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Periodic traveling waves for a nonlocal integro-differential model *

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Abstract

We establish the existence, uniqueness and stability of periodic traveling wave solutions to an intrego-differential model for phase transitions.

1 Introduction

In this paper, we are concerned with the following integro-differential model for phase transitions

$$u_t - Du_{xx} - d(J*u-u) - f(u,t) = 0, \tag{1.1}$$

where $x \in \mathbb{R}$ and D, d are nonnegative constants with $D + d \neq 0$; $J*u(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)dy$ is the convolution of J and $u(x,t)$; $J \in C^1(\mathbb{R}) \bigcap L^1(\mathbb{R})$; $f(u,\cdot)$ is $T-$periodic, i.e., $f(u,t+T) = f(u,t)$ for all $u, t \in \mathbb{R}$; and $f(\cdot,t)$ is bistable. Other conditions on J and f are specified below. A typical example of f is the cubic potential function $f = \rho(1-u^2)(2u-\gamma(t))$, where $\rho > 0$ is a constant, $\gamma(t)$ is $T-$periodic and $0 < \gamma(t) < 2$.

When $d = 0$, equation (1.1) is the classical Allen-Cahn equation [12] for which the results are known. Therefore, we will assume $d > 0$ throughout. Equation (1.1) can be considered as a nonlocal version of the Allen-Cahn equation which incorporates spatial long range interaction. When $d = 0$ and $f(u,\cdot) = f(u)$ is independent of t, the traveling wave solution of the form $u(x,t) = U(x-ct)$ is studied in [12] and [13](see also their references). The nonlocal autonomous case is studied in [5]. X. Chen [9] applied a "squeezing" technique, due to a strong comparison principle, to study the existence, uniqueness and stability of traveling wave solutions for a variety of autonomous nonlocal evolution equations, which includes the Allen-Cahn reaction-diffusion equation, neural networks, the continuum Ising model, and a thalamic model. When $f(u,t)$ is T-periodic, periodic traveling wave solutions of the bistable reaction diffusion are studied in [2]. In this paper, we will establish similar results to those in [2] but for the more general equation (1.1).

We assume in this paper that

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H1) $f \in C^{2,1}(\mathbb{R} \times \mathbb{R})$ is periodic in t with period T, i.e., there is a $T > 0$ such that $f(u,t) = f(u,t+T)$ for all $u, t \in \mathbb{R}$.

H2) The period map $P(\alpha) := w(\alpha, T)$, where $w(\alpha,t)$ is the solution to

$$w_t = f(w,t), \qquad \text{for all} \quad t \in \mathbb{R}, \quad w(\alpha, 0) = \alpha, \tag{1.2}$$

has exactly three fixed points α^-, α^0, α^+, satisfying $\alpha^- < \alpha^0 < \alpha^+$. In addition, they are non-degenerate and $\alpha^\pm$ are stable and α^0 is unstable, that is,

$$\frac{d}{d\alpha} P(\alpha^\pm) < 1 < \frac{d}{d\alpha} P(\alpha^0). \tag{1.3}$$

H3) $J(x) \in C^1(\mathbb{R})$ is nonnegative, $\int_{\mathbb{R}} J(x)\,dx = 1$, and $\int_{\mathbb{R}} |J'(x)|\,dx < \infty$.

In the case $D = 0$, we need the following additional condition,

H4)

$$\sup\{f_u(u,t) : u \in [W^-(t), W^+(t)], t \in [0,T]\} < d, \tag{1.4}$$

where $W^\pm(t) = w(\alpha^\pm, t)$.

We are concerned with the periodic traveling wave solutions of (1.1) connecting the two periodic stable solutions $W^\pm(t)$, that is, the solutions of the form $u(x,t) = U(x - ct, t)$, with $U(x, t+T) = U(x,t)$, for all x, $t \in \mathbb{R}$, and $\lim_{x\to\infty} U(\pm x, t) = W^\pm(t)$ uniformly, where c is some real constant (called the wave speed). We claim that the long time behavior of the solutions of (1.1) coupled with the initial condition

$$u(x,0) = g(x), \tag{1.5}$$

is governed by the periodic traveling wave solutions $u(x,t) = U(x - ct, t)$ of (1.1). If we work in the traveling wave frame and let $\xi = x - ct$, we are led to study the following problem

$$U_t - cU_\xi - DU_{\xi\xi} - d(J * U - U) - f(U,t) = 0, \tag{1.6}$$

$$U(\pm\infty, t) = \lim_{\xi \to \pm\infty} U(\xi, t) = w(\alpha^\pm, t), \text{ uniformly in } t \in \mathbb{R}, \tag{1.7}$$

$$U(\cdot, T) = U(\cdot, 0), \ U(0,0) = \alpha^0. \tag{1.8}$$

The following theorems are our main results concerning the existence, uniqueness and stability of the periodic traveling solutions.

Theorem 1.1. *Assume (H1), (H2) and (H3) hold. In the case $D = 0$, we also assume (H4). Then there exist a unique smooth function $U(\xi, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a unique constant $c \in \mathbb{R}$ such that (1.6) - (1.8) hold. Moreover $U(\cdot, t)$ is strictly increasing.*

Theorem 1.2. *The periodic traveling wave solution $u(x,t) = U(x-ct,t)$, where $U(\xi,t)$ and c are as in Theorem 1.1, is uniformly and asymptotically stable.*

Remark 1.1. *(1) If $u(x,t) = U(x-ct,t)$ is a periodic traveling wave solution of (1.1), so is $U(x-\xi-ct,t)$, for any $\xi \in \mathbb{R}$. Therefore, the stability mentioned above is that of the family of spatial translation. Periodic traveling wave solutions are unique modulo a spatial shift.*

(2) In the autonomous case, there are discontinuous traveling wave solutions if (H_4) fails and the traveling wave solutions are not asymptotically stable. For the periodic case the existence, uniqueness and stability remain open in general, that is, without (H_4).

The paper is organized as follows. In Section 2 we study the uniqueness and monotonicity of the wave. In Section 3, we use a homotopy method to prove the existence of the solution to (1.6)-(1.8). And finally in Section 4 we study the uniform and asymptotic stability of the periodic traveling wave solution.

2 Uniqueness of Periodic Traveling Waves

In this section, we will establish the uniqueness of smooth periodic traveling wave solutions of (1.6)-(1.8) and prove that the wave is strictly monotone in the spatial direction.

For a metric space X, denote

$$C_{unif}(X) = \{u : X \to \mathbb{R}, u \text{ is bounded and uniformly continuous on } X\}, \tag{2.1}$$

and denote $\|u\| = \sup_{x\in X} |u(x)|$. First we need the following comparison principle.

Lemma 2.1. *(Comparison Principle). Suppose that R_1 is a union of open intervals, $R_2 = \mathbb{R}\setminus R_1$, and that, for some $\tau < t_0$, $u(x,t) \in C_{unif}(\mathbb{R}\times[\tau,t_0])$ has the required derivatives. Assume that $u(x,t) \geq 0$ for all $x \in R_2$ and $t \in (\tau,t_0]$, and $u(x,t)$ satisfies*

$$u_t - Du_{xx} - d(J*u-u) - bu_x - cu \geq 0 \tag{2.2}$$

on $R_1 \times (\tau,t_0]$, where D and d are nonnegative constants with $D + d \neq 0$, $b = b(x,t)$, $c = c(x,t)$ are bounded continuous functions on $R_1 \times (\tau,t_0]$. If $u(x,\tau) \geq 0$ for all $x \in \mathbb{R}$, then $u(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t \in (\tau,t_0]$. Moreover, if $u(x,t)$ is not identically 0 on $R_1 \times (\tau,t_0]$, then $u(x,t) > 0$, for all $x \in R_1$ and $t \in (\tau,t_0]$.

Proof. We may assume $\tau = 0$. We take $d > 0$ since the result is standard for $d = 0$. By the assumption that $u(x,t) \in C_{unif}(\mathbb{R}\times[0,t_0])$, $\inf_{x\in\mathbb{R}} u(x,t)$ is continuous on $[0,t_0]$. If the conclusion of the lemma is not true, there exist constants $\epsilon > 0$ and $T_0 > 0$ such that $u(x,t) > -\epsilon e^{2Kt}$, for all $x \in \mathbb{R}$ and $0 < t < T_0$, and

$\inf_{x\in\mathbb{R}}\{u(x,T_0)\} = -\epsilon e^{2KT_0}$, where $K = 2(D+4d+b_0+4c_0)$, $b_0 = \sup\{|b(x,t)| : x \in R_1, t \in [\tau, t_0]\}$, and $c_0 = \sup\{|c(x,t)| : x \in R_1, t \in [\tau, t_0]\}$. Let $z(x)$ be a smooth function such that $1 \le z(x) \le 3$, $z(0) = 1$, $z(\pm\infty) = \lim_{x\to\infty} z(\pm x) = 3$, and $|z'(x)| \le 1$, $|z''(x)| \le 1$. Define $w_\sigma(x,t) = -\epsilon(\frac{3}{4} + \sigma z(x))e^{2Kt}$ for $\sigma \in [0,1]$. Notice that $w_1(x,t) < u(x,t)$ and $w_0(x,t) = -\frac{3}{4}\epsilon e^{2Kt}$ for $(x,t) \in R_1 \times (0,t_0]$. There is a minimum $\sigma^* \in [\frac{1}{8}, 1)$ such that $w_{\sigma^*}(x,t) \le u(x,t)$, for $x \in \mathbb{R}$ and $t \in [0,T_0]$ and there exists $(x_1,t_1) \in R_1 \times (0,T_0]$ such that $u(x_1,t_1) = w_{\sigma^*}(x_1,t_1)$. Therefore, at (x_1,t_1),

$$\begin{aligned} 0 \ &\ge\ (u-w_{\sigma^*})_t - D(u-w_{\sigma^*})_{xx} - d(J*(u-w_{\sigma^*}) - (u-w_{\sigma^*})) \\ &\quad -b(u-w_{\sigma^*})_x - c(u-w_{\sigma^*}) \\ &\ge\ \epsilon e^{2Kt_1}[\frac{7}{8}K - D - 4d - b_0 - 4c_0] > 0, \end{aligned}$$

by the choice of K, which is a contradiction. Therefore $u(x,t) \ge 0$ for all $x \in \mathbb{R}$ and $t \in (0,t_0]$. Suppose $u(x,t)$ is not identically zero on $R_1 \times (0,t_0]$ and there is a point $(x_2,t_2) \in R_1 \times (0,t_0]$ such that $u(x,t)$ achieves the minimum 0. By a similar argument to the above we deduce that $(J*u-u)(x_2,t_2) = 0$. Therefore $u \equiv 0$, which is a contradiction. That completes the proof. □

Now we are ready to state the uniqueness theorem.

Theorem 2.2. *Suppose (H1), (H2) and (H3) hold. Then problem (1.6)- (1.8) admits at most one smooth solution.*

Proof. The proof is similar to that in [2]. Let (U,c) and $(\overline{U},\overline{c})$ be any two solutions of (1.6)-(1.8) with $c \ge \overline{c}$. We prove $U = \overline{U}$ and $c = \overline{c}$, we divide the proof into six steps.

1. By periodicity of $U(\xi,t)$ and the comparison principle, we have

$$w(-M^-, kT+t) \le U(\xi, kT+t) = U(\xi,t) \le w(M^+, kT+t),$$

where $M^\pm = \sup_{\xi\in\mathbb{R}} \pm U(\xi,0)$. Letting $k \to \infty$ gives $W^-(t) \le U(\xi,t) \le W^+(t)$. By Lemma 2.1, we have

$$W^-(t) < U(\xi,t) < W^+(t). \tag{2.3}$$

2. Define $\nu^\pm = -\frac{1}{T}\int_0^T f_u(W^\pm(t),t)dt$. Without loss of generality, we may assume $\nu^+ \ge \nu^-$. The other case can be proved similarly. Let $\nu = (\nu^+ - \nu^-)/\,2$ and $a^\pm(t) = \exp(\frac{\nu^\pm t}{2} + \int_0^t f_u(W^\pm(\tau),\tau)d\tau)$. Notice that $P'(\alpha^\pm) = \exp(\int_0^T f_u(W^\pm(\tau),\tau)d\tau) < 1$. We have $\nu^\pm > 0$ and $a^\pm(T) < 1$. Moreover, there exist two constants C_1 and C_2 such that

$$C_2 a^-(t) \le a^+(t)e^{\nu t} \le C_1 a^-(t). \tag{2.4}$$

For $\eta > 0$ and $t \in [0,T]$, let $I^\pm_\eta(t) := [W^\pm(t) - \eta, W^\pm(t) + \eta]$ and define

$$\delta_0 = \frac{\sup\{\eta\colon |f_u(u,t) - f_u(W^\pm(t),t)| \le \nu^\pm/\,4, \text{ for } t \in [0,T], u \in I^\pm_\eta(t)\}}{2\|a^+(\cdot)\|_{C^0([0,T])} + 2\|a^-(\cdot)\|_{C^0([0,T])}}$$

and let $\zeta(\xi)$ be a smooth function such that $0 \le \zeta(\xi) \le 1$, $\zeta(\xi) = 0$ for $\xi \le -2$, and $\zeta(\xi) = 1$ for $\xi \ge 2$. Let $a(\xi,t) = e^{\nu t}a^+(t)\zeta(\xi) + a^-(t)(1-\zeta(\xi))$. Define

$$\xi_0 = \inf\big\{\hat{\xi} \ge 2\colon |d(J*\zeta-\zeta)(\pm\xi)(a^+(t)e^{\nu t}-a^-(t))| \le \frac{\nu^\pm}{4}\min\{a^+(t)e^{\nu t}, a^-(t)\}, \\ \text{and } |U(\pm\xi,t) - W^\pm(t)| < \delta_0/2, \forall\ \xi \ge \hat{\xi},\ t \in [0,T]\big\}.$$

This ξ_0 is well defined since $\lim_{\xi\to\infty} U(\pm\xi,t) = W^\pm(t)$ uniformly in t and $\lim_{x\to\infty}(J*\zeta-\zeta)(\pm x) = 0$.

For each $\delta \in (0,\delta_0/2]$, define $U_\delta(\xi,t) = U(\xi,t)+\delta a(\xi,t)$. Then, on $(\xi_0,+\infty)$,

$$\begin{aligned}
L^cU_\delta(\xi,t) &:= U_{\delta t} - cU_{\delta\xi} - DU_{\delta\xi\xi} - d(J*U_\delta - U_\delta) - f(U_\delta,t) \\
&= f(U,t) - f(U+\delta a^+(t),t) + [\nu^+/2 + f_u(W^+(t),t)+\nu]\delta a^+(t)e^{\nu t} \\
&\quad -\delta d[a^+(t)e^{\nu t} - a^-(t)](J*\zeta-\zeta) \\
&= \delta a^+e^{\nu t}[\nu^+/2+\nu+f_u(W^+(t),t) - \int_0^1 f_u(U+\delta\theta a^+(t),t)d\theta] \\
&\quad -\delta d[a^+(t)e^{\nu t} - a^-(t)](J*\zeta-\zeta) \\
&\ge (\nu^+/4+\nu)\delta a^+(t)e^{\nu t} - d\delta[a^+(t)e^{\nu t} - a^-(t)](J*\zeta-\zeta) \\
&\ge 0. \qquad (2.5)
\end{aligned}$$

where we have used the fact that $a(\xi,t) = a^+(t)e^{\nu t}$ on $(\xi_0,+\infty)$ and the definitions of ξ_0 and δ_0. Similarly, we have $L^cU_\delta(\xi,t) \ge 0$, on $(-\infty,-\xi_0)$. That is, $U_\delta(\xi,t)$ is a super solution on $((-\infty,-\xi_0)\bigcup(\xi_0,+\infty))\times\mathbb{R}^+$.

3. Since $\lim_{\xi\to\infty}\overline{U}(\pm\xi,t) = W^\pm(t)$ uniformly in t, by (2.3), there exists a large constant $\hat{z}_0$ such that

$$\overline{U}(\xi - z + (c-\overline{c})t,t) \le \begin{cases} U(\xi,t), & \text{if } \xi\in[-\xi_0,\xi_0]; \\ U(\xi,t)+\delta_0, & \text{if } \xi\notin[-\xi_0,\xi_0]; \end{cases}$$

for all $t\in[0,T]$ and $z \ge \hat{z}_0$. Define $\hat{\delta} := \inf\{\delta > 0 : \overline{U}(\xi - z, 0) \le U(\xi,0) + \delta, \text{ for all } z \ge \hat{z}_0, \xi\in\mathbb{R}\}$. Obviously, $\hat{\delta} \le \delta_0$. We claim that $\hat{\delta} = 0$. In fact, for $z > z_0$, $L^c\overline{U}(\xi - z + (c-\overline{c})t,t) = 0$. And on $[-\xi_0,\xi_0]\times(0,T]$,

$$U_{\hat{\delta}}(\xi,t) = U(\xi,t)+\hat{\delta}a(\xi,t) \ge U(\xi,t) \ge \overline{U}(\xi - z + (c-\overline{c})t,t), \qquad (2.6)$$

and

$$U_{\hat{\delta}}(\xi,0) = U(\xi,0)+\hat{\delta}a(\xi,0) = U(\xi,0)+\hat{\delta} \ge \overline{U}(\xi - z, 0) \qquad (2.7)$$

for all $\xi\in\mathbb{R}$. By Lemma 2.1, we have

$$\overline{U}(\xi - z + (c-\overline{c})t,t) \le U_{\hat{\delta}}(\xi,t)$$

for all $z \ge \hat{z}_0$, $(\xi,t)\in\mathbb{R}\times(0,T]$. Since $z \ge \hat{z}_0$ is arbitrary, we have

$$\overline{U}(\xi - z, T) \le U_{\hat{\delta}}(\xi,T)$$

for all $z \geq \hat{z}_0$. By the periodicity of $U(\xi, \cdot)$, we have

$$\overline{U}(\xi - z, T) \leq U(\xi, 0) + \hat{\delta} a(\xi, T)$$

for all $z \geq \hat{z}_0$, $\xi \in \mathbb{R}$. Therefore,

$$\overline{U}(\xi - z, 0) \leq U(\xi, 0) + \hat{\delta} \max\{a^+(T)e^{\nu T}, a^-(T)\}$$

for all $z \geq \hat{z}_0$, and $\xi \in \mathbb{R}$. This contradicts the definition of $\hat{\delta}$ since $a^{\pm}(T) < 1$. Therefore,

$$\overline{U}(\xi - z, 0) \leq U(\xi, 0)$$

for all $\xi \in \mathbb{R}$ and $z \geq \hat{z}_0$.

4. By the comparison principle (Lemma 2.1), $\overline{U}(\xi - z + (c - \overline{c})t, t) \leq U(\xi, t)$, for all $\xi \in \mathbb{R}$, $t \geq 0$, and $z \geq \hat{z}_0$. Therefore by periodicity, $\overline{U}((c - \overline{c})kT - z, 0) \leq U(0, kT) = \alpha^0$. Letting $k \to \infty$, we deduce that $c = \overline{c}$ since $\overline{U}((c - \overline{c})kT - z, 0) \to \alpha^+$ if $c > \overline{c}$.

5. Define $z_0 = \inf\{\hat{z}_0 \ : \ \overline{U}(\xi - z, 0) \leq U(\xi, 0), \text{ for all } \xi \in \mathbb{R}, z \geq \hat{z}_0\}$. Similar to the proof in step 3, we can show that $\overline{U}(\xi - z_0, 0) = U(\xi, 0)$ for $\xi \in \mathbb{R}$.

6. We prove that $z_0 = 0$. If not, $\overline{U}(\xi - z, 0) < U(\xi, 0)$ for all $\xi \in \mathbb{R}, z > z_0$. By the comparison principle and periodicity,

$$U(\xi - z + z_0, 0) = \overline{U}(\xi - z_0 - z + z_0, 0) = \overline{U}(\xi - z) < U(\xi, 0),$$

since $\overline{U}(\xi - z_0, 0) = U(\xi, 0)$. Therefore $U(\xi, 0)$ is strictly increasing. Since $U(z_0, 0) = \overline{U}(0, 0) = \alpha_0 = U(0, 0)$, we deduce that $z_0 = 0$. That completes the proof. □

Corollary 2.3. *Under the conditions of Theorem 2.2, any smooth solution to (1.6)-(1.8) is strictly increasing.*

3 Existence of Periodic Traveling Waves

In this section, we are going to establish the existence of the periodic traveling wave solution to (1.6)-(1.8) by a homotopy argument.

Assume (U_0, c_0) is the unique solution of the following problem, corresponding to the parameter $\theta = \theta_0 \leq 1$,

$$U_t - cU_\xi - [1 - \theta(1 - D)]U_{\xi\xi} - \theta d(J * U - U) - f(U, t) = 0, \quad (3.1)$$

$$U(\pm\infty, t) = \lim_{\xi \to \pm\infty} U(\xi, t) = w(\alpha^{\pm}, t), \text{ uniformly in } t \in \mathbb{R}, \quad (3.2)$$

$$U(\cdot, T) = U(\cdot, 0), \ U(0, 0) = \alpha^0, \quad (3.3)$$

satisfying $U_{0\xi} > 0$, $U_{0\xi}(\xi, t) \to 0$ uniformly in t as $\xi \to \pm\infty$.

Let

$$X_0 = \{v \ : \ v \in C_{unif}(\mathbb{R} \times \mathbb{R}), v(\cdot, t + T) = v(\cdot, t) \text{ and } \lim_{x \to \infty} v(\pm x, t) = 0, \forall\, t \in \mathbb{R}\}.$$

and $L = L(U_0, c_0, \theta_0)$ be the linearization of the operator in (3.1)-(3.3) defined by

$$D(L) = X_2 := \{v \in X_0 \ : \ v_{\xi\xi}, v_\xi, v_t \in X_0\},$$

$$Lv = v_t - [1 - \theta_0(1 - D)]v_{\xi\xi} - \theta_0 d(J * v - v) - c_0 v_\xi - f_u(U_0, t)v \tag{3.4}$$

for $v \in D(L)$. We first establish some lemmas.

Lemma 3.1. *L has 0 as a simple eigenvalue.*

Proof. Clearly, $p = U_{0\xi}$ is an eigenfunction corresponding to the eigenvalue 0. We only need to prove the simplicity. Suppose $\phi(\xi, t) \in X_0$ is another eigenfunction with eigenvalue 0. We prove that $\phi = zp$, for some constant $z \in \mathbb{R}$.

Let $\nu^\pm$ be defined as in Section 2. Without loss of generality we assume $\nu^+ \geq \nu^-$. Let $\nu = (\nu^+ - \nu^-)/\,2$ be as in Section 2. Suppose $\zeta(\xi)$ is a smooth function such that $\zeta(\xi) \equiv 0$, for $\xi < 0$; $\zeta(\xi) \equiv 1$, for $\xi > 4$; and $0 \leq \zeta(\xi) \leq 1$, $0 \leq \zeta'(\xi) \leq 1$, and $|\zeta''(\xi)| \leq 1$, for all $\xi \in \mathbb{R}$. Define

$$A(\xi, t) = \zeta(\xi)a^+(t)e^{\nu t} + (1 - \zeta(\xi))a^-(t), \tag{3.5}$$

$$B(t) = \int_0^t \max\{a^+(\tau)e^{\nu\tau}, a^-(\tau)\}\, d\tau, \tag{3.6}$$

$$K = \frac{\nu^+ - \nu^-/2 + 1 + (D+d) + 2c_0 + 2\|f_u\|}{\min_{(\xi,t)\in[-\xi_0,\xi_0]\times[0,T]} U_{0\xi}(\xi, t)}, \tag{3.7}$$

where $\|f_u\| = \sup\{|f_u(u, t)| \ : \ u \in [W^-(t), W^+(t)],\ t \in [0, T]\}$ and ξ_0 is a large constant to be chosen later. Let $\Psi(\xi, t) = KB(t)U_{0\xi}(\xi, t) + A(\xi, t)$, then $\Psi(\xi, 0) = 1$. We claim that

$$L\Psi(\xi, t) = KB_t U_{0\xi}(\xi, t) + LA(\xi, t) \geq 0. \tag{3.8}$$

We divide the proof by considering three intervals $(-\infty, -\xi_0)$, $[-\xi_0, \xi_0]$, and (ξ_0, ∞). We assume $\xi_0 > 4$.

On (ξ_0, ∞), $A(\xi, t) = a^+(t)e^{\nu t}$, therefore

$$\begin{aligned} LA(\xi, t) = [\nu^+/\,2 + f_u(W^+(t), t) + \nu - f_u(U_0(\xi, t), t)]a^+(t)e^{\nu t} \\ - \theta_0 d(J * \zeta - \zeta)[a^+(t)e^{\nu t} - a^-(t)]. \end{aligned}$$

Notice that $(J * \zeta - \zeta)(\xi) \to 0$, and $U_0(\xi, t) \to W^+(t)$ as $\xi \to \infty$. We deduce, by (2.4), that we can choose ξ_0 large enough such that

$$LA(\xi, t) \geq 0, \qquad \text{on } \ (\xi_0, \infty) \times \mathbb{R}^+.$$

Similarly we have

$$\begin{aligned} LA(\xi, t) \ &= \ [\nu^-/\,2 + f_u(W^-(t), t) - f_u(U_0(\xi, t), t)]a^-(t) \\ &\quad -\theta_0 d[J * \zeta - \zeta][a^+(t)e^{\nu t} - a^-(t)] \quad \text{on } (-\infty, -\xi_0). \end{aligned}$$

Therefore there exists $\xi_0 >> 1$ such that

$$LA(\xi,t) \geq 0, \qquad \text{on } (-\infty,-\xi_0)\times\mathbb{R}^+.$$

We fix ξ_0 large enough such that $LA(\xi,t)\geq 0$, on $((-\infty,-\xi_0)\bigcup(\xi_0,\infty))\times\mathbb{R}^+$. On $[-\xi_0,\xi_0]$,

$$\begin{aligned}
&|LA(\xi,t)|\\
&= |A_t-[1-\theta_0(1-D)]A_{\xi\xi}-\theta_0 d(J*A-A)-c_0A_\xi-f_u(U_0(\xi,t),t)A(\xi,t)|\\
&\leq \max\{a^+(t)e^{\nu t},a^-(t)\}\{\nu^+-\nu^-/2+[1-\theta_0(1-D)]+\theta_0 d+2c_0+2\|f_u\|\}
\end{aligned}$$

Therefore $L\Psi(\xi,t)\geq 0$, on $[-\xi_0,\xi_0]$ by (3.8) and the choice of K in (3.7).

By the comparison principle, we have

$$\phi(\xi,t)\leq \Psi(\xi,t)\|\phi(\xi,0)\|_\infty.$$

Letting $t=kT$ and letting $k\to\infty$, we have

$$|\phi(\xi,0)|\leq KB(\infty)\|\phi(\xi,0)\|_\infty U_{0\xi}(\xi,0),$$

where $B(\infty)=\lim_{t\to\infty}B(t)$. The limit exists since $a^\pm(t)\to 0$ exponentially and (2.4) holds.

Let $z_* := \sup\{z \;:\; \phi(\xi,0)\geq zU_{0\xi}(\xi,0),$ for all $\xi\in\mathbb{R}\}$. We claim that $\phi(\xi,0)=z_*U_{0\xi}(\xi,0)$, for all $\xi\in\mathbb{R}$. If not, there exists a point ξ_0 such that $\phi(\xi_0,0)>z_*U_{0\xi}(\xi_0,0)$. Then by the comparison principle, $\phi(\xi,T)>z_*U_{0\xi}(\xi,T)$. Replacing ϕ by $\phi-z_*U_{0\xi}$, we can assume $z_*=0$. So, $\phi(\xi,0)>0$, for all $\xi\in\mathbb{R}$. Choose $\overline{\xi}$ such that $KB(\infty)\sup_{|\xi|\geq\overline{\xi}}U_{0\xi}(\xi,0)<1/4$ and choose ϵ such that $\phi(\xi,0)>\epsilon U_{0\xi}(\xi,0)$, on $[-\overline{\xi},\overline{\xi}]$. Then

$$\phi(\xi,0)-\epsilon U_{0\xi}(\xi,0)\geq -\epsilon\sup_{|\xi|\geq\overline{\xi}}U_{0\xi}(\xi,0),$$

and therefore,

$$\phi(\xi,t)-\epsilon U_{0\xi}(\xi,t)\geq -\epsilon\Psi(\xi,t)\sup_{|\xi|\geq\overline{\xi}}U_{0\xi}(\xi,0).$$

Letting $t=kT$ and letting $k\to\infty$, we have

$$\phi(\xi,0)-\epsilon U_{0\xi}(\xi,0)\geq -\frac{1}{4}\epsilon U_{0\xi}(\xi,0),$$

which contradicts the definition of z_*, and completes the proof. $\square$

Since $J*u-u$ is a bounded operator on X_0, we know that 0 is an isolated eigenvalue of L for $\theta_0<1$. Now consider the adjoint operator $L^*=L^*(U_0,c_0,\theta_0)$ of L. Since the comparison principle holds for L, we know that 0 is an isolated eigenvalue for L^* with a positive eigenfunction (see Section 11.4 and theorem 9.11 in [17]). We denote by $\phi^*(x,t)$ the positive eigenfunction of L^* corresponding to the eigenvalue 0.

Lemma 3.2. *With θ_0, U_0, and c_0 as above with $\theta_0 < 1$, there exists $\eta > 0$ such that for each $\theta \in [\theta_0, \theta_0 + \eta)$, (3.1)- (3.3) has a solution $(U(\theta, \xi, t), c(\theta))$.*

Proof. Consider the operator $G : (X_2 \times \mathbb{R}) \times \mathbb{R} \to X_0 \times \mathbb{R}$ defined by

$$\begin{aligned} G(w,\theta) =& ((U_0+v)_t - [1-\theta(1-D)](U_0+v)_{\xi\xi} - \theta d(J*(U_0+v) - (U_0+v)) \\ & - (c_0+c)(U_0+v)_\xi - f(U_0+v,t), v(0,0)) \end{aligned}$$

for $w = (v, c) \in X_2 \times \mathbb{R}$. Then G is of class C^1, $G(0, \theta_0) = (0, 0)$ and

$$\frac{\partial G}{\partial w}(0,\theta_0) = \begin{bmatrix} L & U_{0\xi} \\ \delta & 0 \end{bmatrix}.$$

where δ is the δ-function. We show that $\frac{\partial G}{\partial w}(0, \theta_0)$ is invertible. Consider the equation on $X_0 \times \mathbb{R}$:

$$\frac{\partial G}{\partial w}(0,\theta_0) \begin{bmatrix} v \\ c \end{bmatrix} = \begin{bmatrix} h \\ b \end{bmatrix}, \qquad \text{for } \begin{bmatrix} h \\ b \end{bmatrix} \in X_0 \times \mathbb{R},$$

i.e.,

$$Lv + cU_{0\xi} = h, \tag{3.9}$$
$$v(0,0) = b. \tag{3.10}$$

By the Fredholm Alternative, (3.9) is solvable if and only if $h - cU_{0\xi} \perp \phi^*$, i.e.,

$$\int_0^T \int_{\mathbb{R}} [h\phi^* - cU_{0\xi}\phi^*]\, dxdt = 0. \tag{3.11}$$

Since $U_{0\xi} > 0$ and $\phi^* > 0$, c is uniquely determined by (3.11). After we determine c, the solution v of (3.9) is determined up to a term $kU_{0\xi}$, where k is a constant. Then (3.10) determines k uniquely. Therefore $\frac{\partial G}{\partial w}(0, \theta_0)$ is invertible. The lemma now follows from the Implicit Function Theorem. □

Lemma 3.3. *Suppose that for $\theta \in [0, \overline{\theta})$, where $\overline{\theta} \le 1$, there exists a solution $(U(\theta, \xi, t), c(\theta))$ of (3.1)- (3.3). Then $\|U(\theta,\cdot,\cdot)\|_{L^\infty(\mathbb{R}\times[0,T])}$, $\|U_\xi(\theta,\cdot,\cdot)\|_{L^\infty(\mathbb{R}\times[0,T])}$ and $\|U_t(\theta,\cdot,\cdot)\|_{L^\infty(\mathbb{R}\times[0,T])}$ are uniformly bounded for $\theta \in [0, \overline{\theta})$.*

Proof. For the case $\overline{\theta} < 1$, the conclusion of the lemma follows from classical parabolic estimates. Therefore we take $\overline{\theta} = 1$, and prove the lemma for θ near 1. We only prove the uniform boundedness of $U_\xi(\theta, \xi, t)$; all others are similar. Let $v(\theta, \xi, t) := U_\xi(\theta, \xi, t)$ and $M = \sup_{\xi,t\in\mathbb{R}} |J' * U(\xi, t)|$. Then $v(\theta, \xi, t)$ satisfies

$$v_t - [1-\theta(1-D)]v_{\xi\xi} + \theta dv - c(\theta)v_\xi - f_u(U(\theta,\xi,t),t)v = \theta dJ' * U.$$

Define $l(\theta) := \theta d - \sup\{f_u(u,t) \;:\; u \in [W^-(t), W^+(t)],\, t \in [0,T]\}$. For $\theta \in [0, 1)$ such that $l(\theta) > 0$, we have, by the comparison principle for parabolic equations,

$$v(\theta,\xi,t) \le e^{-l(\theta)t} \sup_{\xi\in\mathbb{R}} |v(\theta,\xi,0)| + (1 - e^{-l(\theta)t})M/\, l(\theta).$$

By periodicity, we deduce that $v(\theta, \xi, t)$ is uniformly bounded for $\theta \in [0, 1)$ with $l(\theta) > 0$. □

Lemma 3.4. *Suppose that there is a sequence θ_j such that $\lim_{\theta_j\to\overline{\theta}} U(\theta_j,\xi,t) = U(\overline{\theta},\xi,t)$ uniformly with respect to $(\xi,t)\in\mathbb{R}\times[0,T]$ for some function $U(\overline{\theta},\xi,t)$. Then $\{c(\theta_j)\}$ is bounded.*

Proof. First we prove the following statement. Suppose $(\overline{V},\overline{C})$ satisfies, for some $\overline{\xi}>0$,

$$\begin{aligned}
&\overline{V}_t-[1-\theta(1-D)]\overline{V}_{\xi\xi}-\theta d(J*\overline{V}-\overline{V})-\overline{C}\,\overline{V}_\xi-f(\overline{V},t)\le 0,\\
&\qquad\text{in } (-\infty,\overline{\xi})\times(0,T], &(3.12)\\
&\overline{V}(-\infty,t)<W^-(t),\ \text{for } t\in[0,T],\quad \overline{V}(\xi,0)\le\overline{V}(\xi,T),\ \text{on } (-\infty,\overline{\xi}), &(3.13)\\
&\overline{V}(0,0)\ge\alpha^0,\quad \overline{V}(\xi,t)<U(\theta,\xi,t),\quad \text{in } [\overline{\xi},\infty)\times[0,T], &(3.14)
\end{aligned}$$

and $\overline{V}(\xi,0)$ is monotonically increasing. Then $c(\theta)\le\overline{C}$.

In fact, if $c(\theta)>\overline{C}$, then $U(\theta,\xi,t)$ satisfies

$$\begin{aligned}
L^{\overline{C}}U(\theta,\xi,t) &:= U_t-(1-\theta)U_{\xi\xi}-\theta(J*U-U)-\overline{C}U_\xi-f(U,t)\\
&= (c(\theta)-\overline{C})U_\xi>0.
\end{aligned}$$

Let $m_0=\inf\{m\ :\ U(\theta,\xi,0)>\overline{V}(\xi-m,0),\ \text{for } \xi\in\mathbb{R}\}$. Then by assumption, m_0 is well defined and $m_0\ge 0$. Moreover, there exists a point $\xi_0\in(-\infty,\overline{\xi})$ such that $U(\theta,\xi_0,0)=\overline{V}(\xi_0-m_0,0)$. Applying the strong comparison principle on $(-\infty,\overline{\xi})\times[0,T]$, we get $U(\theta,\xi,t)>\overline{V}(\xi-m_0,t)$, for all $\xi\in\mathbb{R}$, $t\in[0,T]$. This is a contradiction since $U(\theta,\xi_0,T)=U(\theta,\xi_0,0)=\overline{V}(\xi_0-m_0,0)\le\overline{V}(\xi_0-m_0,T)$, and the claim is proved.

We denote θ_j by θ. Let $\zeta(s)=[1+\tanh(s/2)]/2$, $W_1(t)=w(\alpha^+-\epsilon,t)$ and $W_2(t)=w(\alpha^--\epsilon,t)$, where ϵ is a small constant to be chosen. Let $\overline{V}(\xi,t)=W_1(t)\zeta(\xi+\xi_0)+W_2(t)(1-\zeta(\xi+\xi_0))$, where ξ_0 is a constant such that $\zeta(\xi_0)=\frac{\alpha^0-\alpha^-+\epsilon}{\alpha^+-\alpha^-}$. Since $W_i(T)>W_i(0)$, for $i=1,2$, we have $\overline{V}(\cdot,T)\ge\overline{V}(\cdot,0)$. Moreover, $\overline{V}_\xi>0$, $\overline{V}(\infty,0)=\alpha^+-\epsilon$, and $\overline{V}(-\infty,0)=\alpha^--\epsilon$. Since $\lim_{\theta\to\overline{\theta}}U(\theta,\xi,t)=U(\overline{\theta},\xi,t)$ uniformly and $U(\theta,+\infty,t)=W^+(t)$, we can choose $\overline{\xi}$ sufficiently large such that $U(\theta,\xi,t)>\overline{V}(\xi,t)$, for $(\xi,t)\in[\overline{\xi},\infty)\times[0,T]$. For $\xi<\overline{\xi}$,

$$\begin{aligned}
L^{\overline{C}}(\overline{V}) =&\overline{V}_t-[1-\theta(1-D)]\overline{V}_{\xi\xi}-\theta d(J*\overline{V}-\overline{V})-f(\overline{V},t)-\overline{C}V_\xi\\
=&-\zeta(1-\zeta)(W_1-W_2)[\overline{C}+(1-\theta(1-D))(1-2\zeta)]\\
&-\theta d(W_1-W_2)(J*\zeta-\zeta)+[\zeta f(W_1,t)+(1-\zeta)f(W_2,t)\\
&-f(W_1\zeta+W_2(1-\zeta),t)]\\
=&-\zeta(1-\zeta)(W_1-W_2)[\overline{C}+(1-\theta(1-D))(1-2\zeta)\\
&-(W_1-W_2)f_{uu}(\sigma,t)/2-\theta d/(1-\zeta)]-\theta d(W_1-W_2)(J*\zeta-\zeta),
\end{aligned}$$

where we use the Taylor's expansion

$$\zeta f(W_1,t)+(1-\zeta)f(W_2,t)-f(W_1\zeta+W_2(1-\zeta),t)=\zeta(1-\zeta)(W_1-W_2)^2f_{uu}(\sigma,t)$$

for some $\sigma \in [W_2, W_1]$. If we choose $\overline{C} = 1 + D + \frac{1}{2}\sup\{(W^+(t) - W^-(t) + 2)|f_{uu}(u,t)| \ : \ u \in [W^-(t) - 1, W^+(t) + 1], t \in [0,T]\} + \sup_{\xi \le \overline{\xi}} d/\left(1 - \zeta(\xi)\right)$, then $L^{\overline{C}}(\overline{V}) < 0$ for $\xi < \overline{\xi}$.

Therefore $c(\theta) \le \overline{C}$ by our earlier observation. We can get a lower bound estimate similarly. □

We are ready to obtain a solution to (3.1)-(3.3).

Theorem 3.5. *Under the conditions of Theorem 1.1, there exists a solution $(U(\theta,\xi,t), c(\theta))$ to (3.1)-(3.3) for all $\theta \in [0,1]$.*

Proof. By the result in [2], there exists a solution (U_0, c_0) to (3.1)-(3.3) corresponding to $\theta = 0$, such that $U_{0\xi} > 0$ and $\lim_{\xi\to\infty} U_{0\xi} = 0$ uniformly with respect to t. By Lemma 3.2, there exists an interval $[0, \overline{\eta})$ such that for all $\theta \in [0, \overline{\eta})$ system (3.1) - (3.3) has a solution $(U(\theta,\xi,t), c(\theta))$ with the required properties. Suppose $[0,\eta)$ is the maximal interval such that (3.1)-(3.3) admits a solution for each $\theta \in [0,\eta)$. Then we claim that $\eta = 1$ and (3.1)-(3.3) admits a solution for each $\theta \in [0,1]$. By Lemma 3.3 and Helly's theorem, we can choose a subsequence θ_j such that $\lim_{j\to\infty}\theta_j = \eta$, and $\lim_{j\to\infty} U(\theta_j,\xi,t)$ exists uniformly for all $\xi \in \mathbb{R}$ and each rational t. By Lemma 3.3 again, $\|U_t(\theta,\xi,t)\|$ is uniformly bounded for all $\theta \in [0,\eta)$. Therefore there exists a uniformly continuous function $U(\eta,\xi,t)$ such that $\lim_{j\to\infty} U(\theta_j,\xi,t) = U(\eta,\xi,t)$ uniformly for all $(\xi,t) \in \mathbb{R}\times[0,T]$. Moreover, by choosing a subsequence if necessary, the derivatives of $U(\theta_j,\xi,t)$ converge to the corresponding derivatives of $U(\eta,\xi,t)$ uniformly on any compact set of $\mathbb{R}\times[0,T]$. Therefore by Lemma 3.4, we can choose a subsequence of $\{\theta_j\}$ (we label it the same) such that $c(\theta_j) \to c(\eta)$. Therefore $(U(\eta,\xi,t), c(\eta))$ is a solution to (3.1)-(3.3) corresponding to parameter η, with the same properties as (U_0, c_0). Therefore, either $\eta = 1$, or we can extend the existence interval to $[0, \eta+\epsilon)$ for some $\epsilon > 0$, which would contradict the maximality of η. Therefore, for all $\theta \in [0,1]$, (3.1)-(3.3) has a solution. □

4 Stability of the Periodic Traveling Waves

In this section, we study the stability and asymptotic stability of the periodic traveling wave solutions $U(x-ct,t)$ obtained in Section 3.

We denote by u(x,t;g) the solution to the initial value problem

$$u_t - Du_{xx} - d(J*u - u) - f(u,t) = 0, \text{ in } \mathbb{R}\times(0,\infty), \tag{4.1}$$

$$u(x,0) = g(x), \text{ on } \mathbb{R}, \tag{4.2}$$

where $g(\cdot) \in L^\infty(\mathbb{R})$. For the existence and uniqueness of (4.1) and (4.2), we have

Lemma 4.1. *For any $g(\cdot) \in L^\infty(\mathbb{R})$, there exists a unique solution $u(x,t;g) \in C^1([0,\infty), L^\infty(\mathbb{R}))$ of (4.1) and (4.2). Moreover, $u(\cdot,t;g)$ is continuous from $[0,\infty)\times C_{unif}(\mathbb{R})$ to $C_{unif}(\mathbb{R})$.*

Proof. The case $D \neq 0$ follows from standard parabolic theory. We only need to consider the case where $D = 0$. Write (4.1) and (4.2) as

$$u(x,t) = g(x) + \int_0^t (d(J * u - u) + f(u,t))\, dt. \tag{4.3}$$

Then the local existence and uniqueness follow from the contraction mapping theorem in the usual way. Let $M = \sup_{x\in\mathbb{R}} |g(x)|$. Then $w(\pm M, t)$ are super- and sub-solutions of (4.1) respectively. By the comparison principle,

$$w(-M,t) \leq u(x,t;g) \leq w(M,t)$$

for $t > 0$. Global existence follows since $w(\pm M, t)$ are bounded. The continuous dependence can be easily proved using (4.3). □

We claim that the asymptotic behavior of the solutions to (4.1) and (4.2) is governed by the periodic traveling wave solution $U(x - ct, t)$. We have the following result:

Theorem 4.2. *(1) (Uniform Stability) For any $\epsilon > 0$, there is a $\delta > 0$ such that for any $g \in C_{unif}(\mathbb{R})$ with $\|g(\cdot) - U(\cdot, 0)\| < \delta$, one has*

$$\|u(\cdot, t; g) - U(\cdot - ct, t)\| < \epsilon \tag{4.4}$$

for all $t > 0$.

(2). (Asymptotic Stability) For any $g \in C_{unif}(\mathbb{R})$ satisfying

$$\liminf_{x\to\infty} g(x) > W^0(0), \quad \limsup_{x\to-\infty} g(x) < W^0(0), \tag{4.5}$$

where $W^0(t) = w(\alpha^0, t)$ and $w(\alpha^0, t)$ is the solution of (1.2). Then there is $\xi_0 \in \mathbb{R}$ such that

$$\|u(\cdot, t; g) - U(\cdot - ct + \xi_0, t)\| \to 0 \tag{4.6}$$

exponentially as $t \to \infty$.

In order to prove the theorem we need the following lemmas. The first lemma use the monotonicity of $U(\cdot, t)$ to construct super- and sub- solutions.

Lemma 4.3. *There exist $\beta_1 > 0, \delta_1 > 0$ and $\sigma_1 > 0$ such that, for any $\delta \in (0, \delta_1)$, $\tau \in \mathbb{R}^+$ and $\xi_0 \in \mathbb{R}$, $v^\pm(x,t)$ are super- and sub- solutions of (4.1), respectively, on $[\tau, \infty]$, where*

$$v^\pm(x,t) = U(x - c(t-\tau) + \xi_0 \pm \sigma_1\delta(1 - e^{-\beta_1(t-\tau)}), t) \pm \delta e^{-\beta_1(t-\tau)} \tag{4.7}$$

for $x \in \mathbb{R}$ and $t \in [\tau, \infty)$.

Proof. The proof of the lemma is similar to that of Lemma 2.2 in [9]. We omit it. □

The next lemma is an analog of the strong comparison principle of parabolic equations. This is the key lemma to apply the "squeezing" technique employed in [9] to prove the stability.

Lemma 4.4. *There is a positive function $\eta(\cdot,t)$ satisfying $0 \le \eta(\cdot,t) \le 1$ for $t \in [0,T]$ such that $\eta(\cdot,t)$ is non-increasing and for any super-solution $u_1(x,t)$ and sub-solution $u_2(x,t)$ of (4.1) on $\mathbb{R}^+$ satisfying $u_1(x,\tau) \ge u_2(x,\tau)$ for all $x \in \mathbb{R}$ and for some $\tau \in \mathbb{R}$, and $|u_i(x,t)| \le K_0 = \sup_{t\in\mathbb{R}}\{|W^-(t)|+1, |W^+(t)|+1\}$ for all $x \in \mathbb{R}$ and $t \ge \tau$, the following holds*

$$u_1(x,t) - u_2(x,t) \ge \eta(M, t-\tau)\int_z^{z+1}[u_1(y,\tau) - u_2(y,\tau)]dy \tag{4.8}$$

for all $x \in \mathbb{R}$ with $|x-z| \le M$ and $t \ge \tau$.

For the proof of this lemma, we refer the reader to a similar result in [9].

To prove the stability, we first need to show that, for given initial data as in (4.5), the solution with this initial data first forms a vague front of periodic traveling waves as the system evolves. In order to prove that, we need to construct various super- and sub- solutions.

Lemma 4.5. *Let $\zeta(s) = \frac{1}{2}(1 + \tanh\frac{s}{2})$. For any given $T_0 > 0$ and $m_\pm \in \mathbb{R}$ with $m_- < m_+$, there exist positive constants K, C, ϵ_0, and a positive function $\rho(\cdot)$ satisfying $\lim\limits_{\epsilon\to 0}\rho(\epsilon) = 0$, such that, for all $0 < \epsilon \le \epsilon_0$ and $h \in \mathbb{R}$,*

$$\begin{aligned}(1)\ v_1^\pm(x,t) =& w(m_\pm,t)\zeta(\epsilon(x-h)+Ct) \\ &+ w(m_\mp,t)(1-\zeta(\epsilon(x-h)+Ct) \pm \rho(\epsilon)e^{Kt}\end{aligned} \tag{4.9}$$

are super- and sub-solutions of (4.1) on $[0,T_0]$, respectively, where $w(m_\pm,t)$ are solutions of (1.2) with $w(m_\pm,0) = m_\pm$, and

$$\begin{aligned}(2)\ v_2^\pm(x,t) =& w(m_\mp,t)\zeta(\epsilon(x-h)-Ct) \\ &+ w(m_\pm,t)(1-\zeta(\epsilon(x-h)-Ct) \pm \rho(\epsilon)e^{Kt}\end{aligned} \tag{4.10}$$

are super- and sub-solutions of (4.1) on $[0,T_0]$, respectively.

Proof. We only prove that $v_1^+(x,t)$ is a super-solution. The other claims can be proved similarly. Denote $v(x,t) = w(m_+,t)\zeta(\epsilon(x-h)+Ct) + w(m_-,t)(1-$

$\zeta(\epsilon(x-h)+Ct)$. Then

$$\begin{aligned}
v_{1t}^{+}-&Dv_{1xx}^{+}-d(J*v_1^{+}-v_1^{+})-f(v_1^{+},t)\\
=&f(w(m_+,t),t)\zeta(\epsilon(x-h)+Ct)+f(w(m_-,t),t)(1-\zeta(\epsilon(x-h)+Ct))\\
&-f(v_1^{+},t)+C(w(m_+,t)-w(m_-,t))\zeta(\epsilon(x-h)+Ct)\\
&(1-\zeta(\epsilon(x-h)+Ct))+\rho(\epsilon)Ke^{Kt}\\
&+(w(m_+,t)-w(m_-,t))[D\epsilon^2\zeta(\epsilon(x-h)+Ct)(1-\zeta(\epsilon(x-h)+Ct))\\
&(1-2\zeta(\epsilon(x-h)+Ct))-d(J*\zeta(\epsilon(x-h)+Ct)-\zeta(\epsilon(x-h)+Ct))]\\
=&[f(w(m_+,t),t)\zeta(\epsilon(x-h)+Ct)+f(w(m_-,t),t)(1-\zeta(\epsilon(x-h)+Ct))\\
&-f(v(x,t),t)]+[f(v(x,t),t)-f(v_1^{+}(x,t),t)]+\rho(\epsilon)Ke^{Kt}\\
&+C(w(m_+,t)-w(m_-,t))\zeta(\epsilon(x-h)+Ct)(1-\zeta(\epsilon(x-h)+Ct))\\
&-D\epsilon^2(w(m_+,t)-w(m_-,t))\zeta(\epsilon(x-h)+Ct)(1-\zeta(\epsilon(x-h)+Ct))\\
&(1-2\zeta(\epsilon(x-h)+Ct))-d(w(m_+,t)-w(m_-,t))(J*\zeta(\epsilon(x-h)+Ct)\\
&-\zeta(\epsilon(x-h)+Ct))\\
=&I+II+III+IV+V+VI
\end{aligned}\tag{4.11}$$

By Taylor's expansion,

$$\begin{aligned}
I=&f_{uu}(u^*(x,t),t)(u^{**}(x,t)-w(m_-,t))\\
&(w(m_+,t)-w(m_-,t))\zeta(\epsilon(x-h)+Ct)(1-\zeta(\epsilon(x-h)+Ct)),
\end{aligned}$$

where $u^*(x,t)$, $u^{**}(x,t)$ are between $w(m_-,t)$ and $w(m_+,t)$). Therefore there exists a constant M_1, independent of ϵ such that

$$|I|\le M_1(w(m_+,t)-w(m_-,t))\zeta(\epsilon(x-h)+Ct)(1-\zeta(\epsilon(x-h)+Ct)). \tag{4.12}$$

Let

$$\rho(\epsilon)=\sup\{|VI|\ :\ x,t,h,C\in\mathbb{R}\}. \tag{4.13}$$

Since $w(m_\pm,t)$ is bounded and $\zeta(\cdot)$ is uniformly continuous, it is easy to see that $\lim_{\epsilon\to 0}\rho(\epsilon)=0$. For II , let $M_2=\sup\{|f_u(u,t)|\ :\ u\in[w(m_-,t)-1,w(m_+,t)+1],\,t\in\mathbb{R}\}$, and $K=1+M_2$. Choose ϵ_0 such that $\rho(\epsilon_0)e^{KT_0}\le 1$. Then, for $0<\epsilon\le\epsilon_0$,

$$|II|\le\int_0^1|f_u(v+\rho(\epsilon)\theta e^{Kt},t)|\,d\theta\rho(\epsilon)e^{Kt}\le M_2\rho(\epsilon)e^{Kt}. \tag{4.14}$$

Now choose $C=M_1+D$. Then, by (4.11) - (4.14),

$$v_{1t}^{+}\ -\ Dv_{1xx}^{+}-d(J*v_1^{+}-v_1^{+})-f(v_1^{+},t)\ge 0.$$

□

Lemma 4.6. *Suppose that $g \in C_{unif}(\mathbb{R})$ satisfies*

$$\liminf_{x\to\infty} g(x) > W^0(0) \ \textit{and}\ \limsup_{x\to-\infty} g(x) < W^0(0). \tag{4.15}$$

Then, for any $\delta > 0$, there are constants $H > 0$ and $T_0 > 0$ such that

$$U(x-H,T_0) - \delta \le u(x,T_0;g) \le U(x+H,T_0) + \delta. \tag{4.16}$$

Proof. Without loss of generality, we assume, for some $0 < \delta_0 < 1$, that

$$W^-(0) - \delta_0 \le g(x) \le W^+(0) + \delta_0.$$

By assumption (H2), for $\delta << 1$, there is a $T_0 > 0$ such that

$$W^+(T_0) - \delta/4 < w(m_+,T_0) < W^+(T_0) + \delta/4 \tag{4.17}$$

for $m_+ = W^0(0) + \delta_0$ or $m_+ = W^+(0) + 2\delta_0$, and that

$$W^-(T_0) - \delta/4 < w(m_-,T_0) < W^-(T_0) + \delta/4 \tag{4.18}$$

for $m_- = W^0(0) - \delta_0$ or $m_- = W^-(0) - 2\delta_0$, where $w(m_\pm,t)$ are as in Lemma 4.6.

For T_0 fixed as above, by Lemma 4.5, there are $K > 0$, $C > 0$ and $\epsilon_0 > 0$ such that, for all $0 < \epsilon \le \epsilon_0$,

$$\begin{aligned} v^+(x,t) =& w(W^+(0)+2\delta_0,t)\zeta(\epsilon(x+h)+Ct) \\ &+ w(W^0(0)-\delta_0,t)(1-\zeta(\epsilon(x+h)+Ct)) + \rho(\epsilon)e^{Kt} \end{aligned}$$

and

$$\begin{aligned} v^-(x,t) =& w(W^0(0)+\delta_0,t)\zeta(\epsilon(x-h)-Ct) \\ &+ w(W^-(0)-2\delta_0,t)(1-\zeta(\epsilon(x-h)-Ct)) - \rho(\epsilon)e^{Kt} \end{aligned}$$

are super- and sub-solutions on $\mathbb{R} \times [0,T_0]$, respectively.

Fix $\epsilon < \epsilon_0$ small enough such that

$$\rho(\epsilon)e^{KT_0} < \delta/4. \tag{4.19}$$

By (4.15), there is an h large enough such that

$$v^-(x,0) \le g(x) \le v^+(x,0). \tag{4.20}$$

Hence, by the comparison principle,

$$v^-(x,t) \le u(x,t;g) \le v^+(x,t) \tag{4.21}$$

for all $x \in \mathbb{R}$ and $t \in [0,T_0]$. Since $\lim_{x\to\infty} U(x - CT_0, T_0) = W^\pm(T_0)$, by (4.16)-(4.21), there exists H large enough such that

$$U(x-H,T_0) - \delta \le v^-(x,T_0) \le u(x,T_0;g) \le v^+(x,T_0) \le U(x+H,T_0) + \delta \tag{4.22}$$

for all $x \in \mathbb{R}$. This completes the proof. □

The following lemma is the "squeezing technique" employed in [9].

Lemma 4.7. *There exists $\epsilon^* > 0$ such that if $u(x,t)$ is a solution of (4.1), and if for some $\tau \in \mathbb{R}^+$, $\xi \in \mathbb{R}$, $\delta \in (0, \frac{\delta_1}{2})$, and $h > 0$, one has*

$$U(x - c\tau + \xi, \tau) - \delta \le u(x, \tau) \le U(x - c\tau + \xi + h, \tau) + \delta \tag{4.23}$$

for all $x \in \mathbb{R}$, then for every $t \ge \tau + 1$, there exist $\hat{\xi}(t)$, $\hat{\delta}(t) \ge 0$ and $\hat{h}(t) \ge 0$ satisfying

$$\begin{aligned} \hat{\xi}(t) \quad &\in \quad [\xi - \sigma_1\delta, \xi + h + \sigma_1\delta], && (4.24)\\ 0 \le \hat{\delta}(t) \quad &\le \quad e^{-\beta_1(t-\tau-1)}[\delta + \epsilon^* \min\{h, 1\}], && (4.25)\\ 0 \le \hat{h}(t) \quad &\le \quad [h - \sigma_1\epsilon^* \min\{h, 1\}] + 2\sigma_1\delta, && (4.26) \end{aligned}$$

such that (4.23) holds with τ, ξ, δ and h being replaced by t ($\ge \tau + 1$), $\hat{\xi}(t)$, $\hat{\delta}(t)$ and $\hat{h}(t)$ respectively, where β_1, δ_1, and σ_1 are as in Lemma 4.3.

Proof. The proof is similar to that of Lemma 3.3 in [9]. In the proof of that lemma, the only properties used are given by Lemma 4.3 and Lemma 4.4. For details, see [9] and [23]. □

Proof of Theorem 4.2. (1). Let $\epsilon > 0$ be given. Since $U(\cdot, \cdot)$ is uniformly continuous on $\mathbb{R} \times [0, T]$, there is a constant $k_0 > 0$ such that

$$|U(x + k, t) - U(x, t)| < \epsilon/2, \tag{4.27}$$

for all $x \in \mathbb{R}$, $t \in [0, T]$ and all k with $|k| \le k_0$.

Let β_1, δ_1 and σ_1 be given as in Lemma 4.3. Choose $\delta > 0$ such that $\delta < \min\{\delta_1, \epsilon/2, k_0/\sigma_1\}$. Then for any $g \in C_{unif}(\mathbb{R})$ satisfying $\|g(\cdot) - U(\cdot, 0)\| < \delta$, by Lemma 4.3, we have

$$\begin{aligned} U(x - ct - \sigma_1\delta(1 - e^{-\beta_1 t}), t) - \delta e^{-\beta_1 t} \le & u(x, t; g)\\ \le & U(x - ct + \sigma_1\delta(1 - e^{-\beta_1 t}), t) + \delta e^{-\beta_1 t} \end{aligned}$$

for $x \in \mathbb{R}$ and $t \in [0, \infty)$. By (4.27) and the choice of δ, we have

$$\|u(\cdot, t; g) - U(\cdot - ct, t)\| < \epsilon,$$

for all $t > 0$.

(2). Let ϵ^* be given as in Lemma 4.7 and β_1, δ_1 and σ_1 be given as in Lemma 4.3. Let $\overline{\delta} = \min\{\delta_1/2, \epsilon^*/4\}$, and $\overline{\gamma} = \sigma_1\epsilon^* - 2\sigma_1\overline{\delta}$. Let t_0 be chosen such that $e^{-\beta_1(t_0-1)}(\overline{\delta} + \epsilon^*) \le (1 - \overline{\gamma})\overline{\delta}$. By Lemma 4.6, there are $\xi_0 \in \mathbb{R}$, $h > 0$ and $T_0 > 0$ such that

$$U(x - cT_0 + \xi_0, T_0) - \overline{\delta} \le u(x, T_0; g) \le U(x - cT_0 + \xi_0 + h, T_0) + \overline{\delta} \tag{4.28}$$

for all $x \in \mathbb{R}$. First, we may assume $0 < h \leq 1$. In fact, if $h > 1$, we can choose integer $N > 0$ such that $0 \leq h - N\overline{\gamma} \leq 1$. Applying Lemma 4.7 repeatedly, we conclude that

$$\begin{aligned} U(x - c(kt_0 + T_0) + \xi_k, kt_0 + T_0) - \overline{\delta}_k \leq u(x, kt_0 + T_0; g) \\ \leq U(x - c(kt_0 + T_0) + \xi_k + h_k, kt_0 + T_0) + \overline{\delta}_k \end{aligned} \tag{4.29}$$

for all $x \in \mathbb{R}$, where $\xi_k \in [\xi_{k-1} - \sigma_{k-1}\overline{\delta}_{k-1}, \xi_{k-1} + \sigma_{k-1}\overline{\delta}_{k-1} + h_{k-1}]$, $\overline{\delta}_k \leq (1-\overline{\gamma})^k\overline{\delta}$, $h_k \leq h_{k-1} - \overline{\gamma}$, and $\delta_0 = \overline{\delta}$. Therefore (4.28) holds with ξ_0, $\overline{\delta}$, h, and T_0 being replaced by ξ_N, $\overline{\delta}_N$, h_N, and $T_N = Nt_0 + T_0$, respectively.

Now we assume $h \leq 1$ and (4.28) holds. Define $T_k = kt_0$, $\overline{\delta}_k = (1-\overline{\gamma})^k\overline{\delta}$, and $h_k = h_{k-1} - \overline{\gamma}$. Then we can show by induction that (4.29) still holds. Define $\delta(t) = \overline{\delta}_k$, $\xi(t) = \xi_k - \sigma_1\overline{\delta}_k$, and $h(t) = h_k + 2\sigma_1\overline{\delta}_k$, for $t \in [T_k + T_0, T_{k+1} + T_0]$ and $k = 0, 1, \ldots$. Then, by Lemma 4.3,

$$U(x - ct + \xi(t), t) - \delta(t) \leq u(x, t; g) \leq U(x - ct + \xi(t) + h(t), t) + \delta(t) \tag{4.30}$$

for $x \in \mathbb{R}$ and $t \geq T_0$. Note that $\delta(t) \to 0$, $h(t) \to 0$ and $\xi(t) \to \xi(\infty)$ exponentially as $t \to \infty$. Therefore,

$$u(x, t; g) \to U(x - ct + \xi(\infty), t)$$

exponentially as $t \to \infty$.

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