Electronic Journal of Differential Equations, Vol. **1999**(1999), No. 19, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Log-concavity in some parabolic problems *

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Abstract

We improve a concavity maximum principle for parabolic equations of the second order, which was initially established by Korevaar, and then we use this result to investigate some boundary value problems. In particular, we find structural conditions on the equation, and suitable conditions on the domain of the problem and on the boundary data, that suffice to yield spatial log-concavity of the (positive) solution. Examples and applications are provided, and some unsolved problems are pointed out. We also survey some classical as well as recent contributions to the subject.

1 Introduction

In the last two decades we have seen many new results on qualitative properties of solutions to elliptic and parabolic problems. One of the issues that have been investigated is how the shape of the underlying domain influences the shape of the solution. There is a vast literature addressing symmetry questions, and there are also several papers investigating convexity properties of solutions.

In the early 80's, R. Finn posed the question whether capillary surfaces on convex domains are convex, and he gave this problem to N. J. Korevaar as a Ph.D. thesis. Korevaar wrote two papers on convexity that have become classical ([10], [11]), and he remarked that:

1. Convexity of the domain alone, usually does not induce convexity of the solution. A typical condition to be added to the problem in order to obtain convexity of the solution is that *the contact angle at the boundary must be zero.*

2. Even if the differential operator is very simple, and it is not the minimal surface operator, there are counterexamples. For instance, the negative first eigenfunction $-u_1$ of the Laplace operator, with homogeneous Dirichlet boundary conditions on the disc, is not convex. However, it turns out that $-\log u_1$ is convex ([11], Remark 2.7; see also [6], Remark 3.4).

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* *1991 Mathematics Subject Classifications:* 35B50, 35K55, 35K65, 35B05.
Key words and phrases: Concavity maximum principle, Log-concavity, Parabolic equation.

Submitted March 23, 1999. Published May 28, 1999.
The first author was supported by DAAD and by Progetto Finanziato dall'Università di Cagliari.

3. It would be interesting to study convexity *of the level sets* of the solution, instead of convexity of the solution itself.

In this paper, in order to study convexity of level sets of a positive function u we study the convexity of the auxiliary function $v := -\log u$. This auxiliary function is chosen because it leads to a particularly useful structure of the transformed equation. Our main results are Theorem 4.1 (a concavity maximum principle) and Theorem 4.2 (its application to initial boundary value problems). For a review of other techniques we refer to [6] (see also [3], [4] and [7]). For an approach using viscosity solutions see [1].

2 Parabolic maximum principle

Parabolic inequalities, like (1) below, satisfy a nondegeneracy condition in the sense that the coefficient of u_t is nonzero. This property, which of course does not depend on the ellipticity constants of the matrix (a^{ij}), allows the following (well-known) maximum principle to hold.

Theorem 2.1 (Weak maximum principle for parabolic equations) *Let $w \in C^2(\mathcal{G})$ be a classical solution of*

$$w_t \le a^{ij}(x,t)\, w_{ij} + b^i(x,t)\, w_i - c(x,t)\, w \tag{1}$$

in the set $\mathcal{G} := \Omega \times (0,T\,]$, Ω a bounded domain in $\mathbb{R}^N$, $T > 0$. The coefficients a^{ij}, b^i and c are supposed to be real valued functions on $\mathcal{G}$ satisfying $(a^{ij}) \ge 0$ and $\inf_{(x,t)\in\mathcal{G}} c(x,t) > -\infty$. Inequality (1) must hold pointwise.
If $\limsup w(x,t) \le 0$ as (x,t) approaches the parabolic boundary *of $\mathcal{G}$ (i.e., the set $\Omega \times \{0\} \cup \partial\Omega \times [\,0,T\,]$), then we have $w \le 0$ in all of $\mathcal{G}$.*

Proof. Choose $m < \inf c$ and consider $\tilde{w} := w\, e^{mt}$. By substitution into (1) we have

$$\tilde{w}_t \le a^{ij}\, \tilde{w}_{ij} + b^i\, \tilde{w}_i - \tilde{c}\, \tilde{w},$$

where $\tilde{c} = c - m > 0$. Suppose, contrary to the claim, that w becomes somewhere positive. The function $\tilde{w}$ would still be positive there, and a maximizing sequence would converge to some point (x_0, t_0) outside the parabolic boundary, due to the boundary behaviour of w. Since $\tilde{w}$ is smooth at (x_0, t_0), a standard computation contradicts the inequality above and the claim follows. ◊

Remark. If the assumptions of the theorem are strengthened, namely if in addition we suppose that $\lambda\,|\xi|^2 \le a^{ij}(x,t)\,\xi_i\,\xi_j \le \Lambda\,|\xi|^2$ for all $(x,t) \in \mathcal{G}$, $\xi \in \mathbb{R}^N$, with suitable $\lambda, \Lambda > 0$ (*uniform parabolicity*), and $\sup_{(x,t)\in\mathcal{G}} |b^i(x,t)| < +\infty$ for $i = 1, \ldots, N$, then the *strong maximum principle* holds, i.e., if $u = 0$ at some point outside the parabolic boundary then u is identically zero (see, for instance, [13] or [14]).
The weak maximum principle, instead, has the following noteworthy properties:

1. No nondegeneracy condition is imposed to the matrix (a^{ij}).
2. No boundedness of the coefficients a^{ij} and b^i is assumed.

Note, further, that the coefficients in (1) need not be smooth.

Corollary 2.2 (Weak maximum principle on an infinite cylinder) *Let $w \in C^2(\mathcal{S})$ be a classical solution of (1) in the infinite cylinder $\mathcal{S} := \Omega \times \mathbb{R}^+$, Ω a bounded domain in $\mathbb{R}^N$. Suppose that $(a^{ij}) \geq 0$, and that $\inf_{(x,t)\in\Omega\times(0,T)} c(x,t) > -\infty$ for every (finite) $T > 0$.*
If $\limsup w(x,t) \leq 0$ as (x,t) approaches $\partial\mathcal{S}$, then we have $w \leq 0$ in all of $\mathcal{S}$.

Proof. If we assume $w > 0$ at a certain $(x,T) \in \mathcal{S}$ then we reach a contradiction to the preceding theorem. ♢

Remark. Note that we did not assume $w \in C^0(\overline{\mathcal{G}})$ or $C^0(\overline{\mathcal{S}})$. In fact, in our applications the function w will be a concavity function (see below), and the concavity function associated to an unbounded function is not well defined on the boundary.

3 Known convexity results for parabolic equations

Let us recall that, in a classical paper [2], Brascamp and Lieb showed that the heat equation preserves log-concavity of the initial data. Their method is based on analyzing the heat kernel representation and on using the Brunn-Minkowsky inequality. In this paper we use different methods, which are based on the maximum principle, and which are known to yield an alternative proof of Brascamp and Lieb's result (see [11]).

Korevaar ([10], [11]) introduced the elliptic and the parabolic *concavity function*, which may be defined as

$$C(x,y) = v(z) - \frac{v(x)+v(y)}{2} \tag{2}$$

or

$$C(x,y,t) = v(z,t) - \frac{v(x,t)+v(y,t)}{2}, \tag{3}$$

respectively, where $z = (x+y)/2$. If $v(x)$ is a *continuous* function on a convex domain Ω, then v is convex if and only if $C(x,y) \leq 0$ in Ω^2. Correspondingly if $v(x,t)$ is a *continuous* function on a convex cylinder $\Omega \times (0,T)$, then it is convex w.r.t. x for every fixed $t \in (0,T)$ if and only if $C(x,y,t) \leq 0$ in $\Omega^2 \times (0,T)$.

By means of (3), Korevaar investigates in [11] the equation

$$v_t = a^{ij}(t,\nabla v)\, v_{ij} - b(x,t,v,\nabla v) \quad \text{in } \Omega \times (0,T), \tag{4}$$

where ∇v is the spatial gradient. Under structural assumptions on (a^{ij}) and b he proves that C can attain a positive maximum, if it exists, only on the parabolic boundary of $\Omega^2 \times (0,T)$, i.e. for $t = 0$ or when x or $y \in \partial\Omega$. The structural assumptions were $(a^{ij}) > 0$, $b_v \geq 0$, and concavity of b with respect to (x, v).

Of course, when dealing with a parabolic problem, one may also consider convexity of the solution with respect to the $(N+1)$-dimensional variable (x,t). This leads to a study of the function

$$C(x, y, t, s) = v(z, r) - \frac{v(x,t) + v(y,s)}{2},$$

where z is as before and $r = (t+s)/2$. Porru and Serra [12] use this function to investigate equation (4) but with (a^{ij}) independent of t. They prove a convexity maximum principle of parabolic type, i.e., the function $C(x,y,t,s)$ cannot attain a positive maximum when (x,t) and (y,s) are not on the parabolic boundary of $\Omega \times (0,T)$. This conclusion is reached under the following structural assumptions on (4):

1. parabolicity, but not strict parabolicity of the equation;
2. boundedness of b_v from below;
3. concavity of b with respect to (x, t, v).

Kennington [9] and Kawohl [5] also consider $C(x, y, t, s)$, but they regard the parabolic equation as a degenerate elliptic equation in $\mathbb{R}^{N+1}$, and apply the corresponding concavity maximum principle by Kennington [8]. They prove convexity of level sets of the solution to initial value problems of the form

$$\begin{aligned} u_t = \Delta u - f(u) \quad & \text{in } \Omega \times \mathbb{R}^+, \\ u = 1 \quad & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = 1 \quad & \text{in } \Omega, \end{aligned}$$

by applying the concavity maximum principle to a suitable transformed fuction $v := g(u)$. The result follows under suitable assumptions on f. Due to the elliptic nature of the technique, an estimate of C is needed also for $t \to +\infty$.

4 Space concavity

A concavity maximum principle.

In this subsection we generalize Korevaar's result cited above, concerning convexity in space, in the sense that we drop the assumption of strict parabolicity of the equation and we require b to satisfy a Lipschitz condition from below instead of $b_v \geq 0$. We do not require any other smoothness of (a^{ij}) and b. After this extension, the result can also be more easily compared to Porru and Serra's result of convexity in space-time.

The function $b(x, t, v, p)$ is said to satisfy a *Lipschitz condition from below* with respect to v if there exists a constant $L \in \mathbb{R}$ such that

$$b(x, t, u, p) - b(x, t, v, p) \geq -L\,(u - v) \tag{5}$$

for all x, t, u, v, p such that $u > v$.

Theorem 4.1 *Let $v \in C^2(\mathcal{S})$ be a solution of (4) in the infinite cylinder $\mathcal{S} := \Omega \times \mathbb{R}^+$, Ω a convex bounded domain in $\mathbb{R}^N$. Suppose $(a^{ij}) \geq 0$ and let b satisfy (5) and be concave in (x, v).*
If $\limsup C(x,y,t) \leq 0$ as $(x,y,t) \to \partial(\Omega^2 \times \mathbb{R}^+)$, then $C \leq 0$ in all of $\Omega^2 \times \mathbb{R}^+$.

Proof. We construct a parabolic inequality for $C(x,y,t)$ in the domain $\Omega^2 \times \mathbb{R}^+$, and the result will follow from the classical maximum principle (Corollary 2.2). A similar technique was used in [4] for the elliptic case. Korevaar, instead, argued by contradiction at a point where C attained a positive maximum. Let $\tilde{A} = (\tilde{a}^{hk}(x,y,t))$ be the $2N \times 2N$ matrix given by

$$\tilde{A} = \begin{pmatrix} A & A \\ A & A \end{pmatrix},$$

where $A = (a^{ij}(t, \nabla v(z,t)))$. By differentiating C we find:

$$\begin{aligned} \tilde{a}^{hk}\, C_{hk} \quad = \quad & a^{ij}(t, \nabla v(z,t))\, v_{ij}(z,t) - a^{ij}(t, \nabla v(z,t))\, v_{ij}(x,t)/2 \\ & -a^{ij}(t, \nabla v(z,t))\, v_{ij}(y,t)/2, \end{aligned}$$

where $i, j = 1, \ldots, N,\ h, k = 1, \ldots, 2N$ and the summation convention is in effect. Since $\nabla v(z,t) - \nabla v(x,t) = 2\, \nabla_x C(x,y,t)$, and $\nabla v(z,t) - \nabla v(y,t) = 2\, \nabla_y C(x,y,t)$, we may write

$$\begin{aligned} \tilde{a}^{hk}\, C_{hk} + b^h\, C_h \quad = \quad & a^{ij}(t, \nabla v(z,t))\, v_{ij}(z,t) - a^{ij}(t, \nabla v(x,t))\, v_{ij}(x,t)/2 \\ & -a^{ij}(t, \nabla v(y,t))\, v_{ij}(y,t)/2, \end{aligned} \tag{6}$$

where the (not necessarily bounded) coefficients b^h for $h = 1, \ldots, N$ are given by

$$\begin{aligned} b^h(x,y,t) \quad = \quad & \frac{v_h(z,t) - v_h(x,t)}{|\nabla v(z,t) - \nabla v(x,t)|^2} \times \\ & \sum_{i,j=1}^{N} \Big(a^{ij}(t, \nabla v(z,t)) - a^{ij}(t, \nabla v(x,t)) \Big) v_{ij}(x,t) \end{aligned}$$

if $\nabla v(z,t) \neq \nabla v(x,t)$, and $b^h = 0$ if $\nabla v(z,t) = \nabla v(x,t)$. The expression of b^h when $h = N+1, \ldots, 2\,N$ is analogous. It is worthwile to stress the fact that equality holds in (6) for algebraic reasons, and smoothness of a^{ij} is by no means involved.

Now we may use equation (4) and obtain

$$\begin{aligned} \tilde{a}^{hk}\, C_{hk} + b^h\, C_h \quad = \quad & v_t(z,t) + b(z, t, v(z,t), \nabla v(z,t)) \\ & -v_t(x,t)/2 - b(x, t, v(x,t), \nabla v(x,t))/2 \\ & -v_t(y,t)/2 - b(y, t, v(y,t), \nabla v(y,t))/2. \end{aligned}$$

On this expression we operate as follows:

i) We replace $v_t(z,t) - v_t(x,t)/2 - v_t(y,t)/2$ by $C_t(x,y,t)$.

ii) We replace $\nabla v(x,t)$ and $\nabla v(y,t)$, which appear as arguments of b, by $\nabla v(z,t)$. This needs some modification of the coefficients b^h, which is not relevant for the conclusion (the technique is the same as before).

iii) We replace $b(z,t,v(z,t),\nabla v(z,t))$ by

$$b(z,t,(v(x,t)+v(y,t))/2,\nabla v(z,t)) + c\,C(x,y,t),$$

where c is a suitable function, which is bounded from below by virtue of (5).

Taking into account the concavity of b with respect to (x,v), we obtain the following parabolic inequality:

$$\tilde{a}^{hk}\,C_{hk} + b^h\,C_h \geq C_t + c\,C.$$

Therefore the claim follows from Corollary 2.2. ♢

Log-concavity in space.

We apply the concavity maximum principle proved above and study spatial log-concavity of positive solutions to the following problem:

$$\begin{aligned} u_t = a^{ij}(t)\,u_{ij} - f(t,u,\nabla u) \quad &\text{in } \mathcal{S} := \Omega\times\mathbb{R}^+, \\ u(x,0) = u_0(x)&, \\ u(x,t) = 0 \quad &\text{for } x\in\partial\Omega, \end{aligned} \tag{7}$$

where the coefficients a^{ij} are real valued functions on $\mathbb{R}^+$ satisfying $(a^{ij}) \geq 0$, $u_0 \in C^0(\overline{\Omega})$ is a given log-concave function vanishing on $\partial\Omega$, and Ω is a strictly convex bounded domain in $\mathbb{R}^N$, in the sense that $\partial\Omega$ is of class C^2 and has positive Gauss curvature.

As usual, we say that u_0 is *log-concave* if it is *positive* in Ω and $-\log u_0(x)$ is convex. We set $v(x,t) := -\log u(x,t)$ and derive from (7) an equation satisfied by v. Then we show that $v(x,t)$ is convex in x for every given t, provided f satisfies suitable conditions.

In order to obtain such a conclusion by means of the concavity maximum principle we need to know that v is convex near $\partial\Omega$. This follows from the boundary conditions imposed on u, through the noteworthy properties of the log function, and thanks to the strict convexity of the domain Ω, provided that

$$\nabla u(x,t) \neq 0 \quad \text{for all } x\in\partial\Omega \text{ and } t\geq 0. \tag{8}$$

More precisely, a function $u(x)$ which is positive in a strictly convex domain Ω, and which vanishes on $\partial\Omega$ with nonvanishing gradient there, is always concave (hence log-concave) near $\partial\Omega$ in the tangential directions. On the other side, since $\nabla u \neq 0$ on $\partial\Omega$, u is also log-concave near $\partial\Omega$ in the normal direction. The interested reader may consult [11], Lemma 2.4, or [4], Lemma 3.2, for details.

The final conclusion is the following:

Theorem 4.2 *Let $u \in C^2(\mathcal{S}) \cap C^1(\overline{\mathcal{S}})$ be a positive classical solution of problem (7), where u_0 and Ω are as above. Assume (8). If $f(t,u,p)$ is of class $C^2(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N)$ and such that*

$$f_u + \frac{p \cdot \nabla_p f}{u} - \frac{f}{u} \quad \geq \quad -L, \tag{9}$$

$$f_u + \frac{p \cdot \nabla_p f}{u} - \frac{f}{u} \quad \leq \quad u\, f_{uu} + 2\, p \cdot \nabla_p\, f_u + \frac{p_i\, p_j\, f_{p_i p_j}}{u}, \tag{10}$$

for every $t > 0$, $u > 0$, $p \in \mathbb{R}^N$, and with a suitable constant $L > 0$, then u is log-concave in space for all t.

Proof. By computation we find that $v(x,t) := -\log u(x,t)$ satisfies the equation $v_t = a^{ij}(t)\, v_{ij} - b(t,v,\nabla v)$, where

$$b(t,v,\nabla v) = a^{ij}(t)\, v_i\, v_j - e^v\, f(t, e^{-v}, -e^{-v}\, \nabla v).$$

Furthermore, the derivative b_v is given by

$$b_v(t,v,\nabla v) = f_u(t, e^{-v}, p) + e^v\, p \cdot \nabla_p\, f(t, e^{-v}, p) - e^v\, f(t, e^{-v}, p),$$

where we set, for shortness, $p := -e^{-v}\, \nabla v$, therefore it is bounded from below provided (9) holds. Finally, we have

$$b_{vv}(t,v,\nabla v) = f_u + \frac{p \cdot \nabla_p f}{u} - \frac{f}{u} - u\, f_{uu} - 2\, p \cdot \nabla_p\, f_u - \frac{p_i\, p_j\, f_{p_i p_j}}{u},$$

where f and its derivatives are all evaluated at (t, e^{-v}, p). Hence by (10) we have $b_{vv} \leq 0$ and the assumptions of Theorem 4.1 are satisfied. As remarked above, the parabolic concavity function associated to v satisfies $\limsup C(x,y,t) \leq 0$ as $(x,y,t) \to \partial(\Omega^2 \times \mathbb{R}^+)$. This and the concavity maximum principle imply $C(x,y,t) \leq 0$ for all $(x,y,t) \in \Omega^2 \times \mathbb{R}^+$. ♢

Remarks.

1. We have preferred to state the theorem in an infinite cylinder for the sake of simplicity. Of course the result holds, *mutatis mutandis*, as long as the solution exists.

2. Let us just mention that a solution u of (7) has to be *positive* in $\Omega \times \mathbb{R}^+$, and it has to satisfy (8), provided f_u is bounded from below and one of these conditions is verified:

   (a) The equation in (7) is uniformly parabolic, $f(t,0,0) \leq 0$ for all $t > 0$, $\nabla_p f$ is bounded, and $\nabla u_0 \neq 0$ on $\partial\Omega$. In this case the null constant turns out to be a subsolution of the same problem, therefore u must be positive by the strong comparison principle, and ∇u does not vanish on Ω by Hopf's Lemma.

(b) The function f does not depend on t, and there exists a positive solution $w(x)$ of the stationary problem $\Delta w = f(w, \nabla w)$ satisfying $w \leq u_0$, $\nabla w \neq 0$ on $\partial\Omega$. In this case $w(x,t) := w(x)$ is a subsolution.

3. Theorem 4.2 still holds with a log-concave u_0 whose gradient vanishes at the boundary, provided we can approximate u_0 in the C^0-norm by a sequence of log-concave u_{0k} satisfying $\nabla u_{0k} \neq 0$ on $\partial\Omega$ and such that the corresponding problems (7) have positive solutions u_k in $\mathcal{S}$. This can be seen as follows. Since u_k is log-concave,
$$u_k^2\Big(\frac{x+y}{2}, t\Big) \geq u_k(x,t)\, u_k(y,t) \quad \text{in } \Omega^2 \times \mathbb{R}^+.$$
This inequality is preserved under the pointwise limit as $k \to +\infty$. Since u is supposed to be positive, the corresponding inequality for u is equivalent to log-concavity.

4. For the same reason, and under similar conditions, we can even take $u_0 \in C^0(\Omega)$, i.e., not continuous up to the boundary.

5. The result can be extended to the case when Ω is convex but not strictly convex by means of a similar argument, provided the solution u of problem (7) is positive. It suffices that there exists a sequence of strictly convex domains Ω_k and log-concave initial data u_{0k} to which Theorem 4.2 is applicable, and such that the corresponding solutions u_k converge pointwise to u.

6. It is always possible to approximate a log-concave u_0 in the C^0-norm by a sequence of log-concave u_{0k} satisfying $\nabla u_{0k} \neq 0$ on the boundary. See the appendix for an explanation.

Examples and applications.

Of course, the assumptions of Theorem 4.2 are satisfied by the linear heat equation $u_t = \Delta u$. More generally, if f does not depend on the gradient, then (9)-(10) reduce to
$$-L \leq f_u - f/u \leq u\, f_{uu}, \tag{11}$$
which is satisfied, for instance, by the equation $u_t = \Delta u - u^\alpha$ with $\alpha \geq 1$, and by $u_t = \Delta u - u \log^\beta(1+u)$ with $\beta \geq 1$.

We may ask whether, for some special f, the two equalities hold in (11). The answer is positive: indeed, the function $f(u) := (m - \log u)\, u + q$, $m, q \in \mathbb{R}$, satisfies $-1 = f_u - f/u = u\, f_{uu}$. Theorem 4.2 is therefore applicable to the corresponding equations, and in particular to
$$u_t = \Delta u + u \log u.$$
Note that the cited result by Korevaar is not applicable to such equation, because in this case b_v is negative.

In case $f = g(u) + h(|\nabla u|)$ then (9)-(10) hold provided $g(s)$ and $h(s)$ satisfy

$$-L \le g' - g/s \le s\, g'',$$
$$0 \le h' - h/s \le s\, h''.$$

If, in particular, $f = u^\alpha + |\nabla u|^\beta$ then Theorem 4.2 is not directly applicable for $\beta < 2$, since f does not meet the smoothness requirements. However, an inspection of the proof shows that the conclusion still holds provided $\alpha, \beta \ge 1$.

Similarly, one sees that for $f = u^\alpha\, |\nabla u|^\beta$ the conclusion of Theorem 4.2 holds when $\alpha + \beta \ge 1$.

Let us remark that we admit the dependence on time and the degeneracy of the matrix (a^{ij}), so that Theorem 4.2 is applicable to equations like $u_t = (1 + \sin t)\, \Delta u$.

Once we know that the solution to (7) is log-concave in space, we deduce that *the level sets* $E(c,t) := \{\, x \in \Omega \mid u(x,t) \ge c \,\}$ *are convex* for all c and t. In particular, the set where $u(\cdot, t)$ attains its maximum (the *hot spot*) is convex for every given t. If u blows up at a finite time T, then the set where $u(x,T) = +\infty$ is convex.

5 Further remarks and open problems

Let us remark that there are some interesting equations to which our result does not apply. This is due to different factors. For instance, the following equations do not satisfy (9)-(10):

$$u_t = \Delta u + u^\alpha \quad \text{(with the + sign)};$$
$$u_t = \Delta u + e^u;$$
$$u_t = \Delta u + u^\alpha \log u \quad \text{with } \alpha \ne 1 \text{ and positive};$$
$$u_t = \Delta u + u^\alpha - |\nabla u|^\beta, \quad \alpha, \beta > 0;$$
$$u_t = \Delta u - u^\alpha + |\nabla u|^\beta, \quad \alpha, \beta > 0.$$

The equation $u_t = \Delta u - e^u$, instead, satisfies (9)-(10) but the null constant is a supersolution (not a subsolution), and the stationary solution with homogeneous Dirichlet boundary conditions is negative.

We also want to point out that the transformation $v := -\log u$ is not the only conceivable one to investigate convexity of the level sets of a positive u, but it turns out to be particularly well-featured for that purpose. Among the other transformations that are used, one of the most simple is given by $v = -u^\alpha$ with $\alpha \in (0,1)$. This leads to the notion of *power concavity* (see [8], [6], [7]). Even the general transformation $v := g(u)$, with a decreasing and convex g, has been investigated ([11], [4], [6]).

Therefore one may ask what happens if we consider problem (7) with an initial datum which is, for instance, power concave. We have tried to derive an equation for $v := -u^\alpha$ with $\alpha \in (0,1)$ and to apply the concavity maximum principle (Theorem 4.1). Unfortunately, even for the special case $u_t = \Delta u - f(u)$, it turns out that the equation for v, which takes the form $v_t = \Delta v - b(v, \nabla v)$ with

$$b(v, \nabla v) = \frac{\alpha - 1}{\alpha\, v}\, |\nabla v|^2 - \alpha\, (-v)^{1-1/\alpha}\, f((-v)^{1/\alpha}),$$

does not satisfy the assumptions of Theorem 4.1, since the first term in b is convex with respect to v for $v < 0$. Similarly, the substitution $v = u^\alpha$ with $\alpha < 0$ leads to

$$v_t = \Delta v - \frac{\alpha - 1}{\alpha\, v}\, |\nabla v|^2 - \alpha\, v^{1-1/\alpha}\, f(v^{1/\alpha}),$$

and again the assumptions of Theorem 4.1 are violated.

6 Appendix: an approximation lemma

The purpose of assumption (8) in Theorem 4.2 is that of ensuring spatial log-concavity of the solution u near the parabolic boundary. If we only require that $\nabla u(x, t) \neq 0$ for $x \in \partial\Omega$ and $t > 0$, i.e., if we admit $\nabla u(z_0, 0) = 0$ for some $z_0 \in \partial\Omega$, then we cannot *a priori* exclude the existence of a sequence (x_k, y_k, t_k) such that $x_k, y_k \to z_0 \in \partial\Omega$, $t_k \to 0$ and $C(x_k, y_k, t_k) > \varepsilon > 0$ for all k.

Such kind of *corner pathology* is quite subtle and it is sometimes neglected. For instance, Lemma 3.11 in [6] did not take into consideration an analogous question for the elliptic case, and was later fixed in [4], Lemma 3.2. The paper by Korevaar [11] also does not discuss this point in the parabolic case.

As remarked before (Section 4, Remark 3), a possibility to overcome this difficulty consists of approximating the initial datum u_0, which is supposed to be log-concave but whose gradient may now vanish on the boundary, by a sequence of log-concave u_{0k} that satisfy $\nabla u_{0k} \neq 0$ on $\partial\Omega$.

In this appendix we show that such an approximation is always possible.

Lemma 6.1 *Let Ω be a convex (but not necessarily strictly convex) bounded domain in $\mathbb{R}^N$. Let $u_0 \in C^0(\overline{\Omega})$ be a log-concave function in Ω that vanishes on $\partial\Omega$. In order to simplify the presentation, we assume that $\partial\Omega$ is of class C^1 and that $u_0 \in C^1(\Omega)$.*
Then there exists a sequence of log-concave u_{0k} such that

(a) $\sup_\Omega |u_{0k}(x) - u_0(x)| \to 0$ *as* $k \to +\infty$*;*

(b) $u_{0k}(x) \equiv a_k \operatorname{dist}(x, \partial\Omega)$ *for some* $a_k > 0$ *and all* x *sufficiently close to* $\partial\Omega$*;*

(c) *in particular,* $\nabla u_{0k} \neq 0$ *on* $\partial\Omega$ *for each* k.

Proof. The structure of the proof is the following. We consider the function $v(x) := -\log u_0(x)$, which is convex and unbounded from above by assumption. For each integer $k > \min_\Omega v$, we modify v near $\partial\Omega$ and we obtain a suitable v_k. Finally, we show that $u_{0k} := e^{-v_k(x)}$ verifies (a)-(b). Of course, (c) is a consequence of (b).

Define $\Omega_k := \{\, x \in \Omega \mid v(x) > k \,\}$. For k large, Ω_k is nonempty and has a smooth boundary $\partial\Omega_k$ whose distance from $\partial\Omega$ is positive. For each $y \in \partial\Omega_k$ we consider the tangent plane to the graph of v at y. Such tangent plane is the graph of a function that we denote by $\pi_y(x)$. Define $\tilde{v}_k$ on Ω by setting

$$\tilde{v}_k(x) = \begin{cases} v(x), & \text{if } x \in \Omega_k; \\ \sup\limits_{y \in \partial\Omega_k} \pi_y(x), & \text{if } x \in \Omega \setminus \Omega_k. \end{cases}$$

Then $\tilde{v}_k$ is a convex function attaining a *finite* value on $\partial\Omega$. Now choose $b_k \in \mathbb{R}$ so large that $\tilde{v}_k(x) = v(x) \geq -\log \operatorname{dist}(x, \partial\Omega)) - b_k$ in Ω_k. For instance, we may take $b_k := -\min_\Omega v - \log \operatorname{dist}(\Omega_k, \partial\Omega)$. Define

$$v_k(x) := \max\Big(\tilde{v}_k(x), -\log \operatorname{dist}(x, \partial\Omega) - b_k\Big),$$

which is convex because the function $\operatorname{dist}(x, \partial\Omega)$ is concave, hence log-concave, and the maximum of two convex functions is still convex.

Since $\tilde{v}_k$ is bounded on all of Ω, we have $v_k(x) = -\log \operatorname{dist}(x, \partial\Omega) - b_k$ near $\partial\Omega$ and therefore the function $u_k(x) := e^{-v_k(x)}$ coincides with $a_k \operatorname{dist}(x, \partial\Omega)$ for $\operatorname{dist}(x, \partial\Omega)$ small and $a_k := e^{b_k}$.

It remains to check (a). By definition, u_{0k} coincides with u_0 in Ω_k. Outside Ω_k we have $v_k(x) \geq \tilde{v}_k(x) \geq k$, hence $u_k(x) \leq e^{-k}$. This implies $\sup_\Omega |u_{0k}(x) - u_0(x)| \leq e^{-k} + \sup_{\Omega\setminus\Omega_k} |u_0(x)|$. Since Ω_k invades Ω as $k \to +\infty$, and since u_0 is supposed to vanish on $\partial\Omega$, claim (a) follows and the proof is complete. $\diamondsuit$

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