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The effect of thin coatings on second harmonic generation *

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Abstract

The effect of thin coatings of nonlinear material on second harmonic generation is studied in this paper. Asymptotic expansions of the fields inside a thin nonlinear coating are performed. The convergence of these formal expansions is established. Our asymptotic analysis reveals the physical nature of this nonlinear problem. It also provides an effective method for overcoming the computational difficulties that arise in thin nonlinear coatings.

1 Introduction

Significant technological advances have been made recently in nonlinear optics, due to rapid developments of laser technology and of nonlinear materials. Applications of nonlinear optics are everywhere, for example, lasers, spectroscopy, optical switching, optical disc storages, optical computing, and communications [9]. A particularly remarkable application is to generate powerful coherent radiation at a frequency that is twice that of available lasers, so-called second harmonic generation (SHG). The underlying physics is simple: an intense incident (pump) beam induces a response through a nonlinear polarization in a given medium, the medium then reacts to modify the field in a nonlinear fashion. The former and latter processes are governed by the constitutive equations and a nonlinear system of Maxwell's equations respectively. In this paper, we shall restrict our attention to second order nonlinear effects, which are the simplest and representative to other nonlinear effects. In principle, the computation of the field intensity distributions of SHG in a nonlinear medium may be done by combining a method of finite elements and a fixed-point iteration scheme. However, this numerical method fails completely when the thickness of the coating becomes exceedingly small due to numerical instabilities.

The present work is devoted to a mathematical study of optical second harmonic generation in thin nonlinear coatings. We shall adopt a model for the SHG derived by Bao and Dobson in [2], which is a two-point boundary-value

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problem for a nonlinear system of ordinary differential equations (ODEs). Our method is based on asymptotic expansions of the fields in the nonlinear coatings. An advantage of working on this simple model is that an explicit asymptotic analysis becomes available, which provides a complete and rigorous characterization of the physical nature of thin nonlinear coatings. It also provides an effective method for the computation and analysis of the nonlinear model which involves a small scale. This paper is our first attempt to model and analyze the effects of thin nonlinear coatings. In an upcoming paper, we shall investigate thin nonlinear coatings on a periodic structure (grating). In that case, the model problem becomes a system of nonlinear partial differential equations.

We refer the reader to Engquist and Nédélec [6] and Bendali and Lemrabet [5] for recent results and references on effects of a thin coating on the scattering by a linear medium. Results on mathematical analysis and numerical solutions of this and other related SHG models may be found in Bao and Chen [1], Bao and Dobson [2], [3]. See Shen [8] for a detailed discussion on various physical aspects of SHG. The reader is also referred to the book [7] for a description of mathematical problems in nonlinear diffractive optics which arise in industrial applications.

We present the model problem in Section 2. A formal asymptotic expansion of the fields in the nonlinear coating is performed in Section 3. Section 4 is concerned with the well-posedness of the model, as well as a convergence analysis of the asymptotic expansions.

2 Model problem

Assume that the media are nonmagnetic with constant magnetic permittivity everywhere, that no external charge or current is present in the field, and that the electric and magnetic fields are time harmonic, *i.e.*,

$$\mathbf{E} = \mathbf{E}(\mathbf{r})e^{-i\omega t}\,, \quad \mathbf{H} = \mathbf{H}(\mathbf{r})e^{-i\omega t}$$

where $\mathbf{r} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

We are interested in studying optical wave interaction in nonlinear media, or in particular SHG which deals with the phenomenon of two wave mixing. Suppose that there is a pumping wave with same frequency $\omega_1 = \omega$. Consider the two waves fields $\mathbf{E}(\mathbf{r}, \omega_1)$ and $\mathbf{E}(\mathbf{r}, \omega_2)$, where $\omega_2 = 2\omega_1$. To simplify our notation, we denote $\mathbf{E}(\omega_i) = \mathbf{E}(\mathbf{r}, \omega_i)$. The Maxwell equations yield the following coupled system [8]:

$$\begin{aligned}
\left[\nabla \times (\nabla\times) - \omega_1^2 \varepsilon_1 \cdot\right] \mathbf{E}(\omega_1) &= 4\pi\omega_1^2 \chi^{(2)}(\omega_1 = -\omega_2 + \omega_1) : \mathbf{E}^*(\omega_1)\mathbf{E}(\omega_2)\,, \\
\left[\nabla \times (\nabla\times) - \omega_2^2 \varepsilon_2 \cdot\right] \mathbf{E}(\omega_2) &= 4\pi\omega_2^2 \chi^{(2)}(\omega_2 = \omega_1 + \omega_1) : \mathbf{E}(\omega_1)\mathbf{E}(\omega_1)\,,
\end{aligned}$$

where $k_1^2 = \omega_1^2 \varepsilon_1$, $k_2^2 = \omega_2^2 \varepsilon_2$, and E^* denotes the complex conjugate. Through the nonlinear coupling, energy can be transfered back and forth between fields at each frequency. Clearly, the nonlinear nature of the medium is described by $\chi^{(2)}$, the second order nonlinear susceptibility tensor. More discussions on nonlinear susceptibility tensors as well as the derivation of the above system from the Maxwell equations may be found in [8]. Note that the presence of new frequency components is the most striking difference between nonlinear and linear optics.

Throughout, we make the following general assumptions: the fields are transverse; the medium is stratified; the surface is flat; and a normally incident pump beam. Since the medium is stratified, the fields vary only in one direction. The transversality assumption further reduces the Maxwell system to a system of nonlinear Helmholtz equations. The formulation of the model can be simplified because of the general assumptions made in the Introduction. Since the medium is stratified, the fields vary only in one direction. The transversality assumption essentially allows us to reduce the Maxwell system to a system of Helmholtz equations. For instance, by choosing the polarization properly, one may assume that $\mathbf{E}(\mathbf{r},\omega_1)=E(z,\omega_1)\vec{y}$ and $\mathbf{E}(\mathbf{r},\omega_2)=E(z,\omega_2)\vec{x}$.

Let us specify the geometry of the model. Assume that a slab of nonlinear material of thickness h is placed in between two linear materials. Suppose that the whole space is filled with material in such a way that the indexes of refraction $k_1(z)$ and $k_2(z)$ at frequencies ω_1 and $\omega_2=2\omega_1$, respectively, satisfy

$$k_j(z)=\begin{cases} k_{j1} & z\geq h,\\ k_{j0} & 0<z<h,\\ k_{j2} & z\leq 0,\end{cases}$$

for $j=1,2$. Here k_{j1} and k_{j2} are fixed constants with the properties: $k_{j1}>0$, $Re(k_{j2})>0$, and $Im(k_{j2})\geq 0$. However k_{j0} may be a bounded function.

Assume that a plane wave with the electric field $(0,u^{in}e^{ik_{11}z},0)$ is incident on the slab of nonlinear material from the above. Using the jump conditions, as derived in [2], we have the following two-point boundary-value problem:

$$\begin{gathered} u''+k_1^2u=\chi_1\,\overline{u}v \quad \text{in } (0,h),\\ v''+k_2^2v=\chi_2\,u^2 \quad \text{in } (0,h),\\ u'(0)=ik_{12}u(0),\\ u'(h)=-ik_{11}u(h)+2ik_{11}e^{ik_{11}h}u^{in},\\ v'(0)=ik_{22}v(0),\\ v'(h)=-ik_{21}v(h),\end{gathered} \tag{2.1}$$

where $u=E(z,\omega_1), v=E(z,\omega_2)$ are the pump and second-harmonic fields, and $\chi_1(z)$, $\chi_2(z)$ characterize the nonlinearity of the medium at frequencies ω_1,ω_2, respectively.

3 Asymptotic expansions

We first scale the model problem. Let the functions u_h and v_h be defined by

$$\begin{aligned} u_h(y)&=u(hy) \quad \text{for } y\in[0,1],\\ v_h(y)&=v(hy) \quad \text{for } y\in[0,1]\,.\end{aligned}$$

It is then easily seen from (2.1) by noticing $\frac{d}{dz} = \frac{1}{h}\frac{d}{dy}$ that u_h and v_h satisfy the equations

$$\begin{aligned} \tfrac{1}{h^2}u_h'' + k_1^2 u_h &= \chi_1 \overline{u_h} v_h \quad \text{in } (0,1), \\ \tfrac{1}{h^2}v_h'' + k_2^2 v_h &= \chi_2 u_h^2 \quad \text{in } (0,1) \end{aligned} \tag{3.1}$$

along with the boundary conditions

$$\begin{aligned} u_h'(0) &= ihk_{12}u_h(0), \\ u_h'(1) &= -ihk_{11}u_h(1) + 2ihk_{11}e^{ik_{11}h}u^{in}, \\ v_h'(0) &= ihk_{22}v_h(0), \\ v_h'(1) &= -ihk_{21}v_h(1)\,. \end{aligned} \tag{3.2}$$

Assume that for h small, u_h and v_h have the formal expansions

$$u_h(y) = u_0(y) + hu_1(y) + h^2u_2(y) + 0(h^3), \tag{3.3}$$
$$v_h(y) = v_0(y) + hv_1(y) + h^2v_2(y) + 0(h^3). \tag{3.4}$$

Our goal in this section is to determine the functions u_j and v_j for $j = 0, 1, 2$. This may be done by substituting (3.3) for u_h and (3.4) for v_h into (3.1) and (3.2) and equating the coefficients of each power of h.

From the coefficients of h^{-2} in (3.1) and from the coefficients of h^0 in (3.2), we obtain

$$\begin{gathered} u_0'' = 0 \;\text{ in } (0,1)\,, \\ u_0'(0) = 0, \quad u_0'(1) = 0 \end{gathered}$$

and

$$\begin{gathered} v_0'' = 0 \;\text{ in } (0,1)\,, \\ v_0'(0) = 0, \quad v_0'(1) = 0\,. \end{gathered}$$

Hence, both u_0 and v_0 are constants in $[0,1]$.

Similarly the coefficients of h^{-1} in (3.1) and h in (3.2) yield the following problems for u_1 and v_1:

$$\begin{gathered} u_1'' = 0 \;\text{ in } (0,1)\,, \\ u_1'(0) = ik_{12}u_0, \\ u_1'(1) = -ik_{11}u_0 + 2ik_{11}u^{in} \end{gathered} \tag{3.5}$$

and

$$\begin{gathered} v_1'' = 0 \;\text{ in } (0,1)\,, \\ v_1'(0) = ik_{22}v_0, \\ v_1'(1) = -ik_{21}v_0\,. \end{gathered} \tag{3.6}$$

We can now determine the constants u_0 and v_0. In fact, from (3.5) it follows that u_1' is a constant in $[0,1]$. Using the boundary conditions, it is evident that

$$u_0 = \frac{2k_{11}u^{in}}{k_{12}+k_{11}}\,. \tag{3.7}$$

Similarly, from the equations in (3.6), we see that v_1' is also a constant in $[0,1]$. In addition, the boundary conditions in (3.6) yield

$$v_0 = 0.$$

Using (3.6) again, we deduce that v_1 is a constant in $[0,1]$.

Remark 3.1. In general, it is well known [8] that nonlinear optical effects of SHG are very weak. This physical property is justified by the fact that $v_0 = 0$ in the formal expansion for v_h. It also indicates that higher order terms must be used to obtain useful nonlinear properties of the SHG field.

We next examine the problems for u_2 and v_2.

$$\begin{gathered} u_2'' + k_1^2 u_0 = 0 \ \text{ in } (0,1)\,,\\ u_2'(0) = ik_{12}u_1(0),\\ u_2'(1) = -ik_{11}u_1(1) - 2k_{11}^2 u^{in} \end{gathered} \tag{3.8}$$

and

$$\begin{gathered} v_2'' = \chi_2 u_0^2 \ \text{ in } (0,1)\,,\\ v_2'(0) = ik_{22}v_1(0),\\ v_2'(1) = -ik_{21}v_1(1) = -ik_{21}v_1(0)\,. \end{gathered} \tag{3.9}$$

From (3.5) and (3.6) we know that

$$\begin{gathered} u_1(y) = ik_{12}u_0 y + u_1(0),\\ v_1(y) = v_1(0)\,. \end{gathered} \tag{3.10}$$

Using the explicit forms of u_2' and v_2',

$$\begin{gathered} u_2'(y) = -u_0 \textstyle\int_0^y k_1^2(x)\,dx + ik_{12}u_1(0),\\ v_2'(y) = u_0^2 \textstyle\int_0^y \chi_2(x)\,dx + ik_{22}v_1(0), \end{gathered}$$

we arrive at

$$u_1(0) = \frac{1}{i(k_{11}+k_{12})}\Big(k_{11}k_{12}u_0 - 2k_{11}^2 u^{in} + u_0 \int_0^1 k_1^2(x)dx\Big)$$

and

$$v_1(0) = \frac{iu_0^2 \int_0^1 \chi_2(x)\,dx}{k_{21}+k_{22}}\,.$$

The calculation of u_2, v_2 may be carried out similarly. Equating the coefficients of powers h in (3.1) and h^3 in (3.2) yields

$$\begin{gathered}
u_3'' + k_1^2 u_1 = \chi_1 \overline{u_0} v_1 = \tfrac{i\chi_1 |u_0|^2 u_0}{k_{21}+k_{22}} \int_0^1 \chi_2(x)dx \quad \text{in } (0,1)\,, \\
u_3'(0) = ik_{12}u_2(0), \\
u_3'(1) = -ik_{11}u_2(1) - 2ik_{11}^3 u^{in}
\end{gathered}$$

and

$$\begin{gathered}
v_3'' + k_2^2 v_1 = 2\chi_2 u_0 u_1 \quad \text{in } (0,1)\,, \\
v_3'(0) = ik_{22}v_2(0), \\
v_3'(1) = -ik_{21}v_2(1)\,.
\end{gathered}$$

Thus

$$\begin{gathered}
u_2(y) = -u_0 \int_0^y \int_0^x k_1^2(t)\,dtdx + ik_{12}u_1(0)y + u_2(0), \\
v_2(y) = u_0^2 \int_0^y \int_0^x \chi_2(t)\,dtdx + ik_{22}v_1(0)y + v_2(0)\,.
\end{gathered} \tag{3.11}$$

It suffices to determine $u_2(0)$ and $v_2(0)$. From

$$u_3'(y) = -\int_0^y k_1^2(x)u_1(x)dx + \frac{i|u_0|^2 u_0}{k_{12}+k_{21}} \int_0^y \chi_1(x)dx \int_0^1 \chi_2(x)dx + ik_{12}u_2(0),$$

we deduce, by using the boundary condition of u_3 at $y=1$, that

$$\begin{aligned}
&ik_{12}u_2(0) + ik_{11}u_2(1) \\
&= \quad -2ik_{11}^3 u^{in} + \int_0^1 k_1^2(x)u_1(x)dx - \frac{i|u_0|^2 u_0}{k_{21}+k_{22}} \int_0^1 \int_0^1 \chi_1(x)dx \int_0^1 \chi_2(x)\,dx\,.
\end{aligned}$$

But from (3.11)

$$u_2(1) = u_2(0) - u_0 \int_0^1 \int_0^x k_1^2(t)\,dtdx + ik_{12}u_1(0).$$

Thus

$$\begin{aligned}
u_2(0) \quad = \quad & \frac{1}{i(k_{11}+k_{12})}\Big(-2ik_{11}^3 u^{in} + \int_0^1 k_1^2(x)u_1(x)dx \\
& -\frac{i|u_0|^2 u_0}{k_{21}+k_{22}} \int_0^1 \chi_1(x)dx \int_0^1 \chi_2(x)dx \\
& +ik_{11}u_0 \int_0^1 \int_0^y k_1^2(x)\,dxdy + k_{11}k_{12}u_1(0)\Big),
\end{aligned}$$

which allows the calculation of u_2 in $[0,1]$.

Finally, in order to determine $v_2(0)$, we use

$$v_3'(y) = -\int_0^y k_2^2(x)v_1(x)dx + 2u_0 \int_0^y \chi_2(x)u_1(x)dx + ik_{22}v_2(0)$$

together with the boundary condition of v_3 at $y=1$. By a simple computation, we get

$$\begin{aligned} v_2(0) &= \frac{1}{i(k_{21}+k_{22})}\Big(-ik_{21}u_0^2\int_0^1\int_0^x \chi_2(t)dtdx + k_{21}k_{22}v_1(0) \\ &\quad -2u_0\int_0^1 \chi_2(x)u_1(x)dx + v_1\int_0^1 k_2^2(x)dx\Big). \end{aligned}$$

Therefore, we have completed the formal asymptotic expansions for u_h, v_h up to $O(h^3)$.

4 Convergence analysis

In this section, we validate the formal asymptotic expansions performed in Section 3 by estimating $\|u_h-(u_0+hu_1+h^2u_2)\|_{H^1(0,1)}$ and $\|v_h-(hv_1+h^2v_2)\|_{H^1(0,1)}$, where u_0, u_1, u_2, v_1, v_2 are the functions defined in Section 3. We begin with two stability results for the model problem.

Lemma 4.1 *Let m be an integer. If w_h satisfies*

$$\begin{aligned} w_h'' + k_1^2\, h^2 w_h = 0(h^m) \quad &in\ (0,1), \\ w_h'(0) - ik_{12}hw_h(0) = 0(h^m), \\ w_h'(1) + ik_{11}hw_h(1) = 0(h^m), \end{aligned} \tag{4.1}$$

then

$$h^{1/2}\|w_h\|_{L^2(0,1)} + \|w_h'\|_{L^2(0,1)} = 0(h^{m-1/2}). \tag{4.2}$$

Proof. From

$$w_h(y) = \int_0^y w_h'(x)\,dx + w_h(0),$$

it is easy to show that

$$\|w_h\|^2_{L^2(0,1)} \le \|w_h'\|^2_{L^2(0,1)} + 2|w_h(0)|^2. \tag{4.3}$$

Multiplying (4.1) by $\overline{w_h}$ and integrating by parts over $[0,1]$,

$$\begin{aligned} &\int_0^1 |w_h'|^2 - h^2\int_0^1 k_1^2|w_h|^2 + ih(k_{12}|w_h(0)|^2 + k_{11}|w_h(1)|^2) \\ &= \int_0^1 0(h^m)\overline{w}_h + 0(h^m)\overline{w}_h(0) + 0(h^m)\overline{w}_h(1)\,. \end{aligned} \tag{4.4}$$

Taking the imaginary part of (4.4) yields that

$$|w_h(0)| + |w_h(1)| \le C\Big(h^{m-1} + h^{\frac{m-1}{2}}\|w_h\|^{1/2}_{L^2(0,1)}\Big)\,,$$

where the constant C independent of h.

By taking the real part of (4.4) and using Poincaré's inequality (4.3), we arrive at

$$\|w_h'\|_{L^2(0,1)} \leq C\Big(h^m\|w_h\|_{L^2(0,1)} + h^{2m-1}\Big).$$

Combining the above estimates for $\|w_h'\|_{L^2(0,1)}$ and $|w_h(0)|$ together with (4.3), we obtain

$$\|w_h\|_{L^2(0,1)} \leq Ch^{m-1} \text{ and } \|w_h'\|_{L^2(0,1)} \leq Ch^{m-\frac{1}{2}},$$

which completes the proof. □

The next result is useful in our well-posedness study of the model. The proof, which is essentially identical to that of Lemma 4.1, is omitted here.

Lemma 4.2 *Let w_h be defined as the solution of the boundary-value problem*

$$\begin{aligned} w_h'' + k_1^2\, h^2 w_h &= h^2\, f \quad \text{in } (0,1)\,, \\ w_h'(0) - ik_{12}hw_h(0) &= 0\,, \\ w_h'(1) + ik_{11}hw_h(1) &= 0\,, \end{aligned} \tag{4.5}$$

where $f \in L^2(0,1)$. Then there is a positive constant C independent of h, such that

$$\|w_h\|_{H^1(0,1)} \leq C\, h\, \|f\|_{L^2(0,1)}. \tag{4.6}$$

We now present a well-posedness result for the boundary-value problem (3.1)-(3.2).

Theorem 4.1 *There is a constant $h_0 > 0$, such that for any $0 < h < h_0$, Problem (3.1)-(3.2) admits a unique solution (u_h, v_h) in $H^1(0,1) \times H^1(0,1)$.*

Remark 4.1. This existence and uniqueness result was first established by Bao and Dobson in [2] for an arbitrary h but sufficiently small nonlinear susceptibilities, *i.e.*,

$$\|\chi_1\|_{L^\infty(0,1)}\|\chi_2\|_{L^\infty(0,1)} << 1\ .$$

Proof. For the sake of completeness, we sketch the proof here. Following [2], the result is proved by the fixed point theorem.

Let w_h be defined by

$$w_h = u_h - \frac{2k_{11}u^{in}}{k_{12}+k_{11}} e^{i(k_{11}-k_{12})h} e^{ik_{12}hy}. \tag{4.7}$$

Denote

$$f_h = \frac{2k_{11}u^{in}}{k_{12}+k_{11}} e^{i(k_{11}-k_{12})h} e^{ik_{12}hy}\ .$$

It is easy to verify that w_h and v_h satisfy respectively the boundary-value problems

$$\begin{aligned} w_h'' + k_1^2\, h^2 w_h &= h^2\chi_1\overline{w}_h v_h + h^2\chi_1\overline{f}_h v_h \quad \text{in } (0,1), \\ w_h'(0) - ik_{12}hw_h(0) &= 0, \\ w_h'(1) + ik_{11}hw_h(1) &= 0 \end{aligned}$$

and

$$\begin{aligned} v_h'' + k_2^2 h^2 v_h &= h^2\chi_2 w_h^2 + h^2\chi_2 w_h f_h + h^2 \chi_2 f_h^2 \quad \text{in } (0,1)\,, \\ v_h'(0) - ik_{22} h v_h(0) &= 0\,, \\ v_h'(1) + ik_{21} h v_h(1) &= 0\,. \end{aligned}$$

Let $A_j^h : H^1(0,1) \mapsto H^{-1}(0,1)$ be the linear operator defined by $B_j^h(u_1,u_2) = (A_j^h u_1, u_2)$, where

$$B_j^h(u_1,u_2) = \int_0^1 u_1' u_2' - \int_0^1 k_j^2 u_1 \overline{u}_2 + ik_{j1} u_1(1)\overline{u}_2(1) + ik_{j2} u_1(0)\overline{u}_2(0).$$

Lemma 4.2 yields that

$$\|(A_j^h)^{-1}\| \le C_j h^{-1}, \tag{4.8}$$

where the constant C_j is independent of h.

Thus, the existence and uniqueness of a solution (u_h, v_h) to the boundary-value problem (3.1)-(3.2) is equivalent to the well-posedness of the fixed point problem

$$\begin{aligned} w_h &= F(w_h) \\ &= (A_1^h)^{-1}\Big(h^2\chi_1 \overline{w}_h ((A_2^h)^{-1}(h^2\chi_2 w_h^2 + h^2 w_h f_h + f_h^2)) \\ &\quad + f_h((A_2^h)^{-1}(h^2\chi_2 w_h^2 + h^2 w_h f_h + f_h^2))\Big). \end{aligned}$$

The rest of Theorem 4.1's follows immediately from the contraction mapping principle by using Estimate (4.8) and the arguments given in [2]. □

Next we prove strong convergence in $H^1(0,1)$ of u_h and v_h to u_0 and 0, respectively.

Theorem 4.2 *Assume that (u_h, v_h) is the unique solution of the problem (3.1)(3.2). Then*

$$\begin{aligned} u_h &\longrightarrow u_0 \ \textit{in } H^1(0,1), \\ v_h &\longrightarrow 0 \ \textit{in } H^1(0,1). \end{aligned}$$

Proof. By Theorem 4.1, u_h and v_h are uniformly bounded in H^1 with respect to h. Multiplying the equation satisfied by v_h on $[0,1]$ and integrating by parts over $[0,1]$, we get

$$\int_0^1 |v_h'|^2 - h^2 \int_0^1 k_2^2 |v_h|^2 + ih(k_{22}|v_h(0)|^2 + k_{21}|v_h(1)|^2) = h^2 \int_0^1 \chi_2 u_h^2 \overline{v_h}. \tag{4.9}$$

Thus

$$\int_0^1 |v_h'|^2 \le Ch^2 + Ch^2 |\int_0^1 \chi_2 u_h^2 \overline{v_h}|. \tag{4.10}$$

Since u_h is uniformly bounded in H^1 with respect to h, the Sobolev imbedding theorem implies that $\|u_h\|_{L^\infty(0,1)}$ is uniformly bounded in h. Hence

$$|\int_0^1 \chi_2 u_h^2 \overline{v_h}| \leq C\,\|u_h\|_{L^\infty(0,1)}\|u_h\|_{L^2(0,1)}\|v_h\|_{L^2(0,1)}.$$

Substituting the above estimate into (4.10), we obtain

$$\int_0^1 |v_h'|^2 \leq Ch^2\,, \tag{4.11}$$

with the constant C independent of h. Therefore, v_h' converges to 0 strongly in $L^2(0,1)$. Hence v_h converges strongly to a constant v_0 in $H^1(0,1)$.

By taking the imaginary part in (4.9), we have for a constant C independent of h that

$$|v_h(0)|^2 + |v_h(1)|^2 \leq Ch. \tag{4.12}$$

Since the convergence of v_h to v_0 is in $\mathcal{C}^0(0,1)$, the above inequality shows that $v_0 = 0$. Now let w_h be defined by (4.7). By arguments similar to the above, we can show that w_h converges to 0 strongly in $H^1(0,1)$. Hence u_h converges strongly to the constant u_0 given by (3.7) in $H^1(0,1)$. □

We are now in position to prove our main result of this paper.

Theorem 4.3 *There exist positive constants h_0, C_1, C_2, such that the estimates*

$$\begin{aligned} h^{1/2}\|u_h - (u_0 + hu_1)\|_{L^2(0,1)} + \|u_h' - (u_0' + hu_1' + h^2u_2')\|_{L^2(0,1)} &\leq C_1h^{5/2}, \\ h^{1/2}\|v_h - hv_1\|_{L^2(0,1)} + \|v_h' - (hv_1' + h^2v_2')\|_{L^2(0,1)} &\leq C_2h^{5/2} \end{aligned}$$

hold for any $0 < h < h_0$. Here C_1, C_2 are independent of h.

Proof. Let w_h and t_h be defined by

$$w_h = u_h - (u_0 + hu_1 + h^2u_2), \quad t_h = v_h - (hv_1 + h^2v_2).$$

It is easy to see that w_h and t_h satisfy respectively the boundary-value problems

$$\begin{gathered} w_h'' + k_1^2\,h^2w_h = h^2\chi_1\overline{u}_hv_h + 0(h^3) \quad \text{in } (0,1), \\ w_h'(0) - ik_{12}hw_h(0) = 0(h^3), \\ w_h'(1) + ik_{11}hw_h(1) = 0(h^3) \end{gathered}$$

and

$$\begin{gathered} t_h'' + k_2^2\,h^2t_h = h^2\chi_2(u_h^2 - u_0^2) + 0(h^3) \quad \text{in } (0,1), \\ t_h'(0) - ik_{22}ht_h(0) = 0(h^3), \\ t_h'(1) + ik_{21}ht_h(1) = 0(h^3). \end{gathered}$$

From (4.11) and (4.12) we may deduce that

$$\|v_h\|_{L^\infty(0,1)} \leq \|v_h\|_{H^1(0,1)} \leq Ch,$$

where C is a constant independent of h. Similarly, we have

$$\|u_h - u_0\|_{L^\infty(0,1)} \leq \|u_h - u_0\|_{H^1(0,1)} \leq \tilde{C}h,$$

with the constant $\tilde{C}$ independent of h.

Using the above estimates, we obtain that

$$\begin{aligned} \|\chi_1 \overline{u}_h v_h\|_{L^2(0,1)} &= 0(h), \\ \|\chi_2(u_h^2 - u_0^2)\|_{L^2(0,1)} &= 0(h). \end{aligned}$$

Hence we have the equations for w_h, t_h that

$$\begin{aligned} w_h'' + k_1^2\, h^2 w_h &= 0(h^3)\,, \\ t_h'' + k_2^2\, h^2 t_h &= 0(h^3)\,. \end{aligned}$$

A direct application of Lemma 4.1 yields the estimates in the statement of the theorem. The proof is now complete. □

5 Numerical experiments

The method presented here is compared to a finite element method stated in [2]. In Figure 1, the L^2 norm of the difference of the fundamental fields by using these two methods is plotted versus the depth of the nonlinear optical film; while in Figure 2, we plot the difference of the second harmonic fields versus the depth of the nonlinear medium. Here, the material parameters and the incident field are chosen exactly the same as in the example of [2] with appropriate units: $u^{in} = 100$, $\omega_1 = 0.3774$, $h = 0.1$, $\varepsilon_{11} = \varepsilon_{21} = 1.0$, $\varepsilon_{10} = \varepsilon_{20} = 5.6169$, $\varepsilon_{12} = -43.5375 + 1.98i$, $\varepsilon_{22} = -10.5459 + 0.8385i$, and $\chi_1 = \chi_2 = 3 \times 10^{-7}$.

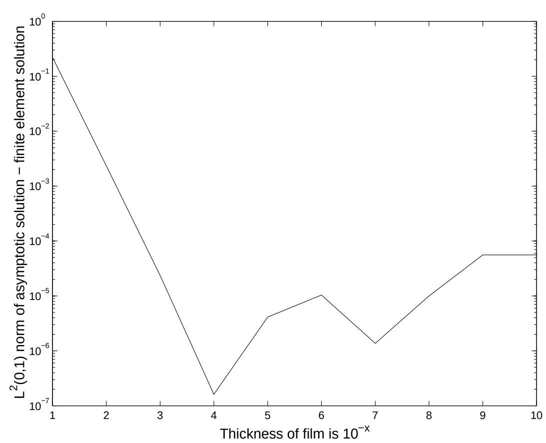

Figure 1: Comparison of asymptotic and finite element solutions: The pump (fundamental) fields.

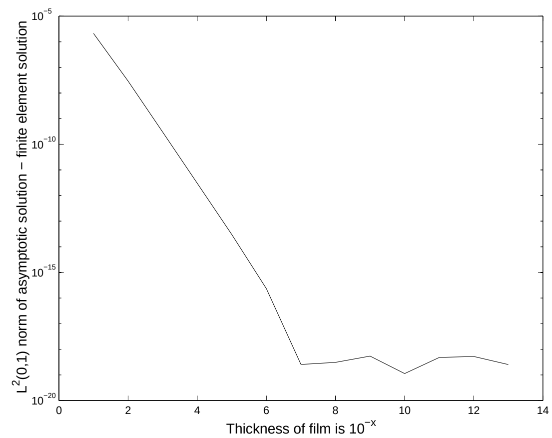

Figure 2: Comparison of asymptotic and finite element solutions: The second harmonic fields.

Our results indicate that the numerical solutions by using asymptotic and finite element methods agree well when h (the depth) is sufficiently small. However, in such a situation, our asymptotic method is certainly the better one because of its simple explicit form. In comparison, the method in [2] which is a combination of the finite element method and a fixed-point approach requires solving systems of linear equations iteratively. The size of the systems increases as h decreases. In addition, numerical instabilities can occur as h becomes exceedingly small.

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