1)\, \|\nabla w\|_{r,\Omega}^{r-p+1}. \tag{2.11}$$

Since ϕ_1 can be used as a test function in (2.5), we obtain

$$\int_\Omega |\nabla w + g|^{p-2}(\nabla w + g)\cdot\nabla\phi_1\, dx = \int_\Omega h\cdot\nabla\phi_1\, dx.$$

Inserting (2.8) we arrive at

$$\begin{aligned}\int_\Omega |\nabla w|^r\, dx =& \int_\Omega |\nabla w|^{p-2}\nabla w\cdot H_1\, dx + \int_\Omega h\cdot\nabla\phi_1\, dx\\ &+ \int_\Omega \big(|\nabla w|^{p-2}\nabla w - |\nabla w + g|^{p-2}(\nabla w + g)\big)\cdot\nabla\phi_1\, dx\\ =& I_1 + I_2 + I_3.\end{aligned} \tag{2.12}$$

We begin with estimating I_1. This is the crucial step of our argument. By using Theorem 2.1 again with $\varepsilon = p-2$, we obtain $\phi_2 \in W^{1,r/(p-1)}(\Omega, \mathbf{R}^m)$ and $H_2 \in L^{r/(p-1)}(\Omega, \mathbf{R}^{m\times n})$ such that

$$|\nabla w|^{p-2}\nabla w = \nabla\phi_2 + H_2 \tag{2.13}$$

$$\int_\Omega H_2\cdot\nabla\varphi\, dx = 0, \qquad \text{when } \varphi \in W_0^{1,r/(r-p+1)}(\Omega, \mathbf{R}^m), \tag{2.14}$$

and

$$\|H_2\|_{r/(p-1),\Omega} \le c_1|p-2|\, \|\nabla w\,\|_{r,\Omega}^{p-1}, \qquad c_1 = C_r(\Omega, m). \tag{2.15}$$

Using (2.13), (2.14), (2.8), (2.9) and (2.15) we have

$$\begin{aligned} I_1 &= \int_\Omega \big(\nabla\phi_2 + H_2\big)\cdot H_1\, dx = \int_\Omega H_1\cdot H_2\, dx\\ &= \int_\Omega \big(|\nabla w|^{r-p}\nabla w - \nabla\phi_1\big)\cdot H_2\, dx\\ &= \int_\Omega |\nabla w|^{r-p}\nabla w\cdot H_2\, dx\\ &\le c_1|p-2|\, \|\nabla w\|_{r,\Omega}^r.\end{aligned}$$

The same reasoning shows that

$$I_1 \leq c_1 |r-p| \, \|\nabla w\|_{r,\Omega}^r$$

and hence

$$I_1 \leq c_1 \min\bigl\{|p-2|, |r-p|\bigr\} \|\nabla w\|_{r,\Omega}^r. \tag{2.16}$$

By virtue of (2.11) we may estimate I_2 and I_3 in the same way as in [IS]. That is

$$\begin{aligned} I_2 + I_3 \leq & C\bigl(\|\nabla w\|_{r,\Omega}^{r-p+1} \|h\|_{r/(p-1),\Omega} \\ & + \|g\|_{r,\Omega}^{p-1} \|\nabla w\|_{r,\Omega}^{r-p+1} + \|g\|_{r,\Omega} \|\nabla w\|_{r,\Omega}^{r-1}\bigr). \end{aligned}$$

Recalling Young's inequality we conclude that, for every $\theta > 0$,

$$I_2 + I_3 \leq \theta \, \|\nabla w\|_{r,\Omega}^r + C_\theta\bigl(\|g\|_{r,\Omega}^r + \|h\|_{r/(p-1),\Omega}^{r/(p-1)}\bigr). \tag{2.17}$$

Using (2.12), (2.16) and (2.17) we obtain

$$\bigl(1 - c_1 \min\{|p-2|, |r-p|\} - \theta\bigr) \|\nabla w\|_{r,\Omega}^r \leq C_\theta\bigl(\|g\|_{r,\Omega}^r + \|h\|_{r/(p-1),\Omega}^{r/(p-1)}\bigr).$$

In particular, if $c_1|p-2| < 1$, then (2.5) holds. Estimates for the constant $c_1 = C_r(\Omega, m)$ can be found in [I] and formula (2.10) in [IS]. Using these estimates it is easy to see that we may choose $c_1 = c(\eta, m, \Omega)$. This completes the proof.

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