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CORRECTORS FOR FLOW IN A PARTIALLY FISSURED MEDIUM

M. RAJESH

Abstract. We prove a corrector result for the homogenization of flow in a partially fissured medium. The homogenization problem was studied by Clark and Showalter [3] using the two-scale convergence technique.

1. Introduction

A fissured medium consists of a porous and permeable *matrix* interlaced, on a fine scale, by a system of highly permeable *fissures*. Fluid flow in such a medium takes place, primarily, through the fissures. The fissured medium is said to be *totally fissured* if the matrix is broken up into disjoint cells by the fissures. In this case, there is no direct flow through the matrix but only an exchange of fluids between the cells and the surrounding fissures. If, on the other hand, the matrix is connected there is a global flow through the matrix as well. This is the *partially fissured case*.

As remarked by Clark and Showalter [3], an exact microscopic model for flow in a fissured medium, written as a classical interface problem, is both analytically and numerically intractable. One way to get around this difficulty is to model the flow on two separate scales, one microscopic and the other macroscopic. The problem, then, can be studied as a problem in homogenization. Such a model for flow in a partially fissured medium was considered by Douglas, Peszyńska and Showalter [6] assuming the diffusion operator to be linear. Clark and Showalter [3] extend the results of [6] to the case where the diffusion operator is quasilinear. The corresponding homogenization problem was solved under weak monotonicity conditions and using the two-scale convergence method.

Correctors for the homogenization of quasilinear equations

$$-\operatorname{div}\left(a\left(\frac{x}{\varepsilon},\nabla u_\varepsilon\right)\right)=f \tag{1.1}$$

were obtained by Dal Maso and Defranceschi [5] under some strong monotonicity conditions on the function a. Later, the proof of the corrector result was greatly simplified using the two-scale convergence method by Allaire [1].

Based on these ideas we prove a corrector result for the flow in a partially fissured medium under strong monotonicity conditions on the diffusion operator.

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The plan of the paper is as follows. In Section 2, we describe the micro-model for flow in a partially fissured medium. In Section 3, we recall the homogenization results obtained by Clark and Showalter in [3] under weak monotonicity of the diffusion operator. In Section 4, we present our results on correctors. Strong monotonicity conditions are required here.

2. The micro-model

We present, here, the micro-model for flow in a partially fissured medium as described in [3].

Let Ω be a bounded open set in R^N. $Y = [0,1]^N$ denotes the unit cube and $Y = Y_1 \bigcup Y_2$, where Y_1 and Y_2 represent the local structure of the fissure and matrix respectively. Let $\chi_j(y)$ denote the characteristic function of Y_j (j=1, 2) extended Y-periodically to all of R^N. We shall assume that the sets $\{y \in R^N : \chi_j(y) = 1\}$ (j=1, 2) are smooth (connectedness will not be required in view of the coercivity conditions to be assumed on the coefficients in the differential operators). The domain Ω is thus divided into the two subdomains, Ω_1^ε and Ω_2^ε, representing the fissures and the matrix respectively, and are given by

$$\Omega_j^\varepsilon = \left\{ x \in \Omega : \chi_j(\frac{x}{\varepsilon}) = 1 \right\}, \; j = 1, 2.$$

Henceforth, we will denote $\chi_j(\frac{x}{\varepsilon})$ by χ_j^ε.
Let $\Gamma_{1,2}^\varepsilon = \partial\Omega_1^\varepsilon \bigcap \partial\Omega_2^\varepsilon \bigcap \Omega$ denote the interface of Ω_1^ε with Ω_2^ε which is interior to Ω. $\Gamma_{1,2} = \partial Y_1 \bigcap \partial Y_2 \bigcap Y$ denotes the corresponding interface in the reference cell Y. We set $\Omega_3^\varepsilon \equiv \Omega_2^\varepsilon, Y_3 \equiv Y_2$, and $\chi_3 \equiv \chi_2$, to be used to simplify notation at times.

Let $\mu_j : R^N \times R^N \longrightarrow R^N$ ($j = 1, 2, 3$) be functions which satisfy the following hypothesis:

1. $\mu_j(\, . \, , \xi)$ is measurable and Y-periodic for all $\xi \in R^N$.
2. $\mu_j(y \, , \, .)$ is continuous for a.e. $y \in Y$.
3. There exist positive constants k, C, c_0 and $1 < p < \infty$ such that for every ξ, $\eta \in R^N$ and a.e. $y \in Y$

$$|\mu_j(y\,,\,\xi)| \le C|\,\xi\,|^{p-1} + k \tag{2.1}$$

$$(\mu_j(y\,,\,\xi) - \mu_j(y\,,\,\eta))\,.\,(\xi - \eta)) \ge 0 \tag{2.2}$$

$$\mu_j(y\,,\,\xi)\,.\,\xi \ge c_0|\,\xi\,|^p - k. \tag{2.3}$$

q will denote the conjugate exponent of p, viz. $q = p/p - 1$. Let $c_j \in C_\sharp(Y)$ ($j = 1, 2, 3$) be continuous Y-periodic functions on R^N such that

$$0 < c_0 \le c_j \le C. \tag{2.4}$$

The flow potential of the fluid in the fissure Ω_1^ε is denoted by the function $u_1^\varepsilon(x,t)$ and the corresponding flux by $-\mu_1\left(\frac{x}{\varepsilon}, \nabla u_1^\varepsilon\right)$. The flow potential in the matrix is represented as the sum of two parts, one component $u_2^\varepsilon(x,t)$ with the flux $-\mu_2\left(\frac{x}{\varepsilon}, \nabla u_2^\varepsilon\right)$ which accounts for the global diffusion through the pore system of the matrix, and the second component $u_3^\varepsilon(x,t)$ with flux $-\varepsilon\mu_3\left(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon\right)$ and corresponding high frequency spatial variations which lead to local storage in the matrix. The "total flow potential" in the matrix is then $\alpha u_2^\varepsilon + \beta u_3^\varepsilon$ (here $\alpha + \beta = 1$ with $\alpha \ge 0, \beta > 0$). The exact microscopic model for diffusion in a partially fissured medium is given by the system

$$c_1(\frac{x}{\varepsilon})\frac{\partial u_1^\varepsilon}{\partial t} - \operatorname{div}\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right) = 0 \quad \text{in } \Omega_1^\varepsilon \tag{2.5}$$

$$c_2(\frac{x}{\varepsilon})\frac{\partial u_2^\varepsilon}{\partial t} - \operatorname{div}\mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u_2^\varepsilon\right) = 0 \quad \text{in } \Omega_2^\varepsilon \tag{2.6}$$

$$c_3(\frac{x}{\varepsilon})\frac{\partial u_3^\varepsilon}{\partial t} - \varepsilon\operatorname{div}\mu_3\left(\frac{x}{\varepsilon}\,,\,\varepsilon\nabla u_3^\varepsilon\right) = 0 \quad \text{in } \Omega_2^\varepsilon \tag{2.7}$$

$$\alpha u_2^\varepsilon + \beta u_3^\varepsilon = u_1^\varepsilon \quad \text{on } \Gamma_{1,2}^\varepsilon \tag{2.8}$$

$$\alpha\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right).\nu_1^\varepsilon = \mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u_2^\varepsilon\right).\nu_1^\varepsilon \tag{2.9}$$

$$\beta\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right).\nu_1^\varepsilon = \varepsilon\mu_3\left(\frac{x}{\varepsilon}\,,\,\nabla\varepsilon u_3^\varepsilon\right).\nu_1^\varepsilon \tag{2.10}$$

where the last two conditions hold on $\Gamma_{1,2}^\varepsilon$. We have the homogeneous Neumann condition on the external boundary

$$\mu_1\left(\frac{x}{\varepsilon}\,,\,\nabla u_1^\varepsilon\right).\nu_1^\varepsilon = 0 \quad \text{on } \partial\Omega_1^\varepsilon\cap\partial\Omega \tag{2.11}$$

$$\mu_2\left(\frac{x}{\varepsilon}\,,\,\nabla u_2^\varepsilon\right).\nu_2^\varepsilon = 0 \quad \text{on } \partial\Omega_2^\varepsilon\cap\partial\Omega \tag{2.12}$$

$$\mu_3\left(\frac{x}{\varepsilon}\,,\,\varepsilon\nabla u_3^\varepsilon\right).\nu_2^\varepsilon = 0 \quad \text{on } \partial\Omega_2^\varepsilon\cap\partial\Omega \tag{2.13}$$

where ν_j^ε denotes the outward normal on $\partial\Omega_j^\varepsilon, j=1,2$.
The system is completed by the initial conditions

$$u_j^\varepsilon(0\,,\,.) = u_j^0 \in L^2(\Omega),\ 1\le j\le 3. \tag{2.14}$$

Remark 2.1: Condition (2.8) is the continuity of flow potential across the interface. Conditions (2.9), (2.10) determine the partition of flux across the interface.□

We now describe the variational formulation needed to study the well posedness of the Cauchy problem. The *state space* is the Hilbert space

$$H_\varepsilon \equiv L^2(\Omega_1^\varepsilon)\times L^2(\Omega_2^\varepsilon)\times L^2(\Omega_2^\varepsilon)\ (= L^2(\Omega_1^\varepsilon)\times L^2(\Omega_2^\varepsilon)^2)$$

equipped with the inner product

$$([u_1,\,u_2,\,u_3],[\phi_1,\,\phi_2,\,\phi_3])_{H_\varepsilon} = \sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})\,u_j(x)\,\phi_j(x)\,dx.$$

Define the *energy space*

$$B_\varepsilon \equiv H_\varepsilon \cap \{[\overrightarrow{u}]\in W^{1,p}(\Omega_1^\varepsilon)\times W^{1,p}(\Omega_2^\varepsilon)^2 : u_1 = \alpha u_2 + \beta u_3 \text{ on } \Gamma_{1,2}^\varepsilon\}$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$. B_ε is a Banach space with the norm

$$\|\,[u_1,\,u_2,\,u_3]\|_{B_\varepsilon} \equiv \sum_{j=1}^{3}\|\,\chi_j^\varepsilon\,u_j\,\|_{L^2(\Omega)} + \sum_{j=1}^{3}\|\,\chi_j^\varepsilon\,\nabla\,u_j\,\|_{L^p(\Omega)}.$$

Define the operator $A_\varepsilon\,:\,B_\varepsilon \longrightarrow B_\varepsilon^{'}$ (where $B_\varepsilon^{'}$ denotes the dual of B_ε) by,

$$A_\varepsilon\,([\,u_1,\,u_2,\,u_3\,])([\,\phi_1,\,\phi_2,\phi_3\,]) \equiv \sum_{j=1}^{2}\int_{\Omega_j^\varepsilon}\mu_j(\tfrac{x}{\varepsilon},\,\nabla u_j\,).\nabla\phi_j\,dx + \int_{\Omega_2^\varepsilon}\mu_3(\,\tfrac{x}{\varepsilon},\,\varepsilon\nabla u_3\,).\varepsilon\nabla\phi_3\,dx$$

for $[u_1,\,u_2,\,u_3]$, $[\phi_1,\,\phi_2,\,\phi_3]\in B_\varepsilon$. Let

$$V_\varepsilon \equiv \{\overrightarrow{u^\varepsilon}\in L^p([0,T];B_\varepsilon) : (\overrightarrow{u^\varepsilon})^{'} \in L^q([0,T];B_\varepsilon^{\prime})\}\,.$$

For $\varepsilon > 0$, the Cauchy problem is equivalent to finding a solution $\overrightarrow{u^\varepsilon} \in V_\varepsilon$ to the problem

$$\begin{aligned} \frac{d\overrightarrow{u^\varepsilon}}{dt} + A_\varepsilon \overrightarrow{u^\varepsilon} &= 0 \text{ in } L^q([0,T];B_\varepsilon^{'}) && (2.15) \\ \overrightarrow{u^\varepsilon}(0) &= \overrightarrow{u^0} \text{ in } H_\varepsilon && (2.16) \end{aligned}$$

and this problem is well-posed, thanks to the conditions (2.1)-(2.3) (cf. Showalter [8]). We end with an identity(cf. [3]),

$$\frac{1}{2}\|\overrightarrow{u^\varepsilon}(T)\|^2_{H_\varepsilon} - \frac{1}{2}\|\overrightarrow{u^\varepsilon}(0)\|^2_{H_\varepsilon} + \int_0^T A_\varepsilon(\overrightarrow{u^\varepsilon})(\overrightarrow{u^\varepsilon})dt = 0. \tag{2.17}$$

3. Homogenization

We recall the results on the homogenization of flow in a partially fissured medium here (cf. [3]) in the form of propositions. For a proof of these results, refer Clark and Showalter (cf. [3]).

We recall the definition of two-scale convergence (cf. [1], [3]).

Definition 3.1. *A function, $\psi(t,x,y) \in L^q([0,T]\times\Omega, C_\sharp(Y))$, which is Y-periodic in y and satisfies*

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_\Omega \psi\left(t,x,\frac{x}{\varepsilon}\right)^q\,dx\,dt = \int_0^T\int_\Omega\int_Y \psi(t,x,y)^q\,dy\,dx\,dt$$

is called an admissible *test function.* □

Definition 3.2. *A sequence f^ε in $L^p([0,T]\times\Omega))$ two-scale converges to a function $f(t,x,y) \in L^p([0,T]\times\Omega\times Y)$ if for any admissible test function $\psi(t,x,y)$,*

$$\lim_{\varepsilon\longrightarrow 0}\int_0^T\int_\Omega f^\varepsilon(t,x)\,\psi\left(t,x,\frac{x}{\varepsilon}\right)\,dx\,dt = \int_0^T\int_\Omega\int_Y f(t,x,y)\,\psi(t,x,y)\,dy\,dx\,dt$$

We write $f^\varepsilon \overset{2-s}{\longrightarrow} f$.□

Proposition 3.1. *Let $\overrightarrow{u^\varepsilon}$ be the solution of the Cauchy problem (2.5)-(2.14). The following estimate holds*

$$\sum_{j=1}^{2}\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|^p_{p,\Omega_T} + \|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|^p_{p,\Omega_T} \le \frac{C}{2c}\sum_{j=1}^{3}\|u_j^0\|^2_{2,\Omega}. \tag{3.1}$$

□

Proposition 3.2. *Let $\overrightarrow{u^\varepsilon}$ be the solution of the Cauchy problem (2.5)-(2.14). There exist functions u_j in $L^p([0,T];W^{1,p}(\Omega))$, $j=1,2$ and functions U_j in $L^p([0,T]\times\Omega;W^{1,p}_\sharp(Y_j)/R)$, $j=1,2,3$ such that, for a subsequence of $\overrightarrow{u^\varepsilon}$, (to be indexed by ε*

again) the following hold:

$$\begin{aligned}
\chi_j^\varepsilon u_j^\varepsilon &\stackrel{2-s}{\longrightarrow} \chi_j(y)u_j(t,x), \;\; j=1,2\\
\chi_2^\varepsilon u_3^\varepsilon &\stackrel{2-s}{\longrightarrow} \chi_2(y)U_3(t,x,y)\\
\chi_j^\varepsilon \nabla u_j^\varepsilon &\stackrel{2-s}{\longrightarrow} \chi_j(y)(\nabla_x u_j(t,x)+\nabla_y U_j(t,x,y)), \;\; j=1,2\\
\varepsilon\chi_2^\varepsilon \nabla u_3^\varepsilon &\stackrel{2-s}{\longrightarrow} \chi_2(y)\nabla_y U_3(t,x,y)\\
\chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon) &\stackrel{2-s}{\longrightarrow} \chi_j(y)\mu_j(y,\nabla_x u_j+\nabla_y U_j), \;\; j=1,2\\
\chi_2^\varepsilon \mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon) &\stackrel{2-s}{\longrightarrow} \chi_2(y)\mu_3(y,\nabla_y U_3)\\
\chi_j^\varepsilon u_j^\varepsilon(T,x) &\stackrel{2-s}{\longrightarrow} \chi_j(y)u_j(T,x), \;\; j=1,2\\
\chi_2^\varepsilon u_3^\varepsilon(T,x) &\stackrel{2-s}{\longrightarrow} \chi_2(y)U_3(T,x,y) \;\; \textit{and}\\
u_1(t,x) &= \alpha u_2(t,x)+\beta U_3(t,x,y) \;\; \textit{for all } y\in\Gamma_{1,2}. \quad \square
\end{aligned}$$

Proposition 3.3. *The functions u_1,u_2,U_1,U_2,U_3 satisfy the homogenized system*

$$\begin{aligned}
&-\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} c_j(y)u_j\frac{\partial\phi_j}{\partial t}\,dy\,dx\,dt-\int_0^T\int_\Omega\int_{Y_2} c_3(y)U_3\,\frac{\partial\Phi_3}{\partial t}\,dy\,dx\,dt\\
&-\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)u_j^0\,\phi_j(0,x)\,dy\,dx-\int_\Omega\int_{Y_2} c_3(y)u_3^0\,\Phi_3(0,x,y)\,dy\,dx\\
&+\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\nabla_x u_j+\nabla_y U_j).(\nabla_x\phi_j+\nabla_y\Phi_j)\,dy\,dx\,dt\\
&+\int_0^T\int_\Omega\int_{Y_2}\mu_3(y,\nabla_y U_3).(\nabla_y\Phi_3)\,dy\,dx\,dt=0
\end{aligned}\tag{3.2}$$

for all

$$\begin{aligned}
\phi_j(t,x) &\in L^p([0,T];W^{1,p}(\Omega)), \;\; j=1,2\\
\Phi_j(t,x,y) &\in L^p([0,T]\times\Omega;W_\sharp^{1,p}(Y_j)), \;\; j=1,2,3
\end{aligned}$$

satisfying

$$\begin{aligned}
\frac{\partial\phi_j}{\partial t} &\in L^q([0,T];W^{-1,q}(\Omega)), \;\; j=1,2\\
\frac{\partial\Phi_j}{\partial t} &\in L^q([0,T]\times\Omega;(W_\sharp^{1,p}(Y_j))'), \;\; j=1,2,3
\end{aligned}$$

$$\beta\Phi_3(t,x,y)=\phi_1(t,x)-\alpha\phi_2(t,x) \;\; \textit{for all } y\in\Gamma_{1,2} \;\; \textit{and,}$$

$$\phi_1(T,x)=\phi_2(T,x)=\Phi_3(T,x,y)=0. \quad \square$$

The strong form of the homogenized problem has the following description. Define the *state space*,

$$H\equiv L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega\times Y_2)$$

with the scalar product

$$(\overrightarrow{\psi}, \overrightarrow{\phi})_H = \sum_{j=1}^{2} \int_\Omega \int_{Y_j} c_j(y)\psi_j(x)\phi_j(x)\, dy\, dx + \int_\Omega \int_{Y_2} c_3(y)\Psi_3(x,y)\Phi_3(x,y)\, dy\, dx$$

for every $\overrightarrow{\psi} = [\psi_1, \psi_2, \Psi_3]$, $\overrightarrow{\phi} = [\phi_1, \phi_2, \Phi_3] \in H$. Define the *energy space*,

$$\begin{aligned} B \equiv \{[\phi_1, \phi_2, \Phi_3] \in \quad & H \cap W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times L^2(\Omega; W^{1,p}_\sharp(Y_2)/R) \\ : \quad & \beta\Phi_3(x,y) = \phi_1(x) - \alpha\phi_2(x,y) \text{ for all } y \in \Gamma_{1,2}\} \end{aligned}$$

and the corresponding *evolution space* $V \equiv \mathrm{L}^p([0,T]; B)$.

Proposition 3.4. *$\overrightarrow{u} = [u_1, u_2, U_3] \in V$ and is the solution of the strong homogenized system,*

$$\begin{aligned} (\int_{Y_1} c_1(y)\, dy)\frac{\partial u_1}{\partial t}(t,x) &+ \frac{1}{\beta}\frac{\partial}{\partial t}(\int_{Y_2} c_3(y)U_3(t,x,y)\, dy) \\ &= \operatorname{div}_x(\int_{Y_1} \mu_1(y, \nabla_x u_1 + \nabla_y U_1)\, dy) \end{aligned} \tag{3.3}$$

$$\begin{aligned} (\int_{Y_2} c_2(y)\, dy)\frac{\partial u_2}{\partial t}(t,x) &- \frac{\alpha}{\beta}\frac{\partial}{\partial t}(\int_{Y_2} c_3(y)U_3(t,x,y)\, dy) \\ &= \operatorname{div}_x(\int_{Y_2} \mu_2(y, \nabla_x u_2 + \nabla_y U_2)\, dy) \end{aligned} \tag{3.4}$$

$$c_3(y)\frac{\partial U_3(t,x,y)}{\partial t} - \operatorname{div}_y \mu_3(y, \nabla U_3(t,x,y)) = 0 \tag{3.5}$$

where $U_3(t,x,y)$ and $\mu_3(y, \nabla_y U_3(t,x,y)).\nu$ are Y-periodic and,

$$\beta U_3(t,x,y) = u_1(t,x) - \alpha u_2(t,x) \text{ for } y \in \Gamma_{1,2} \tag{3.6}$$

with boundary conditions

$$\int_{Y_1} \mu_1(y, \nabla_x u_1 + \nabla_y U_1)\, dy.\nu = 0 \text{ on } \partial\Omega \tag{3.7}$$

$$\int_{Y_2} \mu_2(y, \nabla_x u_2 + \nabla_y U_2)\, dy.\nu = 0 \text{ on } \partial\Omega \tag{3.8}$$

and initial conditions

$$u_j(0,x) = u_j^0(x) \ j = 1,2;\ U_3(0,x,y) = u_3^0(x). \tag{3.9}$$

The functions $U_j(t,x,y)$ solve the cell problems*,*

$$\operatorname{div}_y \mu_j(y, \nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)) = 0 \text{ for } y \in Y_j \tag{3.10}$$

$$\mu_j(y, \nabla_x u_j(t,x) + \nabla_y U_j(t,x,y)).\nu = 0 \text{ on } \Gamma_{1,2} \text{ and} \tag{3.11}$$

Y-periodic on $\Gamma_{2,2}$, for $j = 1,2$. In the above, t, x are treated as parameters and the cell equations are solved.□

For $\xi \in R^N$, define the following functions;

$$\lambda_j(\xi) = \int_{Y_j} \mu_j(y, \xi + \nabla_y V_j^\xi(y)\, dy,\ j = 1,2 \tag{3.12}$$

where V_j^ξ is the Y-periodic solution of

$$\begin{aligned} \operatorname{div}_y \mu_j(y, \xi + \nabla_y V_j^\xi(y)) &= 0 \text{ in } Y_j &(3.13)\\ \mu_j(y, \xi + \nabla_y V_j^\xi(y)).\nu &= 0 \text{ on } \Gamma_{1,2} &(3.14)\end{aligned}$$

Then, because of (3.10), (3.11), the righthandsides in (3.3), (3.4) can be replaced by the functions $\operatorname{div}_x \lambda_1(\nabla_x u_1(t,x))$ and $\operatorname{div}_x \lambda_2(\nabla_x u_2(t,x))$ respectively. Also the lefthandsisdes of (3.7), (3.8) can be replaced by $\lambda_1(\nabla_x u_1).\nu$ and $\lambda_2(\nabla_x u_2).\nu$ respectively.

Remark 3.1: We note that the functions λ_j can be interpreted as the integrands in the $\Gamma -$ limit of the functionals

$$F_{j,\varepsilon}(\nabla v) = \int_\Omega \chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla v)\, dx\,.$$

In fact, $\Gamma - \lim F_{j,\varepsilon}(\nabla v) = \int_\Omega \lambda_j(\nabla v)\, dx$ (cf. DalMaso [4]). Further, the functions $\lambda_j,\ j = 1,2$ satisfy conditons (2.1)-(2.3) for the same p but maybe for different constants $\tilde{k}, \tilde{C}, \tilde{c_0}$ (cf. [5], [2]). □

Proposition 3.5. *The following* energy identity *holds (cf.* [3]*),*

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{2} \int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\, dy\, dx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\, dy\, dx\\ &\quad -\frac{1}{2}\sum_{j=1}^{2} \int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\, dy\, dx - \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|u_3^0(x)|^2\, dy\, dx\\ &\quad +\sum_{j=1}^{2}\int_0^T\int_\Omega\int_{Y_j} \mu_j(y, \nabla_x u_j + \nabla_y U_j).(\nabla_x u_j + \nabla_y U_j)\, dy\, dx\, dt\\ &\quad +\int_0^T\int_\Omega\int_{Y_2} \mu_3(y, \nabla_y U_3).\nabla_y U_3\, dy\, dx\, dt = 0.\end{aligned}$$

4. Correctors

We now prove corrector results for the gradient of flows under stronger hypotheses on μ_j's than (2.1)-(2.3). Let $k_1, k_2 > 0$ be constants and assume for j=1,2, 3:

$$\mu_j(\,.\,\xi) \text{ is measurable and Y-periodic for all } \xi \in R^N \tag{4.1}$$

For $\xi, \eta \in R^N$ with $|\xi| + |\eta| > 0$ and a.e. $y \in Y$,

$$\begin{aligned} \mu_j(y,0) &= 0 &(4.2)\\ |\mu_j(y,\xi) - \mu_j(y,\eta)| &\le k_1(|\xi| + |\eta|)^{p-2}|\xi - \eta| &(4.3)\\ (\mu_j(y,\xi) - \mu_j(y,\eta)).(\xi - \eta) &\ge k_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 &(4.4)\end{aligned}$$

The above hypotheses will, henceforth, be known as (**H**).

Remark 4.1: Note that (4.2) and (4.3) imply

$$|\mu_j(y,\xi)| \le k_1|\xi|^{p-1} \tag{4.5}$$

and, (4.2) and (4.4) imply

$$\mu_j(y,\xi).\xi \geq k_2|\xi|^p. \tag{4.6}$$

Thus, the new hypotheses are indeed stronger than the original hypotheses on μ_j's. Moreover,

$$\begin{aligned} (\mu_j(y,\xi)-\mu_j(y,\eta)).(\xi-\eta) &\geq k_2|\xi-\eta|^p \text{ if } p\geq 2 &(4.7)\\ |\mu_j(y,\xi)-\mu_j(y,\eta)| &\leq k_1|\xi-\eta|^{p-1} \text{ if } 1<p<2. &(4.8)\end{aligned}$$

These inequalities follow from (4.4) and (4.3) and triangle inequality in R^N. □

Remark 4.2: An example of μ_j satisfying (4.2)- (4.4) is $\mu_j = |\xi|^{p-2}\xi$, i.e. the corresponding diffusion operator is the p-Laplacian. More generally, let **C** denote the class of functions

$$f \in C^0(\overline{\Omega}\times R^N; R^N)\cap C^1(\Omega\times R^N\setminus\{0\}; R^N)$$

which satisfy condition (4.2) and the following

$$\begin{aligned} \sum_{j,j=1}^{N} |\frac{\partial f_j}{\partial \eta_i}|(x,\eta) &\leq \Gamma|\eta|^{p-2}\\ \sum_{j,j=1}^{N} |\frac{\partial f_j}{\partial \eta_i}|(x,\eta)\xi_i\xi_j &\geq \gamma|\eta|^{p-2}|\xi|^2\end{aligned}$$

for all $x\in\Omega, \eta\in R^N\setminus\{0\}$ and $\xi\in R^N$ and Γ,γ are positive constants. Then for μ_j 's in the class **C** the conditions (**H**) are satisfied (cf. Damascelli [7]).□

Let $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ be the solution of the Cauchy problem (2.5)- (2.14) and let $[u_1,u_2,U_1,U_2,U_3]$ be as in Section 3. We will denote $[0,T]\times\Omega$ by Ω_T. Define the sequence of functions

$$\begin{aligned} \xi_j(t,x,y) &\equiv \chi_j(y)(\nabla_x u_j(t,x)+\nabla_y U_j(t,x,y)),\ j=1,2, &(4.9)\\ \xi_3(t,x,y) &\equiv \chi_2(y)\nabla_y U_3(t,x,y) &(4.10)\end{aligned}$$

and let,

$$\xi_j^\varepsilon(t,x)\equiv \xi_j(t,x,\frac{x}{\varepsilon}),\ j=1,2,3. \tag{4.11}$$

Our main theorems are the following:

Theorem 4.1. *If the functions, $\nabla_y U_j,\ j=1,2,3$ are admissible (cf. Definition 2.1), then*

$$\begin{aligned} \overline{\lim}_{\varepsilon\to 0}\|\chi_j(\frac{x}{\varepsilon})\left(\nabla u_j^\varepsilon(t,x)-\xi_j^\varepsilon(t,x)\right)\|_{p,\Omega_T} &\longrightarrow 0,\ j=1,2,\\ \overline{\lim}_{\varepsilon\to 0}\|\chi_2(\frac{x}{\varepsilon})\left(\varepsilon\nabla u_3^\varepsilon(t,x)-\xi_3^\varepsilon(t,x)\right)\|_{p,\Omega_T} &\longrightarrow 0.\end{aligned}$$

Theorem 4.2. *If the functions, $\nabla_y U_j,\ j=1,2,3$ are admissible (cf. Definition 2.1), then*

$$\begin{aligned} \overline{\lim}_{\varepsilon\to 0}\|\chi_j(\frac{x}{\varepsilon})\left(\mu_j\left(\frac{x}{\varepsilon},\nabla u_j^\varepsilon\right)-\mu_j\left(\frac{x}{\varepsilon},\xi_j^\varepsilon(t,x)\right)\right)\|_{q,\Omega_T} &\longrightarrow 0,\ j=1,2,\\ \overline{\lim}_{\varepsilon\to 0}\|\chi_2(\frac{x}{\varepsilon})\left(\mu_3\left(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon\right)-\mu_3\left(\frac{x}{\varepsilon},\xi_3^\varepsilon(t,x)\right)\right)\|_{q,\Omega_T} &\longrightarrow 0.\end{aligned}$$

Remark 4.3: Theorem 4.1 gives strong convergence of the gradients of the flow of the Cauchy problem to the gradients of the flow of the homogenized problem by adding a corrector, whereas the gradients of the flow of the Cauchy problem, themselves, only weakly converge to the gradients of the flow of the homogenized problem in L^p. Theorem 4.2 gives an analogous result for the flux terms.□

We first calculate some limits and prove some estimates in order to prove the theorems. For that we need some more notations, functions and quantities which we will use hereafter. We will use M to denote a generic constant which does not depend on ε, but probably on p, k_1, k_2, c_0, C, and the L^2 norm of the initial vector $\overrightarrow{u^0}$. We will also set $\Omega_3^\varepsilon \equiv \Omega_2^\varepsilon$ and $Y_3 \equiv Y_2$. Let $0 < \kappa < 1$ be a constant and $\Phi_j(t,x,y)$ be admissible test functions such that

$$\sum_{j=1}^{3} \|\nabla_y U_j - \Phi_j\|^p_{p,[0,T]\times\Omega\times Y_j} \leq \kappa.$$

Note that,

$$\Phi_j(t,x,\frac{x}{\varepsilon}) \stackrel{2-s}{\longrightarrow} \Phi_j(t,x,y)$$

for j=1,2,3. Define the functions:

$$\eta_j^\varepsilon(t,x) = \chi_j(\frac{x}{\varepsilon})(\nabla_x u_j(t,x) + \Phi_j(t,x,\frac{x}{\varepsilon})),\ j=1,2 \tag{4.12}$$

$$\eta_3^\varepsilon(t,x) = \chi_2(\frac{x}{\varepsilon})\Phi_3(t,x,\frac{x}{\varepsilon}). \tag{4.13}$$

Then we note that the functions $\eta_j^\varepsilon(t,x)$ and $\mu_j^\varepsilon(\frac{x}{\varepsilon}, \eta_j^\varepsilon(t,x))$ arise from admissible test functions and we have the following two-scale convergence (cf. [3]),

$$\begin{aligned}
\eta_j^\varepsilon &\stackrel{2-s}{\longrightarrow} \eta_j(t,x,y) \equiv \chi_j(y)(\nabla_x u_j(t,x) + \Phi_j(t,x,y)),\ \ j=1,2,\\
\eta_3^\varepsilon &\stackrel{2-s}{\longrightarrow} \eta_3(t,x,y) \equiv \chi_2(y)\Phi_3(t,x,y)\\
\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon) &\stackrel{2-s}{\longrightarrow} \chi_j(y)\,\mu_j(y\,,\,\eta_j(t,x,y)),\ j=1,2,3.
\end{aligned}$$

Lemma 4.1. *(cf. [5], lemma 3.1.) Let $1 < p < 2$ and $\phi_1,\ \phi_2 \in L^p(\Omega_T)^N$. Then,*

$$\begin{aligned}
\|\phi_1 - \phi_2\|^p_{p,\Omega_T} \leq\ & \left[\int_0^T \int_\Omega |\phi_1 - \phi_2|^2 (|\phi_1| + |\phi_2|)^{p-2} \chi\, dx\, dt\right]^{\frac{p}{2}}\\
& \times \left[\int_0^T \int_\Omega (|\phi_1| + |\phi_2|)^p\, dx\, dt\right]^{\frac{2-p}{2}}
\end{aligned}$$

where χ denotes the characteristic function of the set

$$\{(t,x) \in [0,T] \times \Omega : |\phi_1|(t,x) + |\phi_2|(t,x) > 0\}$$

Proof: Multiply and divide the integrand in left hand side by $(|\phi_1|+|\phi_2|)^{(2-p)p/2}$ and apply Hölder's inequality to get the result. □

Lemma 4.2.

$$\sum_{j=1}^{2} \|\chi_j(y)(\nabla_x u_j + \nabla_y U_j)\|_p^p + \|\chi_2(y)\nabla_y U_3\|_p^p \leq \frac{C}{2k_2} \sum_{j=1}^{3} \|u_j^0\|^2_{2,\Omega}$$

Proof: Follows from the energy identity (Proposition 3.5) and (4.6). □

Lemma 4.3. *Let $\xi_i, \eta_i, \xi_i^\varepsilon, \eta_i^\varepsilon, i = 1,2,3$ be functions as defined above. Then,*

$$\overline{\lim}_{\varepsilon\to 0} \int_0^T \int_{\Omega_i^\varepsilon} \Big(\mu_i(\frac{x}{\varepsilon}, \nabla u_i^\varepsilon) - \mu_i(\frac{x}{\varepsilon}, \eta_i^\varepsilon)\Big) .(\nabla u_i^\varepsilon - \eta_i^\varepsilon)\, dx\, dt$$
$$\leq \textstyle\sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)) .(\xi_j - \eta_j)\, dy\, dx\, dt$$

for i=1,2 and

$$\overline{\lim}_{\varepsilon\to 0} \int_0^T \int_{\Omega_2^\varepsilon} \Big(\mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon)\Big) .(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\, dx\, dt$$
$$\leq \textstyle\sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} (\mu_j(y,\xi_j) - \mu_j(y,\eta_j)) .(\xi_j - \eta_j)\, dy\, dx\, dt$$

Proof: Denote the integrals appearing in the left-handsides of the above relations by $l_1^\varepsilon, l_2^\varepsilon$ and l_3^ε respectively. Then for i=1,2,3, using (2.17), we obtain,

$$\begin{aligned}
l_i^\varepsilon &\leq \sum_{j=1}^3 l_j^\varepsilon \\
&= \frac{1}{2}\sum_{j=1}^3 \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^0(x)|^2\, dx - \frac{1}{2}\sum_{j=1}^3 \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(T,x)|^2\, dx \\
&\quad - \sum_{j=1}^2 \int_0^T \int_{\Omega_j^\varepsilon} \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon).(\nabla u_j^\varepsilon - \eta_j^\varepsilon)\, dx\, dt \\
&\quad - \int_0^T \int_{\Omega_j^\varepsilon} \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon).(\varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon)\, dx\, dt \\
&\quad - \sum_{j=1}^2 \int_0^T \int_{\Omega_j^\varepsilon} \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon).\eta_j^\varepsilon\, dx\, dt - \int_0^T \int_{\Omega_2^\varepsilon} \mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon).\eta_3^\varepsilon\, dx\, dt
\end{aligned}$$

We now use the two-scale convergence properties of various functions discussed so far to pass to the limit. We get,

$$\begin{aligned}
\overline{\lim}_{\varepsilon\to 0} \sum_{j=1}^3 l_j^\varepsilon &= \frac{1}{2}\sum_{j=1}^3 \int_\Omega \int_{Y_j} c_j(y)|u_j^0(x)|^2\, dy\, dx \\
&\quad - \underline{\lim}_{\varepsilon\to 0} \frac{1}{2}\sum_{j=1}^3 \int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(T,x)|^2\, dx \\
&\quad - \sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} \mu_j(y,\eta_j).(\xi_j - \eta_j)\, dy\, dx\, dt \\
&\quad - \sum_{j=1}^3 \int_0^T \int_\Omega \int_{Y_j} \mu_j(y,\xi_j).\eta_j\, dy\, dx\, dt
\end{aligned}$$

The right hand side can be written as

$$\frac{1}{2}\sum_{j=1}^{3}\int_\Omega\int_{Y_j} c_j(y)|u_j^0(x)|^2\,dy\,dx - \underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx$$
$$+\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt$$
$$-\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\mu_j(y,\xi_j).\xi_j\,dy\,dx\,dt$$

which, using Proposition 3.5 to replace the last expression, is nothing but,

$$\frac{1}{2}\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\,dy\,dx + \frac{1}{2}\int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\,dy\,dx$$
$$-\underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx$$
$$+\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt$$

However, by standard arguments,

$$\sum_{j=1}^{2}\int_\Omega\int_{Y_j} c_j(y)|u_j(T,x)|^2\,dy\,dx + \int_\Omega\int_{Y_2} c_3(y)|U_3(T,x,y)|^2\,dy\,dx$$
$$\le \underline{\lim}_{\varepsilon\to 0}\sum_{j=1}^{3}\int_{\Omega_j^\varepsilon} c_j(\frac{x}{\varepsilon})|u_j^\varepsilon(x,T)|^2\,dx$$

This completes the proof. □

Lemma 4.4. *Let ξ_j,η_j,κ be as before. Then,*

$$\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right).(\xi_j-\eta_j)\,dy\,dx\,dt \le M\kappa^{\delta(p)}$$

where

$$\delta(p) = \begin{cases} 1 & \text{if } 1<p<2,\\ \frac{2}{p} & \text{if } p\ge 2.\end{cases}$$

Proof: Let the left hand side of the estimate be denoted by S.
Case 1: $1<p<2$. Using (4.8) we get,

$$\begin{aligned} S &\le \sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}|\left(\mu_j(y,\xi_j)-\mu_j(y,\eta_j)\right)||(\xi_j-\eta_j)|\,dy\,dx\,dt\\ &\le k_1\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j}|\xi_j-\eta_j|^p\,dy\,dx\,dt\\ &\le M\kappa \end{aligned}$$

Case 2: $2 \le p$. Using (4.3) and Hölder's inequality we get,

$$
\begin{aligned}
S &\le \sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j} |\mu_j(y,\xi_j)-\mu_j(y,\eta_j)||\xi_j-\eta_j|\,dy\,dx\,dt \\
&\le k_1\sum_{j=1}^{3}\int_0^T\int_\Omega\int_{Y_j} |\xi_j-\eta_j|^2(|\xi_j|+|\eta_j|)^{p-2}\,dy\,dx\,dt \\
&\le k_1\sum_{j=1}^{3}\|\xi_j-\eta_j\|_p^2\left(\int_0^T\int_\Omega\int_{Y_j}(|\xi_j|+|\eta_j|)^p\,dy\,dx\,dt\right)^{\frac{p-2}{p}} \\
&\le k_1\sum_{j=1}^{3}\|\xi_j-\eta_j\|_p^2\,(\|\xi_j\|_p+\|\eta_j\|_p)^{p-2} \\
&\le k_1\left(\sum_{j=1}^{3}\|\xi_j-\eta_j\|_p^p\right)^{\frac{2}{p}}\left(\sum_{j=1}^{3}(\|\xi_j\|_p+\|\eta_j\|_p)^p\right)^{\frac{p-2}{p}} \\
&\le k_1\left(\sum_{j=1}^{3}\|\xi_j-\eta_j\|_p^p\right)^{\frac{2}{p}}\left(\sum_{j=1}^{3}(2\|\xi_j\|_p+\|\xi_j-\eta_j\|_p)^p\right)^{\frac{p-2}{p}} \\
&\le k_1 2^{\frac{(p-2)(p-1)}{p}}\left(\sum_{j=1}^{3}\|\xi_j-\eta_j\|_p^p\right)^{\frac{2}{p}}\left(\sum_{j=1}^{3}\left(2^p\|\xi_j\|_p^p+\|\xi_j-\eta_j\|_p^p\right)\right)^{\frac{p-2}{p}}
\end{aligned}
$$

Therefore, by the estimate for the second term proved in lemma 4.2, we get the result.□

Theorem 4.3.

$$
\begin{aligned}
\overline{\lim}_{\varepsilon\to 0}\|\chi_j(\frac{x}{\varepsilon})\left(\nabla u_j^\varepsilon(t,x)-\eta_j^\varepsilon(t,x)\right)\|_{p,\Omega_T}^p &\le M\kappa^{r(p)}\ , \\
\overline{\lim}_{\varepsilon\to 0}\|\chi_2(\frac{x}{\varepsilon})\left(\varepsilon\nabla u_3^\varepsilon(t,x)-\eta_3^\varepsilon(t,x)\right)\|_{p,\Omega_T}^p &\le M\kappa^{r(p)}
\end{aligned}
$$

where

$$
r(p)=\begin{cases}\frac{p}{2} & \text{if } 1<p<2,\\ \frac{2}{p} & \text{if } p\ge 2.\end{cases}
$$

Proof: Case 1: $1<p<2$. We use lemma 4.1 with the functions $\chi_j^\varepsilon\nabla u_j^\varepsilon$ and η_j^ε, $j=1,2$ to get,

$$
\begin{aligned}
&\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p \\
&\quad\le (\textstyle\int_0^T\int_{\Omega_j^\varepsilon}|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^2(|\nabla u_j^\varepsilon|+|\eta_j^\varepsilon|)^{p-2}\,dx\,dt)^{\frac{p}{2}}(\int_0^T\int_{\Omega_j^\varepsilon}(|\nabla u_j^\varepsilon|+|\eta_j^\varepsilon|)^p\,dxdt)^{\frac{2-p}{2}}
\end{aligned}
$$

Therefore, using strong monotonicity (4.4), we get,

$$
\begin{aligned}
\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p \le k&\left(\int_0^T\int_{\Omega_j^\varepsilon}\left(\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\right).\left(\nabla u_j^\varepsilon-\eta_j^\varepsilon\right)\,dx\,dt\right)^{\frac{p}{2}} \\
&\times(\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p+\|\eta_j^\varepsilon\|_p^p)^{\frac{2-p}{2}}
\end{aligned}
$$

where $k = 2^{\frac{(p-1)(2-p)}{2}}/k_2^{\frac{p}{2}}$. Similarly,

$$\|\chi_2^\varepsilon \varepsilon\nabla u_3^\varepsilon - \eta_3^\varepsilon\|_{p,\Omega_T}^p \le k\left(\int_0^T\int_{\Omega_2^\varepsilon}\Big(\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\Big).(\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon)\,dx\,dt\right)^{\frac{p}{2}} \times(\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p+\|\eta_3^\varepsilon\|_p^p)^{\frac{2-p}{2}}$$

Let,

$$\begin{aligned}
S_1^\varepsilon &= \sum_{j=1}^{2}\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p+\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon\|_{p,\Omega_T}^p\,,\\
S_2^\varepsilon &= \sum_{j=1}^{2}\int_0^T\int_{\Omega_j^\varepsilon}\Big(\mu_j(\frac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\frac{x}{\varepsilon},\eta_j^\varepsilon)\Big).(\nabla u_j^\varepsilon-\eta_j^\varepsilon)\,dx\,dt\\
&\quad+\int_0^T\int_{\Omega_2^\varepsilon}\Big(\mu_3(\frac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\frac{x}{\varepsilon},\eta_3^\varepsilon)\Big).(\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon)\,dx\,dt \text{ and },\\
S_3^\varepsilon &= \sum_{j=1}^{2}\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p+\|\eta_j^\varepsilon\|_p^p+\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p+\|\eta_3^\varepsilon\|_p^p.
\end{aligned}$$

Then, by Hölder's inequality, $S_1^\varepsilon \le k(S_2^\varepsilon)^{\frac{p}{2}}\times(S_3^\varepsilon)^{\frac{2-p}{2}}$.
Note that η_j^ε arise from admissible test functions. Therefore,

$$\begin{aligned}
\lim_{\varepsilon\to0}\sum_{j=1}^{3}\|\eta_j^\varepsilon\|_p^p &= \sum_{j=1}^{3}\|\eta_j\|_{p,[0,T]\times\Omega\times Y}^p\\
&\le \sum_{j=1}^{3}2^{p-1}(\|\xi_j\|_{p,[0,T]\times\Omega\times Y}^p+\sum_{j=1}^{3}\|\eta_j-\xi_j\|_{p,[0,T]\times\Omega\times Y}^p)\\
&\le M
\end{aligned}$$

where the last estimate follows from lemma 4.2. Also by (2.17) and (4.6), we get,

$$\sum_{j=1}^{2}\|\chi_j^\varepsilon\nabla u_j^\varepsilon\|_p^p+\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon\|_p^p\le\frac{1}{2k_2}\|\overrightarrow{u^0}\|_{H_\varepsilon}^2\le M$$

From this we conclude that, $\overline{\lim}_{\varepsilon\to0}S_3^\varepsilon\le M$. Therefore, taking limsup as $\varepsilon\to0$ and using lemmas 4.3 and 4.4, we get

$$\overline{\lim}_{\varepsilon\to0}S_1^\varepsilon\le M\kappa^{\frac{p}{2}}.$$

This concludes the proof in this case.
Case 2: $2\le p$. From (4.7), we get,

$$|\nabla u_j^\varepsilon-\eta_j^\varepsilon|^p \le \tfrac{1}{k_2}\left(\mu_j(\tfrac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\tfrac{x}{\varepsilon},\eta_j^\varepsilon)\right).(\nabla u_j^\varepsilon-\eta_j^\varepsilon)$$

Therefore, by integrating with respect to t in $[0,T]$ and x in Ω_j^ε, we get,

$$\|\chi_j^\varepsilon\nabla u_j^\varepsilon-\eta_j^\varepsilon\|_{p,\Omega_T}^p\le\tfrac{1}{k_2}\int_0^T\int_{\Omega_j^\varepsilon}(\mu_j(\tfrac{x}{\varepsilon},\nabla u_j^\varepsilon)-\mu_j(\tfrac{x}{\varepsilon},\eta_j^\varepsilon)).(\nabla u_j^\varepsilon-\eta_j^\varepsilon)\,dx\,dt$$

Similarly,

$$\|\chi_2^\varepsilon\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon\|_{p,\Omega_T}^p\le\tfrac{1}{k_2}\int_0^T\int_{\Omega_2^\varepsilon}(\mu_3(\tfrac{x}{\varepsilon},\varepsilon\nabla u_3^\varepsilon)-\mu_3(\tfrac{x}{\varepsilon},\eta_3^\varepsilon)).(\varepsilon\nabla u_3^\varepsilon-\eta_3^\varepsilon)\,dx\,dt$$

We note that if S_1^ε and S_2^ε are defined as in the previous case, then $S_1^\varepsilon \le \frac{1}{k_2} S_2^\varepsilon$. Passing to the limit, as before, we reach our conclusions. □

Theorem 4.4.

$$\begin{aligned} \overline{\lim}_{\varepsilon\to 0} \|\chi_j^\varepsilon \mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon)\|_{q,\Omega_T}^q &\le M\kappa^{s(p)} \\ \overline{\lim}_{\varepsilon\to 0} \|\chi_2^\varepsilon \mu_3(\frac{x}{\varepsilon}, \varepsilon\nabla u_3^\varepsilon) - \mu_3(\frac{x}{\varepsilon}, \eta_3^\varepsilon)\|_{q,\Omega_T}^q &\le M\kappa^{s(p)} \end{aligned}$$

where

$$s(p) = \begin{cases} \frac{p}{2} & \text{if } 1 < p < 2, \\ \frac{2}{p-1} & \text{if } p \ge 2. \end{cases}$$

Proof: We will prove only the first of these estimates, the other is proved similarly.

If $1 < p < 2$, by (4.3) and triangle inequality in R^N, we get,

$$\int_0^T \int_{\Omega_j^\varepsilon} |\mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon)|^q \,dx\,dt \quad \le k_1 \int_0^T \int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^{q(p-1)} \,dx\,dt$$

Since $q(p-1) = p$, using the theorem 4.3, the estimate follows easily. Let $2 \le p$. Then,

$$\begin{aligned} &\int_0^T \int_{\Omega_j^\varepsilon} |\mu_j(\frac{x}{\varepsilon}, \nabla u_j^\varepsilon) - \mu_j(\frac{x}{\varepsilon}, \eta_j^\varepsilon)|^q \,dx\,dt \\ &\quad \le k_1 \int_0^T \int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^q \left(|\nabla u_j^\varepsilon| + |\eta_j^\varepsilon|\right)^{(p-2)q} \,dx\,dt \end{aligned}$$

The right hand side, by Hölder's inequality,

$$\begin{aligned} &\le k_2 2^{p-1} \left(\int_0^T \int_{\Omega_j^\varepsilon} |\nabla u_j^\varepsilon - \eta_j^\varepsilon|^p \,dx\,dt\right)^{\frac{1}{p-1}} \left(\int_0^T \int_{\Omega_j^\varepsilon} (|\nabla u_j^\varepsilon|^p + |\eta_j^\varepsilon|^p) \,dx\,dt\right)^{\frac{p-2}{p-1}} \\ &\le M \|\chi_j^\varepsilon \nabla u_j^\varepsilon - \eta_j^\varepsilon\|_{p,\Omega_T}^{\frac{p}{p-1}}. \end{aligned}$$

So, again using theorem 4.3, we get the desired result.□

Proof of Theorems 4.1 and 4.2: Since, $\nabla_y U_j$'s are assumed to be admissible test functions, we can take $\Phi_j \equiv \nabla_y U_j$. Thus, κ can be taken arbitrarily small and therefore, Theorem 4.1 follows from Theorem 4.3. Similarly, Theorem 4.2 follows from Theorem 4.4. □

Remark 4.4: The functions $\nabla_y U_j(t,x,y)$ will be admissible if we have C^1 regularity of U_j in the variable y. Even if the functions U_j are not admissible, Theorems 4.3 and 4.4 are corrector results in their own right.

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M. Rajesh
The Institute of Mathematical sciences
C.I.T. Campus, Taramani
Madras-600 113, INDIA
E-mail address: rajesh@imsc.ernet.in