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Antimaximum principle for elliptic problems with weight *

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Abstract

This paper is concerned with the antimaximum principle for the linear problem with weight $-\Delta u = \lambda m(x)u + h(x)$, under Dirichlet or Neumann boundary conditions. We investigate the following three questions: Where exactly can this principle hold? If it holds, does it hold uniformly or not? If it holds uniformly, what is the exact interval of uniformity? We will in particular obtain a variational characterization of this interval of uniformity.

1 Introduction

This paper is concerned with the antimaximum principle (in brief AMP) for the problem

$$-\Delta u = \lambda m(x)u + h(x) \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial\Omega\,. \tag{1.1}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^N$ and $Bu = 0$ represents either the Dirichlet or the Neumann homogeneous boundary conditions.

Let us first consider the case where there is no weight, i.e. $m(x) \equiv 1$ in Ω. It is then a standard consequence of the maximum principle that if $\lambda < \lambda_1$ (where λ_1 denotes the principal eigenvalue of $-\Delta$ under the corresponding boundary conditions) and if h is a nonnegative function that is not identically zero, then the solution u of (1.1) is strictly positive in Ω. Clément and Peletier [5] investigated the situation where $\lambda > \lambda_1$ and proved the following AMP: given a nonnegative function h, not identically zero, there exists $\delta = \delta(h) > 0$ such that if $\lambda_1 < \lambda < \lambda_1 + \delta$, then any solution u of (1.1) is strictly negative in Ω. To describe this situation, we will say that the AMP holds at the right of the eigenvalue λ_1. It is also shown in [5] that δ can be taken independent of h for the Neumann problem in dimension $N = 1$. We will say in this latter case that the AMP holds *uniformly* and denote by δ_1 the largest δ admissible.

Recent works dealing with this AMP (without weight) include [3] (irregular domains), [13] (exact L^p space where h should be taken), [14] (extension to operators of higher order), [7] (connection with the Fučik spectrum), [8] (extension to the p-Laplacian), [2] (variational characterization of δ_1).

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When a weight $m(x)$ is introduced in (1.1), the situation gets more involved. Indeed, generally, *two* principal eigenvalues (i.e. eigenvalues associated to non-negative eigenfunctions) are present. Moreover, as we will see in Remark 5.4 below, the connection between AMP and Fučik spectrum does not hold anymore (even when the weight does not change sign). The only work we know which deals with the AMP in the presence of weight is that of Hess [10]. It is proved there, in the Dirichlet case, with a weight m in $C(\overline{\Omega})$ which changes sign in Ω, that the AMP holds at the right of the positive principal eigenvalue and at the left of the negative principal eigenvalue.

Our purpose in this paper is to answer rather completely for (1.1) the following three questions: (i) Where exactly can the AMP hold? (ii) If it holds, does it hold uniformly or not? (iii) If it holds uniformly, what is the exact interval of uniformity (i.e. the value of δ_1 above)?

To give an idea of our results, let us consider the Neumann problem, with a weight m in $L^\infty(\Omega)$ which changes sign in Ω. Suppose first $\int_\Omega m \neq 0$, say $\int_\Omega m < 0$. It is then known that there are two principal eigenvalues: 0 and a positive one which we denote by λ^* (cf. [4], [12] as well as Section 2 below). We show that the AMP holds at the right of λ^* and at the left of 0. Moreover it is nonuniform when $N \geq 2$ and uniform when $N = 1$. In the latter case the intervals of uniformity are exactly $\lambda^* < \lambda \leq \bar{\lambda}(m)$ and $-\bar{\lambda}(-m) \leq \lambda < 0$, where

$$\bar{\lambda}(m) := \inf \left\{ \begin{array}{c} \int_\Omega (u')^2 \,,\ u \in H^1(\Omega)\,,\ \int_\Omega mu^2 = 1\,, \\ \text{and } u \text{ vanishes somewhere in } \overline{\Omega} \end{array} \right\}. \tag{1.2}$$

We also show in this latter case that the AMP still holds at the right of $\bar{\lambda}(m)$ and at the left of $-\bar{\lambda}(-m)$, of course now non uniformly. Suppose now $\int_\Omega m = 0$. In this singular case, 0 is the unique principal eigenvalue. We show that the AMP holds at the right *and* at the left of 0. Moreover it is nonuniform when $N \geq 2$ and uniform when $N = 1$. In the latter case the intervals of uniformity are exactly $0 < \lambda \leq \bar{\lambda}(m)$ and $-\bar{\lambda}(-m) \leq \lambda < 0$, with $\bar{\lambda}(m)$ as in (1.2). In this latter case also the AMP still holds (non uniformly) at the right of $\bar{\lambda}(m)$ and at the left of $-\bar{\lambda}(-m)$.

We will also see, as a final answer to question (i) above, that the AMP can not hold far away to the right of $\bar{\lambda}(m)$ or to the left of $-\bar{\lambda}(-m)$ (cf. Theorem 3.6). This is true for all N, with a suitable extension of definition (1.2) to higher dimension (cf. formula (3.1)).

Our methods of proof are rather different from those in the linear papers [5], [10] (which however deal with more general non-selfadjoint operators). Our present approach is more in the line of the nonlinear works [8], [2]. We observe in particular that an expression analogous to (1.2) was introduced in [2] in the context of the p-Laplacian. However in [2] (and also in [7]), the answers to questions (ii), (iii) above were derived from information on the Fučik spectrum. As already mentioned, the connection between AMP and Fučik spectrum does not hold anymore in the presence of weight, and a different approach has to be introduced. In this respect the comparison between λ^* and $\bar{\lambda}$ in Lemma 3.1 as well as the argument of completing a square in the proof of Theorem 3.6 are the crucial steps which lead to sharp answers to questions (i), (ii), (iii) above.

Our results relative to the Neumann problem, as briefly described above, are stated in detail in Section 3 and proved in Section 4 (for a general operator in divergence form). The somewhat simpler case of the Dirichlet problem is briefly considered in Section 5. The main differences in that case concern the spectrum itself (since there is no singular case of the type $\int_\Omega m = 0$) and the fact that the AMP is nonuniform for all dimensions. In Section 2 we collect some preliminary results on the principal eigenvalues of a selfadjoint Neumann problem with weight.

2 Principal eigenvalues in the Neumann case

A large part of this paper is concerned with the Neumann problem

$$Lu = \lambda m(x)u + h(x) \quad \text{in } \Omega, \quad \partial u/\partial \nu_L = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

Here Ω is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$, L is an uniformly elliptic symmetric expression of the form

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

with real-valued coefficients $a_{ij} \in C^{0,1}(\overline{\Omega})$, and $\partial/\partial\nu_L$ represents the conormal derivation on $\partial\Omega$ associated to L. The real-valued functions m and h belong respectively to $L^\infty(\Omega)$ and $L^p(\Omega)$, where $p = 1$ if $N = 1$ and $p > N$ if $N \geq 2$. Unless otherwise stated, we will always assume in addition to the above that m changes sign in Ω, i.e.

$$\text{meas}\,\{x \in \Omega; m(x) > 0\} > 0 \text{ and meas}\,\{x \in \Omega;\, m(x) < 0\} > 0\,. \tag{2.2}$$

Also, without loss of generality, changing λ in (2.1) if necessary, we can assume

$$|m(x)| < 1 \text{ a.e. in } \Omega. \tag{2.3}$$

Solutions of (2.1) are understood in the weak sense: $u \in H^1(\Omega)$ with

$$a(u,v) = \lambda \int_\Omega muv + \int_\Omega hv \qquad \forall v \in H^1(\Omega), \tag{2.4}$$

where $a(u,v)$ denotes the Dirichlet form associated to L:

$$a(u,v) := \sum_{i,j=1}^{N} \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}.$$

By the L^p regularity theory, such a solution u belongs to $W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$.

Our purpose in this preliminary section is to collect some results relative to the principal eigenvalues of the associated problem

$$Lu = \lambda m(x)u \quad \text{in } \Omega, \quad \partial u/\partial \nu_L = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Most of those results can be found in [4], [12], although not in the same form nor with the same degree of generality. For the sake of completeness and for later reference, some proofs will be sketched.

The fundamental tool is the following form of the strong maximum principle.

Proposition 2.1 *Let u be a solution of the problem*

$$Lu + a_0(x)u = h \quad \text{in } \Omega, \quad \partial u/\partial \nu_L = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

with $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$, h as above and $h \gneqq 0$. Then u satisfies

$$u > 0 \quad \text{in } \overline{\Omega}. \tag{2.7}$$

Proof. As already observed, $u \in C^1(\overline{\Omega})$ so that (2.7) makes sense. Taking $-u^-$ as testing function in (2.6), one deduces $u \geq 0$ in Ω. Theorem 8.19 in [9] then implies $u > 0$ in Ω. The conclusion (2.7) can then be derived by using the Hopf boundary lemma as given e.g. in Proposition 1.16 from [6]. One should observe here that this last step involves the verification of the fact that the weak solution u satisfies the pointwise equality $\partial u/\partial \nu_L = 0$ on $\partial\Omega$. This can be achieved through a standard argument based on integration by parts. Q.E.D.

We are thus interested in the principal eigenvalues of (2.5). Clearly 0 is a principal eigenvalue, with the nonzero constants as eigenfunctions. We also observe that if $\lambda \in \mathbb{R}$ is a principal eigenvalue with eigenfunction $u \gneqq 0$, then $u > 0$ in $\overline{\Omega}$. (This follows from Proposition 2.1, by writing equation (2.5) as $Lu \pm \lambda u = \lambda(m \pm 1)u$ and using (2.3)).

The following expression will play a central role in our approach:

$$\lambda^*(m) := \inf \left\{ a(u,u);\, u \in H^1(\Omega) \text{ and } \int_\Omega mu^2 = 1 \right\}. \tag{2.8}$$

Proposition 2.2 *(i) Suppose $\int_\Omega m < 0$. Then $\lambda^*(m) > 0$ and $\lambda^*(m)$ is the unique nonzero principal eigenvalue; moreover the interval $]0, \lambda^*(m)[$ does not contain any eigenvalue. (ii) Suppose $\int_\Omega m \geq 0$. Then $\lambda^*(m) = 0$; moreover, if $\int_\Omega m = 0$, then 0 is the unique principal eigenvalue.*

Proposition 2.2 of course also applies to the weight $-m$. In particular, if $\int_\Omega m > 0$, then $-\lambda^*(-m)$ is the unique non zero principal eigenvalue of (2.5).

Proof of Proposition 2.2 In case (i), using Lemma 2.3 below, one sees that the infimum (2.8) is achieved, so that $\lambda^*(m) > 0$. Replacing u by $|u|$ if necessary, one observes that this infimum is achieved at a function $u \gneqq 0$. By Lagrange multipliers, u solves an equation of the form (2.5) for some $\lambda \in \mathbb{R}$. Taking u as testing function in this equation, one concludes that $\lambda = \lambda^*(m)$, which shows that $\lambda^*(m)$ is a principal eigenvalue. In case (ii), if $\int_\Omega m > 0$, then the infimum (2.8) is achieved at a suitable nonzero constant, so that $\lambda^*(m) = 0$. The case $\int_\Omega m = 0$ requires a little more care. (Note that in this case, the infimum (2.8) is *not* achieved). We pick $\psi \in H^1(\Omega)$ with $\int_\Omega m\psi > 0$ and put

$u = u_\epsilon := (1+\epsilon\psi)/[\int_\Omega m(1+\epsilon\psi)^2]^{1/2}$ in (2.8), where $\epsilon > 0$ is chosen sufficiently small so that the denominator in the definition of u_ϵ does not vanish. An easy calculation shows that $a(u_\epsilon, u_\epsilon) \to 0$ as $\epsilon \to 0$, which yields $\lambda^*(m) = 0$.

The fact that if $\int_\Omega m < 0$, then $]0, \lambda^*(m)[$ does not contain any eigenvalue easily follows from the definition (2.8) of $\lambda^*(m)$. Finally the two statements relative to the uniqueness of the principal eigenvalues follow from the more general Proposition 2.4 below. Q.E.D.

Lemma 2.3 *Assume $\int_\Omega m < 0$. Then there exists a constant $c > 0$ such that $a(u,u) \geq c\int_\Omega u^2$ for all $u \in H^1(\Omega)$ with $\int_\Omega mu^2 = 1$*

Proof. Assume by contradiction that for each $k = 1, 2, \ldots$, there exists $u_k \in H^1(\Omega)$ with $\int_\Omega mu_k^2 = 1$ and $a(u_k, u_k) \leq (1/k)\int_\Omega u_k^2$. One distinguishes two cases: either $||u_k||_{L^2}$ remains bounded (for a subsequence), or $||u_k||_{L^2} \to +\infty$ (for a subsequence). In this latter case one considers the normalization $v_k = u_k/||u_k||_{L^2}$. It is then easily verified that each of the two cases leads to a contradiction with $\int_\Omega m < 0$. Q. E. D.

In the following, we will generally only deal with the case $\int_\Omega m \leq 0$. The case $\int_\Omega m > 0$ can be reduced to this one by considering the weight $-m$. The two propositions below concern problem (2.1), with $\lambda \notin [0, \lambda^*(m)]$ in the first one, and $\lambda \in [0, \lambda^*(m)]$ in the second one.

Proposition 2.4 *Suppose $\int_\Omega m \leq 0$. If $\lambda \notin [0, \lambda^*(m)]$, then problem (2.1) with $h \geq 0$ has no solution $u \gneqq 0$.*

Proof. Assume that there exists a solution $u \gneqq 0$ of (2.1) for some $\lambda \in \mathbb{R}$ and some $h \geq 0$. Applying Proposition 2.1, we get $u > 0$ in $\overline{\Omega}$, and so u can be written as $u = e^{-v}$ with say $v \in C^1(\overline{\Omega})$. We pick $w \in H^1(\Omega) \cap L^\infty(\Omega)$ and take $e^v w^2$ as testing function in (2.1). A simple calculation using the idea of "completing a square" yields

$$\lambda \int_\Omega mw^2 = a(w,w) - \int_\Omega he^v w^2 - \int_\Omega < A(\nabla w + w\nabla v), (\nabla w + w\nabla v) >, \quad (2.9)$$

where A denotes the matrix $(a_{ij}(x))$ of the coefficients of L and $<,>$ the scalar product in $\mathbb{R}^N$. Consequently

$$\lambda \int_\Omega mw^2 \leq a(w,w) \quad (2.10)$$

for all $w \in H^1(\Omega) \cap L^\infty(\Omega)$. Since one can clearly restrict oneself to this class of functions in the definition (2.8) of $\lambda^*(m)$, one deduces from (2.10) that $\lambda \leq \lambda^*(m)$, and also that $-\lambda \leq \lambda^*(-m)$. Since $\lambda^*(-m) = 0$ by that part of Proposition 2.2 which is already proved, we can conclude $\lambda \in [0, \lambda^*(m)]$. Q. E. D.

Remark 2.5 The calculation in the above proof will be used again in Section 4. It is inspired from [11].

Proposition 2.6 *Suppose* $\int_\Omega m \le 0$. *Then the problem (2.1) with* $h \gneqq 0$ *does not admit any solution if* $\lambda = 0$ *or* $\lambda^*(m)$. *It admits a unique solution, which is* > 0 *in* $\overline{\Omega}$, *if* $0 < \lambda < \lambda^*(m)$.

Proof. The nonexistence results follow easily by taking for testing function in (2.1) the corresponding eigenfunctions. Suppose now $0 < \lambda < \lambda^*(m)$, and let u be the unique solution of (2.1). Clearly $u \not\equiv 0$. We claim that $u \ge 0$. Indeed, if this is not so, then $u^- \not\equiv 0$, and by taking $-u^-$ as testing function in (2.1), one gets

$$a(u^-, u^-) = \lambda \int_\Omega m(u^-)^2 - \int_\Omega hu^-. \tag{2.11}$$

If $a(u^-, u^-) = 0$, then u is a constant < 0, which is easily seen to be impossible. If $a(u^-, u^-) > 0$, then (2.11) implies $\int_\Omega m(u^-)^2 > 0$ and consequently, by the definition of $\lambda^*(m), a(u^-, u^-) \ge \lambda^*(m) \int_\Omega m(u^-)^2$; combining with (2.11) then yields again a contradiction. So $u \gneqq 0$, and by Proposition 2.1, we conclude $u > 0$ in $\overline{\Omega}$. Q. E. D.

Finally, for later reference, we mention the following result whose proof can be carried out exactly as that of Theorem 1.13 in [6].

Proposition 2.7 *Suppose* $\int_\Omega m \le 0$. *The principal eigenvalues 0 and* $\lambda^*(m)$ *are simple.*

Remark 2.8 The above results can easily be adapted to the simpler case where m does not change sign in Ω, say $m \gneqq 0$. In this case $\lambda^*(m) = 0$ and 0 is the unique principal eigenvalue. Problem (2.1) with $h \gneqq 0$ has no solution $u \ge 0$ if $\lambda > 0$, and no solution at all if $\lambda = 0$; its (unique) solution is > 0 in $\overline{\Omega}$ if $\lambda < 0$.

3 Antimaximum principle in the Neumann case

We consider in this section problem (2.1) with the same assumptions on Ω, L, m and h as in Section 2. The following expression will play an important role in our study of the AMP:

$$\bar{\lambda}(m) := \inf \left\{ \begin{array}{l} a(u,u): \ u \in H^1(\Omega)\,,\ \int_\Omega mu^2 = 1 \\ \text{and } u \text{ vanishes on some ball in } \overline{\Omega} \end{array} \right\}. \tag{3.1}$$

It is easily seen that when $N = 1$, this definition coincides with that given in (1.2). (This follows from the fact that if $u \in H^1(]a, b[)$ and vanishes at x_0 with, say, $x_0 < b$, then u_ϵ defined for $\epsilon > 0$ by $u_\epsilon(x) = u(x)$ if $x < x_0$, $u_\epsilon(x) = 0$ if $x_0 \le x \le x_0 + \epsilon$, $u_\epsilon(x) = u(x - \epsilon)$ if $x > x_0 + \epsilon$, converges to u in $H^1(]a, b[)$ as $\epsilon \to 0$). Clearly $\lambda^*(m) \le \bar{\lambda}(m)$. The following lemma makes more precise the relation between these two members.

Lemma 3.1 *If* $N \ge 2$, *then* $\lambda^*(m) = \bar{\lambda}(m)$. *If* $N = 1$, *then* $\lambda^*(m) < \bar{\lambda}(m)$. *Moreover, in the latter case, there is no eigenvalue in* $]\lambda^*(m), \bar{\lambda}(m)]$.

As in Section 2, we can limit ourselves without loss of generality in the study of (2.1) to the case

$$\int_\Omega m \leq 0\,. \tag{3.2}$$

We recall that if $\int_\Omega m < 0$ and $0 < \lambda < \lambda^*(m)$, then the solution u of (2.1) with $h \gneqq 0$ is > 0 in $\overline{\Omega}$. If $\int_\Omega m = 0$, then no result of the type "$h \gneqq 0$ implies $u \geq 0$" holds. With respect to the AMP, we have the following three results. Theorem 3.2 concerns the AMP in general, and its nonuniformity when $N \geq 2$. Theorem 3.4 characterizes the interval of uniformity when $N = 1$. Theorem 3.5 shows that some form of the AMP still holds outside this interval of uniformity.

Theorem 3.2 *Assume (3.2). (i) Given $h \gneqq 0$, there exists $\delta = \delta(h) > 0$ such that if $\lambda^*(m) < \lambda < \lambda^*(m) + \delta$ or $-\delta < \lambda < 0$, then any solution u of (2.1) satisfies $u < 0$ in $\overline{\Omega}$. (ii) If $N \geq 2$, then no such δ independent of h exists (either at the right of $\lambda^*(m)$ or at the left of 0).*

Remark 3.3 In the case where there is no weight and $L = -\Delta$, the fact that the AMP is nonuniform for $N \geq 2$ (and for all N under the Dirichlet boundary conditions) was already observed in [5] by using Green function. A similar observation was also derived in [2] (see also [7]) in the case of the p-Laplacian (without weight) by using the Fučik spectrum. It is not clear whether the approach based on Green function can be adapted to the context of Theorem 3.2. On the other hand the approach based on the Fučik spectrum can not be adapted, as we will see in Remark 5.4.

Theorem 3.4 *Assume (3.2) and $N = 1$. (i) If $\lambda^*(m) < \lambda \leq \bar\lambda(m)$ or $-\bar\lambda(-m) \leq \lambda < 0$, then any solution u of (2.1) with $h \gneqq 0$ satisfies $u < 0$ in $\overline{\Omega}$. (ii) $\bar\lambda(m)$ and $-\bar\lambda(-m)$ are respectively the largest and the smallest numbers such that the preceding implications hold.*

Theorem 3.5 *Assume (3.2) and $N = 1$. (i) Given $h \gneqq 0$, there exists $\delta = \delta(h) > 0$ such that if $\bar\lambda(m) < \lambda < \bar\lambda(m) + \delta$ or $-\bar\lambda(-m) - \delta < \lambda < -\bar\lambda(-m)$, then any solution u of (2.1) satisfies $u < 0$ in $\overline{\Omega}$. (ii) No such δ independent of h exists (either at the right of $\bar\lambda(m)$ or at the left of $-\bar\lambda(-m)$).*

Our final result makes precise the statement in the introduction that the AMP cannot hold far away to the right of $\bar\lambda(m)$ or to the left of $-\bar\lambda(-m)$.

Theorem 3.6 *Assume (3.2). (i) Given $\varepsilon > 0$, there exists $h \gneqq 0$ such that for any $\lambda \geq \bar\lambda(m) + \varepsilon$, (2.1) has no solution u satisfying $u < 0$ in $\overline{\Omega}$. (ii) Given $\varepsilon > 0$, there exists $h \gneqq 0$ such that for any $\lambda \leq -\bar\lambda(-m) - \varepsilon$, (2.1) has no solution u satisfying $u < 0$ in $\overline{\Omega}$.*

Remark 3.7 Assume (3.2). The following four numbers

$$-\bar\lambda(-m) \leq -\lambda^*(-m) = 0 \leq \lambda^*(m) \leq \bar\lambda(m)$$

thus control the domains of validity of the maximum principle and of the anti-maximum principle.

4 Proofs

Proof of Lemma 3.1 We start with the case $N \geq 2$ and introduce the following functions: for $N \geq 3$

$$v_k(x) := \begin{cases} 1 \text{ if } |x| \geq 1/k, \\ 2k|x| - 1 \text{ if } 1/2k < |x| < 1/k, \\ 0 \text{ if } |x| \leq 1/2k, \end{cases}$$

while for $N = 2$,

$$v_k(x) := \begin{cases} 1 - 2/k \text{ if } |x| \geq 1/k, \\ |x|^{\delta_k} - 1/k \text{ if } (1/k)^{1/\delta_k} < |x| < 1/k, \\ 0 \text{ if } |x| < (1/k)^{1/\delta_k}, \end{cases}$$

where $\delta_k \in]0, 1[$ is chosen so that $(1/k)^{\delta_k} = 1 - 1/k$. A simple calculation shows that v_k converges to the constant function 1 in $H^1_{\text{loc}}(\mathbb{R}^N)$ as $k \to \infty$. Fix now $x_0 \in \Omega$. Then, for any given $u \in H^1(\Omega) \cap L^\infty(\Omega)$, the function $u(x)v_k(x - x_0)$ vanishes on some ball in Ω and converges to u in $H^1(\Omega)$ as $k \to \infty$. This easily yields the conclusion $\lambda^*(m) = \bar{\lambda}(m)$.

We now turn to the proof that for $N = 1$,

$$\lambda^*(m) < \bar{\lambda}(m). \tag{4.1}$$

As observed at the beginning of Section 3, when $N = 1$,

$$\bar{\lambda}(m) = \inf \left\{ \begin{array}{l} a(u,u): \ u \in H^1(\Omega), \ \int_\Omega mu^2 = 1, \\ \text{and } u \text{ vanishes somewhere in } \overline{\Omega} \end{array} \right\}. \tag{4.2}$$

Since $H^1(\Omega)$ is compactly imbedded in $C(\overline{\Omega})$, the infimum in (4.2) is achieved. Replacing u by $|u|$ if necessary, we can assume that it is achieved at $u \geq 0$.

Claim 4.1 *u vanishes at exactly one point x_0 in $\overline{\Omega} = [a, b]$. Moreover $u \in C^1[a, x_0] \cap C^1[x_0, b]$ and*

$$u'(x_0-) < 0 < u'(x_0^+) \tag{4.3}$$

(where (4.3) is modified into $0 < u'(x_0^+)$ if $x_0 = a$, and similarly if $x_0 = b$).

Proof. Part of the argument here is adapted from [2]. We first show that if u vanishes at some $x_0 \in \overline{\Omega}$, then

$$a(u, v) = \bar{\lambda}(m) \int_\Omega muv \tag{4.4}$$

for all $v \in V_{x_0} = \{v \in H^1(\Omega);\ v(x_0) = 0\}$. Indeed one has

$$\bar{\lambda}(m) = \inf \Big\{ a(v, v);\ v \in V_{x_0} \quad \text{and} \quad \int_\Omega mv^2 = 1 \Big\},$$

where the latter infimum is also achieved at u. Applying the standard theorem on Lagrange multipliers in the Hilbert space V_{x_0}, u solves an equation like (4.4) with some multiplier λ instead of $\bar{\lambda}(m)$. But taking $v = u$, one gets $\lambda = \bar{\lambda}(m)$, which yields (4.4). Assume now by contradiction that u vanishes at at least two points x_1 and $x_2 \in \overline{\Omega}$. So u satisfies (4.4) for all $v \in V_{x_1}$ and also for all $v \in V_{x_2}$. Since any $v \in H^1(\Omega)$ can be written as $v_1 + v_2$ with $v_1 \in V_{x_1}$ and $v_2 \in V_{x_2}$, we conclude that u satisfies (4.4) for all $v \in H^1(\Omega)$, i.e. that u is a solution of

$$Lu = \bar{\lambda}(m)mu \quad \text{in } \Omega, \quad \partial u/\partial \nu_L = 0 \quad \text{on } \partial\Omega.$$

Proposition 2.1 then implies $u > 0$ in $\overline{\Omega}$, contradiction. So u vanishes at exactly one point x_0.

Finally, assuming for instance $a < x_0 < b$, equation (4.4) implies that u solves the mixed problem

$$Lu = \bar{\lambda}(m)mu \quad \text{in }]a, x_0[,\ u'(a) = u(x_0) = 0\,.$$

This implies $u \in C^1[a, x_0]$. Moreover $u'(x_0) < 0$ because otherwise u solves the corresponding Neumann problem on $]a, x_0[$ and consequently, by Proposition 2.1, $u > 0$ on $[a, x_0]$, a contradiction. A similar argument on $[x_0, b]$ completes the proof of the claim.

The idea of the proof of (4.1) is now the following. Define, for $\varepsilon \geq 0$, $u_\varepsilon(x) = \max(u(x), \varepsilon)$. Clearly $u_\varepsilon \to u$ in $H^1(\Omega)$ as $\varepsilon \to 0$, and so $\int_\Omega m u_\varepsilon^2 > 0$ for ε sufficiently small. Putting $J(\varepsilon) = a(u_\varepsilon, u_\varepsilon)/\int_\Omega m u_\varepsilon^2$ and $J(0) = a(u,u)/\int_\Omega m u^2$, we will show below that

$$\limsup_{\varepsilon \to 0, \varepsilon > 0} \frac{J(\varepsilon) - J(0)}{\varepsilon} < 0\,. \tag{4.5}$$

This implies in particular $J(\varepsilon) < J(0)$ for $\varepsilon > 0$ sufficiently small. Consequently

$$\lambda^*(m) \leq J(\varepsilon) < J(0) = \bar{\lambda}(m),$$

which yields (4.1).

We will prove (4.5) in the case $x_0 = a$. The argument can be easily adapted if $x_0 > a$. So $u \in C^1[a, b]$, $u > 0$ in $]a, b]$, $u(a) = 0$, $u'(a+) > 0$; moreover L is of the form $Lu = -(p(x)u')'$, with $p(x) \geq p_0 > 0$ on $[a, b]$. Writing $u'(a+) = \alpha$, we first fix $c > a$ such that $u'(x) \geq \alpha/2$ and $\alpha/2\ (x - a) \leq u(x) \leq 2\alpha\ (x - a)$ on $[a, c]$. In the following, $\varepsilon > 0$ will be taken sufficiently small so that $u(x) \geq \varepsilon$ sur $[c, b]$. We have

$$\begin{aligned}
a(u_\varepsilon, u_\varepsilon) &= a(u,u) - \int_{u<\varepsilon} p(u')^2 \\
&\leq a(u,u) - (\varepsilon/2\alpha)p_0(\alpha/2)^2, \\
\int_\Omega m u_\varepsilon^2 &= \int_\Omega m u^2 + \int_{u<\varepsilon} (m u_\varepsilon^2 - m u^2) \\
&= \int_\Omega m u^2 + O(\varepsilon^3),
\end{aligned}$$

and consequently

$$\frac{J(\varepsilon)-J(0)}{\varepsilon} \leq -p_0(\alpha/8)\left(\int_\Omega mu_\varepsilon^2\right)^{-1} + O(\varepsilon^2),$$

which yields (4.5).

To conclude the proof of Lemma 3.1, it remains to see that when $N = 1$, there is no eigenvalue in $]\lambda^*(m), \bar\lambda(m)]$. Let $\lambda > \lambda^*(m)$ be an eigenvalue, with associated eigenfunction u. By Proposition 2.2, u changes sign and consequently vanishes somewhere in $\overline{\Omega}$. Taking u as testing function in $Lu = \lambda mu$, one gets

$$a(u,u) = \lambda \int_\Omega mu^2, \tag{4.6}$$

which implies $\lambda \geq \bar\lambda(m)$. Suppose now by contradiction that $\lambda = \bar\lambda(m)$. Since, by (4.6), $\int_\Omega mu^2 > 0$, we have $\int_\Omega m(u^+)^2 > 0$ or $\int_\Omega m(u^-)^2 > 0$. Consider the first case (the argument is similar in the second case). Taking u^+ as testing function in $Lu = \bar\lambda(m)mu$, one gets

$$a(u^+,u^+) = \bar\lambda(m)\int_\Omega m(u^+)^2,$$

which shows that u^+ is a nonnegative minimizer in (4.2). The claim above then implies that u^+ vanishes at exactly one point, which is impossible since u changes sign. Q. E. D.

Remark 4.2 Functions like v_k in the proof above were used in [7] in the study of the asymptotic behavior of the first curve in the Fučik spectrum. Note that no approximation such as that considered at the beginning of the proof of Lemma 3.1 for $N \geq 2$ is possible when $N = 1$ since, in that case, H^1 convergence implies uniform convergence.

Remark 4.3 Lemma 3.1 still holds, with the same proof, if m does not change sign, with $m \gneqq 0$.

Proof of Theorem 3.2. We first prove part (i) at the right of $\lambda^*(m)$ (the argument at the left of 0 is similar). Assume by contradiction the existence for some $h \gneqq 0$ of sequences $\lambda_k > \lambda^*(m)$ and u_k such that $\lambda_k \to \lambda^*(m)$,

$$Lu_k = \lambda_k mu_k + h \quad \text{in } \Omega, \quad \partial u_k/\partial\nu_L = 0 \quad \text{on } \partial\Omega \tag{4.7}$$

and

$$u_k \geq 0 \text{ somewhere in } \overline{\Omega}. \tag{4.8}$$

We distinguish two cases: either $||u_k||_{L^2}$ remains bounded, or, for a subsequence, $||u_k||_{L^2} \to +\infty$. In the first case, one derives from (4.7) that u_k remains bounded in $W^{2,p}(\Omega)$. Going to the limit in (4.7), one gets a solution u of

$$Lu = \lambda^*(m)mu + h \quad \text{in } \Omega, \quad \partial u/\partial\nu_L = 0 \quad \text{on } \partial\Omega,$$

which is impossible by Proposition 2.6. In the second case, one considers $v_k = u_k/||u_k||_{L^2}$, and arguing in a similar way from

$$Lv_k = \lambda_k m v_k + h/||u_k||_{L^2} \quad \text{in } \Omega, \quad \partial v_k/\partial \nu_L = 0 \quad \text{on } \partial\Omega,$$

one gets that, for a subsequence, $v_k \to v$ in $C^1(\overline{\Omega})$ where $||v||_{L^2} = 1$ and

$$Lv = \lambda^*(m) m v \quad \text{in } \Omega, \quad \partial v/\partial \nu_L = 0 \quad \text{on } \partial\Omega.$$

Consequently v is an eigenfunction associated to $\lambda^*(m)$ and so either $v > 0$ in $\overline{\Omega}$ or $v < 0$ in $\overline{\Omega}$. In the first case, we deduce $v_k > 0$ in $\overline{\Omega}$ for k sufficiently large, which leads to a contradiction with Proposition 2.4. In the second case we deduce $v_k < 0$ in $\overline{\Omega}$ for k sufficiently large, which leads to a contradiction with (4.8). (This argument to derive the AMP is adapted from [8]).

Part (ii) of Theorem 3.2 is a consequence of Theorem 3.6 since (3.2) and $N \geq 2$ imply $\bar{\lambda}(m) = \lambda^*(m)$ and $\bar{\lambda}(-m) = \lambda^*(-m) = 0$. Q. E. D.

Proof of Theorem 3.4. We start with part (i) in the case $\lambda^*(m) < \lambda \leq \bar{\lambda}(m)$ (the case $-\bar{\lambda}(-m) \leq \lambda < 0$ can be treated similarly). Let u be a solution of (2.1) for some $h \gneqq 0$. By Proposition 2.4, u can not be ≥ 0, i.e. $u^- \not\equiv 0$. Taking $-u^-$ as testing function in (2.1), we get

$$a(u^-, u^-) = \lambda \int_\Omega m(u^-)^2 - \int_\Omega h u^-. \tag{4.9}$$

If $a(u^-, u^-) = 0$, then u =Cst< 0 and we are finished. If $a(u^-, u^-) > 0$, then (4.9) implies $\int_\Omega m(u^-)^2 > 0$ and so

$$a(u^-, u^-)/\int_\Omega m(u^-)^2 \leq \lambda. \tag{4.10}$$

Suppose first $\lambda < \bar{\lambda}(m)$. Then (4.10) implies that u^- is not admissible in the definition (1.2) of $\bar{\lambda}(m)$. Consequently u^- does not vanish in $\overline{\Omega}$, i.e. $u < 0$ in $\overline{\Omega}$. Suppose now $\lambda = \bar{\lambda}(m)$. If u^- does not vanish in $\overline{\Omega}$, we are finished as above. If u^- vanishes somewhere in $\overline{\Omega}$, then (4.10) (with $\lambda = \bar{\lambda}(m)$) implies that u^- is a nonnegative minimizer in the definition of $\bar{\lambda}(m)$. By the claim in the proof of Lemma 3.1, u^- vanishes at exactly one point. But (4.9) (with $\lambda = \bar{\lambda}(m)$) implies $\int_\Omega h u^- = 0$, so that u^- vanishes on the set of positive measure where $h > 0$, a contradiction.

Part (ii) of Theorem 3.4 is a consequence of Theorem 3.6. Q. E. D.

Proof of Theorem 3.5. We first prove part (i) at the right of $\bar{\lambda}(m)$ (the argument is similar at the left of $-\bar{\lambda}(-m)$). Assume by contradiction the existence for some $h \gneqq 0$ of sequences $\lambda_k > \bar{\lambda}(m)$ and u_k such that $\lambda_k \to \bar{\lambda}(m)$,

$$Lu_k = \lambda_k m u_k + h \quad \text{in } \Omega\,, \quad \partial u_k/\partial \nu_L = 0 \quad \text{on } \partial\Omega \tag{4.11}$$

and

$$u_k \geq 0 \text{ somewhere in } \overline{\Omega}. \tag{4.12}$$

As in the proof of Theorem 3.2, we distinguish two cases: either $||u_k||_{L^2}$ remains bounded, or, for a subsequence, $||u_k||_{L^2} \to \infty$. In the first case, one obtains that, for a subsequence, u_k converges in $C^1(\overline{\Omega})$ to a solution u of

$$Lu = \bar{\lambda}(m)mu + h \quad \text{in } \Omega\,, \quad \partial u/\partial \nu_L = 0 \quad \text{on } \partial\Omega\,;$$

moreover, by (4.12), $u \geq 0$ somewhere in $\overline{\Omega}$. But this contradicts the fact that the AMP holds for $\lambda = \bar{\lambda}(m)$ (cf. Theorem 3.4). In the second case, one considers $v_k = u_k/||u_k||_{L^2}$ and obtains that, for a subsequence, v_k converges in $C^1(\overline{\Omega})$ to a nonzero solution v of

$$Lv = \bar{\lambda}(m)mv \quad \text{in } \Omega, \quad \partial v/\partial \nu_L = 0 \quad \text{on } \partial\Omega.$$

This again yields a contradiction since by Lemma 3.1, $\bar{\lambda}(m)$ is not an eigenvalue.

Part (ii) of Theorem 3.5 clearly follows from the sharpness of $\bar{\lambda}(m)$ and $-\bar{\lambda}(-m)$ in Theorem 3.4. Q. E. D.

Proof of Theorem 3.6. We prove part (i) (part (ii) is proved similarly). Assume by contradiction that there exists $\varepsilon > 0$ such that for any $h \gneqq 0$ there exists λ with $\lambda \geq \bar{\lambda}(m) + \varepsilon$ such that (2.1) has a solution $u < 0$ in $\overline{\Omega}$. We start with $w \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfying $\int_\Omega mw^2 > 0$ and vanishing on some ball in $\overline{\Omega}$, as in the definition (3.1) of $\bar{\lambda}(m)$. Then we choose $h \gneqq 0$ with $\text{supp}h \cap \text{supp}w = \emptyset$, and finally we consider $\lambda = \lambda_w$ and $u = u_w$ as provided by the above contradictory hypothesis. So $-u > 0$ in $\overline{\Omega}$ and consequently can be written as $-u = e^{-v}$ with $v \in C^1(\overline{\Omega})$. We then take $e^v w^2$ as testing function in (2.1). A simple calculation using the idea of "completing a square", as in the proof of Proposition 2.4, yields a relation analogous to (2.9):

$$\lambda \int_\Omega mw^2 = a(w,w) + \int_\Omega he^v w^2 - \int_\Omega \langle A(\nabla w + w\nabla v), (\nabla w + w\nabla v)\rangle\,.$$

Here the integral involving h vanishes since h and w have disjoint supports. Consequently

$$\bar{\lambda}(m) + \epsilon \leq \lambda_w \leq a(w,w)\Big(\int_\Omega mw^2\Big)^{-1}$$

for all w as above. Taking the infimum with respect to w yields $\bar{\lambda}(m)+\epsilon \leq \bar{\lambda}(m)$, a contradiction. Q. E. D.

Remark 4.4 It is clear from the above proof that the function h in Theorem 3.6 can be taken in $C_c^\infty(\Omega)$, with support of arbitrary small diameter. Similarly the statements (ii) in each of Theorems 3.2, 3.4 and 3.5 still hold if one restricts $h \gneqq 0$ to vary in $C_c^\infty(\Omega)$ with support of arbitrarily small diameter.

Remark 4.5 The above arguments can easily be adapted to the case where m does not change sign in Ω, say $m \gneqq 0$, as in Remark 2.8. In this case the AMP holds at the right of 0. It is nonuniform when $N \geq 2$ and uniform when $N = 1$. In this latter case the interval of uniformity is exactly $0 < \lambda \leq \bar{\lambda}(m)$, with $\bar\lambda(m)$ given by (1.2); moreover the AMP still holds at the right of $\bar\lambda(m)$, in a nonuniform way. Finally, as in Theorem 3.6, the AMP cannot hold far away to the right of $\bar\lambda(m)$.

Remark 4.6 The above results can also be adapted to the case where L is replaced by

$$L_0u := Lu + a_0(x)u,$$

where $a_0 \in L^\infty(\Omega)$ satisfies $a_0 \gneqq 0$. In this case 0 is not anymore an eigenvalue. If m changes sign, the principal eigenvalues are $\lambda^*(m)$ and $-\lambda^*(-m)$, with $\lambda^*(m)$ defined by (2.8) where $a(u,v)$ now stands for the Dirichlet form associated to L_0. $\bar\lambda(m)$ is defined similarly, and the domains of validity of the maximum principle and the antimaximum principle are again controlled by the following four numbers:

$$-\bar\lambda(-m) \le -\lambda^*(-m) < 0 < \lambda^*(m) \le \bar\lambda(m).$$

If m does not change sign, say $m \gneqq 0$, then only the following two numbers play a role:

$$0 < \lambda^*(m) \le \bar\lambda(m).$$

5 Dirichlet boundary conditions

In this section we briefly consider the Dirichlet problem

$$L_0u = \lambda m(x)u + h(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega\,, \tag{5.1}$$

where $L_0u = Lu + a_0(x)u$, with $a_0 \in L^\infty(\Omega)$, $a_0 \ge 0$ in Ω. The assumptions on Ω, L, m and h are the same as in Section 2.

The basic spectral theory for (5.1) is well described in [6]. There are two principal eigenvalues: $\lambda_1(m) > 0$ and $-\lambda_1(-m) < 0$, where

$$\lambda_1(m) := \inf\{a(u,u); u \in H_0^1(\Omega) \text{ and } \int_\Omega mu^2 = 1\},$$

with $a(u,v)$ the Dirichlet form associated to L_0. If $-\lambda_1(-m) < \lambda < \lambda_1(m)$ and $h \gneqq 0$, then (5.1) has a (unique) solution u, which satisfies $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$ (where $\partial/\partial\nu$ represents exterior normal derivation). With respect to the AMP, we have the following two results.

Theorem 5.1 *(i) Given $h \gneqq 0$, there exists $\delta = \delta(h) > 0$ such that if $\lambda_1(m) < \lambda < \lambda_1(m) + \delta$ or $-\lambda_1(-m) - \delta < \lambda < -\lambda_1(-m)$, then any solution u of (5.1) satisfies $u < 0$ in Ω and $\partial u/\partial\nu > 0$ on $\partial\Omega$. (ii) No such δ independent of h exists (either at the right of $\lambda_1(m)$ or at the left of $-\lambda_1(-m)$).*

Theorem 5.2 *(i) Given $\epsilon > 0$ there exists $h \gneqq 0$ such that for any $\lambda \ge \lambda_1(m) + \epsilon$, (5.1) has no solution u satisfying $u < 0$ in Ω. (ii) Similar statement at the left of $-\lambda_1(-m)$.*

As indicated in the introduction, part (i) of Theorem 5.1 in the case of a weight m in $C(\overline{\Omega})$ was proved in [10].

The proof of part (i) of Theorem 5.1 can be carried out by contradiction in a way similar to the proof of Theorem 3.2. Part (ii) of Theorem 5.1 follows from Theorem 5.2. Let us sketch the proof of the latter.

Proof of Theorem 5.2. We only consider part (i). Assume by contradiction that there exists $\epsilon > 0$ such that for any $h \gneqq 0$ there exists λ with $\lambda \geq \lambda_1(m) + \epsilon$ such that (5.1) has a solution u satisfying $u < 0$ in Ω. We start with $w \in C_c^\infty(\Omega)$ satisfying $\int_\Omega mw^2 > 0$. Then we choose $h \gneqq 0$ with supp $h \cap$ supp $w = \emptyset$, and finally we consider $\lambda = \lambda_w$ and $u = u_w$ as provided by the above contradictory hypothesis. So $-u > 0$ in Ω and consequently can be written as $-u = e^{-v}$ with $v \in C^1(\Omega)$. We take $e^v w^2 \in C_c^1(\Omega)$ as testing function in (5.1). By a calculation identical to that in the proof of Theorem 3.6, we get

$$\lambda_1(m) + \epsilon \leq \lambda_w \leq a(w,w)\Big(\int_\Omega mw^2\Big)^{-1}$$

for all w as above. Since the infimum of the right-hand side with respect to w is equal to $\lambda_1(m)$, we reach a contradiction. Q. E. D.

Remark 5.3 Similar results hold when the weight does not change sign in Ω.

Remark 5.4 We insist on the fact that the nonuniformity of the AMP in Theorem 5.1 holds for *any* weight. This should be compared with the recent result in [1] that if $N = 1$ and m has compact support in Ω, then the first curve in the corresponding Fučik spectrum is *not* asymptotic to the horizontal and vertical lines through $(\lambda_1(m), \lambda_1(m))$ (even if m does not change sign). It follows that the qualitative and quantitative connections between "uniformity of the AMP" and "existence of a gap at infinity in the Fučik spectrum between the first curve and the horizontal and vertical lines through $(\lambda_1(m), \lambda_1(m))$", which were observed in [7] when $m(x) \equiv 1$, do not hold any more in general in the presence of a weight.

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