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BRANCHING OF PERIODIC ORBITS FROM KUKLES ISOCHRONES

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Abstract. We study local bifurcations of limit cycles from isochronous (or linearizable) centers. The isochronicity has been determined using the method of Darboux linearization, which provides a birational linearization for the examples that we analyze. This transformation simplifies the analysis by avoiding the complexity of the Abelian integrals appearing in other approaches. As an application of this approach, we show that the Kukles isochrone (linear and nonlinear) has at most one branch point of limit cycles. Moreover, for each isochrone, there are small perturbations with exactly one continuous family of limit cycles.

1. Introduction

In this paper we address the bifurcations of limit cycles (isolated periodic orbits) for polynomial perturbations of polynomial integrable vector fields. When the unperturbed system is isochrone (linearizable), the linearization is known, and is a birational transformation in the phase plane, which is, in general, a Darboux linearization, i.e., a linearizing transformation involving polynomial maps and their complex powers, [5].

Specifically, we consider an autonomous polynomial perturbation (p, q) of a plane vector field in the form

$$\mathcal{X}_\epsilon := (P(x,y) + \epsilon p(x,y))\frac{\partial}{\partial x} + (Q(x,y) + \epsilon q(x,y))\frac{\partial}{\partial y}, \qquad (\mathcal{P}_\epsilon)$$

where

$$P(x,y) = -y + \sum_{2\le i+j\le n} P_{ij}x^i y^j, \quad Q(x,y) = x + \sum_{2\le i+j\le n} Q_{ij}x^i y^j$$

$$p(x,y) = \sum_{i=1}^{n}\sum_{k=0}^{i} a_{i-k,k}x^{i-k}y^k, \quad q(x,y) = \sum_{i=1}^{n}\sum_{k=0}^{i} b_{i-k,k}x^{i-k}y^k,$$

and $\lambda_{ij} = (a_{ij}, b_{ij}, p_{ij}, q_{ij}) \in \mathbb{R}^4$, with ϵ a small parameter. When $\epsilon = 0$, we assume further that the unperturbed vector field $(\mathcal{X}_0)$ has an isochronous center at the origin $O \in \mathbb{R}^2$, i.e., all orbits in a sufficiently small neighborhood $\mathcal{A}$ of the origin are closed and have the same period. The largest such neighborhood is called an *isochronous period annulus*. For a closed orbit $\gamma_0 \in \mathcal{A}$, it is interesting to study the

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creation of limit cycles from γ_0 on passing from $\epsilon = 0$ to small nonzero values of ϵ. Recall that a limit cycle is a periodic orbit isolated in the set of closed orbits of the vector field.

For fixed λ_{ij}, there is a neighborhood U of the origin in $\mathbb{R}^2$ on which the flow associated with $(\mathcal{P}_\epsilon)$ exists for all initial values in U. Assume, furthermore, that U is small enough so that a Poincaré return mapping $\delta(r,\epsilon)$ is defined on U, with the distance coordinate r. The solution $\gamma_\epsilon(t)$ starting at $(r,0)$, $r > 0$, intersects the positive $x-$axis for the first time at some point $(\delta(r,\epsilon),0)$. Let $\Sigma = \{(x,0) \in U, x > 0\}$ denote the transversal section or Poincaré section of U. By transversality the mapping δ is analytic, and can be expanded as the convergent Taylor series

$$\delta(r,\epsilon) = r + \sum_{k\geq 1} \delta_k(\epsilon) r^k. \tag{1-1}$$

On Σ we define the displacement function $d(r,\epsilon) := \delta(r,\epsilon) - r$. Of course, the zeros of $d(r,\epsilon)$ correspond to periodic orbits of $(\mathcal{P}_\epsilon)$ intersecting Σ. Assuming that the period annulus $\mathcal{A}$ is parametrized by r, then $d(r,0) \equiv 0$. We reduce the analysis to that of finding the roots of a suitable bifurcation function derived from the displacement function. This is achieved by investigating the number and position of the periodic orbits in the isochronous period annulus $\mathcal{A}$ that survive after perturbation by giving birth to a continuous family γ_ϵ of limit cycles of the perturbed system.

In section two below, we describe our *isochrone reduction* method for studying both the first order bifurcations of limit cycles in autonomous perturbations of a polynomial isochronous system, and the branching of periodic orbits from isochrones. Section three is entirely devoted to Kukles isochrones. We show that at most one local family of limit cycles bifurcates at first order from these isochrones. Moreover, there exist perturbations which exhibit exactly one such perturbation. Each limit cycle is asymptotic to a circle whose radius is a simple positive zero of the bifurcation function.

2. Isochrone reduction

Under the previous assumptions, consider an element $r_* \in \Sigma$ such that

$$d_\epsilon(r_*,0) = 0, \quad \text{and } d_{r\epsilon}(r_*,0) \neq 0\,, \tag{2-1}$$

i.e., r_* is a simple zero of d_ϵ; the subscripted ϵ and r denote partial derivatives. Thus, by the Implicit Function Theorem, there exits a smooth function $r = \omega(\epsilon)$ defined in some neighborhood of $\epsilon = 0$, such that $\omega(0) = r_*$ and $d(\omega(\epsilon),\epsilon) \equiv 0$. The curve $r = \omega(\epsilon)$ corresponds to a local family of limit cycles emerging from the periodic trajectory γ_{r_*} of the unperturbed system which meets Σ at r_*. A difficulty arises in the calculations and analysis of the partial derivatives of $d(r,\epsilon)$. Of course, for $d_\epsilon(r,0) \equiv 0$, or if one of the zeros is not simple, then higher-order derivatives must be computed. Actually, in $\mathcal{A}$, $d_\epsilon(r,0) = 0$ for all values of r, and so we cannot apply the Implicit Function Theorem. However, from the perturbation of the Taylor series

$$d(r,\epsilon) = \epsilon d_\epsilon(r,0) + O(\epsilon^2) = \epsilon(d_\epsilon(r,0) + O(\epsilon)) = \epsilon B(r,\epsilon), \tag{2-2}$$

with $B(r,\epsilon) := d_\epsilon(r,0) + O(\epsilon)$, we define a reduced displacement function by

$$B(r) := d_\epsilon(r,0), \tag{2-3}$$

for small real values of ϵ. Clearly, if $B(\omega(\epsilon),\epsilon) \equiv 0$ then $d(\omega(\epsilon),\epsilon) \equiv 0$ and the Implicit Function Theorem does apply to B. In other words, a simple zero of B corresponds to the appearance of a local family $r = \omega(\epsilon)$ of periodic orbits. Such a zero, r_*, of B is called *a branch point of periodic orbits* for the system $(\mathcal{P}_\epsilon)$. The corresponding periodic orbit γ_{r_*} is said to *survive* or to *persist* after perturbation.

If r_* is a simple root of $B(r)$ of order k, i.e., $\partial_\epsilon^k d(r_*,0) = 0$, $\partial_r \partial_\epsilon^k (r_*,0) \neq 0$, with $\partial_\epsilon^i d(r_*,0) \equiv 0$, for $i = 0,\dots,(k-1)$, then writing the perturbation Taylor series in the form

$$d(r,\epsilon) = \epsilon^k(\partial_\epsilon^k d(r_*,0)/k! + O(\epsilon)) := \epsilon^k B^k(r,\epsilon) \tag{2-4}$$

yields $B^k(r_*,0) = 0$ and $B_r^k(r_*,0) \neq 0$. Applying the Implicit Function Theorem to B^k, we see that by continuity, there is a number $\epsilon_1 > 0$ and a unique smooth function $r = \omega(\epsilon)$ with $|\epsilon| < \epsilon_1$ such that $\omega(0) = r_*$ and $d(\omega(\epsilon),\epsilon) \equiv 0$. If r_* is a root of multiplicity m, it follows from the Weierstrass Preparation theorem [6] that there at most m distinct smooth functions $r = \omega_i(\epsilon)$.

In the case of an isochronous period annulus the isochronal assumption is essential to our approach. It is well known (see, e.g., [5]) that the origin of $(\mathcal{P}_\epsilon)$ is isochronous if and only if there exists an analytic change of coordinates

$$(\mathcal{T}_l):\ (u(x,y),v(x,y)) = (x + o(|(x,y)|), y + o(|(x,y)|))$$

in its neighborhood, reducing the system to a linear isochrone. Once we know explicitly $(\mathcal{T}_l)$, we reduce the autonomous perturbation of the nonlinear isochrone to that of a linear one; we then derive a simple expression of the bifurcation function B. Practically speaking, consider the perturbed system $(\mathcal{P}_\epsilon)$. Through $(\mathcal{T}_l)$, $(\mathcal{P}_\epsilon)$ is simplified to the weakly linear system

$$\dot v = Av + \epsilon h(v), \tag{$\bar{\mathcal{P}}_\epsilon$}$$

with $v := (X,Y) \in \mathbb{R}^2$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $h(v) := (h_1(v), h_2(v))$.

The determination of the branch points of the periodic orbits of $(\mathcal{P}_\epsilon)$ proceeds as follows. We first reduce the appropriate displacement function to a bifurcation function and apply the Implicit Function Theorem and its related corollaries. We next identify the bifurcation function in terms of the reduced perturbed system; the branch points are its simple zeros.

Theorem 2.1. *Consider a weakly linear system in the form $(\bar{\mathcal{P}}_\epsilon)$. Assume the unperturbed system has a period annulus parametrized by r. A branch point of periodic orbits of $(\bar{\mathcal{P}}_\epsilon)$ is a simple zero of the function*

$$B(r) := \int_0^{2\pi} (h_1(r\cos t, r\sin t)\cos t + h_2(r\cos t, r\sin t)\sin t)\,dt,$$

where r is taken in an interval of $(0,\infty)$.

Proof. Given $d(r,\epsilon)$, the associated displacement function, defined globally on the Poincaré section Σ, the bifurcation function is defined as $B(r) := d_\epsilon(r,0)$ for small values of ϵ. Using a periodic orbit γ with integral curve $\gamma_\epsilon(r,t) := (X(t,r,\epsilon), Y(t,r,\epsilon))$ starting at $(r,0)$ we obtain

$$d_\epsilon(r,0) = \dot{X}(T(r,0),r,0)T_\epsilon(r,0) + X_\epsilon(T(r,0),r,0). \tag{2-5}$$

At $r=0$ we have $\dot{X}(T(r,0),r,0) = -Y(0,r,0) = 0$. Thus, we obtain $d_\epsilon(r,0) = X_\epsilon(T(r,0),r,0)$. Looking for $X_\epsilon(T(r,0),r,0)$ amounts to integrating the variational equation

$$\begin{aligned} \dot{X}_\epsilon &= -Y_\epsilon + h_1(X,Y), \\ \dot{Y}_\epsilon &= X_\epsilon + h_2(X,Y), \\ X_\epsilon(0,r,0) &= Y_\epsilon(0,r,0) = 0\,. \end{aligned} \tag{2-6}$$

In matrix form it is expressed as

$$\begin{aligned} \dot{W} &= AW + H(t), \\ W(0) &= 0\,, \end{aligned} \tag{2-7}$$

where A is as given above, and

$$H(t) = \begin{pmatrix} h_1(X(t,r,\epsilon), Y(t,r,\epsilon)) \\ h_2(X(t,r,\epsilon), Y(t,r,\epsilon)) \end{pmatrix}. \tag{2-8}$$

By the method of variation of constants, we get

$$\begin{aligned} W(T(r,0)) =& (X_\epsilon(T(r,0),r,0), Y_\epsilon(T(r,0),r,0)) \\ =& \Phi(T(r,0)) \int_0^{T(r,0)} \Phi^{-1}(s) H(\gamma(r,s))\, ds, \end{aligned} \tag{2-9}$$

where $\Phi(t)$ denotes the principal fundamental matrix solution of $\dot{W} = AW$ at $t=0$. We have

$$\Phi(t) = e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{2-10}$$

and $H(\gamma(r,t)) = \begin{pmatrix} h_1(r\cos t, r\sin t) \\ h_2(r\cos t, r\sin t) \end{pmatrix}$. Hence, for $T(r,0) = 2\pi$, it follows

$$\begin{pmatrix} X_\epsilon(2\pi,r,0) \\ Y_\epsilon(2\pi,r,0) \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} (h_1(r\cos s, r\sin s)\cos s + h_2(r\cos s, r\sin s)\sin s)\, ds \\ \int_0^{2\pi} (-h_1(r\cos s, r\sin s)\sin s + h_2(r\cos s, r\sin s)\cos s)\, ds \end{pmatrix} \tag{2-11}$$

Thus we obtain

$$\begin{aligned} B(r) = d_\epsilon(r,0) =& X_\epsilon(2\pi,r,0) \\ =& \int_0^{2\pi} (h_1(r\cos s, r\sin s)\cos s + h_2(r\cos s, r\sin s)\sin s)\, ds. \end{aligned} \tag{2-12}$$

□

Perturbations of the linear isochrone. We consider a perturbation of degree n of the linear isochrone and prove the following theorem.

Theorem 2.2. *From the linear isochrone, to first order, no more than $(n-1)/2$, (resp. $(n-2)/2$) continuous families of limit cycles can bifurcate in the direction of any autonomous polynomial perturbation of degree n, where n is odd (resp. even). And we can construct small perturbations with the maximum number of limit cycles. Moreover the limit cycles are asymptotic to the circles whose radii are simple positive roots of the bifurcation function.*

Proof. Using the expressions of p and q as polynomials of degree n in $(\mathcal{P}_\epsilon)$, we compute the bifurcation function B and obtain

$$B(r) = \sum_{i=1}^{n} r^i \sum_{k=0}^{i} \Big(\int_0^{2\pi} (a_{i-k,k}\cos t + b_{i-k,k}\sin t)\cos^{i-k} t \sin^k t\, dt \Big). \tag{2-13}$$

This can be simplified using the well known rules $\int_0^{2\pi} \cos^m t \sin^n t dt = 0$, for m or n odd (including 0). As a result

$$B(r) = r \sum_{s=1, s \text{ odd}}^{N} r^{s-1} c_s, \tag{2-14}$$

where

$$N = \begin{cases} n, & \text{for } n \text{ odd} \\ n-1, & \text{for } n \text{ even} \end{cases} \tag{2-15}$$

and c_s is the nonzero constant

$$c_s = (a_{s0} + b_{0s}) + \sum_{k=1, \text{ k odd}}^{s-2} (b_{s-k,k} + a_{s-k-1,k+1}) \int_0^{2\pi} \cos^{s-k} t \sin^{k+1} t\, dt. \tag{2-16}$$

Therefore the upper bound of the number of simple zeros of $B(r)$ is $M(n) = (n-1)/2$ for n odd and $(n-2)/2$ for n even. Perturbations with the maximum number are constructed as in the cubic case below.

As $\epsilon \longrightarrow 0$, the weakly linear system $(\bar{\mathcal{P}}_\epsilon)$ tends to the linear isochrone whose solution curves are circles $x^2 + y^2 = r^2$. Therefore the periodic orbits (limit cycles) are asymptotic to these circles as $\epsilon \longrightarrow 0$. □

Remark. Note that as an application to first order, no limit cycles can emerge from periodic trajectories of the linear isochrone after a quadratic autonomous perturbation, in agreement with a result in [1] (Section 3.1: The linear isochrone). For a cubic autonomous perturbation, we obtain the following

Corollary 2.3. *From a periodic trajectory γ_0 in the period annulus $\mathcal{A}$ of the linear isochrone, at most one continuous family of limit cycles bifurcate from γ_0 in the direction of the cubic autonomous perturbation (p, q). The maximum number one is attained if and only if the coefficients satisfy the condition $c_0 c_2 < 0$, where c_0 and c_2 are given below. In this instance, this family emerges from the real simple roots of the quadratic function*

$$\Delta(r) := c_0 + c_2 r^2.$$

Proof. The corresponding bifurcation function is given by

$$B(r) = (r\pi)\left((a_{10} + b_{01}) + \frac{r^2}{4}(3a_{30} + a_{12} + b_{21} + 3b_{03})\right). \tag{2-17}$$

Thus, the roots of the quadratic $\Delta(r) := a_{10} + b_{01} + \frac{r^2}{4}(3a_{30} + a_{12} + b_{21} + 3b_{03})$ yield the continuous families of limit cycles that bifurcate from the period annulus at the origin of the linear isochrone. Define

$$c_2 := \frac{1}{4}(3a_{30} + a_{12} + b_{21} + 3b_{03}), \quad c_0 := (a_{10} + b_{01}). \tag{2-18}$$

If $c_0 = 0$ and $c_2 \neq 0$, then the origin is the only root of the polynomial $\Delta(r)$; however, But $c_0 \neq 0$ implies $r^2 = -c_2/c_0$. Therefore, the condition $c_0c_2 < 0$ gives exactly two real roots of opposite signs that must be simple. Only the positive root is accounted for. Moreover one may construct perturbations with condition $c_0c_2 < 0$. Hence the corollary is proven □

In [5], there are several examples of systems for which the isochronous strata are known and an algebraic linearizing transformation is given explicitly. Thus, there are many problems that can be solved using our approach. For the sake of illustration, we choose to address the bifurcations of limit cycles from Kukles isochrones.

3. First order bifurcations from Kukles isochrones

We consider the reduced Kukles system in the form

$$\begin{aligned} \dot{x} &= -\,y\,, \\ \dot{y} &= x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2, \end{aligned} \tag{$\mathcal{K}$}$$

parametrized by $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{R}^6$, see [2,4]. From [7], the following theorem gives the Kukles isochrone, and actually shows that it does possess a birational linearizing transformation as required.

Kukles isochrone. *The origin is an isochronous center of $(\mathcal{K})$ if and only if the system is linear or can be brought, through rescaling of (x, y) and t to the form*

$$\begin{aligned} \dot{x} &= -\,y\,, \\ \dot{y} &= x + 3xy + x^3. \end{aligned} \tag{$\mathcal{K}_0$}$$

Moreover, a rational linearizing change of coordinates of the system $(\mathcal{K}_0)$ is given by

$$(u(x, y), v(x, y)) = \left(\frac{x}{x^2 + y + 1}, \frac{x^2 + y}{x^2 + y + 1}\right). \tag{$\mathcal{T}_l$}$$

First Order Perturbations

Consider a one-parameter cubic autonomous perturbation $(\mathcal{K}_\epsilon)$ of the Kukles nonlinear isochrone $(\mathcal{K}_0)$ in the form

$$\begin{aligned}\dot{x} =& - y + \epsilon p(x,y)\\ \dot{y} =& x + 3xy + x^3 + \epsilon q(x,y),\end{aligned} \qquad (\mathcal{K}_\epsilon)$$

where the parameter $\epsilon \in \mathbb{R}$, and p and q are polynomials of degree 3. From the linearizing change of coordinates $(\mathcal{T}_l)$, and by resetting $(u(x,y), v(x,y)) = (f^*(x,y), g^*(x,y)$, we derive the inverse transformation $(\mathcal{T}_l^{-1})$

$$x(u,v) = f(u,v) = \frac{u}{1-v}, \quad y(u,v) = g(u,v) = \frac{v-(u^2+v^2)}{(1-v)^2}\,. \qquad (3\text{-}1)$$

The system $(\mathcal{K}_\epsilon)$ is transformed via $(\mathcal{T}_l)$ into the weakly linear system

$$\begin{aligned}\dot{u} =& - v + \epsilon\bar{p}(u,v)\,,\\ \dot{v} =& u + \epsilon\bar{q}(u,v)\,,\end{aligned} \qquad (\bar{\mathcal{K}}_\epsilon)$$

where we have

$$\bar{p}(u,v) = f_x^*(u,v)p + f_y^*(u,v)q\,, \quad \bar{q}(u,v) = g_x^*(u,v)p + g_y^*(u,v)q\,, \qquad (3\text{-}2)$$

with

$$\begin{aligned} f_x^*(u,v) =& \frac{\partial f^*}{\partial x}(u,v) = 1 - 2u^2 - v \quad \text{and } f_y^*(u,v) = \frac{\partial f^*}{\partial y}(u,v) = -u(1-v)\,,\\ g_x^*(u,v) =& \frac{\partial g^*}{\partial x}(u,v) = 2u(1-v), \quad \text{and } g_y^*(u,v) = \frac{\partial g^*}{\partial y}(u,v) = (1-v)^2\,.\end{aligned} \qquad (3\text{-}3)$$

Calculation of the bifurcation function yields

$$\begin{aligned} B(r) =& \int_0^{2\pi} (\bar{p}(r\cos t, r\sin t)\cos t + \bar{q}(r\cos t, r\sin t)\sin t)dt\\ =& \sum_{i=1}^{3} r^i \sum_{k=0}^{i} (R_1^{ik} a_{i-k,k} + R_2^{ik} b_{i-k,k}),\end{aligned}$$

with

$$\begin{aligned} R_1^{ik} =& \int_0^{2\pi} \frac{\cos^{i-k+1} t(\sin t - r)^k (1 - 2r^2 + r\sin t)}{(1 - r\sin t)^{i+k}} dt\\ R_2^{ik} =& \int_0^{2\pi} \frac{\cos^{i-k} t(\sin t - r)^{k+1}}{(1 - r\sin t)^{i+k-1}} dt.\end{aligned} \qquad (3\text{-}4)$$

For computational reasons, the expression of $B(r)$ is better expressed as

$$B(r) = (\bar{R}_1 + \bar{R}_2 + \bar{R}_3), \qquad (3\text{-}5)$$

with

$$
\begin{aligned}
\bar{R}_1 =& r(a_{10}R_1^{10} + a_{01}R_1^{11} + b_{10}R_2^{10} + b_{01}R_2^{11})\,,\\
\bar{R}_2 =& r^2(a_{20}R_1^{20} + a_{11}R_1^{21} + a_{02}R_1^{22} + b_{20}R_2^{20} + b_{11}R_2^{21} + b_{02}R_2^{22})\\
\bar{R}_3 =& r^3(a_{30}R_1^{30} + a_{21}R_1^{31} + a_{12}R_1^{32} + a_{03}R_1^{33} + b_{30}R_2^{30} + b_{21}R_2^{31}\\
&+ b_{12}R_2^{32} + b_{03}R_2^{33}).
\end{aligned}
\tag{3-6}
$$

Results from the theory of Residues were used to derive Equation (3-6). The powers of $\cos t$ are odd in the cases

$$
(i,k) \in \begin{cases} \{(2,0);(3,1);(1,1);(2,2);(3,3)\}\,, & \text{for } R_1^{ik} \\ \{(1,0);(2,1);(3,0);(3,2)\}\,, & \text{for } R_2^{ik}, \end{cases}
\tag{3-7}
$$

yielding zero integrals. Thus, we concentrate on the computation of R_l^{ik} with even powers of cosine, that is,

$$
(i,k) \in \begin{cases} \{(1,0);(2,1);(3,0);(3,2)\} & \text{for } R_1^{ij} \\ \{(2,0),(3,1),(1,1),(2,2),(3,3)\} & \text{for } R_2^{ij}\,. \end{cases}
\tag{3-8}
$$

The integrals R_1^{ik} and R_2^{ik} are rational functions of $\sin t$. Through the change of variable $\sin t = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, these integrals consist of terms of the form

$$
T_n^h = \frac{2^{n-h}}{r^n i^{1+h+n}} \int_C \frac{a(z)}{b(z)} dz,
\tag{3-9}
$$

with

$$
\begin{aligned}
a(z) =& z^{n-h-1}(z^2-1)^h \quad \text{(a polynomial in } z \text{ of degree } N = n+h-1),\\
b(z) =& (z-z_1)^n(z-z_2)^n \quad \text{(a polynomial in } z \text{ of degree } M = 2n)\\
=& (z^2 - 2\rho i z - 1)^n,
\end{aligned}
\tag{3-10}
$$

and $z_{1,2} = \mp(\sqrt{1-\rho^2} + \rho i)$ with $\rho = 1/r$. We may therefore apply the following well-known lemma, [3].

Lemma 3.1. *Let $a(z) = \sum_{k=0}^{N} a_k z^{N-k}$ and $b(z) = \sum_{k=0}^{N} b_k z^{N-k}$ be polynomials in z of respective degrees N and M, with $a_0 \neq 0$, $b_0 \neq 0$. Let C be a simple closed contour enclosing all zeros of $b(z)$. Then*

$$
\int_C \frac{a(z)}{b(z)}\, dz = \begin{cases} 2i\pi \frac{a_0}{b_0}, & \textit{if } M - N = 1\,, \\ 0, & \textit{if } M - N \geq 2\,. \end{cases}
\tag{3-11}
$$

This lemma implies that

$$
T_n^h = \begin{cases} 2\pi(-r)^{-n}, & \text{if } n = h, \\ 0, & \text{if } n \geq h+1. \end{cases}
\tag{3-12}
$$

For the remaining cases, corresponding to $n < (h+1)$, we use the fact that

$$
\int_C F_1(z)dz = 2i\pi \operatorname*{Res}_{z=0}\Big(\frac{1}{z^2} F_1\left(\frac{1}{z}\right)\Big) = \frac{(1-z^2)^h}{z^{h+1-n}(z^2 - 2i\rho z - 1)^n}\,,
\tag{3-13}
$$

with $F_1(z) = \frac{a(z)}{b(z)}$. Thus $z = 0$ is a pole of order $m = h + 1 - n$. The residue at this pole is computed by means of classical residue techniques. We then prove the following theorem.

Theorem 3.2. *Define A_0 and A_2 by*

$$A_2 = 4a_{10} + 4b_{01} + 2b_{20} - 15a_{30} + 4b_{21} + a_{11} \text{ and}$$
$$A_0 = 2a_{10} + 2b_{01} + 4b_{20} + 2b_{02} - 8a_{11} - 18a_{30} - 6b_{21} - 2a_{12}\,.$$

From γ_0, a periodic trajectory in the period annulus of the non-linear isochrone $(\mathcal{K}_0)$, one continuous family of limit cycles bifurcates in the direction of the cubic perturbation (p,q) if and only if the coefficients satisfy the condition $A_0A_2 < 0$. When this condition is met, this family emerges exactly from the real simple positive root of the quadratic function

$$\Lambda(r) := A_0 + A_2r^2.$$

Moreover, at most one such family of limit cycles emerge for fixed (p,q).

Proof. Computation of the previous integrals in (3-9) yields

$$T_n^h = \begin{cases} 2\pi(-r)^{-n}, & \text{if } (n,h) \in \{(1,1);(3,3);(5,5)\}\ , \\ 0, & \text{if } (n,h) \in \{(1,0);(3,0);(3,1);(3,2);(5,0);(5,1);(5,2);(5,3);(5,4)\}\ . \end{cases} \tag{3-14}$$

and, respectively, for $(n,h) \in \{(1,2);(1,3);(3,4);(3,5)\}$, we have

$$T_1^2 = \frac{-2\pi}{r^2}; \quad T_1^3 = (-2\pi)\frac{2-r^2}{r^3}; \quad T_3^4 = \frac{-6\pi}{r^4}; \quad T_3^5 = \pi\frac{(24-r^2)}{r^5}\,. \tag{3-15}$$

Thus,

$$B(r) = \left(\bar{R}_1 + \bar{R}_2 + \bar{R}_3\right) = (\pi/\xi)\left(A_2r^2 + A_0\right), \tag{3-16}$$

with

$$\begin{aligned} A_2 =& 4a_{10} + 4b_{01} + 2b_{20} - 15a_{30} + 4b_{21} + a_{11} \text{ and} \\ A_0 =& 2a_{10} + 2b_{01} + 4b_{20} + 2b_{02} - 8a_{11} - 18a_{30} - 6b_{21} - 2a_{12}\,. \end{aligned} \tag{3-17}$$

The branch points of periodic orbits are the positive roots of the quadratic function $\Lambda(r) := A_2r^2 + A_0 = 0$. Hence the result. □

Concluding remarks. The integral $B(r)$ is the first variation of the displacement function with respect to the bifurcation parameter, and its simple zeros are the branch points of periodic orbits. If it vanishes identically the higher variations have to be computed and analyzed. How many variations are sufficient to make the final conclusions about the limit cycles is highly nontrivial. That is the core content of Bautin's result for quadratic systems that inspired [1]. Moreover, we wish to emphasize here that the method of *Isochrone Reduction*, described above, might be applied with much success to various isochronous strata (e.g. quadratic, symmetric cubic, Hamiltonian isochrones) and in the more general case of Darboux linearizable systems. To our knowledge it is the first application of Darboux linearization of isochronous centers to the study of bifurcations of limit cycles.

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