ELECTRONIC JOURNAL OF DIFFERENTIAL EQUATIONS, Vol. **1999**(1999), No. 17, pp. 1–29.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Some properties of Palais-Smale sequences with applications to elliptic boundary-value problems *

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Abstract

When using calculus of variations to study nonlinear elliptic boundary-value problems on unbounded domains, the Palais-Smale condition is not always satisfied. To overcome this difficulty, we analyze Palais-Smale sequences, and use their convergence to justify the existence of critical points for a functional. We show the existence of positive solutions using a minimax method and comparison arguments for semilinear elliptic equations.

§0 Introduction

The goal of this paper is to investigate the existence of positive solutions for a class of elliptic boundary value problems of the form:

$$\Delta u - a(x)u + f(x,u) = 0, \quad u \in W_0^{1,2}(\Omega), \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a connected unbounded domain with smooth boundary $\partial\Omega$. Our approach to (0.1) involves the use of variational method of a mini-max nature. We seek solutions of (0.1) as critical points of the functional J associated with (0.1) and given by

$$J(u) = \int_\Omega [\frac{1}{2}(|\nabla u|^2 + a(x)u^2) - F(x,u)]dx, \tag{0.2}$$

where $F(x,y) = \int_0^y f(x,\eta)d\eta$.

It is assumed that the function $a(x)$ is locally Hölder continuous and satisfies

$$a_1 \geq a(x) \geq a_2 > 0 \quad \text{for all } x \in \bar{\Omega}. \tag{0.3}$$

The basic assumptions for the function f are

(f1) $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and $\lim_{y\to 0} \frac{f(x,y)}{y} = 0$ uniformly in $x \in \bar{\Omega}$.

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* *1991 Mathematics Subject Classifications:* 35J20, 35J25.
Key words and phrases: elliptic equation, Palais-Smale sequence, minimax method.

Submitted December 29, 1998. Published May 14, 1999.
Partially supported by the National Science Council of Republic of China.

(f2) There is a constant a_3 such that $|\frac{\partial f}{\partial y}(x,y)| \leq a_3(1+|y|^{p-1})$ for all $x \in \bar{\Omega}$ and $y \in \mathbb{R}$, where $1 < p < \frac{N+2}{N-2}$ if $N > 2$ and $1 < p < \infty$ if $N = 1, 2$.

(f3) There is a $\lambda > 0$ such that $0 < (\lambda+2)F(x,y) \leq f(x,y)y$ for all $x \in \bar{\Omega}$ and $y \in \mathbb{R}\backslash\{0\}$.

Let $E = W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = (\int_\Omega (a(x)u^2 + |\nabla u|^2)dx)^{1/2}. \tag{0.4}$$

The assumptions listed above imply that $J \in C^1(E, \mathbb{R})$. Moreover, standard arguments from elliptic regularity theory show that critical points of J on E are classical solutions of (0.1). To prove the existence of critical points of functionals like (0.2), one generally needs some compactness as embodied by the Palais-Smale condition (PS) or one of its variants. (PS) says whenever $\{J(u_m)\}$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$, the sequence $\{u_m\}$ possesses a convergent subsequence. Unfortunately, when one deals with elliptic boundary value problems on unbounded domains, (PS) does not always hold. For example, if $\Omega = \mathbb{R}^2$, $a(x) \equiv 1$ and $f(x,y) = |y|^{p-1}y$, it is known that there is a positive solution $u(x)$ of (0.1). The sequence of translates $v_m(x) = u(x + x_m)$ does not possess a (strongly) convergent subsequence in E if $|x_m| \to \infty$ as $m \to \infty$.

Given $\epsilon > 0$, by (f1) and (f2), there is a $C_\epsilon > 0$ such that

$$0 \leq |f(x,u)| \leq \epsilon u + C_\epsilon |u|^p \tag{0.5}$$

and

$$0 \leq F(x,u) \leq \epsilon u^2 + C_\epsilon |u|^{p+1}. \tag{0.6}$$

Hence

$$J(u) = \frac{1}{2}\|u\|^2 + \circ(\|u\|^2) \quad \text{as} \quad \|u\| \to 0 \tag{0.7}$$

and there are positive numbers ρ and σ such that

$$J(u) \geq \sigma \quad \text{for all } u \in E \text{ with } \|u\| = \rho. \tag{0.8}$$

On the other hand, the hypothesis (f3) implies that $F(x,y)$ grows more rapidly than quadratically as $|y| \to \infty$. Hence for any $u \in E\backslash\{0\}$, $J(tu) \to -\infty$ as $t \to \infty$. In other words, $u = 0$ is a strict local minimum but not a global minimum of J. Let $I^b = \{u \in E | J(u) \leq b\}$ and $\Gamma = \Gamma(\Omega) = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) \in I^0\backslash\{0\}\}$. The Mountain Pass Theorem guarantees a critical value β defined by

$$\beta = \beta(\Omega) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \tag{0.9}$$

provided that the Palais-Smale condition is satisfied. Nevertheless, there are some examples for which any sequence $\{v_m\} \subset E$ with $J(v_m) \to \beta$ and $J'(v_m) \to 0$ as $m \to \infty$ possesses no convergent subsequence; in this case there is no solution u of (0.1) with $J(u) = \beta$.

Although the mini-max structure of (0.9) does not guarantee that there is a critical point $u \in E$ with $J(u) = \beta$, we can analyze Palais-Smale sequences to justify if there exist positive solutions of (0.1). Our analysis is based on some comparison arguments which will be described as follows. Let $\{\Omega_k\}$ be a sequence of subsets of Ω such that $(\Omega \cap S_{k+1}) \subset \Omega_k \subset (\Omega \cap S_k)$ and $E_k = W_0^{1,2}(\Omega_k^\circ)$ with the norm

$$\|u\|_k = (\int_{\Omega_k} (a(x)u^2 + |\nabla u|^2)dx)^{1/2},$$

where Ω_k° is the interior of Ω_k and $S_k = \{x \in \mathbb{R}^N \| x| \geq k\}$. For $v \in E_{k+1}$, it can be identified with an element of E_k by extending v to be zero on $\Omega_k^\circ \backslash \Omega_{k+1}^\circ$. The inclusions

$$E_{k+1} \subset E_k \subset \ldots \subset E \tag{0.10}$$

will be used without mentioned explicitly and J_k will be the restriction of J to E_k.

Since our interest in this paper is focused on the positive solutions of (0.1), a well known device will be used by setting $f(x, y) = 0$ if $y < 0$. A sequence $\{u_m\} \subset E$ is called a $(\mathrm{PS})_\mathrm{c}$ sequence if $J(u_m) \to c$ and $J'(u_m) \to 0$ as $m \to \infty$. If any $(\mathrm{PS})_\mathrm{c}$ sequence possesses a convergent subsequence, we say $(\mathrm{PS})_\mathrm{c}$ condition is satisfied. Let $\Lambda(\Omega)$ be the set of positive number c such that there exists a $(\mathrm{PS})_\mathrm{c}$ sequence. The set $\Lambda(\Omega)$ in particular contains all the positive critical values of J. Let $\delta = \delta(\Omega)$ be the infimum of $\Lambda(\Omega)$. It will be shown that $\Lambda(\Omega)$ is a nonempty set and $\delta(\Omega)$ is a positive number. On the restriction J_k, we define the set $\Lambda(\Omega_k)$ and its infimum $\delta_k \equiv \delta(\Omega_k)$ by the same manner.

Theorem 1 *There exists a positive solution u of (0.1) with $J(u) = \delta$, provided that $\delta \notin \Lambda(\Omega_k)$ for some $k \in \mathcal{N}$.*

Remark 1 *The choice of $\{\Omega_k\}$ is not unique. For instance, we may take $\Omega_k \supset \Omega_{k+1}$ and $(\Omega \backslash \widetilde{S}_k) \supset \Omega_k \supset (\Omega \backslash \widetilde{S}_{k+1})$, where $\{\widetilde{S}_{k+1}\}$ is a sequence of compact sets such that $\widetilde{S}_{k+1} \supset \widetilde{S}_k$ and $\cup_{k=1}^\infty \widetilde{S}_k = \mathbb{R}^N$.*

When $\beta > \delta$, it is possible to have multiple solutions for (0.1).

Theorem 2 *There are at least two positive solutions of (0.1) if $\delta < \beta < \delta_k$ for some $k \in \mathcal{N}$.*

A sufficient condition for $\beta(\Omega) = \delta(\Omega)$ is the following

(f4) For fixed $x \in \Omega$, $\frac{f(x,y)}{y}$ is an increasing function of y for $y \in (0, \infty)$ and $\lim_{y\to\infty} \frac{f(x,y)}{y} = \infty$ uniformly in Ω.

In this case, it can be shown that $\{\delta_k\}$ is a nondecreasing sequence and $\delta_k \geq \beta$ for all k. Therefore Theorem 1 can be recast as

Theorem 1' *Assume, in addition to (f1)-(f3), that (f4) is satisfied. Then there exists a positive solution of (0.1) if*

$$\beta < \lim_{k\to\infty} \delta_k. \tag{0.11}$$

Moreover, one can verify that the $(\mathrm{PS})_\beta$ condition is satisfied.

Theorem 3 *Assume (f1)-(f4) are satisfied. Then the $(PS)_\beta$ condition is satisfied if and only if (0.11) holds.*

On the other hand, it is not totally clear yet whether there is a positive solution of (0.1) if $\beta = \lim_{k\to\infty} \delta_k$. Some examples we know of in this direction will be discussed. Also, a different minimax approach from (0.9) will be considered to obtain a positive solution u with $J(u) > \beta$. The detailed description of such a minimax approach will be given at the end of section 5. As a matter of fact, the existence of a positive solution u of (0.1) with $J(u) > \beta$ is an interesting and challenging question. Although we don't have a complete answer, our investigation might serve as a starting point of understanding this question.

There is a seizable literature [ABC, DF, FW, O1, R3, W] on the study of positive solutions of (0.1) for the case $\Omega = \mathbb{R}^N$. The interested readers may consult [N] for more complete references.

In the proofs that follow, we will routinely take $N \geq 3$. The proofs for $N = 1$ or 2 are not more complicated.

§1 Preliminaries

As mentioned in the introduction, the Mountain Pass Theorem cannot be directly applied to obtain the existence of positive solutions of (0.1), since verification of (PS) may not be possible. An alternate approach is to analyze the behavior of Palais-Smale sequences. In this section, several technical results will be established. We begin with the Frechet differentiability of the functional J. A detailed proof of Proposition 1 can be found in [CR].

Proposition 1 *If f satisfies (f1)-(f3) then $J \in C^1(E, \mathbb{R})$.*

Next we prove the boundedness of Palais-Smale sequences.

Lemma 1 *If $\{u_n\}$ is a $(PS)_c$ sequence then there is a constant K (depending on c) such that $\|u_n\| \leq K$ for all n.*

Proof. Since $J'(u_n) \to 0$ as $n \to \infty$, if n is large then

$$\|u_n\|^2 - \int f(x, u_n) u_n dx = J'(u_n)u_n = o(1)\|u_n\|. \tag{1.1}$$

Hence

$$\begin{aligned} c &= J(u_n) + o(1) = J(u_n) - \frac{1}{2} J'(u_n)u_n + o(1)(1 + \|u_n\|) \\ &\geq \left(\frac{1}{2} - \frac{1}{\lambda + 2}\right) \int_\Omega f(x, u_n)u_n dx + o(1)(1 + \|u_n\|), \end{aligned} \tag{1.2}$$

where the last inequality follows from (f3).Substituting (1.1) into (1.2) yields

$$c \geq (\frac{1}{2} - \frac{1}{\lambda+2})\|u_n\|^2 + o(1)(1+\|u_n\|), \tag{1.3}$$

which completes the proof.

Corollary 1 *If $\{u_n\}$ is a $(PS)_c$ sequence then*

$$\lim_{n\to\infty} \sup \|u_n\| \leq \left(\frac{2c(\lambda+2)}{\lambda}\right)^{1/2}. \tag{1.4}$$

Proof. It directly follows from (1.3) and Lemma 1.

Corollary 2 *If $u \in E$, and $J'(u) = 0$ then*

$$J(u) \geq \frac{\lambda}{2(\lambda+2)}\|u\|^2. \tag{1.5}$$

Proof. Note that (1.5) is trivially satisfied when $u \equiv 0$. If $u \not\equiv 0$, (1.5) follows from (1.3) by letting $u_n = u$ for all n.

Lemma 2 *There exists a $(PS)_\beta$ sequence, where β is the mountain pass minimax value defined in (0.9).*

Lemma 2 follows from deformation theory and its proof is omitted. Note that $\beta > 0$ by (0.8) and (0.9). Thus $\Lambda(\Omega)$ is non-empty.

Proposition 2 *If (f1)-(f3) are satisfied then $\delta(\Omega) > 0$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence, where $c > 0$. Applying the Hölder inequality and the Sobolev inequality yields

$$\left(\int_\Omega |u_n|^{p+1}dx\right)^{\frac{1}{p+1}} \leq C_1\|u_n\|^q \cdot \|u_n\|^{1-q} = C_1\|u_n\|, \tag{1.6}$$

where $q = \left(\frac{N}{p+1} - \frac{(N-2)}{2}\right) \in (0,1)$. It follows from Corollary 1 that

$$\left|\int_\Omega f(x,u_n)u_n dx\right| \leq \epsilon\|u_n\|^2 + C_\epsilon C_1^{p+1}\left(\frac{3c(\lambda+2)}{\lambda}\right)^{\frac{p-1}{2}}\|u_n\|^2. \tag{1.7}$$

Choose $\epsilon < \frac{1}{8}$ and $\bar{c} > 0$ such that $C_\epsilon C_1^{p+1}\left(\frac{3\bar{c}(\lambda+2)}{\lambda}\right)^{\frac{p-1}{2}} < \epsilon$. If $c < \bar{c}$ then

$$\left|J'(u_n)\frac{u_n}{\|u_n\|}\right| = \left(\|u_n\|^2 - \int_\Omega f(x,u_n)u_n dx\right)\|u_n\|^{-1} \geq \frac{1}{2}\|u_n\|$$

which implies $\|u_n\| \to 0$ and consequently $J(u_n) \to 0$ as $n \to \infty$. This violates $\lim_{n\to\infty} J(u_n) = c > 0$.Therefore there is no $(PS)_c$ sequence if $c \in (0,\bar{c})$. So $\delta(\Omega) \geq \bar{c} > 0$.

Proposition 3 *If $u \in E$ which satisfies $J'(u) = 0$ and $J(u) > 0$, then u is a positive solution of (0.1).*

To prove Proposition 3, we will use the following proposition which is a direct consequence of maximum principle.

Proposition 4 *If u is a solution of (0.1), $u \geq 0$ in Ω and $u = 0$ at some $x \in \Omega$ then $u \equiv 0$ in Ω.*

Proof of Proposition 3. By elliptic regularity theory, any critical point of J is a classical solution of (0.1). Let $u^-(x) = \max(-u(x), 0)$. Since

$$\int_\Omega (\nabla u \cdot \nabla u^- + a(x)uu^-)dx - \int_\Omega f(x,u)u^- dx = J'(u)u^- = 0, \tag{1.8}$$

it follows that $\|u^-\|^2 = \int_\Omega f(x,u)u^- = 0$. Hence $u \geq 0$ in Ω.

Suppose $u(x) = 0$ for some $x \in \Omega$, then by Proposition 4 we get $u \equiv 0$, which contradicts $J(u) > 0$. Therefore $u > 0$ in Ω.

The next lemma indicates the relationship between Palais-Smale sequences and critical points of J. We refer to [CR] for a detailed proof.

Lemma 3 *Let $\{u_n\}$ be a $(PS)_c$ sequence. Then there exist a $\bar{u} \in E$ and a subsequence $\{u_{n_k}\}$ such that*

$$u_{n_k} \to \bar{u} \text{ weakly in } E \text{ and strongly in } L^{p+1}_{loc}(\Omega), 1<p<(N+2)/(N-2) \tag{1.9}$$

and $u_{n_k} \to \bar{u}$ a.e.. Moreover, $J'(\bar{u}) = 0$ and $J(\bar{u}) \leq c$.

In the remaining of this section, we state some properties of Palais-Smale sequences.

Lemma 4 *Let $\{u_n\}$ be a $(PS)_c$ sequence and $Q_r = \Omega \cap B_r$, where $B_r = \{x||x| < r\}$. Suppose there is an increasing sequence $\{r_n\}$ such that $\lim_{n\to\infty} r_n = \infty$ and*

$$\lim_{n\to\infty} \int_{Q_{2r_n}} |u_n|^2 dx = 0. \tag{1.10}$$

Then

$$\lim_{n\to\infty} \int_{Q_{r_n}} |\nabla u_n|^2 dx = 0, \quad \lim_{n\to\infty} \int_{Q_{r_n}} f(x,u_n)u_n dx = 0, \tag{1.11}$$

and

$$\lim_{n\to\infty} \int_{Q_{2r_n}} |u_n|^{p+1} dx = 0 \quad \text{if } 1 < p < (N+2)/(N-2). \tag{1.12}$$

Proof. Let $\phi_n \in C_0^\infty(\mathbb{R}^n)$ which satisfies $0 \leq \phi_n \leq 1$, $|\nabla \phi_n| \leq 1$ and

$$\phi_n(x) = \begin{cases} 1 & \text{if } x \in B_{r_n} \\ 0 & \text{if } x \notin B_{2r_n}. \end{cases}$$

By Lemma 1 there is a $C_1 > 0$ such that

$$\|\phi_n u_n\|^2 \le \int_\Omega a(x)\phi_n^2 u_n^2 dx + 2\int_\Omega u_n^2 |\nabla\phi_n|^2 dx + 2\int_\Omega \phi_n^2 |\nabla u_n|^2 dx \le C_1 .$$

If n is large then

$$\begin{aligned} & \int_\Omega a(x)u_n^2\phi_n dx + \int_\Omega \nabla u_n \cdot (\phi_n \nabla u_n + u_n \nabla\phi_n)dx - \int_\Omega f(x,u_n)\phi_n u_n dx \\ = \;& J'(u_n)\phi_n u_n = o(1). \end{aligned} \tag{1.13}$$

Applying the Schwarz inequality yields

$$\left|\int_{Q_{2r_n}} u_n \nabla u_n \cdot \nabla\phi_n dx\right| \le \left(\int_{Q_{2r_n}} u_n^2 dx\right)^{\frac{1}{2}} \left(\int_{Q_{2r_n}} |\nabla u_n|^2 dx\right)^{\frac{1}{2}} = o(1). \tag{1.14}$$

From (f1) and (f2), we have

$$\left|\int_{Q_{2r_n}} f(x,u_n)\phi_n u_n dx\right| \le C_2 \int_{Q_{2r_n}} (|u_n|^2 + |u_n|^{p+1})dx \tag{1.15}$$

Invoking the Hölder inequality and the Sobolev inequality yields

$$\begin{aligned} \left(\int_{Q_{2r_n}} |u_n|^{p+1}dx\right)^{\frac{1}{p+1}} &\le \left(\int_{Q_{2r_n}} |u_n|^2 dx\right)^{\frac{q}{2}} \left(\int_{Q_{2r_n}} |u_n|^{\frac{2N}{N-2}}dx\right)^{\frac{(1-q)(N-2)}{2N}} \\ &\le C\|u_n\|^{1-q}\left(\int_{Q_{2r_n}} |u_n|^2 dx\right)^{\frac{q}{2}} = o(1), \end{aligned} \tag{1.16}$$

where $q = \left(\frac{N}{p+1} - \frac{(N-2)}{2}\right) \in (0,1)$. Putting (1.16),(1,10) and (1.15) together gives

$$\lim_{n\to\infty} \int_{Q_{2r_n}} f(x,u_n)\phi_n u_n dx = 0. \tag{1.17}$$

Substituting (1.17), (1.14) into (1.13) yields $\lim_{n\to\infty}\int_{Q_{2r_n}} \phi_n |\nabla u_n|^2 dx = 0$, and consequently (1.11) follows.

Let $\xi : \mathbb{R}^n \to [0,1]$ be a C^∞-function which satisfies

$$\xi(x) = \begin{cases} 0 & \text{if } x \in B_{k+1} \\ 1 & \text{if } x \notin B_{k+2}. \end{cases} \tag{1.18}$$

Lemma 5 *Let $\{u_n\}$ satisfy the hypothesis of Lemma 4 and w_n be the restriction of ξu_n to Ω_k. Then $w_n \in E_k$, and $J_k(w_n) \to c$ and $J_k'(w_n) \to 0$ as $n \to \infty$.*

We omit the proof, since it follows from straightforward calculation.

§2 Existence results

We now prove the existence of positive solutions of (0.1).

Theorem 4 *Suppose there is a $(PS)_c$ sequence such that $c > 0$ and $c \notin \Lambda(\Omega_k)$ for some $k \in \mathcal{N}$, then there is a positive solution u of (0.1) and $c \geq J(u) \geq \delta$.*

Proof Let $\{u_n\}$ be a $(PS)_c$ sequence. By Lemma 3, there exist a $u \in E$ and a subsequence, still denoted by $\{u_n\}$, such that $u_n \to u$ weakly in E, $u_n \to u$ a.e., $J'(u) = 0$ and $J(u) \leq c$. We claim $u \not\equiv 0$. This is true if there exist $r, b > 0$ and $l \in \mathcal{N}$ such that if $n \geq l$ then

$$\int_{Q_r} u_n^2 dx \geq b, \tag{2.1}$$

where Q_r was defined in Lemma 4. Suppose (2.1) is false. Then there exist a sequence $\{r_n\}$ with $\lim_{n\to\infty} r_n = \infty$, and a subsequence, still denoted by $\{u_n\}$, such that $\lim_{n\to\infty} \int_{Q_{2r_n}} u_n^2 dx = 0$. Let ξ be defined as in (1.18) and w_n be the restriction of ξu_n to Ω_k. Invoking Lemma 5 yields $c \in \Lambda(\Omega_k)$. This is contrary to the hypothesis, so (2.1) must hold and $u \not\equiv 0$. Then $J'(u) = 0$ and Corollary 2 shows that $J(u) > 0$. By Proposition 3, u is a positive solution of (0.1). The proof is complete.

Having proved Theorem 4, we next prove two theorems stated in the introduction.

Proof of Theorem 1. By the definition of $\Lambda(\Omega)$, there is a $(PS)_\delta$ sequence, where, by Proposition 2, $\delta = \delta(\Omega) > 0$. Applying Theorem 4 gives a positive solution u of (0.1) with $J(u) = \delta$.

Before proving Theorem 2, we state a technical lemma. Its proof can be found in [CR].

Lemma 6 *Let $\{u_m\}$ be a $(PS)_c$ sequence. Assume that $u \in E$ and $\{u_m\}$ converges to u weakly in E and strongly in $L^s_{loc}(\Omega)$ for $s \in [2, \frac{2N}{N-2})$. If $v_m = u_m - u$, then $\lim_{m\to\infty} J'(v_m) = 0$ and $\lim_{m\to\infty} J(v_m) = c - J(u)$.*

Remark 2 *The arguments used to prove Lemma 3 show that $J'(u) = 0$.*

Proof of Theorem 2. Since $\delta \notin \Lambda(\Omega_k)$, by Theorem 1 there is a positive solution u of (0.1) with $J(u) = \delta$. Invoking Lemma 2 and Lemma 3, we get a $(PS)_\beta$ sequence $\{u_m\}$ which converges to v weakly in E and strongly in $L^s_{loc}(\Omega)$ for $s \in [2, \frac{2N}{N-2})$. Moreover $J'(v) = 0$ and $J(v) \leq \beta$. Since $\beta \notin \Lambda(\Omega_k)$, it follows from the same reasoning as in the proof of Theorem 4 that v is a positive solution of (0.1). Suppose $v = u$. Setting $v_m = u_m - v$, we see from Lemma 6 that $\{v_m\}$ is a $(PS)_{\beta-\delta}$ sequence. Since $0 < \beta - \delta < \delta_k$, repeating the above arguments leads to $\{v_m\}$ converges weakly to some $\bar{v} \in E\backslash\{0\}$. This contradicts that u_m converges weakly to v. So $v \neq u$.

Remark 3 (a) *The proof shows that Theorem 2 still holds if $\delta < \beta$ and $\beta \notin \Lambda(\Omega_k)$, $\delta \notin \Lambda(\Omega_j)$, $\beta - \delta \notin \Lambda(\Omega_i)$ for some $i, j, k \in \mathcal{N}$.*

(b) *In fact, the proof also shows that $J(v) = \beta$, for otherwise, if $J(v) = \alpha < \beta$ then $\{v_m\}$ would be a $(PS)_{\beta-\alpha}$ sequence, which would lead to a contradiction as above.*

§3 A Sufficient Condition for $\delta(\Omega) = \beta(\Omega)$

Although it has been proved in Proposition 2 that $\delta(\Omega) > 0$, it seems to be difficult in general to obtain an optimal lower bound for $\delta(\Omega)$. If (f4) is satisfied, the structure of J is more clear (as will be indicated in Proposition 6) so that we are able to find the exact value of $\delta(\Omega)$. Its applications will be illustrated later.

Proposition 5 *If (f1)-(f4) are satisfied then $\delta(\Omega) = \beta(\Omega)$.*

To prove Proposition 5, we need the following proposition whose proof can be found in [DN].

Proposition 6 *If (f1)-(f4) are satisfied then*

$$\beta = \inf_{\substack{u\in E\\ u\not\equiv 0}} \max_{t\in[0,\infty)} J(tu). \tag{3.1}$$

Proof of Proposition 5. It suffices to show $\delta(\Omega) \ge \beta(\Omega)$ since the reversed inequality is always true. Let $\{u_n\}$ be a $(PS)_c$ sequence with $c > 0$. Then there is an $\epsilon_1 > 0$ such that for large n

$$\|u_n\| \ge \epsilon_1. \tag{3.2}$$

For $u_n \not\equiv 0$, we set $g_n(t) = J(t|u_n|)$. It is clear that $g_n(0) = 0$. Since

$$g_n'(t) = t\|u_n\|^2 - \int_\Omega f(x, t|u_n|)|u_n|dx, \tag{3.3}$$

it follows from (f1) that $g_n'(t) > 0$ if t is positive and sufficiently small. Moreover, we know from (f3) that $\lim_{t\to\infty} g_n(t) = -\infty$. Hence there is a $t_n \in (0,\infty)$ such that

$$g_n'(t_n) = 0 \quad\text{and}\quad g_n(t_n) = \max_{t\in[0,\infty)} g_n(t). \tag{3.4}$$

By Proposition 6

$$\beta \le g_n(t_n). \tag{3.5}$$

Let $R(z) = \{x \in \mathbb{R}^N \,|\, \|x - z\|_\infty \le \frac{1}{2}\}$. We claim there exist a sequence $\{z_n\} \subset \mathcal{Z}^N$ and an $\epsilon_2 > 0$ such that

$$\int_{R(z_n)} |u_n|^{p+1} dx \ge \epsilon_2, \tag{3.6}$$

where u_n is identified with an element of $W^{1,2}(\mathbb{R}^N)$ by extending u_n to be zero on $\mathbb{R}^N\backslash\Omega$. Suppose (3.6) is false. Then

$$s_n \equiv \sup_{z\in\mathcal{Z}^N}\left(\int_{R(z)}|u_n|^{p+1}dx\right)^{\frac{p-1}{p+1}} \to 0 \quad \text{as } n\to\infty. \tag{3.7}$$

Invoking the Sobolev inequality yields

$$\sum_{z\in\mathcal{Z}^N}\left(\int_{R(z)}|u_n|^{p+1}dx\right)^{\frac{2}{p+1}} \le C\sum_{z\in\mathcal{Z}^N}\left(\int_{R(z)}a(x)u_n^2+|\nabla u_n|^2dx\right) = C\|u_n\|^2$$

and

$$\begin{aligned}\|u_n\|_{L^{p+1}}^{p+1} &= \sum_{z\in\mathcal{Z}^N}\left(\int_{R(z)}|u_n|^{p+1}dx\right)^{\frac{p-1}{p+1}}\left(\int_{R(z)}|u_n|^{p+1}dx\right)^{\frac{2}{p+1}} \\ &\le Cs_n\|u_n\|^2.\end{aligned} \tag{3.8}$$

For any given $\epsilon>0$,

$$\begin{aligned}\int_\Omega f(x,|u_n|)|u_n|dx &\le \int_\Omega \epsilon|u_n|^2+C_\epsilon|u_n|^{p+1}dx \\ &\le (\epsilon+CC_\epsilon s_n)\left(\sup_n\|u_n\|^2\right) \le 2\epsilon\left(\sup_n\|u_n\|^2\right)\end{aligned} \tag{3.9}$$

if n is large enough. Hence

$$\lim_{n\to\infty}\int_\Omega f(x,|u_n|)|u_n|dx = 0. \tag{3.10}$$

Assuming for now that

$$J'(|u_n|)|u_n| \to 0 \quad \text{as } n\to\infty, \tag{3.11}$$

we have $\lim_{n\to\infty}\|u_n\|^2 = \lim_{n\to\infty}\left(J'(|u_n|)|u_n| + \int_\Omega f(x,|u_n|)|u_n|dx\right) = 0$. This contradicts (3.2). Consequently (3.6) must hold.

Let $v_n(x) = u_n(x-z_n)$. Since $\|v_n\|$ is bounded, there is a subsequence, still denoted by $\{v_n\}$, such that $v_n\to\bar v$ in $L^{p+1}(R(0))$ and $\int_{R(0)}|\bar v|^{p+1}dx \ge \epsilon_2$. Hence there are positive numbers ϵ_3 and ϵ_4 such that

$$|D_n| \equiv |\{x\in R(z_n)||u_n(x)|\ge\epsilon_3\}| \ge \epsilon_4,$$

where $|D_n|$ is the Lebesgue measure of the set D_n. Then it follows from (3.3) and (3.4) that

$$\begin{aligned}\|u_n\|^2 &= \frac{1}{t_n}\int_\Omega f(x,t_n|u_n|)|u_n| \ge \int_{D_n}\frac{f(x,t_n|u_n|)}{t_n|u_n|}|u_n|^2dx \\ &\ge \frac{f(x,t_n\epsilon_3)}{t_n\epsilon_3}\int_{D_n}|u_n|^2dx \ge \epsilon_3^2\epsilon_4\left(\inf_{x\in D_n}\frac{f(x,t_n\epsilon_3)}{t_n\epsilon_3}\right).\end{aligned}$$

Since $\lim_{y\to\infty}\frac{f(x,y)}{y}=\infty$ uniformly in Ω, $\{t_n\}$ must be bounded. Hence

$$\begin{aligned} g_n(t_n) &= \frac{1}{2}t_n^2 J'(|u_n|)|u_n| + \frac{1}{2}t_n^2\int f(x,|u_n|)|u_n|dx - \int_\Omega F(x,t_n|u_n|)dx \\ &= h(t_n)+o(1), \end{aligned} \tag{3.12}$$

where $h(t)=\frac{1}{2}t^2\int_\Omega f(x,|u_n|)|u_n|dx - \int_\Omega F(x,t|u_n|)dx$. Since

$$h'(t)=\int_\Omega\left(\frac{f(x,|u_n|)}{|u_n|}-\frac{f(x,t|u_n|)}{t|u_n|}\right)t|u_n|^2dx,$$

it follows from (f4) that $h'(t)>0$ if $t\in(0,1)$ and $h'(t)<0$ if $t\in(1,\infty)$. Thus $h(1)=\max_{t\in[0,\infty)}h(t)$. This together with (3.5), (3.11) and (3.12) yields

$$\begin{aligned} \beta(\Omega) &\le \lim_{n\to\infty}\inf g_n(t_n) \le \lim_{n\to\infty}\inf\int_\Omega[\frac{1}{2}f(x,|u_n|)|u_n| - F(x,|u_n|)]dx \\ &= \lim_{n\to\infty}\inf J(|u_n|). \end{aligned} \tag{3.13}$$

Let $u_n^+=\max(u_n,0)$ and $u_n^-=u_n^+-u_n$. Then

$$\begin{aligned} J(|u_n|) &= \frac{1}{2}\|u_n\|^2-\int_\Omega F(x,u_n^+)dx-\int_\Omega F(x,u_n^-)dx \\ &\le \frac{1}{2}\|u_n\|^2-\int_\Omega F(x,u_n^+)dx = J(u_n). \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14) yields $\beta(\Omega)\le\lim_{n\to\infty}J(u_n)=c$. Since c is arbitrary, it follows that $\beta(\Omega)\le\delta(\Omega)$.

It remains to show (3.11) to complete the proof. Note that $J'(|u_n|)|u_n| - J'(u_n)u_n = -\int_\Omega f(x,u_n^-)u_n^-dx$. Clearly, $\|u_n^-\|^2=-J'(u_n)u_n^-\to 0$ as $n\to\infty$. This together with the proof of (3.10) shows that $\lim_{n\to\infty}\int_\Omega f(x,u_n^-)u_n^-dx=0$.

Corollary 3 *If (f1)-(f3) are satisfied then* $\beta(\Omega)\le\beta(\Omega_k)\le\beta(\Omega_{k+1})$.

Proof. It easily follows from (0.10).

Corollary 4 *If (f1)-(f4) are satisfied then* $\delta\le\delta_k\le\delta_{k+1}$.

Proof. It follows from Corollary 3 and Proposition 5.

§4 The $(PS)_\beta$ Condition

One of the applications of Proposition 5 is to show that the $(PS)_\beta$ condition is equivalent to

$$\beta<\lim_{k\to\infty}\beta(\Omega_k) \tag{4.1}$$

if (f1)-(f4) are satisfied. Note that by Proposition 5

$$\delta(\Omega_k)=\beta(\Omega_k). \tag{4.2}$$

So (0.14) is equivalent to (4.1) and Theorem 3 can be restated as follows.

Theorem 3 *Assume (f1)-(f4) are satisfied. Then the $(PS)_\beta$ condition is satisfied if and only if (4.1) holds.*

Proof. We first prove the sufficiency. Let $\{u_n\}$ be a $(PS)_\beta$ sequence. By Lemma 3, there exist a $\bar{u} \in E$ and a subsequence, still denoted by $\{u_n\}$, such that $u_n \to \bar{u}$ weakly in E, $u_n \to \bar{u}$ a.e.,

$$J'(\bar{u}) = 0 \tag{4.3}$$

and

$$J(\bar{u}) \le \beta. \tag{4.4}$$

By Proposition 5, $\delta = \delta(\Omega) = \beta(\Omega) = \beta$. Then it follows from (0.14) and the same reasoning as in the proof of Theorem 4 that

$$J(\bar{u}) \ge \delta = \beta. \tag{4.5}$$

Combining (4.4) with (4.5) gives

$$J(\bar{u}) = \beta. \tag{4.6}$$

Hence

$$\begin{aligned} \beta &= \lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} \inf \int_\Omega \left[\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right] dx \\ &\ge \int_\Omega \left[\frac{1}{2} f(x,\bar{u})\bar{u} - F(x,\bar{u})\right] dx = \frac{1}{2}\|\bar{u}\|^2 - \int_\Omega F(x,\bar{u}) = \beta. \end{aligned} \tag{4.7}$$

Applying Fatou's lemma yields

$$\begin{aligned} &\lim_{n\to\infty} \inf \int_\Omega \left[\frac{1}{\lambda+2} f(x,u_n)u_n - F(x,u_n)\right] dx \\ \ge\ &\int_\Omega \left[\frac{1}{\lambda+2} f(x,\bar{u})\bar{u} - F(x,\bar{u})\right] dx \end{aligned} \tag{4.8}$$

and

$$\lim_{n\to\infty} \inf \int_\Omega f(x,u_n)u_n dx \ge \int_\Omega f(x,\bar{u})\bar{u} dx. \tag{4.9}$$

Suppose inequality (4.9) were strict, it would lead to

$$\begin{aligned} &\lim_{n\to\infty} \inf \int_\Omega \left[\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right] dx \\ \ge\ &\lim_{n\to\infty} \inf \int_\Omega \left(\frac{1}{2} - \frac{1}{\lambda+2}\right) f(x,u_n)u_n dx \\ &+ \lim_{n\to\infty} \inf \int_\Omega \left[\frac{1}{\lambda+2} f(x,u_n)u_n - F(x,u_n)\right] dx \\ >\ &\int_\Omega \frac{1}{2} f(x,\bar{u})\bar{u} dx - \int_\Omega F(x,\bar{u}) dx = \beta, \end{aligned}$$

which would contradict to (4.7). Thus there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty}\int_\Omega f(x,u_{n_k})u_{n_k}dx = \int_\Omega f(x,\bar u)\bar u dx$. This together with $J'(\bar u) = 0$ and $\lim_{k\to\infty} J'(u_{n_k}) = 0$ yields $\lim_{k\to\infty}\|u_{n_k}\|^2 = \|\bar u\|^2$. Therefore $\lim_{k\to\infty}\|u_{n_k} - \bar u\| = 0$.

To prove the necessity, we argue indirectly. Suppose (4.1) is false, then it follows from Corollary 3 that $\beta = \beta(\Omega_k)$ for all k. By Lemma 2 there is a sequence $\{v_n\} \subseteq E_k$ such that $\lim_{n\to\infty} J'_k(v_n) = 0$ and $\lim_{n\to\infty} J_k(v_n) = \beta(\Omega_k) = \beta$. We first claim that

$$\text{there is no } v \in E_k \text{ such that } J'_k(v) = 0 \text{ and } J_k(v) = \beta. \tag{4.10}$$

For otherwise, $\|v\|_k^2 = \int_{\Omega_k} f(x,v)vdx$ which implies that

$$\max_{t\in[0,\infty)} J(tv) = J(v) = J_k(v) = \beta. \tag{4.11}$$

Then (4.11) leads to a contradiction by the following reasoning: Suppose $J'(v) = 0$. It follow from (4.11) and Proposition 3 that $v > 0$ in Ω, which contradicts the fact that $v = 0$ in $\Omega\backslash\Omega_k$. Suppose $J'(v) \neq 0$. Let $\gamma(t) = tv, t \in [0,\infty)$. Then with slight modifications, the deformation theory and the arguments used in the proof of Theorem A.4 of [R1] would give a path $\gamma_1(t), t \in [0,1]$, such that

$$\gamma_1(0) = 0, J(\gamma_1(1)) < 0 \text{ and } \max_{t\in[0,1]} J(\gamma_1(t)) < \beta. \tag{4.12}$$

But (4.12) violates (0.9). Thus the proof of (4.10) is complete.

Next, we claim

$$\lim_{n\to\infty}\int_{\Omega_k\backslash\Omega_j} v_n^2 dx = 0 \quad \text{for all } j > k. \tag{4.13}$$

If not, there exist $m \in \mathcal{N}, \epsilon > 0$ and a subsequence, still denoted by $\{v_n\}$, such that

$$\int_{\Omega_k\backslash\Omega_m} v_n^2 dx \geq \epsilon \quad \text{for all } n. \tag{4.14}$$

Applying Lemma 3 and passing to a subsequence if necessary, we obtain a $\bar v \in E_k$ such that

$$v_n \to \bar v \text{ weakly in } E_k \text{ and strongly in } L^{p+1}_{loc}(\Omega_k), \tag{4.15}$$

$J'_k(\bar v) = 0$ and

$$J_k(\bar v) \leq \beta. \tag{4.16}$$

It follows from (4.14),(4.15) and Corollary 2 that $J_k(\bar v) > 0$. Hence

$$J_k(\bar v) \geq \delta_k = \beta(\Omega_k) = \beta. \tag{4.17}$$

Combining (4.16) with (4.17) yields $J_k(\bar v) = \beta$, which contradicts (4.10). Thus (4.13) must hold. Then it follows from Lemma 4 that

$$\lim_{n\to\infty}\int_{\Omega_k\backslash\Omega_j} (|\nabla v_n|^2 + |v_n|^{p+1})dx = 0 \quad \text{for all } j > k. \tag{4.18}$$

We now prove that

$$J'(v_n) \to 0 \quad \text{as } n \to \infty. \tag{4.19}$$

Let ξ be defined as (1.18). For any $\phi \in E$, it follows from $\xi\phi \in E_k$ that $J'_k(v_n)\xi\phi \to 0$ as $n \to \infty$. By direct calculation,

$$\begin{aligned} J'(v_n)\phi &= J'_k(v_n)\xi\phi + \int_{\Omega\setminus S_{k+2}} (\nabla v_n \cdot \nabla\phi + a(x)v_n\phi - f(x,v_n)\phi)dx \\ &- \int_{\Omega_k\setminus S_{k+2}} [(\nabla v_n \cdot \nabla\phi)\xi + (\nabla v_n \cdot \nabla\xi)\phi + a(x)v_n\xi\phi - f(x,v_n)\xi\phi]dx. \end{aligned}$$

Using (4.13), (4.18) and arguments analogous to the proof of Lemma 5, we obtain that $\sup_{\|\varphi\|=1} |J'(v_n)\varphi| \to 0$ as $n \to \infty$. This completes the proof of (4.19).

Having shown that $J(v_n) \to \beta$ and $J'(v_n) \to 0$ as $n \to 0$,we now prove that there is no subsequence of $\{v_n\}$ which is convergent in E. Suppose there is a subsequence $\{v_{n_j}\}$ such that $v_{n_j} \to w$ in E. Then, by Proposition 1, $J(w) = \beta$ and $J'(w) = 0$. It follows from Proposition 3 that $w > 0$ in Ω. But this is impossible since $v_{n_j} = 0$ in $\Omega\setminus\Omega_k$ for all j.

Corollary 5 *Assume (f1)-(f4) are satisfied. Suppose there is a* $u \in E$ *such that* $J'(u) = 0$ *and* $J(u) = \beta(\Omega)$. *If* $\widetilde{\Omega} \supset \Omega$ *and* $\widetilde{\Omega} \neq \Omega$ *then*

$$\beta(\widetilde{\Omega}) < \beta(\Omega). \tag{4.20}$$

Proof. Suppose (4.20) were false, it would follow from (0.9) and $W_o^{1,2}(\Omega) \subset W_o^{1,2}(\widetilde{\Omega})$ that $\beta(\widetilde{\Omega}) = \beta(\Omega)$. Then by the same reasoning as the proof of (4.10), there were no $v \in E$ such that $J'(v) = 0$ and $J(v) = \beta(\Omega)$.

§5 Examples

We are now considering some examples of existence of positive solutions of (0.1).

Example 1 Let $\Omega \subset \mathbb{R}^N$ and $\Omega \neq \mathbb{R}^N$. If (f4) is satisfied and

$$\beta(\Omega) = \beta(\mathbb{R}^N), \tag{5.1}$$

then by Corollary 5 there is no positive solution u of (0.1) with $J(u) = \beta(\Omega)$.

Remark 4 *(a) When* a *and* f *do not depend on* x*, (5.1) holds if for any* $k \in \mathcal{N}$ *there is a ball of radius* k *contained in* Ω. *As a more concrete example,* Ω *can be a half space, a cone or the union of a cone with a bounded set.*
(b) The question of whether there exists a positive solution u of (0.1) with $J(u) > \beta$ *will be studied at the end of this section and the next section.*

For $z \in \mathbb{R}^N$, we define $D_z = \{x + z | x \in D\}$. In the next three examples it is assumed that $0 \in D \subset \mathbb{R}^N$ and there is a subgroup G of $\mathbb{R}^l$, $l \leq N$, such that

$$D_g = D \quad \text{for all } g \in G. \tag{5.2}$$

Example 2 We study (0.1) for the case $\Omega = D_z$, where D satisfies (5.2). Since the case of D_z is not different from D but merely more complicated in notation, in what follows $\Omega = D$.

In addition to (f1)-(f3), it is assumed that

$$f(x+g,y) = f(x,y) \tag{5.3}$$
$$a(x+g) = a(x) \tag{5.4}$$

for all $g \in G, x \in \Omega$.

Theorem 5 *Let $\Omega = D$, where D satisfies (5.2). Suppose G is a countable set and there is a bounded subset T of D such that $0 \in T$,*

$$D = \cup_{g\in G} T_g, \text{ and } T_g \cap T_{g'} = \phi \text{ if } g, g' \in G \text{ and } g \neq g'. \tag{5.5}$$

Then there exists a positive solution of (0.1).

Proof. By Lemma 2, there is a $(PS)_\beta$ sequence $\{u_n\}$. We claim there exist $\epsilon_1 > 0$ and $m \in \mathcal{N}$ such that

$$\sup_{i\in G} \int_{T_i} |u_n|^{p+1} dx \geq \epsilon_1 \quad \text{if } n \geq m. \tag{5.6}$$

Suppose (5.6) is false. Then there is a subsequence, still denoted by $\{u_n\}$, such that

$$\sup_{i\in G} \int_{T_i} |u_n|^{p+1} dx \to 0 \quad \text{as } n \to \infty. \tag{5.7}$$

Applying the Sobolev inequality, we have

$$\int_\Omega |u_n|^{p+1} dx = \sum_{i\in G} \int_{T_i} |u_n|^{p+1} dx \leq C_1 \left(\sup_{i\in G} \int_{T_i} |u_n|^{p+1} dx \right)^{\frac{p-1}{p+1}} \|u_n\|^2.$$

This together with Lemma 1 and (5.7) yields

$$\int_\Omega |u_n|^{p+1} dx \to 0 \quad \text{as } n \to \infty. \tag{5.8}$$

Taking $\epsilon \in (0, \frac{1}{2})$, we get

$$\begin{aligned} \|u_n\|^2 &= \int_\Omega f(x,u_n) u_n dx + J'(u_n)u_n + o(1) \\ &\leq \epsilon \|u_n\|^2 + C_\epsilon \int_\Omega |u_n|^{p+1} dx + J'(u_n)u_n + o(1). \end{aligned} \tag{5.9}$$

Combining (5.8) with (5.9) yields $\lim_{n\to\infty} \|u_n\| = 0$, which implies $\lim_{n\to\infty} J(u_n) = 0$. This violates $\lim_{n\to\infty} J(u_n) = \beta$ and therefore (5.6) must hold.

Pick $g_n \in G$ such that

$$\int_T |u_n(x+g_n)|^{p+1} dx \geq \frac{\epsilon_1}{2}. \tag{5.10}$$

Let $w_n(x) = u_n(x + g_n)$. It is easy to check that $\{w_n\}$ is a $(PS)_\beta$ sequence. By Lemma 3, there exist a $\bar{u} \in E$ and a subsequence $\{w_{n_k}\}$ such that

$$w_{n_k} \to \bar{u} \text{ in } L^{p+1}(T) \tag{5.11}$$

and

$$J'(\bar{u}) = 0. \tag{5.12}$$

By (5.10) and (5.11), we know $\bar{u} \not\equiv 0$. This together with (5.12) and Proposition 3 shows that $\bar{u}$ is a positive solution of (0.1).

Remark 5 (a) *In the proof of Theorem 5, we may take $\{u_n\}$ to be a $(PS)_\delta$ sequence so that $\bar{u}$ is a positive solution of (0.1) with $J(\bar{u}) = \delta(\Omega)$*
(b) *If (f4) is satisfied then by Proposition 5 there is a positive solution $\bar{u}$ of (0.1) with $J(\bar{u}) = \beta(\Omega)$.*

Lemma 7 *If the hypotheses of Theorem 5 and (f4) are satisfied, then for all k,*

$$\beta(\Omega_k) = \beta(\Omega)\,. \tag{5.13}$$

Proof. Consider $G = \mathcal{Z}$ and let T be defined as in (5.5). Let $\Omega_k = \cup_{|i|\geq k} T_i$ and $\xi \in C^\infty(\Omega)$ such that $\xi(x) = 1$ if $x \in \bigcup\limits_{i\geq k+1} T_i$ and $\xi(x) = 0$ if $x \in \bigcup\limits_{i\leq k} T_i$. Let $u(x)$ be a positive solution of (0.1) with $J(u) = \beta(\Omega)$. By direct computation

$$\lim_{m\to\infty} \max_{t\in[0,\infty)} J_k(tu_m) = \beta(\Omega), \tag{5.14}$$

where $u_m(x) = \xi(x)u(x-m)$. Since $\lim\limits_{t\to\infty} J_k(tu_m) = -\infty$, (5.14) implies $\beta(\Omega_k) \leq \beta(\Omega)$. This together with Corollary 3 yields (5.13).

The proof of the case $G \neq \mathcal{Z}$ is similar. We omit it.

As to use comparison arguments in what follows, we sometime replace $\beta(\Omega_k)$ by $\beta_k(\Omega)$ to distinguish $\beta_k(\Omega)$ from $\beta_k(\widetilde{\Omega})$ when two sets Ω and $\widetilde{\Omega}$ are involved. Also, $\delta_k(\Omega)$ will be used in the same vein.

In the next two examples, it is assumed that $a(x)$ and $f(x, y)$ satisfy (5.3), (5.4) and (f4).

Example 3. Consider $\Omega = D \bigcup \mathcal{B}$, where $D\cup\mathcal{B} \neq D$, D satisfies the hypothesis of Theorem 5 and $\mathcal{B}$ is a bounded set. By Corollary 5 and Remark 5(b),

$$\beta(\Omega) < \beta(D) \tag{5.15}$$

Moreover, it follows from Lemma 7 that

$$\beta(D) = \beta_k(D). \tag{5.16}$$

Since $\mathcal{B}$ is bounded, if k is large enough then

$$\delta_k(\Omega) = \beta_k(\Omega) = \beta_k(D). \tag{5.17}$$

Putting (5.15)-(5.17) together yields $\beta(\Omega) < \delta_k(\Omega)$. Hence there is a positive solution of (0.1).

As a matter of fact, Example 3 is a special case of the following result.

Theorem 6 *Let $\Omega = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{J}_1$ and $\mathcal{J}_2$ be the restrictions of J to $W_0^{1,2}(\mathcal{D}_1)$ and $W_0^{1,2}(\mathcal{D}_2)$ respectively. Suppose there exist $u_1 \in W_0^{1,2}(\mathcal{D}_1)$ and $u_2 \in W_0^{1,2}(\mathcal{D}_2)$ such that $\mathcal{J}_1'(u_1) = 0, \mathcal{J}_1(u_1) = \beta(\mathcal{D}_1)$ and $\mathcal{J}_2'(u_2) = 0, \mathcal{J}_2(u_2) = \beta(\mathcal{D}_2)$. If (f4) is satisfied and $\bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2 \cap S_k = \phi$ for some $k \in \mathcal{N}$, then there is a positive solution u of (0,1).*

Proof. By Corollary 5, we get $\beta(\Omega) < \min(\beta(\mathcal{D}_1), \beta(\mathcal{D}_2))$. Since $\bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2 \cap S_k = \phi$, it follows that $\beta(\Omega_k) = \min(\beta(\mathcal{D}_1 \cap \Omega_k), \beta(\mathcal{D}_2 \cap \Omega_k))$, where we define $\beta(\phi) = +\infty$. Thus $\delta(\Omega_k) = \beta(\Omega_k) \geq \min(\beta(\mathcal{D}_1), \beta(\mathcal{D}_2)) > \beta(\Omega)$, from which we know there is a positive solution of (0.1).

Example 4. Let D satisfy the hypothesis of Theorem 5 and $\mathcal{B}$ be a bounded set. If $D \cap \mathcal{B} \neq \phi$ and $\Omega = D \backslash \mathcal{B}$ then there is no positive solution u of (0.1) such that $J(u) = \beta(\Omega)$. To see this, we argue indirectly. Suppose there is a positive solution of (0.1) with $J(u) = \beta(\Omega)$, it follows from Corollary 5 that

$$\beta(\Omega) > \beta_k(\Omega) = \beta(D). \tag{5.18}$$

Since, for large k, $\beta_k(\Omega) = \beta_k(D) = \beta(D)$, applying Corollary 3 yields $\beta(\Omega) \leq \beta_k(\Omega) = \beta(D)$, which contradicts (5.18).

As illustrated in the above examples, Proposition 5 had been applied as a convenient way to obtain an optimal lower bound for δ_k if (f4) is satisfied. We next consider an example of (0.1) where (f4) will not be assumed. Let

$$\|u\|_\Omega = \left(\int_\Omega (|\nabla u|^2 + u^2) dx \right)^{1/2}. \tag{5.19}$$

By (0.3)

$$\|u\|_\Omega \leq a_4 \|u\|, \tag{5.20}$$

where $a_4 = \max(1, \frac{1}{\sqrt{a_2}})$. For fixed $p \in (1, \frac{N+2}{N-2})$, define

$$\sigma(\Omega) = \inf_{\substack{u \in W_0^{1,2}(\Omega) \\ u \not\equiv 0}} \frac{\|u\|_\Omega}{\|u\|_{L^{p+1}(\Omega)}}. \tag{5.21}$$

It is known that if $\|\bar{u}\|_\Omega = \sigma(\Omega)$ and $\|\bar{u}\|_{L^{p+1}(\Omega)} = 1$ then $u = (\sigma(\Omega))^{\frac{2}{p-1}} |\bar{u}|$ is a positive solution of

$$\Delta u - u + |u|^{p-1} u = 0, \quad x \in \Omega. \tag{5.22}$$

Indeed, (5.22) is a special case of (0.1) and in this case it is not difficult to show that

$$\beta(\Omega) = \left(\frac{1}{2} - \frac{1}{p+1} \right) (\sigma(\Omega))^{2(p+1)/(p-1)}. \tag{5.23}$$

Example 5. Let $B_k = \{x||x| < k\}$ and β_0 be the mountain pass minimax value of J on the subspace $W_0^{1,2}(B_{k_0} \cap \Omega)$ of E. As in (0.5) there is a $C_0 > 0$ such that

$$f(x,y)y \le \frac{a_2}{2}y^2 + C_0|y|^{p+1}. \tag{5.24}$$

Since $\beta \le \beta_0$, by corollary 1, we know $(0,\beta] \cap \Lambda(\Omega_k) = \phi$ if

$$J_k'(w)w \ge \frac{1}{4}\|w\|_k^2 \text{ for all } w \in E_k \text{ with } \|w\|_k < d, \tag{5.25}$$

where $d = [2(\lambda+2)(\beta_0+1)\lambda^{-1}]^{1/2}$. Let a_4 be as in (5.20) and $\sigma = \sigma(\mathbb{R}^N)$. By (5.23)

$$\sigma(\Omega_k) \ge \sigma. \tag{5.26}$$

This together with (5.25), (5.21) and (5.20) implies that

$$\begin{aligned} J_k'(w)w &\ge \frac{1}{2}\|w\|_k^2 - C_0\int_{\Omega_k}|w|^{p+1}dx \ge \frac{1}{2}\|w\|_k^2 - C_0\left(\frac{\|w\|_{\Omega_k}}{\sigma(\Omega_k)}\right)^{p+1} \\ &\ge \left[\frac{1}{2} - C_0\left(\frac{a_4}{\sigma(\Omega_k)}\right)^{p+1}\cdot\|w\|_k^{p-1}\right]\|w\|_k^2. \end{aligned} \tag{5.27}$$

Thus (5.25) holds if

$$\sigma(\Omega_k) \ge a_4\left[4C_0\left(\frac{2(\lambda+2)(\beta_0+1)}{\lambda}\right)^{\frac{p-1}{2}}\right]^{\frac{1}{p+1}}. \tag{5.28}$$

Using the Hölder inequality and the Sobolev inequality yields

$$\begin{aligned} \frac{\|u\|_{\Omega_k}^2}{\|u\|_{L^{p+1}(\Omega_k)}^2} &\ge \frac{\|\nabla u\|_{L^2(\Omega_k)}^2 + \|u\|_{L^2(\Omega_k)}^2}{\|u\|_{L^2(\Omega_k)}^{2q}\|u\|_{L^{\frac{2N}{N-2}}(\Omega_k)}^{2(1-q)}} \\ &\ge C_N^{1-q}\left(1 + \frac{\|\nabla u\|_{L^2(\Omega_k)}^2}{\|u\|_{L^2(\Omega_k)}^2}\right)^q \end{aligned} \tag{5.29}$$

if $u \in W_0^{1,2}(\Omega_k)$, where $q = \left(\frac{N}{p+1} - \frac{(N-2)}{2}\right) \in (0,1)$ and C_N is a constant depending on N only. Define

$$\lambda_1(\Omega_k) = \inf_{\substack{u\in W_0^{1,2}(\Omega_k) \\ u\not\equiv 0}} \frac{\|\nabla u\|_{L^2(\Omega_k)}^2}{\|u\|_{L^2(\Omega_k)}^2}. \tag{5.30}$$

Then (5.29) and (5.21) imply that

$$\sigma(\Omega_k) \ge C_N^{(1-q)/2}[1+\lambda_1(\Omega_k)]^{q/2}. \tag{5.31}$$

If $\lambda_1(\Omega_k)$ is large enough, then (5.28) holds and thus there is a positive solution of (0.1).

In some situations, the following proposition can be used to estimate $\lambda_1(\Omega_k)$. We refer to [Es] for a proof of Proposition 7.

Proposition 7 *Let ω be a bounded open set in $\mathbb{R}^{N-1}$ and $\mathcal{T} = \omega \times \mathbb{R}$. If $u \in W_0^{1,2}(\mathcal{T})$ then*

$$\|\nabla u\|^2_{L^2(\mathcal{T})} \geq \lambda_1(\omega)\|u\|^2_{L^2(\mathcal{T})}. \tag{5.32}$$

If $\Omega_k \subset \mathcal{T}$, it follows from (5.23) that $\sigma(\Omega_k) \geq \sigma(\mathcal{T})$. This together with (5.29) and (5.32) shows that

$$\sigma(\Omega_k) \geq C_N^{(1-q)/2}[1 + \lambda_1(w)]^{q/2}. \tag{5.33}$$

Thus

$$\lambda_1(\omega) \geq \left[\left(4C_0\left(\frac{2(\lambda+2)(\beta_0+1)}{\lambda}\right)^{\frac{p-1}{2}}\right)^{\frac{1}{p+1}} a_4 C_N^{(q-1)/2}\right]^{\frac{2}{q}} - 1$$

is a sufficient condition which ensures the existence of positive solution of (0.1).

Remark 6 *Inequality (5.33) still holds if Ω_k contains several connected components and each component is contained in a cylinder like $\mathcal{T}$.*

We now back to Example 1 where $\Omega \neq \mathbb{R}^n$ and $\beta(\Omega) = \beta(\mathbb{R}^n)$. In this case there is no positive solution u of (0.1) with $J(u) = \beta$. An interesting question is whether there exists a positive solution u of (0.1) with $J(u) > \beta$. This question seems to be quite challenging and hard to give a complete answer. Our aim in the next example is to use a different minimax approach from (0.9) to obtain a positive solution u of (0.1) with $J(u) > \beta$.

Example 6. For simplicity in presentation, we consider the case where a and f do not depend on x. Also, slightly stronger conditions on f will be imposed. In addition to (f4), $f \in C^1$ is replaced by $yf'(y) \in C^1$ in (f1), $(\lambda+2)F(x,y) \leq f(x,y)y$ is replaced by $(\lambda+1)f(y) \leq yf'(y)$ in (f3), and $|(yf'(y))'| \leq a_3(1+|y|^{p-1})$ is added in (f2).

Let $\Omega = \Omega^- \cup O \cup \Omega^+$, where $O \subset \{x||x-x_0|_\infty < 3r\}$ for some $r > 0$ and $x_0 \in \mathbb{R}^N$. Without loss of generality, we may assume that $x_0 = 0$. Suppose that $\Omega^+ \subset [3r,\infty) \times \mathbb{R}^{N-1}$ and $\Omega^- \subset (-\infty, 3r] \times \mathbb{R}^{N-1}$. Moreover, it is assumed that for any $j \in \mathbb{N}$, there exist $\xi_j \in \Omega^-$ and $\eta_j \in \Omega^+$ such that $B_j(\xi_j) \subset \Omega^-$ and $B_j(\eta_j) \subset \Omega^+$, where $B_j(z) = \{x||x-z| < j\}$. Thus $\beta(\Omega) = \beta(\mathbb{R}^N)$ and by Corollary 5 there is no positive solution u of (0.1) with $J(u) = \beta$.

For $k > 3r$, let $\Omega_k^+ = ((k,\infty)\times\mathbb{R}^{N-1})\cap\Omega$ and $\Omega_k^- = ((-\infty,-k)\times\mathbb{R}^{N-1})\cap\Omega$. Let $E_k^+ = W_0^{1,2}(\Omega_k^+)$, $E_k^- = W_0^{1,2}(\Omega_k^-)$ and

$$S = \{u|u \in E\backslash\{0\} \text{ and } J'(u)u = 0\}.$$

For any $\pi_1 > 0$, there exist $z_+ \in E_k^+ \cap S$ and $z_- \in E_k^- \cap S$ such that $\max(J(z_+), J(z_-)) < \beta + \pi_1$. Set $\Gamma_1 = \{\gamma \in C([0,1],S)|\gamma(0) = z_-$ and $\gamma(1) = z_+\}$ and

$$\alpha = \inf_{\gamma\in\Gamma_1}\max_{\theta\in[0,1]} J(\gamma(\theta)). \tag{5.34}$$

We are going to show that there exists a $(\mathrm{PS})_\alpha$ sequence with $\alpha > \beta$, provided that z_+ and z_- are suitably chosen. Furthermore, we have the following existence result.

Theorem 7 *Assume that (f1)-(f4) are satisfied. If $\alpha \notin \Lambda(\Omega_k)$ for some $k > 0$ then there is a positive solution u of (0.1) and $J(u) > \beta$.*

Remark 7 (a) *Suppose up to translation there is a unique positive solution of $\Delta v - v + f(v) = 0$ in $W^{1,2}(\mathbb{R}^N)$. Then there is a positive solution u of (0.1) with $J(u) = \alpha$ if $\alpha < 2\beta$.*
(b) *We refer to [CL,K] for some uniqueness results of positive solutions of $\Delta v - v + f(v) = 0$ in $W^{1,2}(\mathbb{R}^N)$.*

§6 Proof of Theorem 7

To prove Theorem 7, we obtain a Palais-Smale sequence by using (5.34).
Proposition 8 *There exists a $(PS)_\alpha$ sequence, where $\alpha > \beta$.*

We proceed to prove Proposition 8 step by step as in a series of technical lemmas. Let

$$\tilde{I}^a = \{u \in S | J(u) \le a\}. \tag{6.1}$$

Lemma 8 *For any $a > 0$, $\tilde{I}^a$ is a bounded set in E.*

The proof of Lemma 8 is simlar to that of Lemma 1. We omit it.

Lemma 9 *If $\{u_m\} \subset S$ and $\lim\limits_{m\to\infty} J(u_m) = \beta$ then $\lim\limits_{m\to\infty} J'(u_m) = 0$.*

Proof. It follows from Ekeland's variational principle [E,S].

Lemma 10 *There is an $A_1 > 0$ such that if $u \in S$ then*

$$\|u\| \ge A_1. \tag{6.2}$$

Proof. By (f1) and (f2) there is a $C_0 > 0$ such that

$$f(y)y \le \frac{a}{2}y^2 + C_0|y|^{p+1} \tag{6.3}$$

for all $y \in \mathbb{R}$. If $u \in S$ then $u \not\equiv 0$ and

$$0 = J'(u)u \ge \int (|\nabla u|^2 + \frac{a}{2}u^2 - C_0|u|^{p+1})dx > \frac{1}{2} \|u\|^2 - C_2 \|u\|^{p+1},$$

by making use of the Sobolev inequality. Thus (6.2) follows by letting $A_1 = (2C_2)^{\frac{-1}{p-1}}$.

For $\rho > 0$, let $O_\rho = ([-\rho, \rho] \times \mathbb{R}^{N-1}) \cap \Omega$.

Lemma 11 *For any $A_2 > 0$, there is a $\pi_2 = \pi_2(A_2)$ such that if $u \in S$ and $J(u) < \beta + \pi_2$, then*

$$\int_{O_{3r}} (u^2 + |u|^{p+1})dx < A_2. \tag{6.4}$$

Proof. Suppose the assertion of the lemma is false. Then there exist a $b_2 > 0$ and a sequence $\{u_m\} \subset S$ such that $\lim_{m\to\infty} J(u_m) = \beta$ and

$$\int_{O_{3r}} (u_m^2 + |u_m|^{p+1})dx \geq b_2. \tag{6.5}$$

By Lemma 9, $\{u_m\}$ is a $(PS)_\beta$ sequence. Hence along a subsequence

$$u_m \to \bar{u} \text{ weakly in } E \text{ and strongly in } L^{p+1}_{loc} \tag{6.6}$$

for some $\bar{u} \in E$ with $J'(\bar{u}) = 0$. Since (6.5) shows $\bar{u} \not\equiv 0$, it follows that

$$J(\bar{u}) = \beta. \tag{6.7}$$

This is absurd since there is no $u \in S$ with $J(u) = \beta$.

Lemma 12 *For any $A_3 > 0$, there is a $\pi_3 = \pi_3(A_3)$ such that if $u \in S$ and $J(u) < \beta + \pi_3$, then*

$$\int_{O_{2r}} (|\nabla u|^2 + au^2)dx < A_3. \tag{6.8}$$

Proof. Let ϕ_1 be a C^∞-function which satisfies $0 \leq \phi_1 \leq 1$, $|\nabla \phi_1| \leq \frac{2}{r}$, $\phi_1 \equiv 1$ on O_{2r} and $\phi_1 \equiv 0$ on $\Omega \backslash O_{3r}$. Observe that

$$\begin{aligned}
\int_{O_{2r}} (|\nabla u|^2 + au^2)dx &\leq \int_{O_{3r}} \phi_1(|\nabla u|^2 + au^2)dx \\
&\leq \int_{O_{3r}} (|\nabla \phi_1||\nabla u||u| + f(u)\phi_1 u)dx + |J'(u)\phi_1 u| \\
&\leq \frac{2}{r}\,\|u\|\,(\int_{O_{3r}} u^2 dx)^{1/2} + C_3 \int_{O_{3r}} (u^2 + |u|^{p+1})dx + |J'(u)\phi_1 u|.
\end{aligned}$$

By Lemma 8, $\tilde{I}^{\beta+\pi_3}$ is bounded in E. If $\pi_3 \leq \pi_2(A_2)$ and A_2 is sufficiently small then (6.8) follows from Lemma 9 and Lemma 11.

Let φ be a C^∞-function which satisfies $0 \leq \varphi \leq 1$, $|\nabla \varphi| \leq \frac{2}{r}$, $\varphi \equiv 0$ on O_r and $\varphi \equiv 1$ on $\Omega \backslash O_{2r}$. For $u \in S$, if $\{x|\varphi(x)u(x) > 0\}$ has positive measure, then

$$\text{there is a unique } \tau = \tau(\varphi u) \text{ such that } \tau\varphi u \in S. \tag{6.9}$$

Lemma 13 *For any $A_4 \in (0, \frac{\beta}{2})$, there is a $\pi_4 = \pi_4(A_4) > 0$ such that if $u \in S$ and $J(u) < \beta + \pi_4$ then $J(\tau\varphi u) < \beta + A_4$.*

Proof. Given $\rho_0 > 0$. It follows from Lemma 8 and Lemma 12 that

$$\begin{aligned} &|\int_\Omega (|\nabla(\varphi u)|^2 + a(\varphi u)^2)dx - \int_\Omega (|\nabla u|^2 + au^2)dx| \\ \leq &\int_\Omega (1-\varphi^2)(|\nabla u|^2 + au^2)dx \\ &+ \int_\Omega (|\nabla\varphi|^2 u^2 + 2\varphi u|\nabla\varphi\cdot\nabla u|)dx < \frac{\rho_0}{2} \end{aligned} \tag{6.10}$$

if $\pi_4 < \pi_3(A_3)$ and A_3 is sufficiently small. Similarly, invoking Lemma 11 yields

$$|\int_\Omega f(u)udx - \int_\Omega f(\varphi u)\varphi udx| < \frac{\rho_0}{2} \tag{6.11}$$

and

$$|\int_\Omega F(u)dx - \int_\Omega F(\varphi u)dx| < \frac{\rho_0}{2} \tag{6.12}$$

if $\pi_4 < \pi_2(A_2)$ and A_2 is sufficiently small. Putting (6.10)-(6.12) together gives $|J(\varphi u) - J(u)| < \rho_0$ and $|J'(\varphi u)\cdot\varphi u| < \rho_0$. By Lemma 10

$$A_1^2 \leq \int_\Omega (|\nabla u|^2 + au^2)dx. \tag{6.13}$$

This together with (6.8) shows that $\{x|\varphi(x)u(x) > 0\}$ has positive measure if A_3 is sufficiently small. Then $\tau\phi u \in S$ implies that

$$\int_\Omega [|\nabla(\varphi u)|^2 + a(\varphi u)^2 - \frac{f(\tau\varphi u)\varphi u}{\tau}]dx = \frac{1}{\tau^2}J'(\tau\phi u)\tau\phi u = 0,$$

which leads to

$$\begin{aligned} \int_\Omega |f(\varphi u)\varphi u - \frac{f(\tau\varphi u)\varphi u}{\tau}|dx &= |\int_\Omega f(\varphi u)\varphi u - \frac{f(\tau\varphi u)\varphi u}{\tau}dx| \\ &= |J'(\varphi u)\varphi u| < \rho_0\,. \end{aligned}$$

Therefore by (f3)

$$|1-\tau^\lambda|\int_\Omega f(\varphi u)\varphi udx \leq \int_\Omega |f(\varphi u)\varphi u - \frac{f(\tau\varphi u)\varphi u}{\tau}|dx < \rho_0. \tag{6.14}$$

Since (6.11) and (6.13) imply $\int_\Omega f(\varphi u)\varphi udx \geq A_1^2 - \frac{1}{2}\rho_0$, it follows from (6.14) that $|1-\tau^\lambda| < \rho_0(A_1^2 - \frac{1}{2}\rho_0)^{-1}$. Hence there is a $C_4 > 0$ such that $|J(\tau\varphi u) - J(\varphi u)| \leq C_4|\tau - 1|$. This completes the proof, since ρ_0 can be chosen arbitrarily small and C_4 is independent of ρ_0.

Lemma 14 *The function τ, defined in (6.9), is continuous on $E\backslash\{0\}$.*

Proof. Suppose $u \neq 0$ and $\lim_{m\to\infty} \|u_m - u\| = 0$. Let $\tau = \tau(u)$, $v_m = \tau u_m$ and $v = \tau u$. Set $\tau_m = \tau(v_m)$. It suffices to show that

$$\lim_{m\to\infty} \tau_m = 1. \tag{6.15}$$

By (f3)

$$\begin{aligned} |1-\tau_m^\lambda|| \textstyle\int_\Omega f(x,v_m)v_m dx| &\le | \textstyle\int_\Omega [f(x,v_m)v_m - \frac{1}{\tau_m} f(x,\tau_m v_m)v_m]dx| \\ &= |J'(v_m)v_m - \tfrac{1}{\tau_m^2} J'(\tau_m v_m)\tau_m v_m| = o(1). \end{aligned} \tag{6.16}$$

Moreover,

$$\begin{aligned} \int_\Omega f(x,v_m)v_m dx &= \int_\Omega f(x,v)v dx + o(1) \\ &= \int_\Omega [|\nabla v|^2 + av^2]dx + o(1). \end{aligned} \tag{6.17}$$

Combining (6.16) with (6.17) yields (6.15).

Proposition 9. Let $\pi_4 = \pi_4(\frac{\beta}{4})$ be the number defined in Lemma 13. Choose $z_+ \in E_k^+ \cap S$ and $z_- \in E_k^- \cap S$ such that $\max(J(z_+), J(z_-)) < \beta + \frac{\pi_4}{4}$. If α is the minimax value defined by (5.34), then $\alpha \ge \beta + \pi_4$.

Proof. Suppose $\alpha < \beta + \pi_4$. Then there is a $\gamma_0 \in \Gamma_1$ such that

$$\max_{\theta\in[0,1]} J(\gamma_0(\theta)) < \beta + \pi_4. \tag{6.18}$$

Let φ and τ be defined as in (6.9) and $\gamma(\theta) = \tau(\varphi\gamma_0(\theta))\varphi\gamma_0(\theta)$. It follows from Lemma 13 and Lemma 14 that $\gamma \in \Gamma_1$ and

$$\max_{\theta\in[0,1]} J(\gamma(\theta)) < \beta + A_4 < \frac{3}{2}\beta. \tag{6.19}$$

By the definition of φ,

$$\gamma(\theta) = \gamma_+(\theta) + \gamma_-(\theta), \tag{6.20}$$

where $\gamma_+(\theta) \in E_r^+$ and $\gamma_-(\theta) \in E_r^-$. We claim that

$$\text{there is a } \theta_0 \in (0,1) \text{ such that } \gamma_-(\theta_0) \in S \text{ and } \gamma_+(\theta_0) \in S. \tag{6.21}$$

Assuming (6.21) for now, we obtain

$$J(\gamma(\theta_0)) = J(\gamma_+(\theta_0)) + J(\gamma_-(\theta_0)) > \beta + \beta = 2\beta,$$

which contradicts (6.19).

It remains to show (6.21) to complete the proof. Since $\gamma_+(0) = 0$ and $\gamma_+(1) = z_+$, there is a $\theta_1 \in (0,1)$ such that $J'(\gamma_+(\theta_1))\gamma_+(\theta_1) > 0$. This together with $\gamma(\theta_1) \in S$ implies that $J'((\gamma_-(\theta_1))\gamma_-(\theta_1) < 0$. Let

$$\theta_2 = \sup\{\theta | J'(\gamma_-(\theta))\gamma_-(\theta) < 0 \text{ or } \gamma_-(\theta) \in S\}. \tag{6.22}$$

Since $\gamma_-(0) = z_-$ and $\gamma_-(1) = 0$, it follows that $\theta_2 \in (0, 1)$. Using the continuity of J' and γ_- gives $J'(\gamma_-(\theta_2))\gamma_-(\theta_2) = 0$. Since $\gamma(\theta_2) \in S$, it follows that $J'(\gamma_+(\theta_2))\gamma_+(\theta_2) = 0$.

To complete the proof of (6.21), we need to show that $\gamma_-(\theta_2) \neq 0$ and $\gamma_+(\theta_2) \neq 0$. We argue indirectly. If $\gamma_-(\theta_2) = 0$, then either $\gamma_-(\theta) = 0$ for all $\theta \in (\theta_2, 1)$ or there is a $\theta_3 \in (\theta_2, 1)$ such that $J'(\gamma_-(\theta_3))\gamma_-(\theta_3) > 0$. This contradicts (6.22). Suppose $\gamma_+(\theta_2) = 0$. Then there is a $\theta_4 \in (\theta_2, 1)$ such that $J'(\gamma_+(\theta_4))\gamma_+(\theta_4) > 0$. This together with $\gamma(\theta_4) \in S$ yields $J'(\gamma_-(\theta_4))\gamma_-(\theta_4) < 0$, which again violates (6.22). Thus the proof is complete.

To apply minimax methods like (5.34), we need to use deformation theory. We start with the following proposition to establish a deformation theorem on S.

Proposition 10 *S is a $C^{1,1}$ Banach manifold.*

Proof. Let $G(u) = J'(u)u$. It is easy to check that G is a $C^{1,1}$ mapping from E to $\mathbb{R}$. If $u \in S$ then

$$\begin{aligned} G'(u)u &= \int_\Omega [f(u)u - f'(u)u^2]dx \\ &\leq -\lambda \int_\Omega f(u)udx = -\lambda \int_\Omega (|\nabla u|^2 + au^2)dx < 0. \end{aligned}$$

By the Riesz Representation Theorem, there is a unique $g_u \in E\backslash\{0\}$ such that $G'(u)\phi = \langle g_u, \phi\rangle$ for all $\phi \in E$. Define $\Im(u) = \{\phi | \phi \in E \text{ and } \langle g_u, \phi\rangle = 0\}$. Then $E = \text{span}\{g_u\} \oplus \Im(u)$. Let $P_1(u) = \|g_u\|^{-2}\langle g_u, u\rangle g_u$ and $P_2(u) = u - P_1(u)$. Since $\lim_{s\to 0} s^{-1}[G(u) - G(P_1(u) + s\|P_1(u)\|^{-1}P_1(u), P_2(u))] = -G'(u)\|g_u\|^{-1}g_u < 0$, by Implicit Function Theorem there is a neighborhood $\tilde{N}_u$ of $P_2(u)$ and a $C^{1,1}$ mapping $h_u : \tilde{N}_u \cap \Im(u) \to \text{span}\{g_u\}$ such that $G(h_u(\psi), \psi) = 0$ for all $\psi \in \tilde{N}_u \cap \Im(u)$. Let $N_u = \{(h_u(\psi), \psi) | \psi \in \tilde{N}_u \cap \Im(u)\}$. Clearly N_u is an open set in S. Let Φ_u be the projection of N_u to $\Im(u)$. Then Φ_u is one to one and $\Phi_u(N_u) = \tilde{N}_u \cap \Im(u)$. It is easy to check that $\{(N_u, \Phi_u) | u \in S\}$ is a $C^{1,1}$ atlas of S. So S is a $C^{1,1}$ Banach manifold.

Let $T_u(S)$ be the tangent space of S at u. It is easy to check that $T_u(S) = \Im(u)$. Define $T(S) = \bigcup_{u\in S} T_u(S)$. As a standard result in differential geometry, $T(S)$ is a $C^{0,1}$ Banach manifold.

For $u \in S$, let $\partial J(u)$ denote the differential of J at u. Set

$$\|\partial J(u)\|_s = \sup\{|\partial J(u)\phi| \phi \in T_u(S) \text{ and } \|\phi\| \leq 1\} \tag{6.23}$$

and $\tilde{S} = S\backslash\mathcal{K}$, where $\mathcal{K} = \{u | u \in S \text{ and } \|\partial J(u)\|_s = 0\}$. For $u \in S$, an element X_u in $T_u(S)$ is called a pseudo-gradient vector of J at u on S if

$$\|X_u\| \leq 2\|\partial J(u)\|_s \tag{6.24}$$

and

$$\partial J(u)X_u \geq \|\partial J(u)\|_s^2. \tag{6.25}$$

A map $X : \tilde{S} \to T(S)$ is called a pseudo-gradient vector field on $\tilde{S}$ if X is locally Lipschitz continuous and $X(u)$ is a pseudo-gradient vector for J for all $u \in \tilde{S}$.

Proposition 11 *There exists a pseudo-gradient vector field X on $\tilde{S}$.*

Proof. For each $u \in \tilde{S}$, there is a $w \in T_u(S)$ such that $\|w\| = 1$ and $\partial J(u)w > \frac{2}{3}\|\partial J(u)\|_s$. Then $v = \frac{3}{2}\|\partial J(u)\|_s w$ is a pseudo-gradient vector for J at u on S with strict inequality in (6.24) and (6.25). The continuity of J' then shows v is a pseudo-gradient vector for all $z \in O_u \cap S$, where O_u is an open neighborhood of u. As a subset of E, $\tilde{S}$ is a metric space. By a theorem of A. H. Stone [D], S is paracompact. Thus $\{O_u | u \in \tilde{S}\}$ is an open covering of $\tilde{S}$ and it possesses a locally finite refinement. Then the same lines of reasoning as the proof of Lemma A.2 of [R1] completes the proof.

Lemma 15 *For any $A_5 > 0$, there is a $\pi_5 = \pi_5(A_5)$ such that if $u \in S$ and $\|u\| \le A_5$ then $\|J'(u)\| \le \pi_5 \|\partial J(u)\|_s$.*

Proof. For $\phi \in E$ and $\|\phi\| \le 1$, invoking the Hölder inequality and the Sobolev inequality yields

$$\begin{aligned} |\langle g_u, \phi\rangle| &= |\int_\Omega [2\nabla u \cdot \nabla\phi + 2au\phi - f'(u)u\phi - f(u)\phi]dx \\ &\le 2(1 + C_5)\|u\|\|\phi\| + C_5 \int_\Omega |u|^p|\phi|dx \\ &\le 2(1 + \tilde{K})\|u\|\|\phi\| \le 2(1 + \tilde{K})A_5, \end{aligned}$$

where $\tilde{K} = \tilde{K}(A_5)$. Since $\langle g_u, u\rangle \le -\lambda\|u\|^2 \le -\lambda A_1^2$, it follows that

$$\begin{aligned} \|J(u)\| &= \sup_{\|\phi\|\le 1} |J'(u)\phi| = |J'(u)(\phi - \frac{\langle g_u, \phi\rangle}{\langle g_u, u\rangle}u)| \\ &= |\partial J(u)(\phi - \frac{\langle g_u, \phi\rangle}{\langle g_u, u\rangle}u)| \le (\|\phi\| + \frac{2(1 + \tilde{K})A_5}{\lambda A_1^2}\|u\|)\|\partial J(u)\|_s. \end{aligned}$$

Corollary 6 *If $\{u_m\} \subset S$, $J(u_m) \to c$ and $\|\partial J(u_m)\|_s \to 0$ as $m \to \infty$, then $\{u_m\}$ is a $(PS)_c$ sequence.*

Proof. It follows directly from Lemma 8 and Lemma 15.

Now, we use deformation theory to complete the proof of Proposition 8.

Completion of proof of Proposition 8. Suppose there does not exist a $(PS)_\alpha$ sequence. Then by Lemma 15 there exist positive numbers b and $\hat{\epsilon}$ such that $\|\partial J(u)\|_s \ge b$ for all $u \in \tilde{I}^{\alpha+\hat{\epsilon}} \backslash \tilde{I}^{\alpha-\hat{\epsilon}}$. We may assume without loss of generality that $b < 1$ and

$$\hat{\epsilon} < \frac{1}{2}(\alpha - \beta - \frac{\pi_4}{4}). \tag{6.26}$$

Let $Y_1 = \{u \in S | \|\partial J(u)\|_s \le \frac{b}{2\pi_5}$ and $J(u) \le \frac{3\alpha}{2}\}$ and $Y_2 = \{u \in S | \|\partial J(u)\|_s \ge b$ and $J(u) \le \frac{3\alpha}{2}\}$, where $\pi_5 = \pi_5(A_5)$ was defined in Lemma 15 and $A_5 =$

$\frac{3\alpha}{2}(2(\lambda+2)\lambda^{-1})^{1/2}$. Set $Y_3=\{u\in S|\|J'(u)\|\le\frac{b}{2}$ and $J(u)\le\frac{3\alpha}{2}\}$, $Y_4=\{u\in S|\|J'(u)\|\ge b$ and $J(u)\le\frac{3\alpha}{2}\}$ and $A=\inf\{\|u-v\||u\in Y_3$ and $v\in Y_4\}$. It is clear that $\inf\{\|u-v\||u\in Y_1$ and $v\in Y_2\}\ge A>0$. Choose $\epsilon\in(0,\epsilon_1)$, where

$$\epsilon_1=\min(\hat{\epsilon},\frac{b^2}{2},\frac{b}{4}). \tag{6.27}$$

Define $\tilde{I}_a=\{u\in S|J(u)\ge a\}$ and let $Y_5=\tilde{I}^{\alpha-\hat{\epsilon}}\bigcup\tilde{I}_{\alpha+\hat{\epsilon}}$ and $Y_6=\{u\in S|\alpha-\epsilon\le J(u)\le\alpha+\epsilon\}$. For $u\in S$, set $g_1(u)=\frac{\|u-Y_5\|}{\|u-Y_5\|+\|u-Y_6\|}$ and $g_2(u)=\frac{\|u-Y_1\|}{\|u-Y_1\|+\|u-Y_2\|}$. Let $X(u)$ be a pseudo-gradient vector field for J on $\tilde{S}$ and

$$W(u)=-g_1(u)g_2(u)h(\|X(u)\|)X(u), \tag{6.28}$$

where $h(s)=1$ if $s\in[0,1]$ and $h(s)=\frac{1}{s}$ if $s\ge1$.

Consider the Cauchy problem:

$$\frac{d\eta}{dt}=W(\eta),\quad \eta(0,u)=u. \tag{6.29}$$

The basic existence-uniqueness theorem for ordinary differential equations implies that, for each $u\in S$, (6.29) has a unique solution $\eta(t,u)$ which is defined for t in a maximal interval $[0,T(u))$. Moreover, since $\|W(u)\|\le1$ and S is a closed subset of E, an argument analogous to the proof of Theorem A.4 of [R1] shows that $T(u)=+\infty$. Since

$$\frac{d}{dt}J(\eta(t,u))=-\partial J(\eta(t,u))g_1(\eta(t,u))g_2(\eta(t,u))h(\|X(\eta(t,u))\|)X(\eta(t,u)),$$

it follows from (6.25) that $I(\eta(t,u))$ is a non-increasing function of t. Hence

$$\eta(1,\tilde{I}^{\alpha-\epsilon})\subset\tilde{I}^{\alpha-\epsilon}. \tag{6.30}$$

We claim

$$\eta(1,Y_6)\subset\tilde{I}^{\alpha-\epsilon}. \tag{6.31}$$

Indeed, if there is $u\in Y_6$ such that $\eta(1,u)\notin\tilde{I}^{\alpha-\epsilon}$, then, for all $t\in[0,1]$, $\eta(t,u)\in Y_6$. Consequently $g_1(\eta(t,u))=1$ and $g_2(\eta(t,u))=1$. If for some $t\in(0,1)$, $\|X(\eta(t,u))\|\le1$, then $h(\|X(\eta(t,u))\|)=1$ and

$$\frac{d}{dt}J(\eta(t,u))\le-\|\partial J(\eta(t,u))\|_s^2\le-b^2. \tag{6.32}$$

On the other hand, if for some $t\in(0,1)$, $\|X(\eta(t,u))\|>1$, then

$$\begin{aligned}\frac{d}{dt}J(\eta(t,u)) &\le -\|\partial J(\eta(t,u))\|_s^2\|X(\eta(t,u))\|^{-1}\\ &\le -\frac{1}{2}\|\partial J(\eta(t,u))\|_s\le-\frac{b}{2},\end{aligned} \tag{6.33}$$

by making use of (6.24). Since $\eta(t,u) \in Y_6$ for all $t \in [0,1]$, it follows from (6.32) and (6.33) that

$$\begin{aligned} 2\epsilon &\geq J(\eta(0,u)) - J(\eta(1,u)) \\ &= -\int_0^1 \frac{d}{dt} J(\eta(t,u))dt \geq \min(\frac{b}{2}, b^2). \end{aligned} \tag{6.34}$$

Since (6.34) is contrary to (6.27), we conclude that (6.31) must hold. Combining (6.30) with (6.31), we have

$$\eta(1, \tilde{I}^{\alpha+\epsilon}) \subset \tilde{I}^{\alpha-\epsilon}. \tag{6.35}$$

By (5.34) there is a $\gamma \in \Gamma_1$ such that $\max_{\theta\in[0,1]} J(\gamma(\theta)) < \alpha + \epsilon$. Let $\gamma_1(\theta) = \eta(1, \gamma(\theta))$. It follows from (6.35) that

$$\max_{\theta\in[0,1]} J(\gamma(\theta)) \leq \alpha - \epsilon. \tag{6.36}$$

Since $g_1(u) = 0$ if $u \in \tilde{I}^{\alpha-\hat{\epsilon}}$, it follows from (6.28) and (6.29) that $\eta(1,u) = u$ if $u \in \tilde{I}^{\alpha-\hat{\epsilon}}$. In particular, $\max(J(z_+), J(z_-)) < \beta + \frac{\pi_4}{4}$ implies $\gamma_1(0) = \gamma(0), \gamma_1(1) = \gamma(1)$ and consequently $\gamma_1 \in \Gamma_1$. But then (6.36) is contrary to (5.34). The proof is complete.

We are now ready to prove Theorem 7.

Proof of Theorem 7. By Proposition 8 there is a $(\mathrm{PS})_\alpha$ sequence with $\alpha > \beta$. Then we may proceed with the same lines of reasoning as in the proof of Theorem 2 to obtain a positive function $u \in E$ with $J'(u) = 0$ and $\alpha \geq J(u) > \beta$.

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