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Nontrivial solutions to the semilinear Kohn–Laplace equation on $\mathbb{R}^3$ *

Stepan Tersian

Abstract

The existence of nontrivial solutions to the semilinear Kohn–Laplace equation

$$-\Delta_H u + V(P)u = f(u)$$

is considered under appropriate assumptions on $V(P)$ and $f(u)$. Results are obtained using a variational method and a compact-embedding lemma.

1 Introduction

Variational methods have been used for obtaining homoclinic solutions of second-order semilinear ordinary differential equations, and homoclinic type solutions of semilinear elliptic equations on the whole space. See for example Ding and Ni [3], Rabinowitz [8], Omana and Willem [6], and Korman and Lazer [7]. In most of the references, the linear part of the equation is assumed strongly elliptic, which motivates us to consider the degenerate semilinear elliptic case. In particular, we study the existence of a nontrivial solution, $u \in W_H^{1,2}(\mathbb{R}^3)$, to the semilinear Kohn–Laplace or Heisenberg equation

$$-\Delta_H u + V(P)u = f(u), \quad P(x,y,z) \in \mathbb{R}^3\,, \tag{1}$$

where

$$\Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4((x^2+y^2)\frac{\partial^2}{\partial z^2} + y\frac{\partial^2}{\partial z \partial x} - x\frac{\partial^2}{\partial x \partial y})$$

is the Kohn–Laplacian. We assume that $V(P) \in C(\mathbb{R}^n\mathbb{R})$,

$$V(P) > 0, \quad \forall P \in \mathbb{R}^n\,, \quad \text{and} \tag{2}$$

$$V(P) \to +\infty \quad \text{as} \quad |P| \to +\infty\,, \tag{3}$$

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where $|P| = \sqrt{x^2+y^2+z^2}$ is the norm in $\mathbb{R}^3$. We assume that $f(t) \in C(\mathbb{R})$ satisfying

$$f(0)=0, \quad f(t)=o(|t|) \quad \text{as} \quad t\to 0\,, \tag{4}$$

$$f(t)=o(|t|^3) \quad \text{as} \quad t\to\infty\,, \tag{5}$$

$$0<\mu F(t)\equiv \mu\int_0^t f(s)\,ds \le tf(t) \quad \text{with } \mu>2. \tag{6}$$

Let $W_H^{1,2}(\mathbb{R}^3)$ be the Sobolev space associated with the Kohn–Laplacian Δ_H, and

$$X=\{u(P)\in W_H^{1,2}(\mathbb{R}^3): \int_{\mathbb{R}^3}(|\nabla_H u(P)|^2+V(x)|u(P)|^2)\,dP<\infty\}.$$

Hereafter, $\int$ means integration in $\mathbb{R}^3$. Our main result is as follows

Theorem 1 *Let (2)–(6) hold for V and f. Then the equation $-\Delta_H u+V(P)u=f(u)$ has a nontrivial solution $u\in W_H^{1,2}(\mathbb{R}^3)$.*

This paper is organized as follows: In the first section, we state some preliminary results on the Kohn–Laplace operator and prove embedding lemmata which are essential for the forthcoming considerations. We use properties of $\mathbb{R}^3$ as a homogeneous space, with homogeneous dimension 4 with respect to the intrinsic distance, using the Sobolev and Poincare inequalities considered in Biroli and Mosco [1], Biroli, Mosco and Tchou [2], and in Jerison [4]. In the second section, we prove an embedding lemma and an existence result for (1) using the mountain-pass theorem. This idea comes from Omana and Willem [6], where homoclinic solutions of Hamiltonian systems are considered. There are difficulties involved in checking that the corresponding functional satisfies the Palais–Smale condition (as a difference with the one-dimensional case). To overcome these difficulties, we prove a proposition that is analogous to the one in P.L. Lions [5]. In a forthcoming paper by Biroli and Tersian, an extension to semilinear equations related to a general Dirichlet's form will be given.

2 Preliminary results

First, we recall the definition of the Kohn–Laplace operator Δ_H. Let

$$\xi=\tfrac{\partial}{\partial x}+2y\tfrac{\partial}{\partial z}, \qquad \eta=\tfrac{\partial}{\partial y}-2x\tfrac{\partial}{\partial z}\,, \nabla_H=(\xi,\eta)=\sigma\nabla\,,$$

$$\sigma=\begin{bmatrix}1&0&2y\\0&1&-2x\end{bmatrix}, \quad \nabla=(\tfrac{\partial}{\partial x},\tfrac{\partial}{\partial y},\tfrac{\partial}{\partial z}),$$

$$\Delta_H=\nabla_H^2=\operatorname{div}(\sigma^T\sigma\nabla)\,,$$

$$\Delta_H=\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}+4((x^2+y^2)\tfrac{\partial^2}{\partial z^2}+y\tfrac{\partial^2}{\partial z\partial x}-x\tfrac{\partial^2}{\partial x\partial y})\,.$$

The operator Δ_H is elliptic, $P^T\sigma^T\sigma P\ge 0$ for every $P\in\mathbb{R}^3$, but not necessarily strongly elliptic, because the eigenvalues of the matrix

$$\sigma^T\sigma=\begin{bmatrix}1&0&2y\\0&1&-2x\\2y&-2x&4y^2+4x^2\end{bmatrix}$$

are $0, 1, 1 + 4y^2 + 4x^2$ and its rank is 2.

The intrinsic distance $\rho(P, P')$ between $P(x, y, z)$ and $P'(x', y', z')$ associated with the operator Δ_H is defined as

$$\rho(P, P') = \left(((x - x')^2 + (y - y')^2)^2 + (z - z' - 2(x'y - xy'))^2\right)^{1/4}.$$

Under the distance ρ the intrinsic ball $B_\rho(P_0, r)$ is defined as

$$B_\rho(P_0, r) = \{P : \rho(P_0, P) \le r\}.$$

For vectors $P(x, y, z)$ and $P'(x', y', z')$, we define the P' right translation as

$$P \oplus P' = (x + x', y + y', z + z' + 2(x'y - xy')).$$

Let $m(B)$ denote the volume of the Euclidean ball with radius r, $B(P_0, r) \subset \mathbb{R}^3$.

Proposition 2 *(i) $B(0, r) \subset B_\rho(0, r) \subset B(0, r^2)$ for $r \ge 1$; and $B(0, r^2) \subset B_\rho(0, r) \subset B(0, r)$ for $0 < r < 1$.*

(ii) $m(B_\rho(0, r)) = m(B_\rho(P_0, r)) = \pi^2 r^4/2$.

Proof. (i) To show that $B(0, r) \subset B_\rho(0, r)$ for $r \ge 1$, we notice that the projection of both balls on the plane $z = 0$ is the disk

$$D = D(0, r) = \{(x, y) : x^2 + y^2 \le r^2\}.$$

For $z \ge 0, (x, y) \in D$, the boundaries of $B(0, r)$ and $B_\rho(0, r)$ are graphs of the functions $z = \sqrt{r^2 - p^2}$ and $z_\rho = \sqrt{r^4 - p^4}$, where $p^2 = x^2 + y^2$, $0 \le p \le r$. Then we have $z \le z_\rho$ and $B(0, r) \subset B_\rho(0, r)$ for $r \ge 1$.

(ii) Making the change of variables

$$\Phi : \begin{cases} x = p\cos\theta\sqrt{\sin\varphi}, \\ y = p\sin\theta\sqrt{\sin\varphi}, \\ z = p^2\cos\varphi, \end{cases}$$

where $p \ge 0$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, we have

$$m(B_\rho(0, r)) = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho^3\, d\rho\, d\psi\, d\theta = \frac{1}{2}r^4\pi^2.$$

The projection of the ball $B_\rho(P_0, r)$ on the plane $z = 0$ is the disk $D(P_0, r) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 \le r^2\}$ and

$$\begin{aligned} m(B_\rho(P_0, r)) &= 2\int\int_{D(P_0,r)} (r^4 - ((x - x_0)^2 + (y - y_0)^2)^2)^{1/2} dx\, dy \\ &= 2\int\int_D (r^4 - (x^2 + y^2)^2)^{1/2} dx\, dy \\ &= m(B_\rho(0, r)) = \frac{1}{2}r^4\pi^2. \end{aligned}$$

$\diamondsuit$

By Proposition 2 it follows that $\mathbb{R}^3$ with respect to (ρ, m) is a homogeneous space with homogeneous dimension 4, see Biroli and Mosco [1]. The space $\mathbb{R}^3$ for every r can be covered by intrinsic balls of radius r such that each point of $\mathbb{R}^3$ is contained in at most 4 balls.

Let us consider now the space $W_H^{1,2}(\mathbb{R}^3)$, as a completion of the space $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|^2_{W_H^{1,2}(\mathbb{R}^3)} = \int (|\nabla_H u(P)|^2 + |u(P)|^2)\, dP\,.$$

For a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, let

$$W_H^{1,2}(\Omega) := \{u \in L^2(\Omega) : \int_\Omega (|\nabla_H u(P)|^2 + |u(P)|^2)\, dP < \infty\},$$

and let $W_{H,0}^{1,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{W_H^{1,2}(\mathbf{\Omega})} = \int_\Omega (|\nabla_H u(P)|^2 + |u(P)|^2)\, dP\,.$$

The following Poincare inequality is proved by Jerison [4]:

$$\int_{B_\rho(P_0,r)} |u(P) - \bar{u}|^2\, dP \le Cr^2 \int_{B_\rho(P_0,kr)} |\nabla_H u(P)|^2\, dP,$$

where C and $k \ge 1$ are constants independent of P and r, and

$$\bar{u} = \frac{1}{m(B_\rho(P_0,r))} \int_{B_\rho(P_0,r)} u(P)\, dP\,.$$

Using this inequality and a homogeneous covering of $\mathbb{R}^3$ by intrinsic balls we prove the following.

Lemma 3 *The space $W_{H,0}^{1,2}(B_\rho(0,R))$ is compactly embedded in $L^2(B_\rho(0,R))$ and $W_H^{1,2}(B_\rho(0,R))$ is compactly embedded in $L^2(B_\rho(0,R(1+\delta))$ with $\delta > 0$.*

Proof. Let $\{u_k\}$ be a bounded sequence in $W_{H,0}^{1,2}(B_\rho(0,R))$, with $\|u_k\|_{W_H^{1,2}} \le A$, and $u_k \to u$ weakly in $W_{H,0}^{1,2}(B_\rho(0,R))$ and $L^2(B_\rho(0,R))$. We also denote by u_k the extension of u_k to $\mathbb{R}^3$ by 0, which belongs to $W_H^{1,2}(\mathbb{R}^3)$. Let ε be an arbitrary positive number and $\{B_j\}$, $B_j = B_\rho(P_j, r)$, be the covering of $B_R = B_\rho(0,R)$ by intrinsic balls with radius $r = (\frac{\varepsilon}{32AC})^{1/2}$, such that every point of B_R belongs to at most 4 balls B_j. By a result of Jerison [4]

$$\int_{B_j} |u(P) - \bar{u}_j|^2\, dP \le Cr^2 \int_{B_j} |\nabla_H u(P)|^2\, dP\,,$$

where C is a constant independent of u and j, and

$$\bar{u}_j = \frac{1}{m(B_j)} \int_{B_j} u(P)\, dP\,.$$

We have

$$\begin{aligned}
\int_{B_R} u(P)^2\, dP &\le 2\Big(\sum_j \int_{B_j} |u(P)-\bar{u}_j|^2\, dP + \sum_j \frac{1}{m(B_j)}\Big(\int_{B_j} u(P)\, dP)^2\Big) \\
&\le \frac{\varepsilon}{16A} \sum_j \int_{B_j} |\nabla_H u(P)|^2\, dP + \frac{C_1}{\varepsilon^2} \sum_j \Big(\int_{B_j} u(P)\, dP\Big)^2 \\
&\le \frac{\varepsilon}{4A} \int_{B_R} |\nabla_H u(P)|^2\, dP + \frac{C_1}{\varepsilon^2} \sum_j \Big(\int_{B_j} u(P)\, dP\Big)^2, \qquad (7)
\end{aligned}$$

where $C_1 = 2(32AC/\pi)^2$. By (7) for $w_{k,n} = u_k - u_n$,

$$\begin{aligned}
\int_{B_R} w_{k,n}^2\, dP &\le \frac{\varepsilon}{4A} \int_{B_R} |\nabla_H w_{k,n}|^2\, dP + \frac{C_1}{\varepsilon^2} \sum_j \Big(\int_{B_j} w_{k,n}\, dP\Big)^2 \\
&\le \frac{\varepsilon}{2} + \frac{C_1}{\varepsilon^2} \sum_j \Big(\int_{B_j} w_{k,n}\, dP\Big)^2. \qquad (8)
\end{aligned}$$

Since $u_k \to u$ weakly in $L^2(B_R)$ we have that there exists N such that

$$\int_{B_R} w_{k,n} dP \le \frac{\varepsilon^3}{2C_1} \quad \text{for all } k, n > N\,.$$

Then by (8)

$$\int_{B_R} w_{k,n}^2 dP \le \varepsilon \quad \text{for all } k, n > N\,;$$

therefore, $\{u_k\}$ converges in $L^2(B_R)$.

Using the extension by zero of $u \in W_H^{1,2}(B_\rho(0,R))$ on $B_\rho(0,R(1+\delta)) \setminus B_\rho(0,R)$, with $\delta > 0$, we deduce that

$$W_H^{1,2}(B_\rho(0,R)) \subset W_{H,0}^{1,2}(B_\rho(0,R(1+\delta)).$$

By the first part, it follows that the inclusion

$$W_H^{1,2}(B_\rho(0,R)) \subset L^2(B_\rho(0,R(1+\delta))$$

is compact. $\diamondsuit$

Lemma 4 *The embedding $W_H^{1,2}(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)$ is continuous.*

Proof. As the homogeneous dimension of $(\mathbb{R}^3, \rho, m)$ is 4, by the Sobolev inequality [1],

$$\begin{aligned} & \left(\int_{B_\rho(P_0,r)} |u(P)|^4 \, dP \right)^{1/4} \\ & \quad \leq \; C_2 \left(r^2 \int_{B_\rho(P_0,r)} |\nabla_H u(P)|^2 \, dP + \int_{B_\rho(P_0,r)} |u(P)|^2 \, dP \right)^{1/2}, \end{aligned}$$

for $u \in C_0^\infty(B_\rho(P_0, r))$. Covering the Euclidean space $\mathbb{R}^3$ by intrinsic balls $B_j = B_\rho(P_j, r)$ such that each point of $\mathbb{R}^3$ is covered by at most 4 balls for $u \in C_0^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |u(P)|^4 \, dP \; & \leq \; \sum_j C_2 (r^2 \int_{B_j} |\nabla_H u(P)|^2 \, dP + \int_{B_j} |u(P)|^2 \, dP)^2 \\ & \leq \; 2C_2 (\sum_j (\int_{B_j} |\nabla_H u(P)|^2 \, dP)^2 + (\int_{B_j} |u(P)|^2 \, dP))^2) \\ & \leq \; 8C_2 (\int_{\mathbb{R}^3} |\nabla_H u(P)|^2 \, dP + \int_{\mathbb{R}^3} |u(P)|^2 \, dP)^2. \end{aligned}$$

Then $\|u\|_{L^4} \leq C_3 \|u\|_{W_H^{1,2}}$. $\diamondsuit$

3 Existence result

We consider the existence of a nontrivial solutions $u \in W_H^{1,2}(\mathbb{R}^3)$ to the semilinear Kohn–Laplace equation

$$-\Delta_H u + V(P) u = f(u).$$

Suppose that $V(P) \in C(\mathbb{R}^n, \mathbb{R})$ and $f(t) \in C(\mathbb{R})$ satisfy assumptions (2)–(6). By (6) it follows that there exists $m > 0$ such that

$$|F(t) \geq m|t|^\mu, \quad \text{for} |t| \geq 1 \,. \tag{9}$$

Let us consider the operator $L = -\Delta_H + V(x)$ in the space $E = L^2(\mathbb{R}^3)$ under assumptions (2) and (3). Let X be the domain of operator L in E,

$$X = \{ u(P) \in W_H^{1,2}(\mathbb{R}^3) : \int (|\nabla_H u(P)|^2 + V(x) |u(P)|^2) \, dP < \infty \}.$$

It follows that $V(P)$ is uniformly positive; i.e., there exists $a > 0$ such that

$$V(P) \geq a > 0, \quad \forall P \in \mathbb{R}^3. \tag{10}$$

Notice that L is a positive selfadjoint operator in E. Then the graph norm of L in X,

$$\|u\|_Y^2 = \|u\|_{L^2}^2 + \|\nabla_H u\|_{L^2}^2 \,,$$

is equivalent to the norm

$$\|u\|^2 = \int (|\nabla_H u(P)|^2 + V(x)|u(P)|^2)\,dP = \langle Lu, u\rangle\,.$$

Also notice that X is a Hilbert space with the scalar product

$$(u_1, u_2) = \int (\nabla_H u_1 \nabla_H u_2 + V(P)u_1u_2)\,dP\,.$$

Lemma 5 *Suppose $V(x)$ satisfies (2) and (3). Then the embedding of X in E is compact.*

Proof. Let $\{u_k(P)\}$ be a bounded sequence in X, with $\|u_k\| \le A$, and $u_k \to u$ weakly in X. We shall show that $u_k \to u$ strongly in E. Assuming that $u = 0$, we prove that

$$\int u_k^2(P)\,dP \to 0, \quad \text{as} \quad k \to \infty\,. \tag{11}$$

Let $\varepsilon > 0$, $\delta > 0$ and $R > 0$ be such that

$$V(P) \ge \frac{1+A}{\varepsilon} \quad \text{if} \quad |P| \ge R(1+\delta)\,. \tag{12}$$

The operator $S : X \to W_H^{1,2}(B(0,R))$, $Su = u\,|_{B(0,R)}$ is linear and continuous. By Lemma 3 the inclusion $W_H^{1,2}(B(0,R)) \subset L^2(B(0,R(1+\delta)))$ is compact and therefore

$$\int_{B(0,R(1+\delta))} u_k^2(P)\,dP \to 0, \quad \text{as} \quad k \to \infty\,.$$

Let k_0 be such that for $k \ge k_0$

$$\int_{B(0,R(1+\delta))} u_k^2(P)\,dP \le \frac{\varepsilon}{1+A}\,.$$

Then for $k \ge k_0$,

$$\begin{aligned}
\int u_k^2(P)\,dP &= \int_{|P|\ge R(1+\delta)} u_k^2(P)\,dP + \int_{B(0,R(1+\delta))} u_k^2(P)\,dP \\
&\le \frac{\varepsilon}{1+A}(1 + \int_{|P|\ge R(1+\delta)} V(P)u_k^2(P)\,dP) \\
&\le \frac{\varepsilon}{1+A}(1 + \|u_k\|^2) \\
&\le \varepsilon\,.
\end{aligned}$$

◊

Since $W_H^{1,2}(\mathbb{R}^3)$ is not included in $L^\infty(\mathbb{R}^3)$ the approach by Omana and Willem [6] does not work in the case. By Lemmas 3 and 4 we have the embeddings

$$\begin{aligned}
X &\subset L^2(\mathbb{R}^3) = E \quad \text{compactly}, \\
X &\subset W_H^{1,2}(\mathbb{R}^3) \subset L^4(\mathbb{R}^3)\,.
\end{aligned}$$

Now we prove the following proposition that is analogous to the one in P.L. Lions [5].

Lemma 6 *Suppose $g \in C(\mathbb{R})$ satisfies*

$$g(t) = o(|t|) \quad as \quad |t| \to 0$$
$$g(t) = o(|t|^3) \quad as \quad |t| \to +\infty\,.$$

If $\{u_k\}$ is a bounded sequence in $L^4(\mathbb{R}^3)$ and $u_k \to u$ in $L^2(\mathbb{R}^3)$, then

$$\int |g(u_k)(u_k - u)|\, dP \to 0 \quad as \quad k \to +\infty\,.$$

Proof. From the assumptions, for every $\varepsilon > 0$ there exists $\rho > 0$ such that $g(t) \le \varepsilon |t|^3$ for $|t| \ge \rho$, and that there exists $\delta > 0$ such that $|g(t)| \le \varepsilon |t|$ when $|t| \le \delta < \rho$, we have

$$|g(t)| \le \varepsilon(|t| + |t|^3) + C_\varepsilon |t|,$$

where $C_\varepsilon = \delta^{-1} \max_{\delta \le |t| \le \rho} |g(t)|$. Then

$$\begin{aligned}\int |g(u_k)(u_k - u)|\, dP \;&\le\; \varepsilon \int (|u_k|(|u_k| + |u|) + |u_k|^3(|u_k| + |u|))\, dP \\ &\quad + C_\varepsilon \int |u_k||u_k - u|\, dP \\ &\le\; C_1 \varepsilon \int (|u_k|^2 + |u|^2 + |u_k|^4 + |u|^4)\, dP \\ &\quad + C_\varepsilon \int |u_k||u_k - u|\, dP\,.\end{aligned}$$

Then the result follows by the convergence in $L^2(\mathbb{R}^3)$ and the boundedness in $L^4(\mathbb{R}^3)$ of $\{u_k\}$. $\diamondsuit$

Let us consider the functional

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int F(u(x))\, dP\,. \tag{13}$$

It can be proved that $\varphi \in C^1(X, \mathbb{R})$ and

$$\langle \varphi'(u), v\rangle = (u, v) - \int f(x, u(x)) v(x) dP, \quad \forall v \in X\,. \tag{14}$$

The critical points of φ are weak solutions of the equation (1) in the function space $W^{1,2}_H(\mathbb{R}^3)$. We are looking for nontrivial solutions, $u \ne 0$, in $W^{1,2}_H(\mathbb{R}^3)$ of the equation (1).

To prove the existence of nontrivial critical points, $u \ne 0$, we apply the mountain-pass theorem of Ambrosetti and Rabinowitz to the functional φ.

Lemma 7 *If $f \in C(\mathbb{R})$ satisfies (3)–(5), then the functional φ satisfies the Palais–Smale condition in X.*

Proof. Let $\{u_k\}$ be a sequence in X such that

$$|\varphi(u_k)| \leq C,\ \varphi'(u_k) \to 0 \text{ in } X^* \tag{15}$$

as $k \to +\infty$. Then there exists k_0 such that for $k \geq k_0$

$$|\langle \varphi'(u_k), u_k\rangle| \leq \mu \|u_k\|.$$

Then

$$\begin{aligned} C + \|u_k\| &\geq \varphi(u_k) - \frac{1}{\mu}\langle \varphi'(u_k), u_k\rangle \\ &= (\frac{1}{2} - \frac{1}{\mu})\|u_k\|^2 + \frac{1}{\mu}\int (f(u_k)u_k - \mu F(u_k))\, dP \\ &\geq (\frac{1}{2} - \frac{1}{\mu})\|u_k\|^2\,; \end{aligned}$$

so $\{u_k\}$ is bounded in X. By Lemmas 4, 5 and 6 there exists a subsequence denoted again by $\{u_k\}$, such that $u_k \to u \in X$ weakly, $u_k \to u$ in $L^2(\mathbb{R}^3)$ strongly, and

$$\lim_{k\to+\infty} \int |f(u_k)(u_k - u)|\ dP = 0\,. \tag{16}$$

By (15) we have that

$$|\langle \varphi'(u_k), v\rangle| \leq \varepsilon_k \|v\|\,. \tag{17}$$

where $\varepsilon_k \to 0$ as $k \to +\infty$. Then by (16) it follows

$$\lim_{k\to+\infty} (\|u_k\|^2 - \langle u, u_k\rangle) = \lim_{k\to+\infty} \int f(u_k)(u_k - u)\, dP = 0\,.$$

Then $\{u_k\}$ converges to u strongly in X. ♢

Proof of Theorem 1. The existence of a solution follows from the the mountain-pass theorem. It remains to show its geometric conditions.

There exists a constant C such that

$$\|u\|_E \leq C\|u\|, \quad \|u\|_{L^4} \leq C\|u\|\,.$$

Let $0 < \varepsilon < 1/(2C^2)$. Then

$$\begin{aligned} \int F(u)\, dP &\leq \varepsilon(\|u\|_E^2 + \|u\|_{L^4}^4) + C_\varepsilon \|u\|_{L^4}^4 \\ &\leq \varepsilon C^2 \|u\|^2 + (\varepsilon + C_\varepsilon)C^4\|u\|^4 \end{aligned}$$

and

$$\varphi(u) \geq (\frac{1}{2} - \varepsilon C^2)\|u\|^2 - (\varepsilon + C_\varepsilon)C^4\|u\|^4 > 0$$

for small enough $\|u\| = r > 0$.

Let us take a $u_0 \in C_0^\infty(\mathbb{R}^3)$ such that $u_0 > 0$, $\operatorname{supp} u_0 \subset B(0,2)$ and $u_0|_{B(0,1)} = 1$. Then $\|u_0\| > a|B(0,1)| > \rho$. By (9), for $x \in B(0,1)$, we have

$$F(u_0(x)) \geq m|u_0(x)|^\mu \,.$$

For $\gamma > 1$, we have

$$\begin{aligned}
\varphi(\gamma u_0) &= \frac{1}{2}\gamma^2\|u_0\|^2 - \int F(\gamma u_0)\,dP \\
&\leq \frac{1}{2}\gamma^2\|u_0\|^2 - \int_{B(0,1)} F(\gamma u_0)\,dP \\
&\leq \frac{1}{2}\gamma^2\|u_0\|^2 - \gamma^\mu|B(0,1)| < 0\,,
\end{aligned}$$

for sufficiently large γ, because $\mu > 2$. $\diamondsuit$

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Stepan Tersian
Center of Applied Mathematics and Informatics
University of Rousse
8, Studentska
7017 Rousse, Bulgaria
E-mail: tersian@ami.ru.acad.bg