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Adjoint and self-adjoint differential operators on graphs *

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Abstract

A differential operator on a directed graph with weighted edges is characterized as a system of ordinary differential operators. A class of local operators is introduced to clarify which operators should be considered as defined on the graph. When the edge lengths have a positive lower bound, all local self-adjoint extensions of the minimal symmetric operator may be classified by boundary conditions at the vertices.

1 Introduction

Although there is a large body of literature on the spectral theory of linear difference operators associated with a combinatorial graph [3], the study of differential operators on a topological graph has received much less attention. This situation has begun to change, due in large part to quantum-mechanical problems associated with advances in micro-electronic fabrication [2, 7, 8, 10]. In developing physical models one often needs to know when a differential operator is essentially self adjoint on a given domain. This paper provides a description of adjoints, and considers domains of essential self adjointness for a class of differential operators on weighted directed graphs.

These differential operators $\mathcal{L}$ are actually a (possibly infinite) system of ordinary differential operators on intervals whose lengths are given by the edge weights of the graph $\mathcal{G}$. For regular ordinary differential operators acting on $L^2[a,b]$ there is a classical description of adjoints and self-adjoint extensions in terms of boundary conditions [5, pp. 284–297]. This theory has a close connection with the abstract treatment of self-adjoint extensions of symmetric operators [14, pp. 140–141]. The general treatment is somewhat deficient for differential operators on graphs, since the role of the vertices of the graph $\mathcal{G}$ is unclear. When there are infinitely many vertices the description of extensions appears particularly awkward.

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To remedy these problems, we will impose an additional restriction on the domain of an operator $\mathcal{L}$. Let $\phi : \mathcal{G} \to \mathcal{C}$ denote a C^∞ function which has compact support in $\mathcal{G}$ and is constant in an open neighborhood of each vertex. We say that $\mathcal{L}$ is a local operator if for every ϕ, ϕf is in the domain of $\mathcal{L}$ whenever f is. We will see that local operators have domains described via boundary conditions which only compare boundary values at endpoints which are identified with a single vertex of the graph $\mathcal{G}$.

One result uses conditions at the vertices to characterize functions of compact support in the domain of the adjoint of a local operator. The main results assume that the edge lengths of $\mathcal{G}$ have a positive lower bound. In this case there is a complete classification of local self-adjoint operators $\mathcal{L}$ in terms of boundary conditions at the graph vertices when the coefficients of the operator are bounded and satisfy some mild additional regularity assumptions. A final application shows that Schrödinger operators on a graph with $\delta-$ function interactions are essentially self adjoint on a domain of functions of compact support.

2 Local Differential Operators on Graphs

In this work a graph $\mathcal{G}$ will have a countable vertex set $\mathcal{V}$ and a countable set of directed edges e_n. Each edge has a positive weight (length) w_n. Assume further that each vertex appears in at least one, but only finitely many edges. The graph may have loops and multiple edges with the same vertices.

A topological graph may be constructed using the graph data [12, p. 190]. For each directed edge e_n let $[a_n, b_n]$ be a real interval of length w_n, and let $\alpha_m \in \{a_n, b_n\}$. Identify interval endpoints α_m if the corresponding edge endpoints are the same vertex v, in which case we will write $\alpha_m \sim v$. This topological graph, also denoted $\mathcal{G}$, is assumed to be connected. The Euclidean metric on the intervals may be extended to a metric on $\mathcal{G}$ by taking the distance between two points to be the length of the shortest (undirected) path joining them. Notice that every compact set $K \subset \mathcal{G}$ is contained in a finite union of closed edges e_n, since K has a covering by open sets which hit only finitely many edges.

Let $L^2(\mathcal{G})$ denote the Hilbert space $\oplus_n L^2(e_n)$ with the inner product

$$\langle f, g\rangle = \int_{\mathcal{G}} f\overline{g} = \sum_n \int_{a_n}^{b_n} f_n(x)\overline{g_n(x)}\, dx, \quad f = (f_1, f_2, \dots).$$

A differential operator $\mathcal{L}$ acts componentwise on functions $f \in L^2(\mathcal{G})$ in its domain,

$$\mathcal{L}f = \sum_{j=0}^{M} c_j(x) f^{(j)}(x).$$

The leading coefficient c_M is nowhere 0 and c_j is a j times continuously differentiable complex valued function on each interval $[a_n, b_n]$. The associated formal operator is

$$L = \sum_{j=0}^{M} c_j(x) D^j, \quad D = \frac{d}{dx}.$$

The domain of $\mathcal{L}$, denoted $\mathrm{Dom}(\mathcal{L})$, will always include $\mathcal{D}_{\min}$, the linear span of C^∞ functions supported in the interior of a single interval (a_n, b_n). The domain of $\mathcal{L}$ will be contained in $\mathcal{D}_{\max}$ (which depends on L), the set of functions $f \in L^2(\mathcal{G})$ with $f_n, \dots, f_n^{(M-1)}$ continuous and $f_n^{(M-1)}$ absolutely continuous on $[a_n, b_n]$, and $Lf \in L^2(\mathcal{G})$.

A convenient reference for differential operators on $L^2[a,b]$ is [6, pp. 1278–1310]. The development there assumes that $c_j \in C^\infty$, but this distinction is unimportant. In addition, these authors assume a somewhat larger minimal domain for the operators. This is also inconsequential since $\mathcal{L}$ is closable [11, p. 168], and the closure of $\mathcal{L}$ will have a domain [11, pp. 169–171] which includes the functions $f \in \mathcal{D}_{\max}$ which are supported on an interval $[a_n, b_n]$, and which satisfy

$$f_n^{(j)}(a_n) = 0 = f_n^{(j)}(b_n), \quad j = 0, \dots, M-1.$$

If $\mathcal{L}_{\min}$ has the domain $\mathcal{D}_{\min}$, then the adjoint operator $\mathcal{L}_{\min}^*$ will again be a differential operator. By working on one interval $[a_n, b_n]$ at a time, and using the classical theory [6, p. 1294], [11, pp. 169–171], one may obtain the following result.

Lemma 2.1 *A function f is in the domain of the adjoint operator $\mathcal{L}_{\min}^*$, if and only if $f \in \mathcal{D}_{\max}$ for L^+, where*

$$L^+ = \sum_{j=0}^{M} (-1)^j D^j \overline{c_j(x)} = \sum_{j=0}^{M} (-1)^j \sum_{i=0}^{j} \binom{j}{i} \overline{c_j^{(j-i)}(x)} D^i.$$

If $f \in \mathrm{Dom}(\mathcal{L}_{\min}^*)$, then $\mathcal{L}_{\min}^* f = L^+ f$.

If $\alpha_m \in \{a_n, b_n\}$, then the functionals $f^{(j)}(\alpha_m)$, for $j = 0, \dots, M-1$ are continuous [6, pp. 1297–1301] on $\mathrm{Dom}(\mathcal{L})$ when the domain is given the norm $\|f\|_{\mathcal{L}} = [\|f\|_2 + \|\mathcal{L}f\|_2]^{1/2}$. Say that β_v is a vertex functional at v if β_v is a linear combination of $f^{(j)}(\alpha_m)$ for $j = 0, \dots, M-1$, and $\alpha_m \sim v$. A (homogeneous) vertex condition at v is a equation of the form $\beta_v(f) = 0$.

Whether or not $\mathcal{L}$ is local, there will always be a (complex) vector space $\mathcal{B}_v$ of vertex functionals β_v at v such that every function f in $\mathrm{Dom}(\mathcal{L})$ satisfies $\beta_v(f) = 0$. If $\mathcal{L}$ is local and closed, these vertex conditions will give a local description of functions in $\mathrm{Dom}(\mathcal{L})$. Let $\mathcal{D}_{\mathrm{com}}$ be the set of functions of compact support in $\mathcal{D}_{\max}$.

Lemma 2.2 *Suppose that $\mathcal{L}$ is local and closed. If $f \in \mathcal{D}_{\mathrm{com}}$ and $\beta_v(f) = 0$ for all $\beta_v \in \mathcal{B}_v$ and all $v \in \mathcal{V}$, then f is in the domain of $\mathcal{L}$.*

Proof Fix the vertex v, and let $\delta(v)$ be its degree. Consider the range of the linear map from $\mathrm{Dom}(\mathcal{L})$ to $C^{M\delta(v)}$, which sends g to boundary values

$$g^{(j)}(\alpha_m), \quad j = 0, \dots, M-1, \quad \alpha_m \sim v.$$

If this subspace did not include the vector of values $f^{(j)}(\alpha_m)$ there would be a vertex functional at v which annihilated $\mathrm{Dom}(\mathcal{L})$, but not f. Since this contradicts the assumptions on f, there is some $g_v \in \mathrm{Dom}(\mathcal{L})$ satisfying

$$g_v^{(j)}(\alpha_m) = f^{(j)}(\alpha_m), \quad j = 0, \dots, M-1, \quad \alpha_m \sim v.$$

Since $\mathcal{L}$ is local, we may assume that g_v has compact support and vanishes in a neighborhood of every other vertex. Since f has compact support, there is a finite collection of vertices v for which $f^{(j)}(\alpha_m) \neq 0$, for some $0 \le j < M$, and $\alpha_m \sim v$. Thus there is a function $g \in \mathrm{Dom}(\mathcal{L})$ of compact support, such that $f^{(j)}(\alpha_m) = g^{(j)}(\alpha_m)$ for $j = 0, \dots, M-1$, at every endpoint α_m. Since $\mathcal{L}$ is closed and $\mathcal{D}_{\min} \subset \mathrm{Dom}(\mathcal{L})$, we find that $f - g$, and thus f, are in $\mathrm{Dom}(\mathcal{L})$. $\square$

Before turning to the description of the domain for the adjoint of a local operator $\mathcal{L}$, some additional ideas are reviewed.

Suppose $f, g \in \mathcal{D}_{\max}$, with the support of g in an open ball containing at most one vertex v. Then integration by parts [5, p. 285] leads to

$$\langle Lf, g\rangle - \langle f, L^+ g\rangle = [f, g]_v$$

where $[f, g]_v$ is a nondegenerate form in the boundary values of f and g at the $\alpha_m \sim v$.

Consider the second order case $Lf = f'' + c_1 f' + c_0 f$. On $[a_n, b_n]$ we have, without restrictions on the support of f and g,

$$\begin{aligned} \int_{a_n}^{b_n} \Big[\overline{g} Lf - f\overline{L^+ g}\Big] \quad &= \quad f'(b_n)\overline{g}(b_n) - f'(a_n)\overline{g}(a_n) + f(a_n)\overline{g}'(a_n) - f(b_n)\overline{g}'(b_n) \\ &\quad + f(b_n)c_1(b_n)\overline{g}(b_n) - f(a_n)c_1(a_n)\overline{g}(a_n). \end{aligned}$$

If g vanishes outside of a small neighborhood of v, and

$$\sigma_m = \begin{cases} 0, & \alpha_m = b_m\,, \\ 1, & \alpha_m = a_m\,, \end{cases}$$

then

$$[f, g]_v = \sum_m (-1)^{\sigma_m} \Big[f'(\alpha_m)\overline{g}(\alpha_m) - f(\alpha_m)\overline{g}'(\alpha_m) + f(\alpha_m)c_1(\alpha_m)\overline{g}(\alpha_m)\Big]\,,$$

with $\alpha_m \sim v$.

At each v pick an ordering $\alpha_1,\dots,\alpha_{\delta(v)}$ of the $\alpha_m \sim v$, and for $f \in \mathcal{D}_{\max}$ let $\hat{f} \in C^{M\delta(v)}$ be the vector with components

$$\hat{f}_{j\delta(v)+k} = f^{(j)}(\alpha_k), \quad j = 0,\dots,M-1, \quad k = 1,\dots,\delta(v).$$

With respect to this basis there is an invertible $M\delta(v) \times M\delta(v)$ matrix $\mathcal{S}_v$ such that

$$[f,g]_v = \mathcal{S}_v \hat{f} \bullet \hat{g}. \qquad (2.a)$$

where $\bullet$ denotes the usual dot product on $C^{M\delta(v)}$. Single vertex conditions may now be written as

$$\sum b_{j,k} f^{(j)}(\alpha_k) = \sum b_{j,k} \hat{f}_{j\delta(v)+k} = 0,$$

and a maximal independent set of vertex conditions at v may be written more compactly as $B_v \hat{f} = 0$, where B_v is a $K(v) \times M\delta(v)$ matrix with linearly independent rows.

Since the null space $N(B_v) \in \mathcal{C}^{M\delta(v)}$ has dimension $M\delta(v) - K(v)$, there is an $[M\delta(v) - K(v)] \times M\delta(v)$ matrix B_v^+, such that

$$B_v^+ X = 0 \quad \text{if and only if} \quad \mathcal{S}_v^* X \in N(B_v)^\perp, \quad X \in C^{M\delta(v)}. \qquad (2.b)$$

Call any such matrix B_v^+ a complementary matrix to B_v, and the vertex conditions $B_v^+ \hat{f} = 0$ complementary boundary conditions.

3 Domains of adjoint operators

If $\mathcal{L}$ is local, functions in the domain of the adjoint operator $\mathcal{L}^*$ must also satisfy vertex conditions. The treatment of an operator defined on a single interval may be found in [5, pp. 284–297]. We have taken advantage of some refinements worked out in [4].

Find a basis $z_1,\dots,z_{M\delta-K(v)}$ for $N(B_v)$, and let Z_v be the $M\delta(v) \times [M\delta(v) - K(v)]$ matrix whose columns are z_j.

Theorem 3.1 *Suppose that $\mathcal{L}$ is local, and that the vertex conditions at v annihilating the domain of $\mathcal{L}$ are written as*

$$B_v \hat{f} = 0,$$

where B_v is a $K(v) \times M\delta(v)$ matrix, with linearly independent rows.

Then the adjoint $\mathcal{L}^*$ is local and closed. A function $g \in \mathcal{D}_{\text{com}}$ is in the domain of $\mathcal{L}^*$ if and only if $B_v^+ \hat{g} = 0$ for a set of vertex conditions complementary to the conditions $B_v \hat{f} = 0$.

A matrix B_v^+ is complementary to B_v if and only if B_v^+ is $[M\delta(v) - K(v)] \times M\delta(v)$, with linearly independent rows, and the equations

$$B_v^+ [\mathcal{S}_v^*]^{-1} (B_v^*) = 0$$

are satisfied. One such matrix is $B_v^+ = (\mathcal{S}_v Z_v)^*$.

Proof If $g \in \mathrm{Dom}(\mathcal{L}^*)$ then $g \in \mathrm{Dom}(\mathcal{L}^*_{\mathrm{min}})$, so by Lemma 2.1 $\mathcal{L}^* g = L^+ g$, and

$$\langle Lf, g\rangle = \langle f, L^+ g\rangle, \quad f \in \mathrm{Dom}(\mathcal{L}).$$

Since $\mathcal{L}$ is local, any vertex values $\hat{f}$ at v satisfying $B_v \hat{f} = 0$ are the vertex values of some $f \in \mathrm{Dom}(\mathcal{L})$ which has compact support and 0 is in an open neighborhood of every vertex except v. For such f,

$$\langle Lf, g\rangle - \langle f, L^+ g\rangle = 0 = [f, g]_v.$$

By (2.a) we have $\mathcal{S}_v^* \hat{g} \in N(B_v)^\perp$, and by (2.b) the equations $B_v^+ \hat{g} = 0$ are satisfied for any matrix complementary to B_v. Now if ϕ has compact support and constant in neighborhood of each vertex, then $\phi g \in \mathcal{D}_{\mathrm{com}}$ with $B_v^+ \hat{\phi g} = 0$. This implies that $\phi g \in \mathrm{Dom}(\mathcal{L}^*)$ and $\mathcal{L}^*$ is local, and more generally that $g \in \mathcal{D}_{\mathrm{com}}$ is in the domain of $\mathcal{L}^*$ if and only if $B^+ \hat{g} = 0$. In addition, adjoint operators are always closed.

What remains is to characterize the matrices B_v^+ complementary to B_v. The vector $\hat{g}$ will satisfy the vertex conditions of a function in $\mathrm{Dom}(L^*)$ if and only if $\mathcal{S}_v^* \hat{g} \in N(B_v)^\perp$. Since

$$\mathrm{Ran}(Z_v) = N(B_v), \quad N(B_v)^\perp = \mathrm{Ran}(Z_v)^\perp = N(Z_v^*),$$

the condition on $\hat{g}$ is equivalent to $Z_v^* \mathcal{S}_v^* \hat{g} = 0$. Thus we may take $B_v^+ = (\mathcal{S}_v Z_v)^*$.

To recognize more generally when a matrix B_v^+ is complementary to B, start with the fact that this is equivalent to requiring that $\hat{g} \in N(B_v^+)$ if and only if $\mathcal{S}_v^* \hat{g} \in N(B_v)^\perp$, or $\hat{g} \in [\mathcal{S}_v^*]^{-1} N(B_v)^\perp$. Thus we want $N(B_v^+) = [\mathcal{S}_v^*]^{-1} \mathrm{ran}(B_v^*)$, or that B_v^+ is a $[M\delta(v) - K(v)] \times M\delta(v)$ matrix with linearly independent rows such that the equation $B_v^+ [\mathcal{S}_v^*]^{-1} (B_v^*) = 0$ is satisfied. □

The following observation about self-adjoint operators is a corollary of the last result.

Corollary 3.2 *Suppose that $\mathcal{L}$ is self adjoint and local, with vertex conditions $B_v \hat{f}_v = 0$ as in Theorem 3.1. Then each B_v is an $[M\delta(v)/2] \times M\delta(v)$ matrix, and*

$$B_v [\mathcal{S}_v^*]^{-1} (B_v^*) = 0 \qquad (3.a).$$

Conversely, suppose that $L = L^+$, and that vertex conditions $B_v \hat{f}_v = 0$ are given at each vertex so that (3.a) is satisfied. If each B_v is an $[M\delta(v)/2] \times M\delta(v)$ matrix with linearly independent rows, then the operator $\mathcal{L}$ with

$$\mathrm{Dom}(\mathcal{L}) = \{f \in \mathcal{D}_{\mathrm{com}} \mid B_v \hat{f} = 0, \quad v \in \mathcal{V}\}$$

is symmetric, and has no symmetric extensions whose domain is a subset of $\mathcal{D}_{\mathrm{com}}$.

The next lemma will help identify formal operators $L = L^+$ and vertex conditions such that $\mathcal{L}$ will be essentially self adjoint if $\mathrm{Dom}(\mathcal{L}) = \{f \in \mathcal{D}_{\mathrm{com}} \mid B_v \hat{f} = 0\}$. We will need some hypotheses on the coefficients of L, and will require that the lengths w_n of the edges have a positive lower bound.

Lemma 3.3 *Suppose that $w_n \ge C > 0$ for all n, and that vertex matrices B_v with independent rows are given. Assume that the leading coefficient $|c_M|$ of L is bounded below by a positive constant, and that all coefficients of L^+ are uniformly bounded on $\mathcal{G}$.*

Let $\mathrm{Dom}(\mathcal{L}) = \{f \in \mathcal{D}_{\mathrm{com}} \mid B_v \hat{f} = 0, \quad v \in \mathcal{V}\}$, and let $\mathcal{L}^+$ be the restriction of $\mathcal{L}^*$ to $\mathrm{Dom}(\mathcal{L}^+) = \{f \in \mathcal{D}_{\mathrm{com}} \mid B_v^+ \hat{f} = 0, \quad v \in \mathcal{V}\}$ for matrices B_v^+ complementary to B_v.

Assume that there is a positive constant ϵ, and a complex number λ such that

$$\|(\mathcal{L} - \lambda) f\| \ge \epsilon \|f\|, \quad f \in \mathrm{Dom}(\mathcal{L}), \tag{3.b}$$

$$\|(\mathcal{L}^+ - \overline{\lambda})\| \ge \epsilon \|f\|, \quad f \in \mathrm{Dom}(\mathcal{L}^+). \tag{3.c}$$

Then the closure of $\mathcal{L} - \lambda$ has a bounded inverse.

Proof Part of the method of proof is adopted from [11, p. 274]. The inequality (3.b) extends to the closure of $\mathcal{L} - \lambda$, which is therefore injective and boundedly invertible on its range. If the range is not dense there must be a nontrivial vector ψ in $N(\mathcal{L}^* - \overline{\lambda})$. We will assume the existence of ψ, and obtain a contradiction.

Pick a C^∞ function $\eta(x)$ on $(0, C)$ which is 1 in a neighborhood of 0 and vanishes identically for $x > C/4$. Pick any edge e_0, and for $K = 1, 2, 3, \dots$ construct a C^∞ cutoff function ϕ_K on $\mathcal{G}$ as follows. On the set E_0 of (closed) edges containing some point whose distance from a vertex of e_0 is less than or equal to K, let $\phi_K = 1$. On edges $e = [a_n, b_n]$ not in E_0 which share a vertex $v \sim a_n$ (resp. $v \sim b_n$) with an edge in E_1, let $\phi_K = \eta(x - a_n)$ (resp. $\phi_K = \eta(b_n - x)$) where η is defined. Otherwise let $\phi_K = 0$.

Since $\mathcal{L}^*$ is local, $\phi_K \psi \in \mathrm{Dom}(\mathcal{L}^+)$. A computation gives

$$[\mathcal{L}^+ - \overline{\lambda}] \phi_K \psi = \phi_K [\mathcal{L}^+ - \overline{\lambda}] \psi + R_K$$

where the first term on the right hand side is 0. The term R_K is a sum, in which each summand has as a factor $\phi_K^{(j)}$ for $j \ge 1$. Thus we may write

$$R_K = \sum_{j<M} C_j \psi^{(j)},$$

where the C_j vanish outside the support of ϕ_K', and are bounded independent of K.

Let $E(K)$ denote those edges where ϕ_K' is not identically zero. By virtue of the hypotheses on the coefficients of L^+, and the construction of ϕ_K, there is a constant C such that

$$\int_{E_K} |R_K|^2 \le C\Big[\int_{E_K} |L^+\psi|^2 + \int_{E_K} |\psi|^2\Big] \le C[1+|\lambda|^2] \int_{E_K} |\psi|^2.$$

The constant C may be chosen independent of K [9, p. 19]. Thus

$$0 = \lim_{K\to\infty} \|R_K\|^2 = \lim_{K\to\infty} \|[\mathcal{L}^+ - \overline{\lambda}]\phi_K\psi\|^2.$$

But this violates the bound (3.c). Thus the range of $\mathcal{L}-\lambda$ is dense, establishing the result. □

Lemma 3.3 shows that domains of local self-adjoint operators may often be completely classified by means of vertex conditions.

Theorem 3.4 *Suppose that $w_n \ge C > 0$ for all n, and that $L = L^+$. Assume that $|c_M|$ is bounded below by a positive constant, and that all coefficients of L are uniformly bounded.*

If $[M\delta(v)/2] \times M\delta(v)$ vertex matrices B_v are given with linearly independent rows, and satisfying (3.a), and if $\mathcal{L}$ has domain

$$\mathrm{Dom}(\mathcal{L}) = \{f \in \mathcal{D}_{\mathrm{com}} \mid B_v\hat{f} = 0, \quad v \in \mathcal{V}\},$$

then $\mathcal{L}$ is essentially self adjoint. Conversely, every local self-adjoint operator $\mathcal{L}_1$ formally given by such an L whose domain includes $\mathcal{D}_{\min}$ is the closure of one of the operators $\mathcal{L}$.

Proof Since the vertex matrices B_v are self complementary, $\mathrm{Dom}(\mathcal{L}) \subset \mathrm{Dom}(\mathcal{L}^*)$ by Theorem 3.1. Since $L = L^+$, $\mathcal{L}$ is symmetric. It then follows [11, p. 270] that

$$\|(\mathcal{L} \pm i)f\| \ge \|f\|.$$

By Lemma 3.3 the closures of $(\mathcal{L} \pm i)$ are boundedly invertible, so [13, p. 256] $\mathcal{L}$ is essentially self adjoint.

On the other hand, if $\mathcal{L}$ is local and self adjoint, with $\mathcal{D}_{\min} \subset \mathrm{Dom}(\mathcal{L})$, then by Corollary 3.2 and the first part of this theorem there are self complementary vertex matrices B_v, and a domain

$$\mathcal{D}_1 = \{f \in \mathcal{D}_{\mathrm{com}} \mid B_v\hat{f} = 0, \quad v \in \mathcal{V}\}$$

such that $\mathcal{D}_1 \subset \mathrm{Dom}(\mathcal{L})$ and the restriction of $\mathcal{L}$ to $\mathcal{D}_1$ is essentially self adjoint.

4 Schrödinger operators on graphs

For many applications of physical interest, the functions in $\mathrm{Dom}(\mathcal{L})$ will be continuous at the vertices. This condition can be express as a set of $\delta(v) - 1$ independent conditions at each vertex,

$$f_{\alpha_m}(v) = f_{\alpha_{m+1}}(v), \quad m = 1, \dots, \delta(v) - 1.$$

We turn to the example of Schrödinger operators $L = D^2 + p$ where one additional vertex condition will be needed to define a self-adjoint operator.

An independent vertex condition may be written as

$$\sum_{n=1}^{\delta(v)} d_n f'(\alpha_m) = \rho(v) f(v), \tag{3.d}$$

with not all coefficients equal to 0, and where $f(v)$ is the common value of the $f(\alpha_m)$. The example considered after Lemma 2.2 shows that for $L = D^2 + p$

$$[f,g]_v = \sum_n (-1)^{\sigma_n} \Big[f'(\alpha_m)\overline{g}(\alpha_m) - f(\alpha_m)\overline{g}'(\alpha_m) \Big], \quad \alpha_m \sim v.$$

Working directly with this form, it is a simple exercise to characterize the additional vertex conditions with the property that all functions satisfying the vertex conditions are annihilated by the form. The following result is thus obtained.

Corollary 4.1 *Suppose that $w_n \ge C > 0$ for all n, and that $L = D^2$. The operator $\mathcal{L}$ whose vertex conditions $B_v \hat{f} = 0$ include the continuity conditions $f(\alpha_m) - f(\alpha_{m+1}) = 0$ for $1 \le m \le \delta(v) - 1$ at each vertex $v \in \mathcal{G}$, and one additional boundary condition of the form*

$$\gamma \sum_{n=1}^{\delta(v)} (-1)^{\sigma_n} f'(\alpha_m) - \rho f(v) = 0, \quad \rho, \gamma \in R, \quad \rho^2 + \gamma^2 \ne 0,$$

will be essentially self adjoint on $\mathrm{Dom}(\mathcal{L}) = \{ f \in \mathcal{D}_{\mathrm{com}} \mid B_v \hat{f} = 0, \quad v \in \mathcal{V} \}$. *Conversely every local self-adjoint operator $\mathcal{L}_1 = D^2$ whose domain includes $\mathcal{D}_{\min}$ and satisfies the continuity conditions at every vertex is the closure of one of the operators $\mathcal{L}$.*

One may immediately extend this corollary to $L = D^2 + p$ for a real bounded measurable function p by a standard perturbation result [11, p. 287]. For operators on the real axis, these vertex conditions are known as δ(function) interactions. See an extensive treatment of such operators in [1].

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