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Uniform controllability for Kirchhoff and Mindlin-Timoshenko elastic systems *

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Abstract

The Mindlin-Timoshenko operator is a perturbation of the Kirchhoff operator and it is well known that there exist solutions to the exact controllability problem for their associated systems. This article shows that the solution of the controlled problem of the Mindlin-Timoshenko system converges to that of Kirchhoff; that is, we show uniform controllability.

1 Introduction

We are concerned with the boundary controllability problem known as the *Exact-Controllability Problem* (ECP). Suppose we have a well-posed initial-boundary value problem for an evolution equation $Lw = 0$ in a cylindrical domain $Q = \Omega \times [0, +\infty)$, where Ω is a bounded domain in $\mathbb{R}^n$. The ECP deals with the following question: Given Cauchy data in Ω at $t = 0$, can this data be supplemented with appropriate inhomogeneous time-dependent boundary data (boundary controls), prescribed in the lateral boundary of Q, such that the solution of the initial-boundary problem will vanish for $t \geq T_1$?

In this work, we are interested in the evolution equations associated with the Kirchhoff operator

$$L = \rho h \partial_t^2 - \frac{\rho h^3}{12} \Delta \partial_t^2 + \Delta^2$$

and with the Mindlin-Timoshenko operator

$$L_K = \rho h \partial_t^2 - \frac{\rho h^3}{12} \Delta \partial_t^2 + \Delta^2 + \frac{\rho h}{K} \left(\frac{\rho h^3}{12} \partial_t^4 - \Delta \partial_t^2 \right) ,$$

where Δ is the Laplacian Operator acting in the space variables, i.e.,

$$\Delta = \sum_{i=1}^{n} \partial_{x_i}^2 .$$

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When $n = 2$, the equations $Lw = 0$ and $L_K w_K = 0$, in a suitable domain with appropriate initial and boundary data, model vibrations of elastic plates. In such a case $w(x,t)$ represents the vertical displacement of the *middle surface*. The physical constants are: $\rho \equiv$ density, $h \equiv$ plate thickness, $K \equiv$ shear modulus. The Mindlin-Timoshenko equation is obtained by uncoupling the Reissner-Mindlin Plate. See Lagnese [4], Lagnese & Lions [5] and Lagnese, Leugering & Schmidt [6].

Komornik [3] studies the Reissner-Mindlin plate equation and established that the control time is independent of K. A Similar result follows for the corresponding Timoshenko beam equation. To our knowledge, there are no results about convergence of the solution of the control problem for the Reissner-Mindlin system to that of the Kirchhoff operator.

For the Timoshenko beam equation, a solution to this problem is presented in Moreles [10]. In addition to uniform control time, it is shown that the solution of the control problem for the Timoshenko system, converges in a *strong* sense to the solution of the control problem of the Rayleigh system. The latter is the one dimensional version of the Kirchhoff system. In this work, we generalize this result to several space dimensions, in particular for plates.

The Mindlin-Timoshenko operator is hyperbolic and the solution of the ECP is a direct application of Theorem 2 in Littman [8]. The method of solution therein was successfully generalized to elastic operators (beams and plates) in Littman [9]. It is observed that the Kirchhoff operator does not correspond to those of the cited works. However, we show that Littman's method still applies and solve the ECP. Moreover the control time for both systems is the same and independent of K.

Once the ECP is solved for both operators we prove uniformity, that is, we show that the solution of the controlled problem for the Mindlin-Timoshenko operator converges to that of Kirchhoff when $K \to +\infty$. Our proof rests on the results in Moreles [10].

The outline of this article is as follows: Section 2 presents the statement of the main result. The proof is carried out in the remaining sections. To illustrate our result we consider a weak version of the ECP in Lasiecka & Triggiani [7]. In Section 3 we show the first step of Littman's method. More precisely, we show that the solutions of the Cauchy problems for the Mindlin-Timoshenko and Kirchhoff operators are smooth away from the origin if Cauchy data with compact support is provided. This is accomplished by studying the *singular support* of the fundamental solutions. For the Mindlin-Timoshenko operator there is nothing to do: the information about the singular support contained in the principal part follows from the theory of hyperbolic operators. The Kirchhoff operator, however, does require some analysis. We construct the fundamental solution as a Fourier integral operator in the sense of Hörmander [1] and apply the results therein to determine the singular support. In Section 4 we deal with two perturbation problems that arise on Littman's method, and prove the analogue results to those in Moreles [10]. Consequently we deduce uniform controllability.

The Theory of Distributions of L. Schwartz and the the Theory of Hyperbolic Operators will be assumed throughout this article. As a classical reference we have Hörmander's book [2].

2 Main Result

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $W_2^l(\Omega)$ be the Sobolev space of functions $f \in L_2(\Omega)$ for which the distributional derivatives $\partial_x^\alpha f$ are in $L_2(\Omega)$ for $|\alpha| \le l$, and let the norm be denoted by $\|f\|_l$. When $\Omega = \mathbb{R}^n$ we regard W_2^l as H_l and

$$\|f\|_l = \left(\int \Lambda_{2l} \left| \widehat{f} \right|^2 \right)^{1/2} ,$$

where $\widehat{f}$ is the Fourier Transform of f, and

$$\Lambda_s(\xi) = \left(1 + |\xi|^2 \right)^{\frac{s}{2}} , \; s \in \mathbb{R} .$$

Let w^0 and w^1 be functions in $W_2^{m+l}(\Omega)$, and let w^2 and w^3 be functions in $W_2^{m+l-2}(\Omega)$, where m and l are nonnegative integers with $l > \frac{n}{2}$. Let ε be a positive constant. Extend the Cauchy Data to have compact support in an ε-neighborhood of Ω which we call again Ω. For $b \ge 0$ we denote by Q_b the set $\overline{\Omega} \times [b, +\infty)$.

Theorem 1 *(i) Let $T_0 = d\sqrt{\rho h^3/12}$, where d is the diameter of Ω. Then for each $T_1 > 2T_0$ there exist solutions $w(x,t)$ and $w_K(x,t)$ to the Cauchy problems*

$$\begin{gathered} Lw = 0, \quad \text{in } \overline{\Omega} \times \{t \ge 0\} , \\ w(x,0) = w^0(x) , \quad w_t(x,0) = w^1(x) \quad \text{in } \overline{\Omega} \end{gathered}$$

and

$$\begin{gathered} L_K w_K = 0 , \quad \text{in } \overline{\Omega} \times \{t \ge 0\} , \\ \partial_t^j w_K(x,0) = w^j(x) , \quad j = 0,1,2,3 \quad \text{in } \overline{\Omega} \end{gathered}$$

both vanishing in $\overline{\Omega} \times \{t \ge T_1\}$. Moreover, If $|\alpha| \le m$, then $\partial_x^\alpha w_K$ converges to $\partial_x^\alpha w$ when $K \to +\infty$ in the L^∞-norm in bounded subsets of Q_0.

(ii) If we restrict further $w^0 \in W_2^{m+l+1}(\Omega)$, $w^2 \in W_2^{m+l-1}(\Omega)$, then $\partial_x^\alpha \partial_t w_K$ converges to $\partial_x^\alpha \partial_t w$ when $K \to +\infty$ in the L^∞-norm in bounded subsets of Q_0 also for $|\alpha| \le m$.

The functions $w(x,t)$ and $w_K(x,t)$ with the alleged properties are constructed in the proof. Hence to solve the ECP for both equations we just need to read off appropriate boundary conditions to have uniqueness of the resulting initial-boundary-value problem. For instance, the following boundary conditions guarantee uniqueness

$$f_K = w_K\,, \quad g_K = \Delta w_K \quad \text{in } \Gamma\times[0,T]\,;$$
$$f = w\,, \quad g = \Delta w \quad \text{in } \Gamma\times[0,T]\,.$$

Here $\Gamma = \partial\Omega$ is the (smooth) boundary of Ω. Observe that these functions are bounded in Q_0. Moreover $f_K \to f$ and $g_K \to g$ in the L^∞-norm.

Observe that the control time T_1 in the theorem is independent of K, that is, the control time is uniform. We shall see that the functions $w(x,t)$ and $w_K(x,t)$ are smooth in Q_{T_0}. Hence, in this set the convergence is uniform.

Now the drawbacks. From the point of view of Exact Controllability we have a weak result. For instance, in Lasiecka & Triggiani [7] the Kirchhoff system is considered with the following boundary conditions

$$w = 0 \quad \text{in } \Gamma\times[0,T]$$
$$\Delta w = 0 \quad \text{in } \Gamma_0\times[0,T]$$
$$\Delta w = u \quad \text{in } \Gamma_1\times[0,T]\,,$$

where Γ is the disjoint union of Γ_0 and Γ_1. Thus, only one control in part of the boundary is required to drive the system to rest.

In contrast we require controls in the whole boundary. Also, these controls need not be unique, and we offer no criteria for comparison. Furthermore, for solving the initial-boundary-value problem we require on the Cauchy data more regularity than usual. Our offering is convergence in a rather strong sense.

A natural continuation to this work, is to consider the problem under weaker assumptions.

3 The Cauchy Problems

Theorem 2 *Let u_K and u be the solutions of the Cauchy problems*

$$Lu = 0 \quad \textit{in } \mathbb{R}^n\times\{t\ge 0\}\,, \qquad (1)$$
$$u(x,0) = w^0(x)\,, \quad u_t(x,0) = w^1(x) \quad \textit{in } \mathbb{R}^n$$

and

$$L_K u_K = 0 \quad \textit{in } \mathbb{R}^n\times\{t\ge 0\}\,, \qquad (2)$$
$$\partial_t^j u_K(x,0) = w^j(x)\,, \quad j = 0,1,2,3 \quad \textit{in } \mathbb{R}^n\,.$$

Then u_K and u are smooth in Q_{T_0}.

As remarked before, this theorem is obtained as a consequence of the study of the singular support of the fundamental solutions.

The Mindlin-Timoshenko Operator

Since the Mindlin-Timoshenko Operator is hyperbolic the problem is settled. Indeed, the principal part is

$$L_{K4} = \frac{\rho^2 h^4}{12K}\left(\partial_t^2 - \frac{K}{\rho h}\Delta\right)\left(\partial_t^2 - \frac{12}{\rho h^3}\Delta\right).$$

It has two characteristic cones, namely

$$\begin{aligned} \Gamma &= \left\{(x,t)\in\mathbb{R}^{n+1} : t^2 = \frac{12}{\rho h^3}|x|^2\,,\ t\geq 0\right\} \\ \Gamma_1 &= \left\{(x,t)\in\mathbb{R}^{n+1} : t^2 = \frac{K}{\rho h}|x|^2\,,\ t\geq 0\right\} \end{aligned}$$

with dual cones

$$\begin{aligned} \Gamma^0 &= \left\{(x,t)\in\mathbb{R}^{n+1} : t^2 = \frac{\rho h^3}{12}|x|^2\,,\ t\geq 0\right\} \\ \Gamma_1^0 &= \left\{(x,t)\in\mathbb{R}^{n+1} : t^2 = \frac{\rho h}{K}|x|^2\,,\ t\geq 0\right\}. \end{aligned} \tag{3}$$

Thus if G_K is the fundamental solution of L then

$$\text{sing supp}\ G_K \subset \Gamma^0 \cup \Gamma_1^0\,.$$

The Kirchhoff Operator

The principal part of the Kirchhoff Operator is

$$L_4 = \frac{\rho h^3}{12}\Delta\left(\partial_t^2 - \frac{12}{\rho h^3}\Delta\right).$$

Notice that the wave operator $\partial_t^2 - \frac{12}{\rho h^3}\Delta$ has characteristic cone Γ. We shall see that this hyperbolic part of L_4 determines the singularities of the fundamental solution of L, that is, we show that

$$\text{sing supp}\ G \subset \Gamma^0\,.$$

For convenience we use the following notation

$$D_{x_j} \equiv -i\partial_{x_j}\,,\quad D_x = D_{x_1},\dots,D_{x_n}\,,\quad D_t \equiv -i\partial_t\,,\quad \overline{\Delta} = -\Delta\,.$$

and write the Kirchhoff Operator in the form

$$P(D_x, D_t) = \left(1 + \frac{h^2}{12}\overline{\Delta}\right)D_t^2 - \frac{1}{\rho h}\overline{\Delta}^2\,. \tag{4}$$

We shall construct a distribution $G(x,t)$ supported in the half space $\{(x,t)\in\mathbb{R}^{n+1} : t\geq 0\}$ such that

$$\left[\left(1 + \tfrac{h^2}{12}\overline{\Delta}\right)D_t^2 - \tfrac{1}{\rho h}\overline{\Delta}^2\right]G(x,t) = \delta(x,t)\,.$$

Let

$$S(\xi,t)=\frac{1}{2\lambda(\xi)}\left(e^{i\lambda(\xi)t}-e^{-i\lambda(\xi)t}\right)=i\frac{\sin\lambda(\xi)t}{\lambda(\xi)}\,,$$

where

$$\lambda(\xi)=\frac{|\xi|^2}{\sqrt{\rho h+\frac{\rho h^3}{12}\,|\xi|^2}}\,. \tag{5}$$

it satisfies

$$\left[\left(1+\tfrac{h^2}{12}\,|\xi|^2\right)D_t^2-\tfrac{1}{\rho h}\,|\xi|^4\right]S(\xi,t)=0\,,\quad \text{for } t>0\,. \tag{6}$$

$$S(\xi,0)=0\,,\quad D_tS(\xi,0)=1\,.$$

Observe that

$$P(\xi,D_t)=\left(1+\frac{h^2}{12}\,|\xi|^2\right)D_t^2-\frac{1}{\rho h}\,|\xi|^4$$

is obtained from (4) after applying the Fourier Transform with respect to x.

Choose $\chi(\xi)\in\mathcal{D}$ such that $\chi(\xi)=1$ in a neighborhood of 0. Let

$$\begin{aligned}
P_0u(x,t) &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\frac{1}{1+\frac{h^2}{12}\,|\xi|^2}S(\xi,t)\chi(\xi)u(y)\,d\xi\,dy\\
Au(x,t) &= \frac{1}{(2\pi)^n}\int e^{i\varphi(x,y,\xi)}a(\xi)u(y)\,d\xi\,dy \\
Bu(x,t) &= \frac{1}{(2\pi)^n}\int e^{i\psi(x,y,\xi)}a(\xi)u(y)\,d\xi\,dy\,,
\end{aligned}\tag{7}$$

where

$$a(\xi)=\frac{1}{2\lambda(\xi)}\cdot\frac{1}{1+\frac{h^2}{12}\,|\xi|^2}\cdot(1-\chi(\xi)) \tag{8}$$

$$\begin{aligned}
\varphi(x,y,\xi) &= (x-y)\cdot\xi+\lambda(\xi)t\\
\psi(x,y,\xi) &= (x-y)\cdot\xi-\lambda(\xi)t\,.
\end{aligned}$$

Since $\frac{\partial\varphi}{\partial x_i}=\xi_i$ it follows that

$$|\nabla_x\varphi|^2+|\xi|^2\,|\nabla_\xi\varphi|^2\geq|\xi|^2\,. \tag{9}$$

Moreover, $\lambda(\xi)\in S^1$ and so is $\varphi(x,y,\xi)$. Recall that for any real m S^m is the set of symbols of order m.

Therefore, according to Definition 2.3 in Hörmander [1], φ is a *phase function*. Similar argument applies to ψ. Also, it is readily seen that $a(\xi)$ in (8) is in S^{-3}. Then from Theorem 2.4 in Hörmander [1], A and B are Fourier integral operators with symbol $a(\xi)$ and phase functions $\varphi(x,y,\xi)$, $\psi(x,y,\xi)$ respectively.

To study singularities of the fundamental solution, we need the following definitions. Let Ω_φ be the set of all $((x,t),y)\in\mathbb{R}^{n+1}\times\mathbb{R}^n$ such that for some constant C, depending on $((x,t),y)$,

$$1\leq C\,|\nabla\varphi|^2\,,\quad |\xi|>C\,.$$

Then the set Ω_φ is open, hence its complement, denoted by F_φ, is closed. Define similarly Ω_ψ and F_ψ. For subsets X of $\mathbb{R}^n$ define

$$F_\varphi X = \left\{(x,t) \in \mathbb{R}^{n+1} : ((x,t),y) \in F_\varphi \text{ for some } y \in \mathbb{R}^n\right\} .$$

Our claim about the fundamental solution will be a consequence of the following theorem.

Theorem 3 *Let Γ^0 be the cone in (3), and let $u = \delta$ in (7). Define*

$$E\delta(x,t) = P_0\delta(x,t) + A\delta(x,t) - B\delta(x,t)$$

Then $E\delta$ is a distribution such that the singular support of $E\delta$ is a subset of Γ^0

Proof. We shall prove that in the complement of Γ^0 the distribution $E\delta(x,t)$ is smooth. Observe that $P_0\delta(x,t)$ is C^∞ in $\mathbb{R}^{n+1}$; hence we need to consider only $A\delta(x,t)$ and $B\delta(x,t)$.

In light of Corollary 2.7 in Hörmander [1], it suffices to show that if $|x|^2 \neq \frac{12}{\rho h^3}t^2$ for $t \geq 0$ then $(x,t) \notin F_\varphi \operatorname{sing\,supp} \delta$ and $(x,t) \notin F_\psi$ sing supp δ. We prove the assertion for φ. For ψ, the proof is similar.

Our interest is $u = \delta$, the Dirac's Delta distribution. In this case

$$\operatorname{sing\,supp} \delta = \{0\} .$$

Thus $(x,t) \notin F_\varphi \operatorname{sing\,supp} \delta$ is equivalent to $((x,t),0) \notin F_\varphi$ that is $((x,t),0) \in \Omega_\varphi$.

It can be easily shown that

$$\frac{\partial \varphi}{\partial \xi_i} = x_i + \frac{\frac{1}{\sqrt{\rho h}}t}{\left(1 + \frac{h^2}{12}|\xi|^2\right)^{3/2}} \left(2 + \frac{h^2}{12}|\xi|^2\right) \xi_i \, ;$$

thus

$$\nabla_\xi \varphi((x,t),0,\xi) = x + \frac{\frac{1}{\sqrt{\rho h}}t}{\left(1 + \frac{h^2}{12}|\xi|^2\right)^{3/2}} \left(2 + \frac{h^2}{12}|\xi|^2\right) \xi$$

and

$$\left|\nabla_\xi \varphi((x,t),0,\xi)\right| \geq \left| |x| - \frac{\frac{1}{\sqrt{\rho h}}t}{\left(1 + \frac{h^2}{12}|\xi|^2\right)^{3/2}} \left(2 + \frac{h^2}{12}|\xi|^2\right) |\xi| \right| .$$

Consider the function

$$f(r) = \frac{2 + \frac{h^2}{12}r^2}{\left(1 + \frac{h^2}{12}r^2\right)^{3/2}}\, r\, , \quad r \geq 0\, , \quad r = |\xi| \, .$$

Then $f(r) \searrow \frac{\sqrt{12}}{h}$ when $r \to \infty$ and f attains its maximum value at $r = \frac{\sqrt{24}}{h}$ with $f\left(\frac{\sqrt{24}}{h}\right) = \frac{8}{3\sqrt{6}}\frac{\sqrt{12}}{h}$.

If $|x| < \sqrt{\frac{12}{\rho h^3}}\, t$ then $|x| = (1-\varepsilon)\sqrt{\frac{12}{\rho h^3}}\, t$ for some ε, $0<\varepsilon<1$, thus

$$|\nabla_\xi \varphi((x,t),0,\xi)| \geq \frac{t}{\sqrt{\rho h}} f(r) - |x|$$

but $f(r) \geq \frac{\sqrt{12}}{h}$ for $r \geq \frac{\sqrt{24}}{h}$, so

$$|\nabla_\xi \varphi((x,t),0,\xi)| \geq \varepsilon \sqrt{\frac{12}{\rho h^3}}\, t$$

or

$$|\nabla_\xi \varphi((x,t),0,\xi)|\, C \geq 1\,, \quad \text{if} \quad |\xi| > C$$

with $C = \max\left\{ \frac{1}{\varepsilon\sqrt{\frac{12}{\rho h^3}}\, t}, \frac{\sqrt{24}}{h} \right\}$, i.e. $(x,t) \in \Omega_\varphi$.

For the case $|x| > \sqrt{\frac{12}{\rho h^3}}\, t$ we have $|x| = (1+\varepsilon)\sqrt{\frac{12}{\rho h^3}}\, t$ for some $\varepsilon > 0$, thus

$$|\nabla_\xi \varphi((x,t),0,\xi)| \geq |x| - \frac{t}{\sqrt{\rho h}} f(r)$$

Choose $\widetilde{\varepsilon}$ small enough so that

$$\varepsilon - \widetilde{\varepsilon}\left(\frac{8}{3\sqrt{6}} - 1\right) > 0$$

there is $r_{\varepsilon,\widetilde{\varepsilon}} \geq \frac{\sqrt{24}}{h}$ so that, if $r > r_{\varepsilon,\widetilde{\varepsilon}}$ then $f(r) \leq \left[1 + \widetilde{\varepsilon}\left(\frac{8}{3\sqrt{6}} - 1\right)\right] \frac{\sqrt{12}}{h}$. We have

$$|\nabla_\xi \varphi((x,t),0,\xi)| \geq \sqrt{\frac{12}{\rho h^3}}\, t \left(\varepsilon - \widetilde{\varepsilon}\left(\frac{8}{3\sqrt{6}} - 1\right)\right)$$

Again

$$|\nabla_\xi \varphi((x,t),0,\xi)|\, C \geq 1\,, \quad \text{if} \quad |\xi| > C$$

with $C = \max\left\{ \left(\sqrt{\frac{12}{\rho h^3}}\, t \left(\varepsilon - \widetilde{\varepsilon}\left(\frac{8}{3\sqrt{6}} - 1\right)\right)\right)^{-1}, r_{\varepsilon,\widetilde{\varepsilon}} \right\}$. Therefore $(x,t) \in \Omega_\varphi$.

Finally we show that the singular support is contained in Γ^0. Now we use the phase function ψ. We have

$$\nabla_\xi \psi((x,t),0,\xi) = x - \frac{\frac{t}{\rho h}}{\left(1 + \frac{h^2}{12}|\xi|^2\right)^{3/2}} \left(2 + \frac{h^2}{12}|\xi|^2\right) \xi$$

In the cone Γ^0, $|x| = \sqrt{\frac{12}{\rho h^3}}\, t$ so

$$\nabla_\xi \psi((x,t),0,\xi) = x - \frac{h\,|x|}{\sqrt{12}} \frac{f(r)}{r} \xi\,.$$

Let $\xi = \mu x$, with $\mu > 0$ to be chosen later. Then

$$|\nabla_\xi \psi((x,t),0,\xi)| = \left|1 - \frac{h}{\sqrt{12}} f(\mu\,|x|)\right| |x|$$

Since $f(r) \searrow \frac{\sqrt{12}}{h}$ when $r \nearrow \infty$ it follows that for any $C > 0$ we may choose $\mu >> 0$ so that

$$\left|1 - \frac{h}{\sqrt{12}} f\left(\mu \left|x\right|\right)\right| \left|x\right| < \frac{1}{C}$$

that is if $|x| = \sqrt{\frac{12}{\rho h^3}}\, t\,|x|$ then $(x,t) \notin \Omega_\psi$, i.e., $(x,t) \in F_\psi$ and the singular support of $E\delta$ is a subset of Γ^0 as asserted.

Let $H(t)$ be the Heaviside function. From (6) we see that

$$\left[\left(1 + \frac{h^2}{12}\left|\xi\right|^2\right) D_t^2 - \frac{1}{\rho h}\left|\xi\right|^4\right]\left(\frac{i}{1 + \frac{h^2}{12}\left|\xi\right|^2} S(\xi,t)H(t)\right) = \delta(t)$$

This fact together with the previous theorem give us

Corollary 3.1 *If $G(x,t) = iE\delta(x,t)H(t)$, then $G(x,t)$ is a fundamental solution of the Kirchhoff Operator supported in the half space $t \geq 0$. Moreover the singular support of G is a subset of Γ^0.*

Remark. Existence of fundamental solutions for differential operators with constant coefficients is a classical result. In applications some times a more explicit expression is necessary as illustrated in the present work. The Kirchhoff operator is quasi-hyperbolic in the sense of Ortner & Wagner [11, 12]. They are able to find explicit expressions for some of such operators. However, their approach does not seem to apply here.

4 Perturbation Problems

¿From the Cauchy problems (1) and (2), we obtain the first perturbation problem.

After applying the Fourier Transform on the space variables, we obtain

$$\begin{gathered} L(i\xi,t)U = 0 \\ U(\xi,0) = W^0(\xi)\,, \quad U_t(\xi,0) = W^1(\xi) \end{gathered}$$

and

$$\begin{gathered} L_K(i\xi,t)U_K = 0 \\ \partial_t^j U_K(\xi,0) = W^j(\xi)\,, j = 0,1,2,3 \end{gathered}$$

Let $R = U_k - U$. We will establish the following inequalities

$$\begin{aligned} \left|R(\xi,t)\right| &\leq \frac{c(t)}{K}\left[\left|W^0\right| + \left|W^1\right| + \Lambda_{-2}\left(\left|W^2\right| + \left|W^3\right|\right)\right] \\ \left|\partial_t R(\xi,t)\right| &\leq \frac{c(t)}{\sqrt{K}}\left(\Lambda_1\left|W^0\right| + \left|W^1\right| + \Lambda_{-1}\left|W^2\right| + \Lambda_{-2}\left|W^3\right|\right) \qquad (10) \\ \left|\partial_t^2 R(\xi,t)\right| &\leq c(t)\left(\Lambda_2\left|W^0\right| + \Lambda_1\left|W^1\right| + \left|W^2\right| + \Lambda_{-1}\left|W^3\right|\right)\,, \end{aligned}$$

where $c(t)$ is a polynomial on t independent of K. From these estimates, using the fact that $\int \Lambda_{-s}$ converges for $s > n$, and Hölder's inequality, for any multi-index α we have that

$$\begin{aligned}|\partial_x^\alpha (u_K(x,t) - u_0(x,t))| \quad &\leq \quad \frac{c(t)}{K}\Big(\left\|w^0\right\|_{|\alpha|+\frac{s}{2}} + \left\|w^1\right\|_{|\alpha|+\frac{s}{2}} \\ &\qquad + \left\|w^2\right\|_{|\alpha|+\frac{s}{2}-2} + \left\|w^3\right\|_{|\alpha|+\frac{s}{2}-2}\Big)\end{aligned}$$

and

$$\begin{aligned}|\partial_x^\alpha \partial_t (u_K(x,t) - u_0(x,t))| \quad &\leq \quad \frac{c(t)}{\sqrt{K}}\Big(\left\|w^0\right\|_{|\alpha|+\frac{s}{2}+1} + \left\|w^1\right\|_{|\alpha|+\frac{s}{2}} \\ &\qquad + \left\|w^2\right\|_{|\alpha|+\frac{s}{2}-1} + \left\|w^3\right\|_{|\alpha|+\frac{s}{2}-2}\Big).\end{aligned}$$

Consequently, we have the following result.

Theorem 4 *Let $s > n$ and α be a multi-index such that $|\alpha| \leq m$. Assume that w^0 and are in $w^1 \in H^{m+\frac{s}{2}}$, and that w^2 and w^3 in $H^{m+\frac{s}{2}-2}$. Then*

(i) $\partial_x^\alpha u_K$ converges to $\partial_x^\alpha u$ when $K \to +\infty$ in the norm $L^\infty(\mathbb{R}^n \times [0,\tau])$ with $\tau < +\infty$.

(ii) If we restrict further $w^0 \in H^{m+\frac{s}{2}+1}$ and $w^2 \in H^{m+\frac{s}{2}-1}$, then $\partial_x^\alpha \partial_t u_K$ converges to $\partial_x^\alpha \partial_t u$ when $K \to +\infty$ in the norm $L^\infty(\mathcal{R}^n \times [0,\tau])$ with $\tau < +\infty$.

Remarks.

(i) Notice that in Theorem 1 $l \geq \lceil s/2 \rceil$.

(ii) Thanks to Estimate (10) we require less regularity of the Cauchy Data than in Theorem 3.2 in Moreles [10]. Thus, Theorem 4 lead us to a slight improvement of the main result therein.

(iii) Observe that the last estimate in (10) implies weaker convergence when taking second derivatives with respect to t.

Let us discuss the proof of the estimates in (10). It is not difficult to see that

$$U_0(\xi,t) = W^0 \cos\lambda t + \frac{1}{\lambda} W^1 \sin\lambda t$$

with

$$\lambda \equiv \lambda(\xi) = \frac{|\xi|^2}{\sqrt{\rho h + \frac{\rho h^3}{12}|\xi|^2}},$$

as in (5). The remainder R satisfies

$$\begin{aligned} L_K(i\xi,t)R &= \tfrac{\rho^2 h^4}{12K} F \\ \partial_t^j R(\xi,0) &= R^j(\xi)\,, \end{aligned}$$

where $F(\xi,t) = -\partial_t^4 U_0 - \frac{12}{\rho h^3}\left|\xi\right|^2 \partial_t^2 U_0$ and

$$\begin{aligned} R^0(\xi) &= R^1(\xi) = 0 \\ R^2(\xi) &= W^2(\xi) - \partial_t^2 U_0\,(\xi,0)\;, \\ R^3(\xi) &= W^3(\xi) - \partial_t^3 U_0\,(\xi,0)\;. \end{aligned}$$

Then we obtain

$$F(\xi,t) = -\left(\lambda^2 - \frac{12}{\rho h^3}\left|\xi\right|^2\right)\left(\lambda^2 W^0 \cos\lambda t + \lambda W^1 \sin\lambda t\right)\;.$$

Notice that

$$\left|\lambda^2 - \frac{12}{\rho h^3}\left|\xi\right|^2\right| \le c\,; \tag{11}$$

hence, in terms of Λ_s we obtain

$$\left|F(\xi,t)\right| \le c\left(\Lambda_2\left|W^0\right| + \Lambda_1\left|W^1\right|\right)\;. \tag{12}$$

Now we mimic the proof of Theorem 3.2 in Moreles [10] to obtain the estimates (10).

Remark. Estimate (12) follows from the explicit form of F and (11). Compare (12) with the estimate after (3.10) in Moreles [10].

The second perturbation problem follows from the proof of Theorem 2 in Littman [8]. Let us rework such a proof to derive this perturbation problem as well as the uniform time of controllability.

Let $\varphi(t)$ be a cutoff function such that

$$\varphi(t) = \begin{cases} 1 & \text{for } t \le T_0 \\ 0 & \text{for } t \ge T_0 + \varepsilon\,. \end{cases}$$

Let $f = L[u\varphi]$ and $f_K = L_K[u\varphi]$. Then f and f_K are smooth and have support in the strip $T_0 \le t \le T_0 + \varepsilon$.

Theorem 5 *There are smooth solutions V and V_K for*

$$L[V] = f \quad \textit{and} \quad L_K[V_K] = f_K \tag{13}$$

vanishing in a neighborhood of

$$\overline{\Omega} \times \{t=0\} \quad \textit{and} \quad \overline{\Omega} \times \{t = T_1\}\,. \tag{14}$$

Moreover, if $|\alpha|+l \le 3$, then $\partial_x^\alpha \partial_t^l V_K$ converges to $\partial_x^\alpha \partial_t^l V$ uniformly in compacta.

Proof. For each unit vector ω let $I_\omega = \left\{ s : s = x\cdot\omega \,,\, x \in \overline{\Omega} \right\}$. Then we have

$$L(\partial_x,\partial_t)v(x\cdot\omega,t) = \left[\rho h \partial_t^2 - \tfrac{\rho h^3}{12}\partial_s^2\partial_t^2 + \partial_s^4 \right] v(s,t)\,,$$
$$L_K(\partial_x,\partial_t)v_K(x\cdot\omega,t) = \left[\rho h \partial_t^2 - \tfrac{\rho h^3}{12}\partial_s^2\partial_t^2 + \partial_s^4 + \tfrac{\rho h}{K}\left(\tfrac{\rho h^3}{12}\partial_t^4 - \partial_s^2\partial_t^2 \right) \right] v_K(s,t)\,.$$

We obtain the beam operators of Rayleigh and Timoshenko

$$L(\partial_s,\partial_t) = \rho h \partial_t^2 - \tfrac{\rho h^3}{12}\partial_s^2\partial_t^2 + \partial_s^4$$
$$L_K(\partial_s,\partial_t) = \rho h \partial_t^2 - \tfrac{\rho h^3}{12}\partial_s^2\partial_t^2 + \partial_s^4 + \tfrac{\rho h}{K}\left(\tfrac{\rho h^3}{12}\partial_t^4 - \partial_s^2\partial_t^2 \right)$$

which are hyperbolic in the s-direction. For both operators there are two fundamental solutions supported respectively in the cones

$$\Gamma_+^0 = \left\{ (s,t) \in \mathbb{R}^2 : t^2 = \tfrac{\rho h^3}{12}|s|^2 \,,\; s \ge 0 \right\}$$
$$\Gamma_-^0 = \left\{ (s,t) \in \mathbb{R}^2 : t^2 = \tfrac{\rho h^3}{12}|s|^2 \,,\; s \le 0 \right\}\,.$$

Let

$$f(x,t) = \int\limits_{|\omega|=1} f_\omega(s,t)\,d\omega \quad \text{and} \quad f_K(x,t) = \int\limits_{|\omega|=1} f_{K\omega}(s,t)\,d\omega\,.$$

be the plane wave decompositions of f and f_K.

Let $V_{\omega R}$, V_{ω_L} , $V_{K\omega R}$, and $V_{K\omega R}$ be the solutions to

$$\begin{aligned} L(\partial_s,\partial_t)[V_{\omega R}] &= f_\omega \,,\; s \ge 0\,; \\ L(\partial_s,\partial_t)[V_{\omega L}] &= f_\omega \,,\; s \le 0\,; \\ L_K(\partial_s,\partial_t)[V_{K\omega R}] &= f_{K\omega} \,,\; s \ge 0\,; \\ L_K(\partial_s,\partial_t)[V_{K\omega L}] &= f_{K\omega} \,,\; s \le 0\,. \end{aligned} \tag{15}$$

Here we assume that $0 \in \Omega$.

We see that the supports of $V_{\omega R}$, V_{ω_L} , $V_{K\omega R}$, and $V_{K\omega R}$ are bounded away from the sets

$$\overline{\Omega} \times \{t = 0\} \quad \text{and} \quad \overline{\Omega} \times \{t = T_1\}$$

Let

$$V_\omega = V_{\omega R} + V_{\omega_L}\,, \quad V_{K\omega} = V_{K\omega R} + V_{K\omega R} \tag{16}$$

and

$$V(x,t) = \int\limits_{|\omega|=1} V_\omega\,d\omega\,, \quad V_K(x,t) = \int\limits_{|\omega|=1} V_{K\omega}\,d\omega\,.$$

These functions satisfy (13) and (14). ¿From (15), (16), and Theorem 4.2 in Moreles [10], it follow that $\partial_s^m\partial_t^l V_{K\omega}$ converges to $\partial_s^m\partial_t^l V_\omega$ uniformly in compacta independently of ω. This proves the theorem.

Finally let

$$w = u\varphi - V \quad \text{and} \quad w_K = u_K\varphi - V_K\,.$$

Then w and w_K satisfy all the requirements of Theorem 1.

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