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Decay of solutions of a degenerate hyperbolic equation *

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Abstract

This article studies the asymptotic behavior of solutions to the damped, non-linear wave equation

$$\ddot{u}+\gamma\dot{u}-m(\|\nabla u\|^2)\Delta u=f(x,t)\,,$$

which is known as degenerate if the greatest lower bound for m is zero, and non-degenerate if the greatest lower bound is positive. For the non-degenerate case, it is already known that solutions decay exponentially, but for the degenerate case exponential decay has remained an open question. In an attempt to answer this question, we show that in general solutions can not decay with exponential order, but that $\|\dot{u}\|$ is square integrable on $[0,\infty)$. We extend our results to systems and to related equations.

1 Introduction

This article presents a study of the asymptotic behavior of solutions to the initial value problem

$$\begin{gathered}\ddot{u}+\gamma\dot{u}-m(\|\nabla u\|^2)\Delta u=f(x,t)\,,\quad \text{for } x\in\Omega,\ t\geq 0 \\ u(x,0)=g(x)\,,\quad \dot{u}(x,0)=h(x)\,,\quad \text{for } x\in\Omega \\ u(x,t)=0\,,\quad \text{for } x\in\partial\Omega\ \ t\geq 0\,;\end{gathered} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$; γ is a positive constant; m is a non-negative, bounded, and continuous function; $\dot{u}$ denotes the derivative of u with respect to time; and as usual

$$\Delta u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}\,,\quad \|\nabla u\|^2=\sum_{i=1}^{n}\int_\Omega|\frac{\partial u}{\partial x_i}|^2\,dx\,.$$

This equation appears in mathematical physics as the Carrier or Kirchoff equation, when modeling planar vibrations. For a background and physical properties of this model, we refer the reader to [3], [4], [8], [12], and their references.

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When the greatest lower bound for m is positive, (1) is known as non-degenerate, and it has been the subject of many publications (see [6], [9], [10], [12], and [14]). Global solutions have been obtained under various assumptions on the data, of which we are interested in the $(-\Delta)$-analyticity introduced by Pohozaev ([12]). Exponential decay in the non-degenerate case for (1) and for related problems has been obtained in [9] and [2], respectively.

When the greatest lower bound for m is zero, (1) is known as degenerate, and has been considered in just a few publications. In the special case $m(r) = r^\alpha, \alpha \geq 1$, existence of solutions and polynomial decay has been shown in [7] and [10]. For general m, assumed only to be continuous and bounded below by zero, the existence of global solutions was shown by Arosio and Spagnolo ([1]). Their article assumes that the initial data are $(-\Delta)$-analytic, and that $f \equiv 0, \gamma = 0$. Using the same analyticity assumption, and replacing the damping term $\gamma \dot{u}$ by a memory term, existence of global solutions has been proven in [5]. In spite of these developments, decay for degenerate problems remains an open question.

The outline of this article is as follows: Section 1 sketches the proof of existence of global solutions. Section 2 proves exponential decay for the non-degenerate case, and explains why these estimates can not be used in the degenerate case. Section 3 shows that if $m \equiv 0$, the decay is exponential. In general degenerate-problem solutions do not decay exponentially, as we indicate with an example, but $\|\dot{u}\|$ is square integrable on $[0, \infty)$. Section 4 extends our results to related systems.

Notation

For the remainder of this article, H denotes the standard Hilbert space $L^2(\Omega)$, with norm $\|.\|$ and inner product $\langle ., . \rangle$. We define the self-adjoint operator A as the negative Laplacian, with domain

$$\mathcal{D}(A) \subset H^2(\Omega) \cap H_0^1(\Omega)\,,$$

where H^2, H^1 are the usual Hilbert Sobolev spaces. The negative Laplacian, with zero boundary conditions, has eigenvectors denoted as follows

$$A\phi_i = \lambda_i^2 \phi_i\,, \quad \text{with } 0 < \lambda_1 \leq \lambda_2 \ldots\,, \quad \lim_{i \to \infty} \lambda_i = +\infty\,.$$

Furthermore, these eigenvectors can be chosen to form an orthogonal basis for $L^2(\Omega)$, in which functions have Fourier expansions of the form

$$u(x) = \sum_{i=1}^{\infty} u_i \phi_i(x)\,, \quad \text{with } u_i = \langle u, \phi_i \rangle\,.$$

Using this spectral decomposition, powers of A are defined by

$$A^k u(x) = \sum_{i=1}^{\infty} \lambda_i^{2k} u_i \phi_i(x)\,, \quad \text{provided that } \sum_{i=1}^{\infty} \lambda_i^{4k} |u_i|^2 < +\infty\,.$$

Notice that

$$\lambda_1^2\|u\| \leq \|Au\|\,, \quad \lambda_1\|u\| \leq \|\nabla u\| = \|A^{1/2}u\|\,, \tag{2}$$

and that (1) can be rewritten as

$$\begin{gathered}\ddot{u} + \gamma\dot{u} + m(\|A^{1/2}u\|^2)Au = f(x,t) \\ u(x,0) = g(x)\,, \quad \dot{u}(x,0) = h(x)\,. \end{gathered} \tag{3}$$

A function u is said to be A-analytic, or $(-\Delta)$-analytic, if there exists a positive constant η such that $\sum_{i=1}^{\infty} e^{2\eta\lambda_i}|u_i|^2 < \infty$. Notice that every A-analytic function is in the domain of all powers of A. Properties of A-analytic functions and an equivalent definition can be found in [1] and [5].

By a solution on $[0,T)$ we mean a function that satisfies (3) and belongs to $C^2(0,T;H^{-1}) \cap C(0,T;H^1)$.

2 Existence of global solutions

Theorem 1 *Assume that m is bounded or that $\int_0^\infty m = +\infty$; and that $f(.,t)$, g, and h are A-analytic for all $t \geq 0$. Then there exists a global solution to (1).*

Proof Solutions are obtained by the use of energy estimates and the Galerkin method which is a standard technique described in the book by Temam ([13]). To accommodate the fact that m may attain zero values, we follow the procedure used in [1]. However the presence of $\gamma\dot{u}$ and f requires some algebraic manipulations of the type shown in the proof of Theorem 4. Since the proof is basically the same as the one in [1], we shall indicate the main steps in this proof, and refer the reader to the original source.

Step 1. Replace $m(\|\nabla u\|^2)$ by a non-negative bounded continuous function $a(t)$. Then show that if u is a solution to the new equation on the interval $[0,T)$, then u admits a limit as $t \to T^-$, and u and $\dot{u}$ are A-analytic on $[0,T]$.

Step 2. Use the Galerkin method and a compactness imbedding argument to obtain a local solution. Then show that $m(\|\nabla u\|^2)$ is bounded in its domain of definition.

Step 3. Use Zorn's Lemma and the result in step 1 to show that the maximal domain of definition is $[0,+\infty)$.

Remark Uniqueness of solutions has been shown under the additional assumption that m is Lipschitz; see [1].

3 Decay in the non-degenerate case

The following statement is already known for $f = 0$ (see [9]). We present a proof for general f and show how the rate of decay depends on the lower bound for m.

Theorem 2 *Assume that $0 < m_0 \le m(.) \le M_0$, and that $\|f(.,t)\|$ decays exponentially to zero as $t \to \infty$. Then for every solution u of (1), $\|\dot u\|$ and $\|\nabla u\|$ decay exponentially to zero.*

Proof We shall find bounds for $\dot u$ and $A^{1/2}u$ by estimating the energy functional
$$F(t) = \|\dot u\|^2 + \int_0^{\|A^{1/2}u\|^2} m(r)\,dr + \delta\langle \dot u, u\rangle\,, \tag{4}$$
where $\delta = \min\{2\lambda_1\sqrt{m_0}, \gamma/2\}$. This choice of δ ensures that F is non-negative. In fact,
$$F(t) \ge \|\dot u\|^2 + m_0\|A^{1/2}u\|^2 - \|\dot u\|^2 - \frac{\delta^2}{4}\|u\|^2 \ge (m_0 - \frac{\delta^2}{4\lambda_1^2})\|A^{1/2}u\|^2 \ge 0\,.$$
Here and in several expressions to follow, we use inequalities of the form
$$2|\langle v, w\rangle| \le 2\|v\|\,\|w\| \le \alpha\|v\|^2 + \frac{1}{\alpha}\|w\|^2 \quad \forall \alpha > 0\,. \tag{5}$$
Also we will use the following two equations that arise from taking the inner product of each term in (3) with $2\dot u$, and with u, respectively.
$$\tfrac{d}{dt}\|\dot u\|^2 + 2\gamma\|\dot u\|^2 + \tfrac{d}{dt}\int_0^{\|A^{1/2}u\|^2} m(r)\,dr = 2\langle f, \dot u\rangle\,, \tag{6}$$
$$\langle \ddot u, u\rangle + \gamma\langle \dot u, u\rangle + m(\|A^{1/2}u\|^2)\|A^{1/2}u\|^2 = \langle f, u\rangle\,. \tag{7}$$
Now, we differentiate F and build a first-order linear inequality that yields the desired estimates.
$$\begin{aligned} F'(t) &= \frac{d}{dt}\|\dot u\|^2 + \frac{d}{dt}\int_0^{\|A^{1/2}u\|^2} m(r)\,dr + \delta\langle \ddot u, u\rangle + \delta\|\dot u\|^2 \\ &= -(2\gamma - \delta)\|\dot u\|^2 + 2\langle f, \dot u\rangle - \gamma\delta\langle \dot u, u\rangle - \delta m(.)\|A^{1/2}u\|^2 + \delta\langle f, u\rangle\,, \end{aligned}$$
where we have used (6) and (7). Now from (2) and (5) we obtain $2\langle f, \dot u\rangle \le \frac{2}{\gamma}\|f\|^2 + \frac{\gamma}{2}\|\dot u\|^2$ and
$$\delta\langle f, u\rangle \le \frac{\delta}{\lambda_1}\|f\|\,\|A^{1/2}u\| \le \frac{\delta}{2m_0\lambda_1^2}\|f\|^2 + \frac{\delta m_0}{2}\|A^{1/2}u\|^2\,.$$
Using the two inequalities above, and the fact that $m_0 \le m(.) \le M_0$, we obtain
$$\begin{aligned} F'(t) &\le -\gamma\|\dot u\|^2 - \frac{\delta m_0}{2M_0}\int_0^{\|A^{1/2}u\|^2} m(r)\,dr - \delta\gamma\langle \dot u, u\rangle + (\frac{2}{\gamma} + \frac{\delta}{2m_0\lambda_1^2})\|f\|^2 \\ &\le -c_1 F(t) + (\frac{2}{\gamma} + \frac{\delta}{2m_0\lambda_1^2})\|f\|^2\,, \end{aligned}$$
where $c_1 = \min\{\gamma, \frac{\delta m_0}{2M_0}\}$. From this first-order differential inequality, it follows that
$$F(t) \le e^{-c_1 t}\left(F(0) + (\frac{2}{\gamma} + \frac{\delta}{2m_0\lambda_1^2})\int_0^t e^{c_1 s}\|f(.,s)\|^2\,ds\right)\,.$$

From the assumption that $\|f(.,s)\|$ decays exponentially follows the existence of positive constants c_2, c_3, such that $F(t) \leq c_2 e^{-c_3 t}$, with $c_3 < c_1$. Therefore,

$$\|\dot{u}\|^2 \leq c_2 e^{-c_3 t}\,, \quad \|A^{1/2}u\|^2 \leq \frac{c_2}{m_0} e^{-c_3 t}\,, \quad \forall t \geq 0\,,$$

which concludes this proof ♢

Remark The order of exponential decay approaches zero as the lower bound m_0 approaches zero. This is so because the constant c_3 in Theorem 2 satisfies

$$c_3 < c_1 \leq \frac{\lambda_1}{2M_0}(m_0)^{3/2}$$

the right side of which approaches zero as m_0 approaches 0.

4 Decay in the degenerate case

We start with a positive statement about exponential decay.

Theorem 3 *Assume that $m \equiv 0$ and that $\|f(.,t)\|$ decays exponentially to zero as $t \to \infty$. Then for any solution u of (1), $\|\dot{u}\|$ decays exponentially to zero.*

Proof Since $m \equiv 0$, Equation (1) reduces to $\ddot{u} + \gamma\dot{u} = f$. By computing the inner product of $2\dot{u}$ with each term in this equation, we have

$$\begin{aligned} \tfrac{d}{dt}\|\dot{u}\|^2 + 2\gamma\|\dot{u}\| = 2\langle f, \dot{u}\rangle \leq \tfrac{1}{\gamma}\|f\|^2 + \gamma\|\dot{u}\|^2\,, \\ \tfrac{d}{dt}\|\dot{u}\|^2 + \gamma\|\dot{u}\| \leq \tfrac{1}{\gamma}\|f\|^2\,. \end{aligned}$$

This first-order differential inequality and the initial conditions yield the inequality

$$\|\dot{u}\|^2 \leq e^{-\gamma t}\big(\|h\|^2 + \frac{1}{\gamma}\int_0^t e^{\gamma s}\|f(.,s)\|^2\,ds\big)\,.$$

From the assumption that $\|f(.,s)\|$ decays exponentially there exist positive constants c_2, c_3, such that $\|\dot{u}\|^2 \leq c_2 e^{-c_3 t}$, with $c_3 < \gamma$. Which concludes this proof. ♢

The following example shows that decay of solutions is not necessarily exponential.

Example Consider the initial value problem

$$\begin{gathered} \ddot{u} + \dot{u} - m(\|u_x\|^2)\ u_{xx} = 0\,, \quad \text{for } 0 \leq x \leq 2\pi,\ t \geq 1 + \sqrt{2} \\ u(x, 1+\sqrt{2}) = \tfrac{1}{\sqrt{\pi}} e^{1/(1+\sqrt{2})}\sin x\,, \quad \dot{u}(x, 1+\sqrt{2}) = \tfrac{1}{9\sqrt{\pi}} e^{1/(1+\sqrt{2})}\sin x \\ u(0,t) = 0\,, \quad u(2\pi, t) = 0\,, \quad \text{for } t \geq 1+\sqrt{2}\,, \end{gathered}$$

where m is the non-negative and continuous function defined as

$$m(r) = \begin{cases} \frac{1}{16}(\ln r)^2(4 - 4\ln r - (\ln r)^2) & \text{if } 1 \le r \le e^{2/(1+\sqrt{2})}\,, \\ 0 & \text{Otherwise}\,. \end{cases}$$

Then $u(x,t) = \frac{1}{\sqrt{\pi}} e^{1/t} \sin x$ is a solution. Since

$$\dot{u} = -\tfrac{1}{t^2} u\,, \quad \ddot{u} = (\tfrac{1}{t^4} + \tfrac{2}{t^3})u\,, \quad u_x = \frac{1}{\sqrt{\pi}} e^{1/t} \cos x\,, \quad u_{xx} = -u\,,$$

$\|u_x\|^2 = e^{2/t}$, and $m(e^{2/t}) = \frac{1}{t^2} - 2\frac{1}{t^3} - \frac{1}{t^4}$ for $t \ge 1 + \sqrt{2}$, it follows that u satisfies the initial-value problem. Notice that $\|\dot{u}\|$ decays polynomially rather than exponentially as $t \to \infty$. In fact,

$$\|\dot{u}\|^2 = \frac{1}{t^4} e^{2/t} = O(t^{-4})\,.$$

For non-constant m, the convergence of $\|\dot{u}\|$ to zero remains illusive: we are unable to prove it, and unable to give a counter-example. So far, our best result is:

Theorem 4 *If $\|f(.,t)\|$ is square integrable on $[0,\infty)$ and u is a solution to (1), then $\|\dot{u}\|$ is square integrable on $[0,\infty)$.*

Proof The desired integral is obtained by estimating the growth of the energy functional

$$E(t) = \|\dot{u}\|^2 + \int_0^{\|A^{1/2}u\|^2} m(r)\, dr\,. \tag{8}$$

Using (5) and (6), it follows that the derivative of E satisfies

$$E'(t) = -2\gamma\|\dot{u}\|^2 + 2\langle f, \dot{u}\rangle \le -\gamma\|\dot{u}\|^2 + \frac{1}{\gamma}\|f\|^2\,.$$

From this inequality and the Fundamental Theorem of Calculus, we obtain

$$E(t) + \gamma \int_0^t \|\dot{u}\|^2\, ds \le E(0) + \frac{1}{\gamma}\int_0^t \|f(.,s)\|^2\, ds\,.$$

Since by hypothesis $\int_0^\infty \|f\|^2 < \infty$, it follows that $\int_0^\infty \|\dot{u}\|^2 < \infty$, and the proof is complete. ♢

Remark From the physics point of view, Theorems 2 and 3 state that the energy $\|\dot{u}\|^2 + \|\nabla u\|^2$ decays as time goes by. In terms of non-linear dynamics (see [13]), these two theorems indicate that in the space of A-analytic functions, every ball of center zero and finite radius is an absorbent set (under the norm $\|\dot{u}\|^2 + \|\nabla u\|^2$). This means that given a ball of center zero, the orbit of every bounded set enters and stays in this ball after a certain time.

5 Extension of results

Higher order derivatives

To extend the previous analysis to equations that involve powers of A, for example Δ^2 which appears in modeling non-planar vibrations, we introduce the equation

$$\ddot{u} + \gamma\dot{u} + m(\|A^{\alpha/2}u\|^2)A^\alpha u + p(\|A^{\beta/2}u\|^2)A^\beta u = f(x,t)\,, \tag{9}$$

where α and β are non-negative integers, $\alpha > \beta$, and p is a bounded and continuous function. Existence of solutions is proven as in Theorem 1, with the assumption that $0 \le p(.) \le P_0$.

Theorem 4 is proven under the assumption $0 \le p(.) \le P_0$, in which case, the energy functional (4) is redefined to be

$$E(t) = \|\dot{u}\|^2 + \int_0^{\|A^{\alpha/2}u\|^2} m(r)\,dr + \int_0^{\|A^{\beta/2}u\|^2} p(r)\,dr\,.$$

Estimates for this functional require some algebraic manipulations, but otherwise the proof is the same as before.

Exponential decay is proven under the conditions of Theorem 2 and the assumption that $-m_0\delta\lambda_1^{\alpha-\beta} \le p(.) \le P_0$. Notice that p is allowed to assume negative values. When proving this statement, the energy functional (8) remains the same, but extra algebraic manipulations are required.

Systems of equations

Let $\mathbf{u}, \mathbf{f}, \mathbf{g}, \mathbf{h}$ be functions with values in $\mathbb{R}^k$, and m, γ be $k\times k$ diagonal matrices. Then rewrite (3) as

$$\begin{gathered}\ddot{\mathbf{u}} + \gamma\dot{\mathbf{u}} + m(\|A^{1/2}\mathbf{u}\|^2)A\mathbf{u} = \mathbf{f}(x,t)\,, \\ \mathbf{u}(x,0) = \mathbf{g}(x)\,, \quad \dot{\mathbf{u}}(x,0) = \mathbf{h}(x)\,.\end{gathered} \tag{10}$$

Now, components of vectors are denoted with sub-indices; derivatives are computed component-wise,

$$\dot{\mathbf{u}} = (\dot{u}_1, \dots, \dot{u}_k)\,, \quad A\mathbf{u} = (Au_1, \dots, Au_1)\,;$$

inner products and norms are redefined as

$$\langle \mathbf{u}, \mathbf{v}\rangle = \langle u_1, v_1\rangle + \ldots + \langle u_k, v_k\rangle\,, \quad \|\mathbf{u}\|^2 = \|u_1\|^2 + \ldots + \|u_k\|^2\,;$$

functions are A-analytic if their components are A-analytic; and eigenvectors of A have the form $e_i\phi_j$, $i = 1, \dots, k$, $j = 1, 2, \dots$, where e_i's are the standard basis for $\mathbb{R}^k$, and ϕ_j's are the eigenvectors defined in §1.

Because most of the previous estimates hold with very little modification, Theorems 1, 2, 4 are proven along similar lines to the previous proofs. For example the energy functional (4) is rewritten as

$$F(t)=\|\dot{\mathbf{u}}\|^2+\sum_{i=1}^{k}\int_0^{\|A^{1/2}u_i\|^2} m_{ii}(r)\,dr+\delta\langle\dot{\mathbf{u}},\mathbf{u}\rangle\,.$$

Computation of estimates for this functional depends on the constants $m_0=\min_i\inf_r m_{ii}(r)$, $\gamma_0=\min\gamma_{ii}$, and the inequality

$$2\langle\mathbf{v},\mathbf{w}\rangle\le\alpha\|\mathbf{v}\|^2+\frac{1}{\alpha}\|\mathbf{w}\|^2\quad\forall\alpha>0\,.$$

Notice that small modifications of the system (10) lead to a variety of control problems for which decay of solutions is of great interest. For example pre-multiply $\ddot{\mathbf{u}}$ by a diagonal matrix that has some entries equal to zero, and substitute m and γ by positive-definite matrices (instead of diagonal matrices).

Modified Carrier model

For equations in which the powers of A in the coefficient and in the argument of m are not in the ratio two to one, we introduce

$$\ddot{u}+\gamma\dot{u}+m(\|A^{\alpha/2}u\|^2)A^{\beta}u=f(x,t)\,,\tag{11}$$

where α and β are non-negative integers. Global solutions are obtained by the same method as the one used in Theorem 1.

Exponential decay is proven under the assumption that $\|A^{\alpha/2-\beta}f\|$ decays exponentially as $t\to\infty$. In proving this statement, we follow the proof of Theorem 2, with the energy functional

$$F(t)=\|A^{(\alpha-\beta)/2}\dot{u}\|^2+\int_0^{\|A^{\alpha/2}u\|^2} m(r)\,dr+\delta\langle A^{(\alpha-\beta)/2}\dot{u},A^{(\alpha-\beta)/2}u\rangle\,.$$

To estimate $F'(t)$, we use the following two equations that come from taking the inner product of (11) with $2A^{\alpha-\beta}\dot{u}$ and $A^{\alpha-\beta}u$, respectively.

$$\begin{gathered}\tfrac{d}{dt}\|A^{(\alpha-\beta)/2}\dot{u}\|^2+2\gamma\|A^{(\alpha-\beta)/2}\dot{u}\|^2+m(\|A^{\alpha/2}u\|)\tfrac{d}{dt}\|A^{\alpha/2}u\|^2\\ =2\langle A^{(\alpha-\beta)/2}\dot{u},A^{(\alpha-\beta)/2}f\rangle\,,\\ \langle A^{(\alpha-\beta)/2}\ddot{u},A^{(\alpha-\beta)/2}u\rangle+\gamma\langle A^{(\alpha-\beta)/2}\dot{u},A^{(\alpha-\beta)/2}u\rangle+m(\|A^{\alpha/2}u\|^2)\|A^{\alpha/2}u\|^2\\ =\langle A^{\alpha/2}u,A^{\alpha/2-\beta}f\rangle\,.\end{gathered}$$

As in Theorem 2, we obtain positive constants c_2,c_3 such that

$$\|A^{(\alpha-\beta)/2}\dot{u}\|^2\le c_2e^{-c_3t}\,,\quad\|A^{\alpha/2}u\|\le\frac{c_2}{m_o}e^{-c_3t}\,.$$

As in Theorem 4, under the assumption $\int_0^\infty \|A^{(\alpha-\beta)/2}f\|^2 < \infty$, and using the energy functional

$$E(t) = \|A^{(\alpha-\beta)/2}\dot{u}\|^2 + \int_0^{\|A^{\alpha/2}u\|^2} m(r)\,dr\,,$$

we obtain $\int_0^\infty \|A^{(\alpha-\beta)/2}\dot{u}\|^2 < \infty$.

Notice that the larger the difference $\alpha - \beta$, the higher the order of the decaying derivative. Also notice that the earlier example can be used to show that in systems the decay is not necessarily exponential. In fact, the same u and m satisfy (11) with $\alpha = 0$ and $\beta = 1$.

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