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DINI-CAMPANATO SPACES AND APPLICATIONS TO NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We generalize a result due to Campanato [C] and use this to obtain regularity results for classical solutions of fully nonlinear elliptic equations. We demonstrate this technique in two settings. First, in the simplest setting of Poisson's equation $\Delta u = f$ in B, where f is Dini continuous in B, we obtain known estimates on the modulus of continuity of second derivatives D^2u in a way that does not depend on either differentiating the equation or appealing to integral representations of solutions. Second, we use this result in the concave, fully nonlinear setting $F(D^2u, x) = f(x)$ to obtain estimates on the modulus of continuity of D^2u when the L^n averages of f satisfy the Dini condition.

0. Introduction

Let $1 \le q \le \infty$ and let Ω be a domain in $\mathbb{R}^n$. For any Dini modulus of continuity $\omega(t)$ and $u \in L^q(\Omega)$, we define the seminorm

$$[u]'_{k,\omega} = [u]'_{q,k,\omega;\Omega} = \sup_{\substack{x_0\in\overline{\Omega} \\ 0<r\le d(\Omega)}} \left[\frac{1}{r^{kq+n}\omega(r)^q} \inf_{P\in\mathcal{P}_k} \int\limits_{\Omega_r(x_0)} |u(x) - P(x)|^q dx\right]^{1/q},$$

where $\Omega_r(x_0) = \overline{B}_r(x_0) \cap \Omega$ and $\mathcal{P}_k$ denotes the spaces of polynomials of degree less than or equal to k. We define the Dini-Campanato space $\mathcal{M}_q^{k,\omega}(\Omega)$ as the space of functions

$$\mathcal{M}_q^{k,\omega}(\Omega) = \left\{u \in L^q(\Omega) : [u]'_{q,k,\omega;\Omega} < +\infty\right\}.$$

Following Campanato's original proof (in [C]) of the inclusion $\mathcal{L}_k^{(q,\lambda)}(\Omega) \subset C^{k,\alpha}(\Omega)$, for $\alpha = (\lambda - n - kq)/q$, we obtain the regularity result $\mathcal{M}_q^{k,\omega}(\Omega) \subset C^{k,\omega_1}(\Omega)$, under the assumption that $\omega(t)$ is a Dini modulus of continuity and $\omega_1(t) = \int_0^t \frac{\omega(r)}{r}\, dr$. If $\omega(t)$ is a modulus of continuity, the space $C^{k,\omega}(\Omega)$ is defined in obvious generalization of the Hölder spaces, namely the space of all $u \in C^k(\Omega)$ with seminorm

$$[u]_{k,\omega;\Omega} = \sup_{\substack{x,y\in\Omega \\ |\beta|=k}} \frac{|D^\beta u(x) - D^\beta u(y)|}{\omega(|x-y|)} < +\infty\,.$$

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Our present result is more general than Campanato's original inclusion. Indeed, if $u \in \mathcal{L}_k^{(q,\lambda)}(\Omega)$, then $\omega(r) \sim r^\beta$ for some $\beta \in (0,1]$ and $\lambda = kq + n + \beta q$. Yet for $\omega(r) = r^\beta, \omega_1(t) = \int_0^t r^{\beta-1}, dr \sim t^\beta$ and so by our present result $u \in C^{k,\omega_1} = C^{k,\beta}$, whereas by Campanato's original inclusion, $u \in C^{k,\alpha}$ for $\alpha = \frac{\lambda-n-kq}{q} = \frac{(kq+n+\beta q)-n-kq}{q} = \beta$. On the other hand, there are examples of $u \in \mathcal{M}_q^{k,\omega}$, where $u \notin C^{k,\alpha}$ for any $\alpha > 0$. The special case $k = 0$, $q = 1$ was proved by Spanne [Sp].

In [C1],[CC], L. Caffarelli uses polynomial approximation to obtain *pointwise* Hölder estimates for derivatives of viscosity solutions to fully nonlinear elliptic equations. In the special case $\omega(t) \sim t^\alpha$, $C^{1,\alpha}$ estimates involve approximation by affine functions ($q = \infty, k = 1$), while $C^{2,\alpha}$ estimates involve approximation by paraboloids ($q = \infty, k = 2$). Using a generalization of the argument in Chapter 8 of [CC], we use the Dini Campanato inclusion to obtain regularity results for solutions of fully nonlinear elliptic equations. We illustrate this technique in two settings. In Chapter 2, in the simplest setting of Poisson's equation $\Delta u = f$ in B, where f is Dini continuous in B, we obtain known estimates on the modulus of continuity of second derivatives D^2u in a way that does not depend on either differentiating the equation or appealing to integral representations of solutions. In Chapter 3, we use this technique in the concave, fully nonlinear setting $F(D^2u, x) = f(x)$ to obtain estimates on the modulus of continuity of D^2u, when f and the oscillations of F in x are Dini continuous. Here, Dini continuity is measured in the weaker setting of L^n averages instead of the usual L^∞ norm. This condition was proposed by Wang (see his closing remark of Section 1.1) in [W].

Finally, we remark that even in the simplest setting of Poisson's equation, second derivatives of C^2 solutions will not, in general, be Dini continuous even when f is. For example, direct calculation shows that the function

$$u(x) = u(x_1, x_2) = x_1 x_2 \left(\ln \frac{1}{|x|}\right)^{-1}, \quad x \in B = B_{1/2}(0)$$

satisfies

$$\Delta u(x) = \frac{x_1 x_2 \left(\ln \frac{1}{|x|}\right)^{-2}}{|x|^2} \left(\frac{2}{\ln \frac{1}{|x|}} + 4\right) := f(x)$$

in B, where $f(x) = O\Big(\left(\ln \frac{1}{|x|}\right)^{-2}\Big)$ is Dini continuous in B with Dini modulus of continuity $\omega(t) \sim \left(\ln \frac{1}{t}\right)^{-2}$. However direct calculation shows that

$$D_{12}u(x) = \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} = O\Big(\left(\ln \frac{1}{|x|}\right)^{-1}\Big)$$

has modulus of continuity $\sim \left(\ln \frac{1}{t}\right)^{-1}$, which fails the Dini condition, since for any $\varepsilon > 0$

$$\int_0^\varepsilon \frac{\left(\ln \frac{1}{r}\right)^{-1}}{r}\, dr = \int_{\ln \frac{1}{\varepsilon}}^\infty u^{-1}\, du = +\infty\,.$$

It is well known, (see [GT] Chapter 4) that if $u \in C^2(B_r)$ satisfies $\Delta u = f$ in B_r, where $f \in C^\alpha(B_r)$, then $D^2u \in C^\alpha(B_{r/2})$. This "reproducing" regularity occurs not only for $\omega(t) = t^\alpha$ but more generally, for $\omega(t) = t^\alpha \left(\ln \frac{1}{t}\right)^\beta, \alpha \in (0,1)$. This can be seen by noting that both integrals in (13) are $\sim t^\alpha \left(\ln \frac{1}{t}\right)^\beta$, when $\omega(t) = t^\alpha \left(\ln \frac{1}{t}\right)^\beta$. See also [B],[K].

We recall that any modulus of continuity $\omega(t)$ is non-decreasing, subadditive, continuous and satisfies $\omega(0) = 0$. Hence any modulus of continuity $\omega(t)$ satisfies

$$\frac{\omega(r)}{r} \leq 2\frac{\omega(h)}{h}, \qquad 0 < h < r.$$

Indeed, by subadditivity, for $m \in \mathbb{N}$ and $h > 0$, we have $\omega(mh) \leq \ m\omega(h)$. Thus for $0 < h < r$, $\omega(r) = \omega(\frac{r}{h}h) \leq \omega(\lceil \frac{r}{h} \rceil h) \leq \lceil \frac{r}{h} \rceil \omega(h) \leq 2\frac{r}{h}\omega(h)$, where $\lceil a \rceil$ denotes the smallest integer $\geq a$. In particular, it immediately follows that $\omega(t) \leq 2\omega_1(t)$, since for $t > 0, \omega_1(t) = \int_0^t \frac{\omega(r)}{r}\, dr \geq \frac{\omega(t)}{2t} \int_0^t dr = \frac{\omega(t)}{2}$.

1. The Dini-Campanato Inclusion $\mathcal{M}_q^{k,\omega} \subset C^{k,\omega_1}$

We restrict ourselves to domains $\Omega \subset \mathbb{R}^n$ which satisfy the following property (this includes Lipschitz domains).

Definition 1.0. *We say that Ω satisfies property (I) if there exists a constant $A > 0$ such that $\forall x_0 \in \Omega, \forall r \in [0, d(\Omega)]$, the Lebesque measure of $\Omega_r(x_0)$, $|\Omega_r(x_0)|$ satisfies*

$$|\Omega_r(x_0)| \geq Ar^n.$$

Main Theorem. *Let $1 \leq q \leq \infty$. If $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, where Ω satisfies property (I), then $u \in C^{k,\omega_1}(\Omega)$, where $\omega_1(t) = \int_0^t \frac{\omega(r)}{r}\, dr$. That is, the kth order derivatives of u satisfy*

$$|D^k u(x) - D^k u(y)| \leq C(n,k,q,A)\omega_1(|x-y|) \qquad \forall x, y \in \Omega.$$

We begin the proof of the main theorem for the case $1 \leq q < \infty$ with a lemma due to De Giorgi.

Lemma 1.1 (De Giorgi). *If $P(x) \in \mathcal{P}_k$, $q \geq 1$ and E is any measurable subset of $\overline{B}_r(x_0)$ satisfying*

$$|E| \geq Ar^n,$$

then $\exists$ constant $c_1(k,q,n,A)$ such that $\forall$ n-tuple p of non-negative integers, we have

$$\left|[D^p P(x)]_{x=x_0}\right|^q \leq \frac{c_1}{r^{n+|p|q}} \int_E |P(x)|^q\, dx.$$

If $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, one can show that $\forall x_0 \in \Omega, \forall r \in [0, d(\Omega)], \exists P_k(x, x_0, r, u) \in \mathcal{P}_k$ such that

$$\int\limits_{\Omega_r(x_0)} |u(x) - P_k(x, x_0, r, u)|^q\, dx = \inf_{P \in \mathcal{P}_k} \int\limits_{\Omega_r(x_0)} |u(x) - P(x)|^q\, dx. \tag{1}$$

If $P(x)$ is an arbitrary polynomial in $\mathcal{P}_k$, then for convenience we write it in the form

$$P(x) = \sum_{|p| \leq k} \frac{a_p}{p!}(x - x_0)^p$$

and henceforth put $P_k(x, x_0, r)$ for $P_k(x, x_0, r, u)$ and set

$$a_p(x_0, r) = [D^p P_k(x, x_0, r)]_{x=x_0}. \tag{2}$$

Lemma 1.2. *If $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, then $\forall x_0 \in \overline{\Omega}, \forall\, 0 < r \le d(\Omega)$ and integer $h \ge 0$, we have*

$$\int_{\Omega_{\frac{r}{2^{h+1}}}(x_0)} \left|P_k\left(x, x_0, \tfrac{r}{2^h}\right) - P_k\left(x, x_0, \tfrac{r}{2^{h+1}}\right)\right|^q \, dx \le 2^{q+1}[u]'^q \omega\left(\tfrac{r}{2^h}\right)\left(\tfrac{r}{2^h}\right)^{kq+n}. \quad (3)$$

Proof. $\forall x_0 \in \overline{\Omega}$, $0 < r \le d(\Omega)$, integer $h \ge 0$, $x \in \Omega_{\frac{r}{2^{h+1}}}(x_0)$, by (1) and the definition of $\mathcal{M}_q^{k,\omega}(\Omega)$ we have

$$\begin{aligned}
&\int_{\Omega_{\frac{r}{2^{h+1}}}(x_0)} \left|P_k\left(x, x_0, \tfrac{r}{2^h}\right) - P_k\left(x, x_0, \tfrac{r}{2^{h+1}}\right)\right|^q dx \\
&\le 2^q \Big\{ \int_{\Omega_{\frac{r}{2^h}}(x_0)} \left|P_k\left(x, x_0, \tfrac{r}{2^h}\right) - u(x)\right|^q dx + \int_{\Omega_{\frac{r}{2^{h+1}}}(x_0)} \left|u(x) - P_k\left(x, x_0, \tfrac{r}{2^{h+1}}\right)\right|^q dx \Big\} \\
&\le 2^q \left\{ [u]'^q \omega\left(\tfrac{r}{2^h}\right)^q \left(\tfrac{r}{2^h}\right)^{kq+n} + [u]'^q \omega\left(\tfrac{r}{2^{h+1}}\right)^q \left(\tfrac{r}{2^{h+1}}\right)^{kq+n} \right\} \\
&\le 2^{q+1}[u]'^q \omega\left(\frac{r}{2^h}\right)^q \left(\frac{r}{2^h}\right)^{kq+n}. \quad \square
\end{aligned}$$

Lemma 1.3. *If Ω has property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, then $\forall$ two points $x_0, y_0 \in \overline{\Omega}$ and any n-tuple p of integers with $|p| = k, \exists\, c_2 = c_2(k, n, q, A)$ such that*

$$|a_p(x_0, 2|x_0 - y_0|) - a_p(y_0, 2|x_0 - y_0|)|^q \le c_2\, [u]'^q\, \omega(|x_0 - y_0|)^q. \quad (4)$$

Proof. Say $x_0, y_0 \in \overline{\Omega}$ and put $r = |x_0 - y_0|$, $I_r = \Omega(x_0, 2r) \cap \Omega(y_0, 2r)$. Then $\forall x \in \Omega(x_0, r) \subset I_r$, we have, again by (1) and the fact that $\omega(t)$ is a modulus of continuity

$$\begin{aligned}
&\int_{\Omega_r(x_0)} |P_k(x, x_0, 2r) - P_k(x, y_0, 2r)|^q \, dx \\
&\le 2^q \Big\{ \int_{\Omega_{2r}(x_0)} |P_k(x, x_0, 2r) - u(x)|^q \, dx + \int_{\Omega_{2r}(y_0)} |u(x) - P_k(x, y_0, 2r)|^q \, dx \Big\} \\
&\le 2^q \left\{ 2[u]'^q \omega(2r)^q (2r)^{kq+n} \right\} \\
&\le 2^{2q+1+kq+n}[u]'^q\, r^{kq+n} \omega(r)^q.
\end{aligned}$$

Applying Lemma 1.1 to the polynomial $P(x) = P_k(x, x_0, 2r) - P_k(x, y_0, 2r)$, and observing that the k-th derivatives of a polynomial of degree k are constant and hence can be evaluated independent of any particular point, we see that

$$\begin{aligned}
|a_p(x_0, 2r) - a_p(y_0, 2r)|^q &\le \frac{c_1}{r^{n+kq}} \int_{\Omega_r(x_0)} |P_k(x, x_0, 2r) - P_k(x, y_0, 2r)|^q \, dx \\
&\le \frac{c_1}{r^{n+kq}}\, 2^{2q+1+kq+n}[u]'^q\, r^{kq+n} \omega(r)^q \\
&= c_2 [u]'^q \omega(r)^q. \quad \square
\end{aligned}$$

Lemma 1.4. *If Ω has property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, then $\exists$ constant $c_3(q,k,n,A)$ such that $\forall x_0 \in \overline{\Omega}, 0 < r \le d(\Omega)$ and integer $i \ge 0$, $|p| \le k$, we have*

$$\left|a_p(x_0,r) - a_p\left(x_0, \frac{r}{2^i}\right)\right| \le c_3[u]' \sum_{h=0}^{i-1} \left(\frac{r}{2^h}\right)^{k-|p|} \omega\left(\frac{r}{2^h}\right). \tag{5}$$

Proof. With x_0, r, $|p| \le k$ as in the hypotheses, note that by (2) and (3)

$$\begin{aligned}
&\left|a_p(x_0,r) - a_p\left(y_0, \frac{r}{2^i}\right)\right| \le \sum_{h=0}^{i-1} \left|a_p\left(x_0, \tfrac{r}{2^h}\right) - a_p\left(x_0, \tfrac{r}{2^{h+1}}\right)\right| \\
&= \sum_{h=0}^{i-1} \left| D^p \left[P_k\left(x, x_0, \tfrac{r}{2^h}\right) - P_k\left(x, x_0, \tfrac{r}{2^{h+1}}\right)\right]_{x=x_0}\right| \\
&\le \sum_{h=0}^{i-1} \frac{c_1^{1/q}}{\left(\frac{r}{2^{h+1}}\right)^{\frac{n+|p|q}{q}}} \left\{ \int_{\Omega_{\frac{r}{2^{h+1}}}(x_0)} \left|P_k\left(x, x_0, \tfrac{r}{2^h}\right) - P_k\left(x, x_0, \tfrac{r}{2^{h+1}}\right)\right|^q dx \right\}^{1/q} \\
&\le \sum_{h=0}^{i-1} \frac{c_1^{1/q}}{\left(\frac{r}{2^{h+1}}\right)^{\frac{n+|p|q}{q}}} 2^{\frac{q+1}{q}} \left(\tfrac{r}{2^h}\right)^{\frac{n+kq}{q}} \omega\left(\tfrac{r}{2^h}\right) [u]' \\
&\le c_1^{1/q}\, 2^{\frac{n+(k+1)q+1}{q}} [u]' \sum_{h=0}^{i-1} \left(\tfrac{r}{2^h}\right)^{k-|p|} \omega\left(\tfrac{r}{2^h}\right). \quad \square
\end{aligned}$$

Lemma 1.5. *Let Ω have property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, where $\omega(t)$ is a Dini modulus of continuity. Then for every l with $0 \le l \le k$, there exists a system of functions $\{v_p(x_0)\}_{|p|\le l}$, defined in $\overline{\Omega}$ such that $\forall\, 0 < r \le d(\overline{\Omega}), x_0 \in \overline{\Omega}$ and $|p| \le l$, we have, for some constant $c_5 = c_5(q,k,n,A)$*

$$|a_p(x_0,r) - v_p(x_0)| \le c_5[u]'\, r^{k-|p|}\omega_1(r), \qquad \text{where} \qquad \omega_1(t) = \int_0^t \frac{\omega(r)}{r}\, dr. \tag{6}$$

$$\textit{Consequently } \forall x_0 \in \overline{\Omega},\ \lim_{r\to 0} a_p(x_0,r) = v_p(x_0) \ \textit{ uniformly.} \tag{7}$$

Proof. Fix $x_0 \in \overline{\Omega}$, $0 < r \le d(\Omega)$, $|p| \le l$. We will show that the sequence $\{a_p(x_0, \frac{r}{2^i})\}$ converges as $i \to \infty$. Indeed, if i, j are non-negative integers with $j > i$, then by Lemma 1.4 we have

$$\left|a_p\left(x_0, \frac{r}{2^j}\right) - a_p\left(x_0, \frac{r}{2^i}\right)\right| \le c_3[u]' \sum_{h=i}^{j-1} \omega\left(\frac{r}{2^h}\right)\left(\frac{r}{2^h}\right)^{k-|p|}.$$

But since $\omega(t)$ is a Dini modulus of continuity, the integral test, applied to the non-negative, non-increasing sequence $\{\omega(\frac{r}{2^h})\}_{h=0}^{\infty}$ yields that the series $\sum_{h=0}^{\infty}\omega(\frac{r}{2^h})$ converges. Indeed, by the integral test

$$\sum_{h=0}^{\infty} \omega\left(\frac{r}{2^h}\right) \le \omega(r) + \int_1^{\infty} \omega\left(\frac{r}{2^{x-1}}\right) dx = \omega(r) + \frac{1}{\ln 2}\int_0^r \frac{\omega(t)}{t}\, dt \le \Big(2 + \frac{1}{\ln 2}\Big)\omega_1(r).$$

Thus, $\{a_p(x_0, \frac{r}{2^i})\}$ is a Cauchy sequence, and hence convergent. Moreover the limit will be independent of our choice of $r \in [0, d(\Omega)]$. Indeed, if r_1, r_2 satisfy $0 < r_1 \le r_2 \le d(\Omega)$, then by Lemma 1.1 and the definition of $\mathcal{M}_q^{k,\omega}(\Omega)$, we get that

$$\begin{aligned}
&\left|a_p\left(x_0, \tfrac{r_1}{2^i}\right) - a_p\left(x_0, \tfrac{r_2}{2^i}\right)\right|^q \le \left| D^p \left[P_k\left(x, x_0, \tfrac{r_1}{2^i}\right) - P_k\left(x, x_0, \tfrac{r_2}{2^i}\right)\right]_{x=x_0}\right|^q \\
&\le \frac{c_1}{(\frac{r_1}{2^i})^{n+|p|q}} \int_{\Omega_{\frac{r_1}{2^i}}(x_0)} \left|P_k\left(x, x_0, \tfrac{r_1}{2^i}\right) - P_k\left(x, x_0, \tfrac{r_2}{2^i}\right)\right|^q \, dx \\
&\le \frac{c_1 2^q}{(\frac{r_1}{2^i})^{n+|p|q}} \Big\{ \int_{\Omega_{\frac{r_1}{2^i}}(x_0)} \left|P_k\left(x, x_0, \tfrac{r_1}{2^i}\right) - u(x)\right|^q \, dx + \int_{\Omega_{\frac{r_2}{2^i}}(x_0)} \left|u(x) - P_k\left(x, x_0, \tfrac{r_2}{2^i}\right)\right|^q \, dx \Big\} \\
&\le \frac{c_1 2^q}{(\frac{r_1}{2^i})^{n+|p|q}} \left\{ [u]'^q \omega\left(\frac{r_1}{2^i}\right)^q \left(\frac{r_1}{2^i}\right)^{kq+n} + [u]'^q \omega\left(\frac{r_2}{2^i}\right)^q \left(\frac{r_2}{2^i}\right)^{kq+n} \right\} \\
&\le \frac{c_1 2^{q+1} [u]'^q}{(\frac{r_1}{2^i})^{n+|p|q}} \omega\left(\frac{r_2}{2^i}\right) \left(\frac{r_2}{2^i}\right)^{kq+n} \\
&= c_1 2^{q+1} [u]'^q \omega\left(\frac{r_2}{2^i}\right) \frac{r_2^{kq+n}}{r_1^{|p|q+n}} 2^{iq(|p|-k)} \quad \to 0 \text{ as } i \to \infty \quad (\text{even if } |p| = k).
\end{aligned}$$

Thus, $\{a_p(x_0, \frac{r}{2^i})\}$ converges independent of our choice of $r \in [0, d(\Omega)]$ and $\forall x_0 \in \overline{\Omega}$, $0 < r \le d(\Omega)$, $|p| \le l \le k$, $\lim_{i\to\infty} a_p(x_0, \frac{r}{2^i}) = v_p(x_0)$. Furthermore, from (5) in Lemma 1.4 we have

$$\left| a_p(x_0, r) - a_p\left(x_0, \frac{r}{2^i}\right)\right| \le c_3\, [u]'\, r^{k-|p|} \sum_{h=0}^{i-1} \omega\left(\frac{r}{2^h}\right).$$

Letting $i \to \infty$, we get

$$|a_p(x_0, r) - v_p(x_0)| \le c_3\, [u]'\, r^{k-|p|} \sum_{h=0}^{\infty} \omega\left(\frac{r}{2^h}\right) \le c_5\, [u]'\, r^{k-|p|} \omega_1(r). \quad \square \tag{8}$$

Theorem 1.6. *If Ω has property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, where $\omega(t)$ is a Dini modulus of continuity, then the functions $v_p(x)$ with $|p| = k$ have modulus of continuity $\omega_1(t)$ in $\overline{\Omega}$ and $\forall x, y \in \overline{\Omega}$, we have, with $\omega_1(t) = \int_0^t \frac{\omega(r)}{r}\, dr$*

$$|v_p(x) - v_p(y)| \le c_6\, [u]' \omega_1(|x-y|), \tag{9}$$

for some constant $c_6 = c_6(k, q, n, A)$.

Proof. Fix an n-tuple of nonnegative integers $p = (p_1, ..., p_n)$ with $|p| = k$ and let $x, y \in \overline{\Omega}$. Since Ω is connected, we may assume that $r = |x-y| \le \frac{d(\Omega)}{2}$. By Lemma 1.5 (with $2r$ in place of r), Lemma 1.3 and the fact that $\omega_1(mr) \le m\omega_1(r) \forall m \in \mathbb{N}$, we have

$$\begin{aligned}
|v_p(x) - v_p(y)| &\le |v_p(x) - a_p(x, 2r)| + |a_p(x, 2r) - a_p(y, 2r)| + |a_p(y, 2r) - v_p(y)| \\
&\le c_5 [u]'\, \omega_1(2r) + c_2^{1/q} [u]'\, \omega(r) + c_5 [u]' \omega_1(2r) \\
&\le c_6 [u]' \omega_1(|x-y|). \quad \square
\end{aligned}$$

Theorem 1.7. *If Ω has property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, where $\omega(t)$ is a Dini modulus of continuity, then the functions $v_p(x)$ with $|p| \le k-1$ have first order partial derivatives in Ω and $\forall x \in \Omega$, we have*

$$\frac{\partial v_p(x)}{\partial x_i} = v_{(p+e_i)}(x) \quad (i = 1, 2, ..., n). \tag{10}$$

Proof. By Theorem 1.6, the $v_p(x)$ with $|p| = k$ are continuous in $\overline{\Omega}$. Our theorem will be proved by induction under the additional assumption that the $v_{(p+\delta e_i)}(x)$ are continuous in $\overline{\Omega}$ for $\delta = 1, 2, ..., k-|p|$. So let $|p| \le k-1$, $1 \le i \le n$, $x_0 \in \Omega$ and choose r so small that $\overline{B}_{|r|}(x_0) \subset \Omega$. We have

$$\begin{aligned} \frac{a_p(x_0+e_ir, 2|r|) - a_p(x_0, 2|r|)}{r} &= \frac{D^p[P_k(x, x_0+e_ir, 2|r|) - P_k(x, x_0, 2|r|)]_{x=x_0}}{r} \\ &\quad - \sum_{\delta=1}^{k-|p|} \frac{(-1)^\delta}{\delta!} r^{\delta-1} a_{(p+\delta e_i)}(x_0+e_ir, 2|r|). \end{aligned} \tag{11}$$

First, since $k-|p|-1 \ge 0$, applying Lemma 1.1, we see that

$$\begin{aligned} &\left| \frac{D^p[P_k(x, x_0+e_ir, 2|r|) - P_k(x, x_0, 2|r|)]_{x=x_0}}{r} \right|^q \\ &\le \frac{c_1}{(2|r|)^{n+|p|q}} \int\limits_{\Omega_{|r|}(x_0)} |P_k(x, x_0+e_ir, 2|r|) - P_k(x, x_0, 2|r|)|^q \, dx \\ &\le \frac{c_1}{(2|r|)^{n+|p|q}} \, 2^{2q+1+kq+n} \, |r|^{kq+n} [u]'^q \omega(|r|) \\ &= c[u]'^q \omega(|r|) \, |r|^{q(k-|p|)} \quad \to 0 \text{ as } r \to 0. \end{aligned}$$

Second, $\forall \delta$ with $1 \le \delta \le k-|p|$, by (8) we have

$$\begin{aligned} &|a_{(p+\delta e_i)}(x_0+e_ir, 2|r|) - v_{(p+\delta e_i)}(x_0)| \\ &\le |a_{(p+\delta e_i)}(x_0+e_ir, 2|r|) - v_{(p+\delta e_i)}(x_0+e_ir)| + |v_{(p+\delta e_i)}(x_0+e_ir) - v_{(p+\delta e_i)}(x_0)| \\ &\le c_5[u]'(2|r|)^{k-|p|}\omega_1(2|r|) + |v_{(p+\delta e_i)}(x_0+e_ir) - v_{(p+\delta e_i)}(x_0)| \end{aligned}$$

But since the $v_{(p+\delta e_i)}(x)$ are continuous for $\delta = 1, 2, \ldots, k-|p|$, we immediately get

$$\lim_{r\to 0} a_{(p+\delta e_i)}(x_0+e_ir, 2|r|) = v_{(p+\delta e_i)}(x_0) \quad \delta = 1, 2, ..., k-|p|.$$

Hence by (11), we get (uniformly with respect to x_0)

$$\lim_{r\to 0} \frac{a_p(x_0+e_ir, 2|r|) - a_p(x_0, 2|r|)}{r} \left(= \lim_{r\to 0} a_{(p+e_i)}(x_0+e_ir, 2|r|) \right) = v_{(p+e_i)}(x_0).$$

It remains only to verify that

$$\lim_{r\to 0} \frac{v_p(x_0+e_ir) - v_p(x_0)}{r} = \lim_{r\to 0} \frac{a_p(x_0+e_ir, 2|r|) - a_p(x_0, 2|r|)}{r}.$$

Recalling that $|p| \le k-1$, we write

$$\frac{v_p(x_0+e_ir)-v_p(x_0)}{r} = \frac{v_p(x_0+e_ir)-a_p(x_0+e_ir,2|r|)}{r} + \frac{a_p(x_0+e_ir,2|r|)-a_p(x_0,2|r|)}{r} + \frac{a_p(x_0,2|r|)-v_p(x_0)}{r}$$

But by the first inequality in (8), we see that the first and third summands $\to 0$ as $r \to 0$, proving the theorem. $\square$

Theorem 1.8. *If Ω has property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, where $\omega(t)$ is a Dini modulus of continuity, then the function $v_{(0)}(x) \in C^{k,\omega_1}(\overline{\Omega})$ and $\forall x \in \Omega$, $|p| \le k$,*

$$D^p v_{(0)}(x) = v_p(x).$$

Proof. Immediate corollary of Theorems 1.7 and 1.8.

Main Theorem. *If Ω has property (I) and $u \in \mathcal{M}_q^{k,\omega}(\Omega)$, where $\omega(t)$ is a Dini modulus of continuity, then $u \in C^{k,\omega_1}(\Omega)$*

$$[u]_{k,\omega_1;\Omega} \le c_6[u]'_{q,k,\omega;\Omega}.$$

Proof. Recall that if $u \in L^q(\Omega)$, then by Lebesgue's theorem, for almost every $x_0 \in \Omega$ we have

$$\lim_{r\to 0} \frac{1}{|\Omega_r(x_0)|} \int\limits_{\Omega_r(x_0)} |u(x)-u(x_0)|^q\, dx = 0. \tag{12}$$

So choose $x_0 \in \Omega$ for which (12) holds. For almost every $x \in \Omega$, we have

$$\begin{aligned}&|a_{(0)}(x_0,r)-u(x_0)|^q\\ &\le c_6(q)\Big\{|a_{(0)}(x_0,r)-P_k(x,x_0,r)|^q + |P_k(x,x_0,r)-u(x)|^q + |u(x)-u(x_0)|^q\Big\}\end{aligned}$$

and hence integrating over $\Omega_r(x_0)$ gives

$$\begin{aligned}&|a_{(0)}(x_0,r)-u(x_0)|^q \le \frac{c_6}{|\Omega_r(x_0)|}\int\limits_{\Omega_r(x_0)} |a_{(0)}(x_0,r)-P_k(x,x_0,r)|^q\, dx\\ &+ \frac{c_6}{|\Omega_r(x_0)|}\int\limits_{\Omega_r(x_0)} |P_k(x,x_0,r)-u(x)|^q\; dx + \frac{c_6}{|\Omega_r(x_0)|}\int\limits_{\Omega_r(x_0)} |u(x)-u(x_0)|^q\, dx\\ &= I_1+I_2+I_3.\end{aligned}$$

By Lebesgue's theorem, $I_3 \to 0$ as $r \to 0$. Since Ω satisfies property (I), as $r \to 0$

$$I_2 \le \frac{c_6}{Ar^n}\int\limits_{\Omega_r(x_0)} |P_k(x,x_0,r)-u(x)|^q\; dx \le \frac{c_6\, r^{kq+n}}{Ar^n}\omega(r)^q[u]'^q \to 0 \text{ as } r \to 0.$$

Finally, for some constant $c_7 = c_7(A,n,q,k)$, we have

$$\begin{aligned} I_1 \leq & \frac{c_6}{Ar^n} \int_{\Omega_r(x_0)} |a_{(0)}(x_0,r) - P_k(x,x_0,r)|^q \, dx \\ \leq & c_7 \sum_{1\leq |p| \leq k} |a_p(x_0,r)|^q r^{|p|q} \to 0 \quad \text{as } r \to 0. \end{aligned}$$

Thus, $|a_{(0)}(x_0,r) - u(x_0)|^q \to 0$ as $r \to 0$ and so, for almost every $x_0 \in \Omega$

$$\lim_{r\to 0} a_{(0)}(x_0,r) = u(x_0).$$

But then by (7) we have $u(x_0) = \lim_{r\to 0} a_{(0)}(x_0,r) = v_{(0)}(x_0) \in C^{k,\omega_1}(\Omega)$. Since $x_0 \in \Omega$ was arbitrary, $u \equiv v_{(0)}$ and thus $\forall x,y \in \Omega$, $|p| = k$, Theorems 1.6 and 1.8 give, with $c_6 = c_6(k,q,n,A)$

$$|D^p u(x) - D^p u(y)| = |D^p v_{(0)}(x) - D^p v_{(0)}(y)| = |v_p(x) - v_p(y)| \leq c_6 [u]' \omega_1(|x-y|).$$

That is,

$$[u]_{k,\omega_1;\Omega} \leq c_6 [u]'_{q,k,\omega;\Omega} \,,$$

which proves our Main Theorem for the case $1 \leq q < \infty$. $\square$

Remark. For the case $q = \infty$, the proof is the same (yet easier), and the space

$$\mathcal{M}^{k,\omega}_\infty(\Omega) = \left\{ u \in L^\infty(\Omega) : [u]'_{\infty,k,\omega;\Omega} < +\infty \right\}$$

is defined by way of the finite seminorm

$$[u]'_{\infty,k,\omega;\Omega} = \sup_{\substack{x_0\in\overline{\Omega} \\ 0<r\leq d(\Omega)}} \frac{1}{r^k \omega(r)} \inf_{p\in\mathcal{P}_k} \|u-p\|_{L^\infty(\Omega_r(x_0))}.$$

When Ω is convex, it follows by Taylor's formula that $C^{k,\omega}(\Omega) \subset \mathcal{M}^{k,\omega}_q(\Omega)$, and hence when Ω is convex and satisfies property (I), by our main theorem, we have the inclusion $C^{k,\omega}(\Omega) \subset \mathcal{M}^{k,\omega}_q(\Omega) \subset C^{k,\omega_1}(\Omega)$. In our applications, we will use only the case $q = \infty$.

Remark. The inclusion $\mathcal{M}^{k,\omega}_\infty \subset C^{k,\omega_1}$ is sharp in the sense that ω_1 cannot be replaced by a smaller modulus of continuity. In particular (since $\omega(t) \leq 2\omega_1(t)$), $\mathcal{M}^{k,\omega}_\infty \not\subset C^{k,\omega}$. The following example demonstrates this, as well as provides an example of $u \in \mathcal{M}^{k,\omega}_\infty$ where $u \notin C^{k,\alpha} \forall \alpha > 0$.

Example. Let $k = n = 1$, $q = \infty$. Consider the function

$$u(x) = x\left(\ln\frac{1}{|x|}\right)^{-1}, \qquad x \in \Omega = B_{1/2}(0).$$

Note that

$$u'(x) = \left(\ln\frac{1}{|x|}\right)^{-1} + \left(\ln\frac{1}{|x|}\right)^{-2},$$

and hence $u'(x)$ has modulus of continuity $\sim \left(\ln \frac{1}{t}\right)^{-1}$, while $u \in \mathcal{M}^{1,\omega}_{\infty}(B_{1/2}(0))$ for $\omega(t) = \left(\ln \frac{1}{t}\right)^{-2}$. But

$$\omega_1(t) = \int_0^t \frac{\left(\ln \frac{1}{r}\right)^{-2}}{r}\, dr = \left(\ln \frac{1}{t}\right)^{-1}.$$

That is, $Du = u'$ has modulus of continuity $\sim \omega_1(t)$, hence our inclusion above is sharp. To verify that $u \in \mathcal{M}^{1,\omega}_{\infty}$, i.e. that $[u]'_{\infty,1,\omega} < +\infty$, fix $x \in \Omega = B_{1/2}(0)$ and take $r > 0$. For any $y \in B_r(x)$, set $p(y) = T_{1,x}u(y) \in \mathcal{P}_1$. Of course, $u''(x) \leq 3\left(\ln \frac{1}{|x|}\right)^{-2}/|x|$, for all $x \in \Omega$. Now if $|x| \geq 2r$, by Taylor's Theorem, for some $z \in (y, x)$, we have

$$\begin{aligned} |u(y) - p(y)| &= \Big|\frac{u''(z)}{2}(y-x)^2\Big| \leq \frac{3\left(\ln \frac{1}{|z|}\right)^{-2}|y-x|^2}{2|z|} \\ &\leq \frac{3\left(\ln \frac{1}{r}\right)^{-2} r^2}{2r} = \frac{3r\left(\ln \frac{1}{r}\right)^{-2}}{2} = \frac{3}{2} r\omega(r). \end{aligned}$$

On the other hand, if $|x| < 2r$, choose $p(y) = y\left(\ln \frac{1}{r}\right)^{-1} \in \mathcal{P}_1$. Without loss of generality, since u is an odd function, we may consider $x > 0$. By the Mean Value Theorem, we have, for some $z \in (y, r)$

$$\begin{aligned} \sup_{y \in B_r(x)} |u(y) - p(y)| &\leq \sup_{y \in B_{3r}(0)} |u(y) - p(y)| = \sup_{|y| \leq 3r} \left| y\left(\ln \frac{1}{|y|}\right)^{-1} - y\left(\ln \frac{1}{|r|}\right)^{-1} \right| \\ &\leq \sup_{|y| \leq 3r} |y| \left| \frac{\left(\ln \frac{1}{|z|}\right)^{-2}}{|z|}(y - r) \right| \\ &\leq 3r\left(\ln \frac{1}{r}\right)^{-2} = 3r\omega(r), \end{aligned}$$

hence $[u]'_{\infty,1,\omega} \leq 3$, since $x \in B_{1/2}$, $r > 0$ were arbitrary. Thus $u \in \mathcal{M}^{1,\omega}_{\infty}(B_{1/2})$. Note however that $u \notin C^{1,\omega}(B_{1/2})$, since

$$\begin{aligned} \sup_{x \neq y \in B_{1/2}(0)} \frac{|u'(x) - u'(y)|}{\omega(|x-y|)} &\geq \sup_{x \neq 0 \in B_{1/2}(0)} \frac{|u'(x)|}{\omega(|x|)} \\ &= \sup_{x \neq 0 \in B_{1/2}(0)} \frac{\left(\ln \frac{1}{|x|}\right)^{-1} + \left(\ln \frac{1}{|x|}\right)^{-2}}{\left(\ln \frac{1}{|x|}\right)^{-2}} = +\infty\,. \end{aligned}$$

Thus, $[u]'_{\infty,1,\omega} < +\infty$, while $[u]_{1,\omega} = +\infty$ and so in general, even if $\omega(t)$ is a Dini modulus of continuity, the seminorms $[u]'_{q,k,\omega;\Omega}$ and $[u]_{k,\omega;\Omega}$ are not equivalent. Moreover, $u \notin C^{1,\alpha}(B_{1/2}(0))$ for *any* $\alpha > 0$. Since $u'(0) = 0$, we have

$$\begin{aligned} \sup_{x \neq y \in B_{1/2}(0)} \frac{|u'(x) - u'(y)|}{|x-y|^{\alpha}} &\geq \sup_{x \neq 0 \in B_{1/2}(0)} \frac{|u'(x)|}{|x|^{\alpha}} \\ &= \sup_{x \neq 0 \in B_{1/2}(0)} \frac{\left(\ln \frac{1}{|x|}\right)^{-1} + \left(\ln \frac{1}{|x|}\right)^{-2}}{|x|^{\alpha}} = +\infty\,. \end{aligned}$$

2. Interior regularity for $\Delta u = f$

In this section, we give an application of the inclusion $\mathcal{M}^{2,\omega}_{\infty}(B) \subset C^{2,\omega_1}(B)$ in the simplest setting. We use this inclusion to obtain estimates on the modulus of continuity of second derivatives of classical solutions of Poisson's equation $\Delta u = f$ in B, where f is Dini continuous in B, i.e. $f \in C^{0,\omega}(B)$. Using potential theory, various authors (see [ME], [B], [HW]) have shown that if $u \in C^2(\overline{B}_2(x_0))$ satisfies $\Delta u = f$ in $B_2(x_0)$, then for all $0 < r < 1$, we have

$$\sup_{|x-y|\le r} |D^2u(x) - D^2u(y)| \le C\Big\{\int_0^r \frac{\omega(t)}{t}\,dt + r\int_r^c \frac{\omega(t)}{t^2}\,dt\Big\}, \tag{13}$$

where C depends only on $n, \omega, |u|_{0;B_1}$ and $|f|_{0,\omega;B_1}$ and c is independent of r. Of course, when $f \in C^{\alpha}(B_2)$, i.e. $\omega(t) \sim t^{\alpha}$, $0 < \alpha < 1$, the right hand side of (13) is $\le Cr^{\alpha}$. But for general Dini moduli of continuity, neither of the summands in the right hand side of (13) can be omitted, as simple examples show. The usual way of obtaining this estimate is by a lengthy examination of the Newtonian potential of f. By using the Dini-Campanato inclusion, we can obtain this estimate in a simpler way. Specifically, we will show that if $u \in C^2(\overline{B}_1(0))$ satisfies $\Delta u = f \in C^{0,\omega}(B_1(0))$, then $u \in \mathcal{M}^{2,\varphi}_{\infty}(B_{1/2}(0)) \subset C^{2,\varphi_1}(B_{1/2}(0))$, for an appropriate Dini modulus of continuity φ, where φ_1 will be the right hand side of (13). It suffices to show $\exists \delta = \delta(n,\omega) > 0$, such that if $|u|_{0;B_1} \le 1$ and $|f|_{0,\omega} \le \delta$, then $u \in \mathcal{M}^{2,\varphi}_{\infty}(B_{1/2}(0)$. The estimate (2.0) will follow by rescaling. For our solution u, consider the function

$$\tilde{u}(x) = \frac{u(x)}{|u|_{0;B_1} + \delta^{-1}|f|_{\omega;B_1}} := \frac{u(x)}{K}, \quad \text{if } K \ge 1$$

(Otherwise, consider $\tilde{u} = u$.) Note that $\tilde{u}$ satisfies $|\tilde{u}|_{0;B_1} \le 1$ and $\Delta\tilde{u} = \frac{f}{K} := \tilde{f}$ in B_1, where $\tilde{f}$ is Dini continuous in B_1 and $|\tilde{f}|_{\omega;B_1} \le \frac{|f|_{\omega;B_1}}{K} \le \delta$. That $u \in \mathcal{M}^{2,\varphi}_{\infty}(B_{1/2}(0)$ follows from the following lemma.

Lemma 2.1. *Take any $x_0 \in B_{1/2}(0)$. There exists $0 < \mu < 1$ depending only on n, ω and a sequence of paraboloids*

$$P_k(x) = P_{k,x_0}(x) = a_k + b_k \cdot (x - x_0) + \frac{(x-x_0)^t C_k (x-x_0)}{2}$$

such that $\forall k \in \mathbb{N}^+$

$$tr(C_k) = 0$$
$$|u - P_k|_{0;B_{\mu^k}(x_0)} \le \mu^{2k}\varphi(\mu^k),$$

where $P_0 \equiv 0$ and $\varphi(t) = t\int_t^c \frac{\omega(r)}{r^2}\,dr, \quad t \in (0, c/2].$

Proof. In the upper limit of the integral defining φ, we usually take $c \le 1$, depending on the domain of definition of ω. (e.g. if $\omega(t) = t^{\alpha}\left(\ln\frac{1}{t}\right)^{\beta}, \alpha \in (0,1)$, we can take $c = 1$.) Note since ω is nondecreasing, by the definition of $\varphi(t)$, we always have

$$\omega(t) \le \left(\frac{c}{c-t}\right)\varphi(t) \le 2\varphi(t).$$

We may assume $x_0 = 0$ and $f(0) = 0$. First choose μ so small, depending on ω, so that

$$N_1 81\mu c_e \le \frac{1}{2}, \qquad \mu \le \frac{7}{16}, \qquad \omega(\mu) \le \frac{1}{2}$$

and then choose $\delta = \frac{\mu^3}{4N_2}$, where c_e, N_1, N_2 are constants depending only on n. Observe that by considering $\varphi(Kt)$ instead of $\varphi(t)$ (and considering smaller values of t) we may assume $\varphi(1) \ge 1$ and hence the claim holds for $k = 0$, since $P_0 \equiv 0$, $tr(0) = 0$ and $|u|_{0;B_1(0)} \le 1$. Assume it holds for $k = i$. We now show it holds for $k = i + 1$. So for this fixed i, consider the function

$$v(x) = \frac{(u - P_i)(\mu^i x)}{\mu^{2i}\varphi(\mu^i)} \qquad x \in B_1(0),$$

which, by inductive hypothesis, satisfies

$$\Delta v(x) = \frac{\Delta u(\mu^i x) - tr(C_i)}{\varphi(\mu^i)} = \frac{f(\mu^i x)}{\varphi(\mu^i)} := f_i(x) \quad \text{in } B_1(0), \quad |v|_{0;B_1(0)} \le 1.$$

Let $h \in C^\infty(\overline{B}_{7/8}(0))$ be the solution to the Dirichlet problem

$$\begin{gathered} \Delta h = 0 \text{ in } B_{7/8}(0) \\ h = v \text{ on } \partial B_{7/8}(0) \end{gathered}$$

with

$$[h]_{4,0;B_{\frac{7}{16}}(0)} \le \left(\tfrac{16}{7}\right)^4 c_e |v|_{0;\partial B_{7/8}(0)} \le 81 c_e |v|_{0;B_1(0)} \le 81 c_e,$$

for some constant $c_e = c_e(n)$. By Taylor's formula, for

$$T_{2,0}h(x) = h(0) + Dh(0)x + \frac{1}{2}x^t D^2 h(0)x \quad \in \mathcal{P}_2,$$

we have

$$|h - T_{2,0}h|_{0;B_\mu(0)} \le N_1(n)[h]_{4,0;B_\mu(0)}\mu^4 \le N_1[h]_{4,0;B_{\frac{7}{16}}(0)}\mu^4 \le N_1 81\mu^4 c_e\,.$$

Since $f(0) = 0$, the classical a priori estimates yield, for some constant $N_2 = N_2(n)$

$$\begin{aligned} |v - h|_{0;B_{7/8}(0)} \le& |v - h|_{0;\partial B_{7/8}(0)} + N_2 \left(\tfrac{7}{8}\right)^2 |\Delta v - \Delta h|_{0;B_{7/8}(0)} \\ \le& N_2 |f_i|_{0;B_{7/8}(0)} \le N_2 [f]_\omega \frac{\omega(\mu^i)}{\varphi(\mu^i)} \le 2N_2[f]_{0,\omega} \end{aligned}$$

Thus

$$|v - T_{2,0}h|_{0;B_\mu(0)} \le |v - h|_{0;B_\mu(0)} + |h - T_{2,0}h|_{0;B_\mu(0)} \le 2N_2[f]_{0,\omega} + N_1 81\mu^4 c_e.$$

So for $x \in B_{\mu^{i+1}}(0)$, set $P_{i+1}(x) = P_i(x) + \mu^{2i}\varphi(\mu^i)T_{2,0}h\left(\frac{x}{\mu^i}\right) \in \mathcal{P}_2$. Rescaling back, plugging in the definition of v, recalling that μ, δ are small and that $\mu\varphi(\mu^i) \le$

$\varphi(\mu^{i+1})$ we get, $\forall x \in \overline{B}_{\mu^{i+1}}(0)$

$$\begin{aligned}
|u(x) - P_{i+1}(x)| &= \left|u(x) - P_i(x) - \mu^{2i}\varphi(\mu^i)T_{2,0}h\left(\tfrac{x}{\mu^i}\right)\right| \\
&= \mu^{2i}\varphi(\mu^i)\left|v\left(\tfrac{x}{\mu^i}\right) - T_{2,0}h\left(\tfrac{x}{\mu^i}\right)\right| \\
&\le \mu^{2i}\varphi(\mu^i)|v - T_{2,0}h|_{0;B_\mu(0)} \\
&\le \mu^{2i}\varphi(\mu^i)\Big\{2N_2[f]_\omega + N_1 81\mu^4 c_e\Big\} \\
&= \mu^{2(i+1)}\Big\{2N_2[f]_\omega\frac{\varphi(\mu^i)}{\mu^2} + N_1 81\varphi(\mu^i)\mu^2 c_e\Big\} \\
&\le \mu^{2(i+1)}\Big\{2N_2[f]_\omega\frac{\varphi(\mu^{i+1})}{\mu^3} + N_1 81\varphi(\mu^{i+1})\mu c_e\Big\} \\
&\le \mu^{2(i+1)}\varphi(\mu^{i+1})\Big\{\frac{2N_2\delta}{\mu^3} + N_1 81\mu c_e\Big\} \\
&\le \mu^{2(i+1)}\varphi(\mu^{i+1}),
\end{aligned}$$

and hence $|u - P_{i+1}|_{0;B_{\mu^{i+1}(0)}} \le \mu^{2(i+1)}\varphi(\mu^{i+1})$. Moreover, by definition of P_{i+1}, $C_{i+1} = C_i + \varphi(\mu^i)D^2h(0)$ from which it follows

$$tr(C_{i+1}) = tr(C_i) + \varphi(\mu^i)\Delta h(0) = 0,$$

which completes the proof of Lemma 2.1. $\square$

By Lemma 2.1, we know that $\forall\, x_0 \in B_{1/2}(0), \exists\, 0 < \mu < 1$ (depending only on n, ω) and a sequence $\{P_k\} = \{P_{k,x_0}\} \subset \mathcal{P}_2$ such that

$$|u - P_k|_{0;B_{\mu^k}(x_0)} \le \mu^{2k}\varphi(\mu^k) \quad \forall k \ge 0.$$

So, $\forall\, 0 < r \le 1$, choose $k \ge 0$ so large that $\mu^{k+1} < r \le \mu^k$. Since $\{P_k\} \subset \mathcal{P}_2$, we immediately get

$$\begin{aligned}
\inf_{p\in\mathcal{P}_2}|u - p|_{0;B_r(x_0)} &\le \inf_{p\in\mathcal{P}_2}|u - p|_{0;B_{\mu^k}(x_0)} \\
&\le |u - P_k|_{0;B_{\mu^k}(x_0)} \le \mu^{2k}\varphi(\mu^k) = \tfrac{\mu^{2(k+1)}}{\mu^2}\varphi\left(\tfrac{\mu^{k+1}}{\mu}\right) \\
&\le \frac{1}{\mu^3}r^2\varphi(r) = C_1 r^2\varphi(r).
\end{aligned}$$

Since $0 < r \le 1 = d(B_{1/2}(0))$ and $x_0 \in B_{1/2}(0)$, are arbitrary, we have with $q = \infty$

$$[u]'_{2,\varphi;B_{1/2}(0)} = \sup_{\substack{0<r\le d(B_{1/2}(0)) \\ x_0\in B_{1/2}(0)}} \frac{1}{r^2\varphi(r)}\inf_{p\in\mathcal{P}_2}|u - p|_{0;B_r(x_0)\cap B_{1/2}(0)} \le C_1.$$

That is, $u \in \mathcal{M}^{2,\varphi}_\infty(B_{1/2}(0))$. Since φ is a Dini modulus of continuity (since ω is), by our Dini-Campanato inclusion, we have $u \in C^{2,\varphi_1}(B_{1/2}(0))$ and

$$[u]_{2,\varphi_1;B_{1/2}(0)} \le N[u]'_{2,\varphi;B_{1/2}(0)} \le C_2.$$

But by definition of $\varphi(t)$ and Fubini's theorem, we have

$$\begin{aligned}\varphi_1(t) &= \int_0^t \frac{\varphi(r)}{r}\,dr = \int_0^t \int_r^c \frac{\omega(\rho)}{\rho^2}\,d\rho\,dr \\ &= \int_0^t \int_0^\rho \frac{\omega(\rho)}{\rho^2}\,dr\,d\rho + \int_t^c \int_0^t \frac{\omega(\rho)}{\rho^2}\,dr\,d\rho \\ &= \int_0^t \frac{\omega(\rho)}{\rho}\,d\rho + t\int_t^c \frac{\omega(\rho)}{\rho^2}\,d\rho.\end{aligned}$$

3. Interior Regularity for $F(D^2u, x) = f(x)$

In this chapter, we use the inclusion $\mathcal{M}^{2,\omega}_\infty(B) \subset C^{2,\omega_1}(B)$ to estimate the modulus of continuity of second derivatives D^2u of solutions of fully nonlinear elliptic equations $F(D^2u, x) = f(x)$ in $B = B_1(0)$. Here, we assume that f is Dini continuous in B_1, in the weaker L^n (as opposed to L^∞) sense with Dini modulus of continuity $\omega(t)$. That is, we assume that $\forall x_0 \in B_1(0)$

$$\left\{\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |f(x) - f(x_0)|^n\,dx\right\}^{1/n} \le C\omega(r), \quad \forall r < 1.$$

We further assume that $\omega(t)$ satisfies the following property

$$\lim_{\mu\to 0+} \sup_{0\le t\le \frac12} \frac{\mu^{\overline{\alpha}}\varphi(t)}{\varphi(t\mu)} = 0, \qquad \text{where } \varphi(t) := t^{\overline{\alpha}} + \omega(t), \tag{14}$$

where $\overline{\alpha} = \overline{\alpha}(n, \lambda, \Lambda) \in (0, 1)$ is the Hölder exponent given in the Evans-Krylov theorem. This restriction was not required in the linear case, since there, we had solvability of the constant coefficient Dirichlet problem with any order of smoothness. In the fully nonlinear setting however, we have solvability of the constant coefficient Dirichlet problem with order of smoothness only $2 + \overline{\alpha}$.

Remark. As strong a condition as (14) appears, it is satisfied by $\omega(t) = t^\alpha \left(\ln \frac{1}{t}\right)^\beta$, $0 < \alpha < \overline{\alpha}, \beta \in \mathbb{R}$. This enables us to generalize the known result for Hölder continuous f, i.e. $f \in C^{0,\alpha}(B)$, $0 < \alpha < \overline{\alpha}$. Indeed, for $\omega(t) = t^\alpha \left(\ln \frac{1}{t}\right)^\beta$, $0 < \alpha < \overline{\alpha}$, integration by parts gives that $\int_0^t \frac{r^\alpha \left(\ln \frac{1}{r}\right)^\beta}{r}\,dr \le Ct^\alpha \left(\ln \frac{1}{t}\right)^\beta$ and hence by Theorem 3.1 below, D^2u has modulus of continuity $\le C\psi(t)$, where

$$\psi(t) = t^{\overline{\alpha}} + \int_0^t \frac{r^\alpha \left(\ln \frac{1}{r}\right)^\beta}{r}\,dr \le t^{\overline{\alpha}} + Ct^\alpha \left(\ln \frac{1}{t}\right)^\beta \le C_1 t^\alpha \left(\ln \frac{1}{t}\right)^\beta.$$

Taking $\beta = 0$, we recover the well-known result for $f \in C^{0,\alpha}(B), 0 < \alpha < \overline{\alpha}$. Note that $\psi(t)$ is a Dini modulus of continuity. This is not always the case, as our Example 3.1 shows.

More importantly, (14) holds for $\omega(t) = \left(\ln \frac{1}{t}\right)^\beta, \beta < -1$. The significance of this class of moduli of continuity satisfying (14) is that it permits us to consider f whose L^n averages are Dini continuous, yet not in $C^{0,\alpha}(B)$ for *any* $\alpha > 0$. (See Example 3.1 below.) Property (14) fails for Dini moduli of continuity which are "nice" compared to $t^{\overline{\alpha}}$. Indeed (14) implies that $\lim_{t\to 0+}\frac{t^{\overline{\alpha}}}{\omega(t)}=0$, which generalizes the $0<\alpha<\overline{\alpha}$ condition. Hence (14) fails for $\omega(t)=t^{\overline{\alpha}}$, $\omega(t)=t\left(\ln\frac{1}{t}\right)^{\beta},\beta\geq 0$ and most notably for $\omega\equiv 0$. But if $\omega\equiv 0$, then f is constant and by the Evans-Krylov theorem, $D^2u\in C^{0,\overline{\alpha}}$. Furthermore, for sufficiently small $t>0$, $t\left(\ln\frac{1}{t}\right)^{\beta}\leq t^{\alpha},\forall\alpha\in(0,1)$. Hence any f whose L^n averages are $\sim t\left(\ln\frac{1}{t}\right)^{\beta},\beta\geq 0$ will automatically have L^n averages belonging to $C^{0,\alpha}(B),\forall\alpha\in(0,\overline{\alpha})$ and hence by Safonov's result (see [S1]), $D^2u\in C^{0,\alpha}_{loc}(B)$. We cannot conclude however, that if $\omega(t)$ fails (14) then $\omega(t)\leq Ct^{\overline{\alpha}}$, since for example, $\omega(t)=t^{\overline{\alpha}}\ln\frac{1}{t}$, has limit 1 in (14). Even in this case, the regularity of second derivatives is covered by known results, since for sufficiently small $t>0$, $t^{\overline{\alpha}}\ln\frac{1}{t}\leq t^{\alpha}\ \forall\alpha\in(0,\overline{\alpha})$. Thus, property (14) enables us to generalize well-known regularity results for Hölder continuous f (subject to the restriction $0<\alpha<\overline{\alpha}$) and extend these results to a large class of functions whose L^n averages are Dini, yet non-Hölder continuous.

Example 3.1. Consider the uniformly elliptic, concave equation

$$F(D^2u,x)=f(x):=\left(\ln\frac{1}{|x|}\right)^{-2}\quad\text{in } B=B_{1/2}(0).$$

Taking $x_0=0$ (since $f(0)=0$), the inequalities

$$\begin{aligned}C(n)\left(\ln\frac{1}{r}\right)^{-2}&\leq\Big\{\frac{n}{r^n}\int_0^r\rho^{n-1}\left(\ln\frac{1}{\rho}\right)^{-2n}d\rho\Big\}^{1/n}=\Big\{\fint_{B_r(0)}\left(\ln\frac{1}{|x|}\right)^{-2n}dx\Big\}^{1/n}\\&\leq\left(\ln\frac{1}{r}\right)^{-2}\end{aligned}$$

show that f is not Hölder continuous at $x_0=0$ in the L^n sense for *any* $\alpha\in(0,1)$. Here, $\fint$ denotes average. Yet clearly, f is Dini continuous in B in the L^n sense, since for $x_0\in B$, by the subadditivity of the function $\left(\ln\frac{1}{t}\right)^{-2}$ for $t>0$ small, we have

$$\begin{aligned}\Big\{\fint_{B_r(x_0)}\left|\left(\ln\frac{1}{|x|}\right)^{-2}-\left(\ln\frac{1}{|x_0|}\right)^{-2}\right|^n dx\Big\}^{1/n}&\leq\Big\{\fint_{B_r(x_0)}\left(\ln\frac{1}{|x-x_0|}\right)^{-2n}dx\Big\}^{1/n}\\&\leq\left(\ln\frac{1}{r}\right)^{-2}\end{aligned}$$

and $\omega(r)=\left(\ln\frac{1}{r}\right)^{-2}$ is a Dini modulus of continuity which satisfies (14). Hence by our Theorem 3.1 below, locally, D^2u has modulus of continuity $\leq C\psi(t)$, where for sufficiently small $t>0$

$$\psi(t)=t^{\overline{\alpha}}+\int_0^t\frac{\left(\ln\frac{1}{r}\right)^{-2}}{r}dr=t^{\overline{\alpha}}+\left(\ln\frac{1}{t}\right)^{-1}\leq C_1\left(\ln\frac{1}{t}\right)^{-1}.$$

Observe that $\psi(t)$ is not a Dini modulus of continuity.

Now consider the function

$$\tilde{\beta}(x,x_0)=\tilde{\beta}_F(x,x_0)=\sup_{M\in S}\frac{|F(M,x)-F(M,x_0)|}{\|M\|+1},$$

which measures the oscillation of F in x near the point $x=x_0\in B$. For our Theorem 3.1, we must impose some sort of continuity restriction on $\tilde{\beta}(\cdot,x_0)$, since even in the linear case $Lu=tr\left[A(x)D^2u\right]=a_{ij}(x)D_{ij}u=f(x)$ (for Hölder continuity) we require that f and a_{ij} belong to $C^{0,\alpha}$. Hence we require that both f and all $\tilde{\beta}(\cdot,x_0)$ belong to $C^{0,\omega}(B)$ in the L^n sense. The following is a generalization of the argument used by Caffarelli in [C1],[CC] to prove pointwise $C^{2,\alpha}$ estimates for viscosity solutions of $F(D^2u,x)=f(x)$.

Theorem 3.1. *Let F be concave, uniformly elliptic (with ellipticity constants λ and Λ), F and f are continuous in x. Suppose that f, as well as all the oscillations of F in x, belong to $C^{0,\omega}(B_1)$ in the L^n sense, where $\omega(t)$ is a Dini modulus of continuity satisfying property (14). If $u\in C^2(B_1)$ is a solution of $F(D^2u,x)=f(x)$ in $B_1(0)$, then $u\in C^{2,\psi}(B_{1/2}(0))$, where for $0\le t\le 1/2$*

$$\psi(t)=t^{\overline{\alpha}}+\int_0^t\frac{\omega(r)}{r}\,dr,$$

where $\overline{\alpha}=\overline{\alpha}(n,\lambda,\Lambda)\in(0,1)$ is the Hölder exponent given in the Evans-Krylov theorem.

Proof.. Since $\omega(t)$ is a Dini modulus of continuity, assume for definiteness that $\int_0^1\frac{\omega(r)}{r}\,dr<+\infty$. Following routine normalizations (see [CC] p.75), we may assume $|u|_{0;B_1}\le 1$. It suffices to prove $\exists\delta>0$ (small enough) depending only on n,λ,Λ,ω such that if $u\in C^2(B_1)$ is a solution of $F(D^2u,x)=f(x)$ in $B_1=B_1(0)$ and if $\forall x_0\in B_{1/2}(0)$

$$\Big\{\fint_{B_r(x_0)}\tilde{\beta}(x,x_0)^n\,dx\Big\}^{1/n}\le\delta\omega(r),\quad\Big\{\fint_{B_r(x_0)}|f(x)-f(x_0)|^n\,dx\Big\}^{1/n}\le\delta\omega(r)\quad\forall r\le 1,$$

then $u\in C^{2,\psi}(\overline{B_{1/2}(0)})$. It suffices to prove the following lemma.

Lemma 3.2. *Take any $x_0\in B_{1/2}(0)$. There exists $0<\mu<1$ depending only on n,λ,Λ,ω and a sequence of polynomials*

$$P_k(x)=a_k+b_k\cdot(x-x_0)+\frac{1}{2}(x-x_0)^tC_k(x-x_0)$$

such that $F(C_k,x_0)=0$ for all $k\ge 0$, $|u-P_k|_{0;B_{\mu^k}(x_0)}\le\mu^{2k}\varphi(\mu^k)$ for all $k\ge 0$ and

$$|a_k-a_{k-1}|+\mu^{k-1}|b_k-b_{k-1}|+\mu^{2(k-1)}\|C_k-C_{k-1}\|\le 13c_e\mu^{2(k-1)}\varphi(\mu^{k-1}),$$

where $P_0\equiv P_{-1}\equiv 0$, c_e is a universal constant and $\varphi(t)=t^{\overline{\alpha}}+\omega(t)$.

Proof. As before, we assume that $x_0=0$ and that $F(0,0)=0=f(0)$. First choose μ small enough (depending only on n,ω,λ,Λ) such that (14) holds and $\omega(\mu)\le 1/2$, $\mu\le 7/16$. Then choose δ such that

$$2N_1\omega_n^{1/n}\delta(52c_e\varphi_1(1)\,+1)(9c_e+2)\le c_e\mu^{2+\overline{\alpha}},$$

where $c_e = c_e(n,\lambda,\Lambda)$ is the constant in the Evans-Krylov theorem, $N_1 = N_1(n,\lambda,\Lambda)$ is from the Alexandrov estimates and ω_n is the volume of the unit ball. Note that δ depends only on n,λ,Λ,ω. The claim holds for $k=0$ since $P_0 \equiv P_{-1} \equiv 0$, $F(0,0) = 0$ and $|u|_{0;B_1(0)} \le 1$. Assume it holds for $k=i$. We now show it holds for $k=i+1$. So for this fixed i, consider the function

$$v(x) = \frac{(u-P_i)(\mu^i x)}{\mu^{2i}\varphi(\mu^i)} \quad x \in B_1(0),$$

which satisfies $F\big(\varphi(\mu^i)\,D^2v(x) + C_i, \mu^i x\big) = f(\mu^i x)$ and hence $F_i(D^2v,x) = f_i(x)$ in $B_1(0)$, where

$$F_i(M,x) = \frac{F(\varphi(\mu^i)M + C_i,\mu^i x) - F(C_i,\mu^i x)}{\varphi(\mu^i)}, \quad f_i(x) = \frac{f(\mu^i x) - F(C_i,\mu^i x)}{\varphi(\mu^i)}.$$

Now $F_i(M,x)$ is concave in M and has ellipticity constants λ,Λ (since F does), and $F_i(0,x) = 0$. By the Evans-Krylov theorem, $\exists\, h \in C^{2,\overline{\alpha}}_{loc}(\overline{B}_{7/8}(0))$ solving

$$\begin{gathered} F_i(D^2h,0) = 0 \text{ in } B_{7/8}(0) \\ h = v \text{ on } \partial B_{7/8}(0) \end{gathered}$$

and

$$\|h\|^*_{C^{2,\overline{\alpha}}(B_{7/16}(0))} \le c_e|v|_{0;\partial B_{7/8}(0)} \le c_e|v|_{0;B_1(0)} \le c_e$$

where $c_e = c_e(n,\lambda,\Lambda)$. By Taylor's formula, for

$$T_{2,0}h(x) = h(0)\; + Dh(0)x + \frac{1}{2}\,x^tD^2h(0)x \in \mathcal{P}_2,$$

we have

$$\begin{aligned} |h - T_{2,0}h|_{0;B_\mu(0)} &\le [h]_{2,\overline{\alpha};B_\mu(0)}\mu^{2+\overline{\alpha}} \le [h]_{2,\overline{\alpha};B_{\frac{7}{16}}(0)}\mu^{2+\overline{\alpha}} \\ &\le \big(\tfrac{16}{7}\big)^{2+\overline{\alpha}}\, c_e\mu^{2+\overline{\alpha}} \le c_e 27\mu^{2+\overline{\alpha}}. \end{aligned}$$

By the classical Alexandrov estimates, we have, for some constant $N_1 = N_1(n,\lambda,\Lambda)$

$$\begin{aligned} |v-h|_{0;B_{7/8}(0)} &\le |v-h|_{0;\partial B_{7/8}(0)} + N_1\|F_i(D^2h,\cdot) - F_i(D^2v,\cdot)\|_{L^n(B_{7/8})} \\ &= N_1\|F_i(D^2h,\cdot) - f_i\|_{L^n(B_{7/8})} \\ &\le N_1\Big\{\|F_i(D^2h,\cdot) - F_i(D^2h,0)\|_{L^n(B_{7/8})} + \|f_i\|_{L^n(B_{7/8})}\Big\} \\ &\le N_1\Big\{\|\tilde{\beta}_{F_i}(\cdot,0)\|_{L^n(B_1)}(9c_e+1) + \|f_i\|_{L^n(B_1)}\Big\} \end{aligned}$$

We need to estimate both $\|\tilde{\beta}_{F_i}(\cdot,0)\|_{L^n(B_1)}$ and $\|f_i\|_{L^n(B_1)}$. For $x \in B_1(0)$,

$$\begin{aligned} &\tilde{\beta}_{F_i}(x,0) = \sup_{M\in\mathcal{S}} \frac{|F_i(M,x) - F_i(M,0)|}{\|M\|+1} \\ &= \sup_{M\in\mathcal{S}} \left| \frac{\big[F(\varphi(\mu^i)M + C_i,\mu^i x) - F(\varphi(\mu^i)M + C_i,0)\big] - \big[F(C_i,\mu^i x) - F(C_i,0)\big]}{\varphi(\mu^i)(\|M\|+1)} \right| \\ &\le \sup_{M\in\mathcal{S}} \left(\frac{\|\varphi(\mu^i)M + C_i\| + 1 + \|C_i\| + 1}{\|M\|+1} \right) \frac{\widetilde{\beta}(\mu^i x,0)}{\varphi(\mu^i)} \\ &\le \sup_{M\in\mathcal{S}} \left(\frac{\varphi(\mu^i)\|M\| + 2(\|C_i\|+1)}{\|M\|+1} \right) \frac{\widetilde{\beta}(\mu^i x,0)}{\varphi(\mu^i)} \end{aligned}$$

Since ω (hence φ) is a Dini modulus of continuity, the integral test yields

$$\begin{aligned}\|C_i\| \le \sum_{k=1}^{i} \|C_k - C_{k-1}\| &\le 13c_e \sum_{k=1}^{\infty} \varphi(\mu^{k-1}) \\ &\le 13c_e \left(\varphi(1) + \ln\left(\tfrac{1}{\mu}\right)^{-1} \int_0^1 \frac{\varphi(r)}{r}\, dr \right) \le 52 c_e \varphi_1(1).\end{aligned}$$

Hence for $x \in B_1(0)$

$$\begin{aligned}\widetilde{\beta_{F_i}}(x,0) &\le \sup_{M \in \mathcal{S}} \left(\frac{\varphi(\mu^i)\|M\| + 2\big(52c_e\varphi_1(1)+1\big)}{\|M\|+1} \right) \frac{\widetilde{\beta}(\mu^i x, 0)}{\varphi(\mu^i)} \\ &\le 2\left(52c_e\varphi_1(1)+1\right) \frac{\widetilde{\beta}(\mu^i x,0)}{\varphi(\mu^i)}\,,\end{aligned}$$

and thus since $\omega \le \varphi$ and the L^n average of $\tilde{\beta}(\cdot,0)$ is small, we get

$$\begin{aligned}\|\tilde{\beta_{F_i}}(\cdot,0)\|_{L^n(B_1)} &\le 2\left(52c_e\varphi_1(1)+1\right) \frac{\|\widetilde{\beta}(\mu^i\cdot,0)\|_{L^n(B_1)}}{\varphi(\mu^i)} \\ &\le 2\left(52c_e\varphi_1(1)+1\right) \frac{\omega_n^{1/n}\delta\omega(\mu^i)}{\varphi(\mu^i)} \le 2\omega_n^{1/n}\delta\left(52c_e\varphi_1(1)+1\right).\end{aligned}$$

Similarly, for $x \in B_1(0)$

$$\begin{aligned}|f_i(x)| = \frac{|f(\mu^i x) - F(C_i,\mu^i x)|}{\varphi(\mu^i)} &\le \frac{|f(\mu^i x)| + |F(C_i,0) - F(C_i,\mu^i x)|}{\varphi(\mu^i)} \\ &\le \frac{|f(\mu^i x)| + \widetilde{\beta}(\mu^i x,0)(\|C_i\|+1)}{\varphi(\mu^i)} \\ &\le \frac{|f(\mu^i x)| + \widetilde{\beta}(\mu^i x,0)\left(52c_e\varphi_1(1)+1\right)}{\varphi(\mu^i)},\end{aligned}$$

which implies, since the L^n average of f is small

$$\begin{aligned}\|f_i\|_{L^n(B_1)} &\le \frac{\|f(\mu^i\cdot)\|_{L^n(B_1)} + \|\widetilde{\beta}(\mu^i\cdot,0)\|_{L^n(B_1)}\left(52c_e\varphi_1(1)+1\right)}{\varphi(\mu^i)} \\ &\le \frac{\omega_n^{1/n}\delta\omega(\mu^i) + \omega_n^{1/n}\delta\omega(\mu^i)\cdot\left(52c_e\varphi_1(1)+1\right)}{\varphi(\mu^i)} \\ &\le 2\omega_n^{1/n}\delta\left(52c_e\varphi_1(1)+1\right).\end{aligned}$$

Returning to our a priori estimates and recalling that δ is small, we get

$$\begin{aligned}|v-h|_{0;B_{7/8}} &\le N_1\Big\{ \|\tilde{\beta}_{F_i}(\cdot,0)\|_{L^n(B_1)}(9c_e+1) + \|f_i\|_{L^n(B_1)} \Big\} \\ &\le N_1\Big\{ 2\delta\omega_n^{1/n}\left(52c_e\varphi_1(1)+1\right)(9c_e+1) + 2\delta\omega_n^{1/n}\left(52c_e\varphi_1(1)+1\right) \Big\} \\ &\le N_1 2\delta\omega_n^{1/n}\left(52c_e\varphi_1(1)+1\right)(9c_e+2) \\ &\le c_e\mu^{2+\overline{\alpha}},\end{aligned}$$

and hence, since $\mu \le \frac{7}{16}$, we have

$$|v - T_{2,0}h|_{0;B_\mu(0)} \le |v-h|_{0;B_\mu(0)} + |h - T_{2,0}h|_{0;B_\mu(0)} \le 28c_e\mu^{2+\overline{\alpha}}.$$

Now, for $x \in B_{\mu^{i+1}}(0)$, set $P_{i+1}(x) = P_i(x) + \mu^{2i}\varphi(\mu^i)T_{2,0}h\left(\frac{x}{\mu^i}\right) \in \mathcal{P}_2$. Rescaling back, plugging in the definition of v and recalling that $\omega(t)$ satisfies (14), we get

$$\begin{aligned}|u(x) - P_{i+1}(x)| &= \left|u(x) - P_i(x) - \mu^{2i}\varphi(\mu^i)T_{2,0}h\left(\tfrac{x}{\mu^i}\right)\right|\\ &= \mu^{2i}\varphi(\mu^i)\left|v\left(\tfrac{x}{\mu^i}\right) - T_{2,0}h\left(\tfrac{x}{\mu^i}\right)\right|\\ &\le \mu^{2i}\varphi(\mu^i)28c_e\mu^{2+\overline{\alpha}}\\ &\le \mu^{2(i+1)}\varphi(\mu^{i+1}),\end{aligned}$$

i.e. $|u - P_{i+1}|_{0;B_{\mu^{i+1}}(0)} \le \mu^{2(i+1)}\varphi(\mu^{i+1})$, completing the induction step. Note that P_{i+1}'s coefficients satisfy

$$C_{i+1} = C_i + \varphi(\mu^i)D^2h(0), \quad b_{i+1} = b_i + \mu^i\varphi(\mu^i)Dh(0), \quad a_{i+1} = a_i + \mu^{2i}\varphi(\mu^i)h(0).$$

Hence $F(C_{i+1},0) = F(\varphi(\mu^i)D^2h(0) + C_i, 0) = \varphi(\mu^i)F_i(D^2h(0),0) + F(C_i,0) = 0$. Since $\|h\|^*_{C^{2,\overline{\alpha}}(B_{\frac{7}{16}}(0))} \le c_e$, we have

$$\begin{aligned}&|a_{i+1} - a_i| + \mu^i|b_{i+1} - b_i| + \mu^{2i}\|C_{i+1} - C_i\|\\ &\le \mu^{2i}\varphi(\mu^i)\Big(|h(0)| + |Dh(0)| + \|D^2h(0)\|\Big)\\ &\le \mu^{2i}\varphi(\mu^i)\Big(c_e + \frac{16}{7}c_e + \left(\frac{16}{7}\right)^2 c_e\Big)\\ &\le 13c_e\mu^{2i}\varphi(\mu^i).\end{aligned}$$

This completes the proof of Lemma 3.2. $\square$

The above argument holds at any fixed $x_0 \in B_{1/2}(0)$ since for concave F, the Evans-Krylov theorem guarantees the solvability of the Dirichlet problem for $F(D^2h, x_0) = 0$, with universal constant c_e. The same argument which follows Lemma 2.1 now gives us that $u \in \mathcal{M}^{2;\varphi}_\infty(B_{1/2}(0)) \subset C^{2,\varphi_1}(B_{1/2}(0))$. But by definition of $\varphi(t)$, we have

$$\varphi_1(t) = \int_0^t \frac{\varphi(r)}{r}\,dr = \int_0^t r^{\overline{\alpha}-1}\,dr + \int_0^t \frac{\omega(r)}{r}\,dr \sim \psi(t),$$

which completes the proof of Theorem 3.1.

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