Electronic Journal of Differential Equations, Vol. 1999(1999), No. 40, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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EXISTENCE RESULTS FOR HAMILTONIAN ELLIPTIC SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We prove the existence of nontrivial solutions to the system

$$\Delta u = u, \quad \Delta v = v,$$

on a bounded set of $\mathbb{R}^N$, with nonlinear coupling at the boundary given by

$$\partial u/\partial\eta = H_v, \quad \partial v/\partial\eta = H_u\,.$$

The proof is done under suitable assumptions on the Hamiltonian H, and based on a variational argument that is a generalization of the mountain pass theorem. Under further assumptions on the Hamiltonian, we prove the existence of positive solutions.

1. Introduction.

In this paper we study the existence of nontrivial solutions of the elliptic system

$$\begin{aligned} \Delta u &= u \\ \Delta v &= v \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

with nonlinear coupling at the boundary given by

$$\begin{aligned} \frac{\partial u}{\partial\eta} &= H_v(x,u,v) \\ \frac{\partial v}{\partial\eta} &= H_u(x,u,v) \quad x \in \partial\Omega\,. \end{aligned} \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary (say $C^{2,\alpha}$), $\frac{\partial}{\partial\eta}$ is the outer normal derivative and $H : \partial\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a smooth positive function (say C^1) with growth control on H and its first derivatives.

Existence results for nonlinear elliptic systems have created a great deal of interest in recent years, in particular when the nonlinear term appears as a source in the equation with Dirichlet boundary conditions. For this type of result see, among others, [2, 3, 4, 6, 7, 9] and the survey [5]. There are two major classes of systems that can be treated variationally: Hamiltonian and gradient systems. Here we deal with a Hamiltonian problem. Problems with no variational structure can

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1991 *Mathematics Subject Classification.* 35J65, 35J20, 35J55.
Key words and phrases. elliptic systems, nonlinear boundary conditions, variational problems.

Submitted May 30, 1999. Published October 7, 1999.
The first and third author were supported by Univ. de Buenos Aires grant TX047 and by ANPCyT PICT No. 03-00000-00137. The third author was also supported by Fundación Antorchas.

be treated via fixed-point arguments. For example (1.1) with nonlinear boundary conditions and without variational assumptions has been studied in [10].

This work is inspired by the article [6] where the authors study

$$\begin{aligned} -\Delta u &= H_v(x,u,v) \\ -\Delta v &= H_u(x,u,v), \end{aligned}$$

with Dirichlet boundary conditions (see also [12]). They prove existence of strong solutions using the same variational arguments as we use here and, as in our case, they were forced to impose similar growth restrictions on H.

The crucial part here is to find the proper functional setting for (1.1)-(1.2) that allows us to treat our problem variationally. We accomplish this by defining a self-adjoint operator that takes into account the boundary conditions together with the equations and considering its fractional powers that satisfy a suitable "integration by parts" formula. Once we have done this the proof follows the steps used in [6], but we include here the arguments in order to make the paper self contained. For the proof we use a linking theorem in a version due to Felmer [8]. Linking theorems have been a useful tool in obtaining existence results for elliptic problems; see for example [1].

We observe that the techniques used here can be applied to the semilinear system

$$\begin{aligned} -\Delta u + u &= G_v(x,u,v) \\ -\Delta v + v &= G_u(x,u,v) \quad x \in \Omega\,, \end{aligned}$$

with boundary conditions (1.2). However, for clarity of exposition we state and prove our results for (1.1)-(1.2). The general case requires hypotheses on G that are similar to those in [6].

The precise assumptions on the Hamiltonian H are

$$|H(x,u,v)| \le C\left(|u|^{p+1} + |v|^{q+1} + 1\right), \tag{1.3}$$

and for small positive r, if $|(u,v)| \le r$, then

$$|H(x,u,v)| \le C\left(|u|^{\alpha} + |v|^{\beta}\right), \tag{1.4}$$

where the exponents satisfy $p+1 \ge \alpha > p > 0$ and $q+1 \ge \beta > q > 0$ with

$$1 > \frac{1}{\alpha} + \frac{1}{\beta}, \tag{1.5}$$

$$\max\left\{\frac{p}{\alpha} + \frac{q}{\beta};\ \frac{q}{q+1}\frac{p+1}{\alpha} + \frac{p}{p+1}\frac{q+1}{\beta}\right\} < 1 + \frac{1}{N-1}\,, \tag{1.6}$$

$$\frac{p}{p+1}\frac{q+1}{\beta} < 1 \quad \text{and} \quad \frac{q}{q+1}\frac{p+1}{\alpha} < 1\,. \tag{1.7}$$

If $N \ge 4$, we have to impose the additional hypothesis

$$\max\left\{\frac{p}{\alpha};\ \frac{q}{\beta};\ \frac{q}{q+1}\frac{p+1}{\alpha};\ \frac{p}{p+1}\frac{q+1}{\beta}\right\} < \frac{N+1}{2(N-1)}\,. \tag{1.8}$$

When $\alpha = p+1$ and $\beta = q+1$, conditions (1.5), (1.6) and (1.8) become

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{1}{N-1}, \tag{1.9}$$

$$p, q \le \frac{N+1}{N-3} \quad \text{if } N \ge 4. \tag{1.10}$$

Remark 1.1. *These hypothesis and (1.5)-(1.8), imply that there exists* s *and* t *with* $s+t=1$, $s,t>1/4$ *such that*

$$\frac{\alpha-p}{\alpha}>\frac{1}{2}-\frac{2s-1/2}{N-1},\quad \frac{\beta-q}{\beta}>\frac{1}{2}-\frac{2t-1/2}{N-1},$$
$$1-\frac{p(q+1)}{\beta(p+1)}>\frac{1}{2}-\frac{2s-1/2}{N-1},\quad 1-\frac{q(p+1)}{\alpha(q+1)}>\frac{1}{2}-\frac{2t-1/2}{N-1}.$$

On the derivatives of H we impose the following:

$$\begin{aligned}\Big|\frac{\partial H}{\partial u}(x,u,v)\Big| &\le C\left(|u|^p+|v|^{p(q+1)/(p+1)}+1\right),\\ \Big|\frac{\partial H}{\partial v}(x,u,v)\Big| &\le C\left(|u|^{q(p+1)/(q+1)}+|v|^q+1\right).\end{aligned}\tag{1.11}$$

And for R large, if $|(u,v)|\ge R$,

$$\frac{1}{\alpha}\frac{\partial H}{\partial u}(x,u,v)u+\frac{1}{\beta}\frac{\partial H}{\partial v}(x,u,v)v\ge H(x,u,v)>0,\tag{1.12}$$

We observe that from (1.12), it follows that (see [8])

$$|H(x,u,v)|\ge c\left(|u|^\alpha+|v|^\beta\right)-C.\tag{1.13}$$

The main result in this paper is the following Theorem.

Theorem 1.1. *Assume that* $H:\partial\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *satisfies* (1.3)-(1.12). *Then there exists a nontrivial strong solution to* (1.1)-(1.2).

Next, we look for positive solutions. If we also impose

$$\frac{\partial H}{\partial u}(x,u,v),\frac{\partial H}{\partial v}(x,u,v)\ge 0,\qquad \text{for all } u,v\ge 0,\tag{1.14}$$

$$\begin{aligned}\frac{\partial H}{\partial u}(x,u,v)&=0,\quad \text{when } u=0,\\ \frac{\partial H}{\partial v}(x,u,v)&=0,\quad \text{when } v=0,\end{aligned}\tag{1.15}$$

we can prove the following.

Theorem 1.2. *If* $H:\partial\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *satisfies* (1.3)-(1.12) *and* (1.14)-(1.15), *then there exists at least one positive strong solution to* (1.1)-(1.2).

This paper is organized as follows, in Section 2 we establish the functional setting in which the problem will be posed, give the definition of weak solution and prove a regularity result (weak solutions are in fact strong). In Section 3, we prove the existence of a weak solution by means of a minimax Theorem due to Felmer. Finally, in Section 4 we discuss the existence of positive solutions of (1.1)-(1.2).

2. The functional setting

In this section we describe the functional setting that allows us to treat (1.1)-(1.2) variationally.

Let us consider the space $L^2(\Omega)\times L^2(\partial\Omega)$ which is a Hilbert space with inner product, that we will denote by $\langle\cdot,\cdot\rangle$, given by

$$\langle(u,v),(\phi,\psi)\rangle=\int_\Omega u\phi+\int_{\partial\Omega}v\psi.$$

Now, let $A : D(A) \subset L^2(\Omega) \times L^2(\partial\Omega) \to L^2(\Omega) \times L^2(\partial\Omega)$ be the operator defined by

$$A(u, u \mid_{\partial\Omega}) = (-\Delta u + u, \frac{\partial u}{\partial \eta}),$$

where $D(A) = \{(u, u \mid_{\partial\Omega})/u \in H^2(\Omega)\}$. We claim that $D(A)$ is dense in $L^2(\Omega) \times L^2(\partial\Omega)$. In fact, let $(f, g) \in C(\overline{\Omega}) \times C(\partial\Omega)$, take $\varepsilon > 0$ and consider $\Omega_\varepsilon = \{x \in \Omega/\mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. Now we choose $u \in C^2(\overline{\Omega_\varepsilon})$ such that $\|u - f\|_{L^2(\Omega_\varepsilon)}$ is small. As $\partial\Omega$ is smooth, we can extend u to the whole $\overline{\Omega}$ in such a way that $u \in C^2(\overline{\Omega})$ and $\|u - g\|_{L^2(\partial\Omega)}$ is also small. As ε is arbitrary and $C(\overline{\Omega}) \times C(\partial\Omega)$ is dense in $L^2(\Omega) \times L^2(\partial\Omega)$ the claim follows.

We observe that A is invertible with inverse given by

$$A^{-1}(f, g) = (u, u \mid_{\partial\Omega}),$$

where u is the solution of

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

By standard regularity theory, see [11, p. 214], it follows that A^{-1} is bounded and compact. Therefore, $R(A) = L^2(\Omega) \times L^2(\partial\Omega)$ thus in order to see that A (and hence A^{-1}) is selfadjoint it remains to check that A is symmetric [15, p. 512]. To see this, let $u, v \in D(A)$, by Green's formula we have

$$\begin{aligned} \langle Au, v\rangle &= \int_\Omega (-\Delta u + u)v + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v \\ &= \int_\Omega u(-\Delta v + v) + \int_{\partial\Omega} u \frac{\partial v}{\partial \eta} \\ &= \langle u, Av\rangle; \end{aligned}$$

therefore, A is symmetric. Moreover, A (and hence A^{-1}) is positive. In fact, let $u \in D(A)$ and using again Green's formula,

$$\begin{aligned} \langle Au, u\rangle &= \int_\Omega (-\Delta u + u)u + \int_{\partial\Omega} \frac{\partial u}{\partial \eta} u \\ &= \int_\Omega |\nabla u|^2 + u^2 \geq 0 \end{aligned}$$

Therefore, there exists a sequence of eigenvalues $(\lambda_n) \subset \mathbb{R}$ with eigenfunctions $(\phi_n, \psi_n) \in L^2(\Omega) \times L^2(\partial\Omega)$ such that $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ... \nearrow +\infty$ and $\phi_n \in H^2(\Omega)$, $\phi_n \mid_{\partial\Omega} = \psi_n$,

$$\begin{aligned} -\Delta \phi_n + \phi_n &= \lambda_n \phi_n \quad \text{in } \Omega, \\ \frac{\partial \phi_n}{\partial \eta} &= \lambda_n \phi_n \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Let us consider the fractional powers of A, namely for $0 < s < 1$,

$$A^s : D(A^s) \to L^2(\Omega) \times L^2(\partial\Omega), \quad \text{with} \quad A^s u = \sum_{n=1}^{\infty} \lambda_n^s a_n(\phi_n, \psi_n),$$

where $u = \sum a_n(\phi_n, \psi_n)$.

Let $E^s = D(A^s)$, which is a Hilbert space under the inner product

$$(u, \phi)_{E^s} = \langle A^s u, A^s \phi\rangle.$$

Notice that $E^s \subset H^{2s}(\Omega)$. In fact, if we define $A_1 : H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$A_1 u = -\Delta u + u,$$

and $A_2 : H^2(\Omega) \subset D(A_2) \subset L^2(\partial\Omega) \to L^2(\partial\Omega)$ by

$$A_2 u = \frac{\partial u}{\partial \eta},$$

then $\tilde{A} = (A_1, A_2)$ satisfies

$$A = \tilde{A} \mid_{(u,u)} \quad u \in D(A_1) \cap D(A_2),$$

and hence

$$A^s = \tilde{A}^s \mid_{(u,u)} \quad u \in D(A_1^s) \cap D(A_2^s).$$

As $D(A_1) = H^2(\Omega) \subset D(A_2)$ we have, $D(A_1^s) \subset D(A_2^s)$, therefore

$$E^s = D(A^s) = D(A_1^s).$$

Now, by the results of [16, p. 187] (see also [13], [15]), as Ω is smooth, it follows that $E^s = D(A_1^s) \subset H^{2s}(\Omega)$.

So we have the following inclusions

$$E^s \hookrightarrow H^{2s}(\Omega) \hookrightarrow H^{2s-1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega).$$

More precisely, we have the following immersion Theorem,

Theorem 2.1. *Given $s > 1/4$ and $p \geq 1$ so that $\frac{1}{p} \geq \frac{1}{2} - \frac{2s-1/2}{N-1}$ the inclusion map $i : E^s \to L^p(\partial\Omega)$ is well defined and bounded. Moreover, if above we have strict inequality, then the inclusion is compact.*

Let us now set $E = E^s \times E^t$ where $s + t = 1$, s, t given by Remark 1.1 and define $B : E \times E \to \mathbb{R}$ by

$$B((u,v),(\phi,\psi)) = \langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle.$$

E is a Hilbert space with the usual product structure, and hence B is a bounded, bilinear, symmetric form. Therefore, there exists a unique bounded, selfadjoint, linear operator $L : E \to E$, such that

$$B(z,\gamma) = (Lz,\gamma)_E.$$

Now we define

$$\mathcal{Q}(z) = \frac{1}{2}B(z,z) = \frac{1}{2}(Lz,z)_E = \langle A^s u, A^t v\rangle.$$

For future reference we state the following Lemma that gives us a characterization of L,

Lemma 2.1. *The operator L defined above can be written as*

$$L(u,v) = (A^{-s}A^t v, A^{-t}A^s u).$$

Proof. Let $z = (u,v)$, $\eta = (\phi,\psi)$ and $Lz = (w,y)$. Then we have

$$\begin{aligned}(Lz,\eta)_E &= ((w,y),(\phi,\psi))_E = (w,\phi)_{E^s} + (y,\psi)_{E^t}\\ &= \langle A^s w, A^s \phi\rangle + \langle A^t y, A^y \psi\rangle.\end{aligned}$$

On the other hand

$$(Lz,\eta)_E = B(z,\eta) = \langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle.$$

Now if we take $\psi = 0$ we obtain,

$$\langle A^s w, A^s \phi\rangle = \langle A^t v, A^s \phi\rangle,$$

then

$$\langle A^s w - A^t v, A^s \phi\rangle = 0.$$

As A^s is invertible, it follows that $A^s w = A^t v$ and hence $w = A^{-s}A^t v$. Analogously, $y = A^{-t}A^s u$. □

Next, we consider the eigenvalue problem

$$Lz = \lambda z. \tag{2.3}$$

Using Lemma 2.1 we can rewrite (2.3) as

$$A^{-s}A^t v = \lambda u, \tag{2.4}$$

$$A^{-t}A^s u = \lambda v, \tag{2.5}$$

where $z = (u, v)$. As A^s and A^t are isomorphisms, it follows that $\lambda = 1$ or $\lambda = -1$. The associated eigenvectors are

$$\text{for } \lambda = 1,\ (u, A^{-t}A^s u)\ \forall u \in E^s, \tag{2.6}$$

$$\text{for } \lambda = -1,\ (u, -A^{-t}A^s u)\ \forall u \in E^s. \tag{2.7}$$

We can define the eigenspaces

$$E_+ = \{(u, A^{-t}A^s u)\ /\ u \in E^s\}, \tag{2.8}$$

$$E_- = \{(u, -A^{-t}A^s u)\ /\ u \in E^s\}, \tag{2.9}$$

which give a natural splitting $E = E_+ \oplus E_-$.

For future references we state the following Lemma, that gives us expressions for the projections over $E_\pm$.

Lemma 2.2. *The projections $P_\pm : E \to E_\pm$ are given by*

$$P_\pm(u, v) = \frac{1}{2}(u \pm A^{-s}A^t v, v \pm A^{-t}A^s u). \tag{2.10}$$

Proof. Immediate from definitions. □

By (1.3), Remark 1.1 and Theorem 2.1 we can define the functional, $\mathcal{H} : E \to \mathbb{R}$ as

$$\mathcal{H}(u, v) = \int_{\partial\Omega} H(x, u, v).$$

Proposition 2.1. *The functional $\mathcal{H}$ defined above is of class C^1 and its derivative is given by*

$$\mathcal{H}'(u, v)(\phi, \psi) = \int_{\partial\Omega} H_u(x, u, v)\phi + \int_{\partial\Omega} H_v(x, u, v)\psi. \tag{2.11}$$

Moreover, $\mathcal{H}'$ is compact.

Proof. From (1.11) we have

$$\int_{\partial\Omega} \left|\frac{\partial H}{\partial u}(x, u, v)\phi\right| \le C \int_{\partial\Omega} \left(|u|^p + |v|^{p(q+1)/(p+1)} + 1\right) |\phi|.$$

By Hölder inequality and Theorem 2.1 we have

$$\int_{\partial\Omega} \left|\frac{\partial H}{\partial u}(x, u, v)\phi\right| \le C \left(\|u\|_{E^s}^p + \|v\|_{E^t}^{p(q+1)/(p+1)} + 1\right) \|\phi\|_{E^s}.$$

In a similar way we obtain the analogous inequality for H_v.

Thus $\mathcal{H}'$ is well defined and bounded in E. Next, a standard argument gives that $\mathcal{H}$ is Fréchet differentiable with $\mathcal{H}'$ continuous. The fact that $\mathcal{H}'$ is compact comes from Theorem 2.1 (see [14] for the details). □

Now we can define the functional $\mathcal{F} : E \to \mathbb{R}$ as

$$\mathcal{F}(z) = \mathcal{Q}(z) - \mathcal{H}(z). \tag{2.12}$$

$\mathcal{F}$ is of class C^1 and in the next section we prove that it has the structure needed in order to apply the minimax techniques.

Let us now give the definition of weak solution of (1.1)-(1.2).

Definition 2.1. *We say that $z = (u, v) \in E = E^s \times E^t$ is an $(s, t)-$weak solution of* (1.1)-(1.2) *if z is a critical point of $\mathcal{F}$. In other words, for every $(\phi, \psi) \in E$ we have*

$$\langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle - \int_{\partial\Omega} H_u(x, u, v)\phi - \int_{\partial\Omega} H_v(x, u, v)\psi = 0. \tag{2.13}$$

Now, we prove a Theorem that gives us the regularity of (s, t)-weak solutions.

Theorem 2.2. *If $(u, v) \in E^s \times E^t$ is an (s, t)-weak solution of* (1.1)-(1.2) *then $u \in W^{2,(q+1)/q}(\Omega)$, $v \in W^{2,(p+1)/p}(\Omega)$ and (u, v) is in fact a strong solution of* (1.1)-(1.2).

Proof. Let us first consider $\psi = 0$ in (2.13), then

$$\langle A^s \phi, A^t v\rangle - \int_{\partial\Omega} H_u(x, u, v)\phi = 0, \tag{2.14}$$

for all $\phi \in E^s$. If we take $\phi \in H^2(\Omega)$, we have

$$\langle A^s \phi, A^t v\rangle = \langle A\phi, v\rangle = \int_\Omega (-\Delta\phi + \phi)v + \int_{\partial\Omega} \frac{\partial\phi}{\partial\eta} v. \tag{2.15}$$

On the other hand, using (1.11) we find

$$H_u(x, u(x), v(x)) \in L^{(p+1)/p}(\partial\Omega).$$

Then from basic elliptic theory (see [11]), there exists a function $w \in W^{2,(p+1)/p}(\Omega)$ such that

$$\begin{array}{rcll} \Delta w &=& w & \text{in } \Omega, \\ \frac{\partial w}{\partial\eta} &=& H_u(x, u(x), v(x)) & \text{on } \partial\Omega. \end{array}$$

Now, integration by parts gives us

$$0 = \int_\Omega (-\Delta w + w)\phi = \int_\Omega w(-\Delta\phi + \phi) + \int_{\partial\Omega} w\frac{\partial\phi}{\partial\eta} - \int_{\partial\Omega} H_u(x, u, v)\phi. \tag{2.16}$$

Combining (2.14),(2.15) and (2.16), we obtain

$$\langle v - w, A\phi\rangle = \int_\Omega (v - w)(-\Delta\phi + \phi) + \int_{\partial\Omega} (v - w)\frac{\partial\phi}{\partial\eta} = 0,$$

from where it follows that $v = w$. We argue similarly for u. □

3. Proof of Theorem 1.1

We want to apply a minimax Theorem due to Felmer [8] as it is used in [6]. First, we describe this Theorem and then we show how to use it in our situation.

Let E be a Hilbert space with inner product $(\cdot,\cdot)_E$ and norm $\|\cdot\|$. We assume that E has a splitting $E = E_+ \oplus E_-$, not necessarily orthogonal. Let $\mathcal{F} : E \to \mathbb{R}$ be a functional having the following form

$$\mathcal{F}(u) = \frac{1}{2}(Lu,u)_E - \mathcal{H}(u),$$

with

(F1) $L : E \to E$ is a linear, bounded, selfadjoint operator.

(F2) $\mathcal{H}$ is C^1 with $\mathcal{H}'$ is compact.

(F3) There exists two linear bounded invertible operators B_1, $B_2 : E \to E$ such that, if $\omega \geq 0$ then the linear operator

$$\hat{B}(\omega) = P_- B_1^{-1} \exp(\omega L) B_2 : E_- \to E_-,$$

is invertible (here P_- is the projection over E_- given by the splitting).

Let $\rho > 0$ and define

$$S = \{B_1 z \ / \ \|z\| = \rho,\ z \in E_-\}.$$

For $z_- \in E_-$, $z_- \neq 0$ we define

$$Q = \{B_2(\tau z_- + z) \ / \ 0 \leq \tau \leq \sigma,\ \|z\| \leq M,\ z \in E_+\},$$

for $\sigma > \rho / \|B_1^{-1} B_2 z_+\|$ and $M > \rho$.

By ∂Q we denote the boundary of Q relative to the subspace

$$\{B_2(\tau z_- + z) \ / \ \tau \in \mathbb{R},\ z \in E_+\}.$$

Now we can state the following Theorem that gives the existence of critical points of $\mathcal{F}$.

Theorem 3.1. *Let $\mathcal{F} : E \to \mathbb{R}$ be a C^1 functional satisfying the Palais-Smale condition, (F1),(F2) and (F3). Assume moreover that there exists a constant $\delta > 0$ such that*

$$\mathcal{F}(z) \geq \delta \quad \forall z \in S, \tag{S}$$

$$\mathcal{F}(z) \leq 0 \quad \forall z \in \partial Q. \tag{Q}$$

Then $\mathcal{F}$ has a critical point with critical value $C \geq \delta$.

For the proof see [8]. The critical point given by this Theorem has the following minimax characterization. Let us consider the class of functions

$$\Gamma = \{h \in C(E \times [0,1], E) \ / \ h \text{ satisfies } \Gamma 1, \Gamma 2 \text{ and } \Gamma 3\}, \tag{3.1}$$

where

(Γ1) h is given by $h(z,t) = \exp(\omega(z,t)L)z + K(z,t)$, where $\omega : E \times [0,1] \to \mathbb{R}_{\geq 0}$ is continuous and transforms bounded sets into bounded sets, and $K : E \times [0,1] \to E$ is compact.

(Γ2) $h(z,t) = z \ \forall z \in \partial Q,\ \forall t \in [0,1]$.

(Γ3) $h(z,0) = z,\ \forall z \in Q$.

Then the minimax value

$$C = \inf_{h \in \Gamma} \sup_{z \in Q} \mathcal{F}(h(z,1)),$$

is the critical value given in Theorem 3.1.

Now, let us see that the functional $\mathcal{F}$ of Section 2 satisfies the hypothesis of Theorem 3.1. (F1) and (F2) are consequences of Section 2.

Let $\mu, \nu > 0$ be such that,

$$\mu + \nu < \min\{\mu\alpha, \nu\beta\}. \tag{3.2}$$

In order to verify the hypothesis of Theorem 3.1, we define B_1 and B_2 as,

$$B_1(u, v) = (\rho^{\mu-1}u, \rho^{\nu-1}v), \tag{3.3}$$

$$B_2(u, v) = (\sigma^{\mu-1}u, \sigma^{\nu-1}v). \tag{3.4}$$

Here, ρ and σ are positive constants to be determined. We observe that $B_1, B_2 : E \to E$ are linear, bounded, invertible operators. To show that $\hat{B}(\omega)$ is invertible we prove the following formula for $\exp(\omega L)$.

Lemma 3.1. *Let $\omega \in \mathbb{R}$. Then the operator $\exp(\omega L) : E \to E$ is given by*

$$\exp(\omega L)(u, v) = \cosh(\omega)(u, v) + \sinh(\omega)(A^{-s}A^t v, A^{-t}A^s u). \tag{3.5}$$

Proof. We recall that

$$L(u, v) = (A^{-s}A^t v, A^{-t}A^s u).$$

Writing explicitly the exponential as a series and using this identity, the result follows by reordering the terms. □

With this Lemma we can prove that $\hat{B}(\omega)$ is invertible in E_-.

Proposition 3.1. *The operator $\hat{B}(\omega) : E_- \to E_-$ is invertible.*

Proof. Given $z \in E_-$ we have $z = (u, -A^{-t}A^s u)$ with $u \in E^s$. By (3.4), we have

$$B_2(z) = (\sigma^{\mu-1}u, -\sigma^{\nu-1}A^{-t}A^s u).$$

Therefore, using Lemma 3.1 if we write $\exp(\omega L)B_2(z) = (x, y)$ we have

$$x = (\cosh(\omega)\sigma^{\mu-1} - \sinh(\omega)\sigma^{\nu-1})u,$$

$$y = (-\cosh(\omega)\sigma^{\nu-1} + \sinh(\omega)\sigma^{\mu-1})A^{-t}A^s u.$$

Now, applying B_1^{-1}, by (3.3) we obtain, if we write $B_1^{-1}\exp(\omega L)B_2(z) = (\overline{x}, \overline{y})$,

$$\overline{x} = \frac{\cosh(\omega)\sigma^{\mu-1} - \sinh(\omega)\sigma^{\nu-1}}{\rho^{\mu-1}}u,$$

$$\overline{y} = \frac{-\cosh(\omega)\sigma^{\nu-1} + \sinh(\omega)\sigma^{\mu-1}}{\rho^{\nu-1}}A^{-t}A^s u.$$

Finally, we project into E_-. Now, calling $\hat{B}(\omega)z = (\phi, \psi)$ and using the projection formula given by Lemma 2.2, we get

$$\phi = \left[\frac{1}{2}\left(\frac{\sigma^{\mu-1}}{\rho^{\mu-1}} + \frac{\sigma^{\nu-1}}{\rho^{\nu-1}}\right)\cosh(\omega) - \frac{1}{2}\left(\frac{\sigma^{\nu-1}}{\rho^{\mu-1}} + \frac{\sigma^{\mu-1}}{\rho^{\nu-1}}\right)\sinh(\omega)\right]u,$$

$$\psi = -\left[\frac{1}{2}\left(\frac{\sigma^{\mu-1}}{\rho^{\mu-1}} + \frac{\sigma^{\nu-1}}{\rho^{\nu-1}}\right)\cosh(\omega) - \frac{1}{2}\left(\frac{\sigma^{\nu-1}}{\rho^{\mu-1}} + \frac{\sigma^{\mu-1}}{\rho^{\nu-1}}\right)\sinh(\omega)\right]A^{-t}A^s u.$$

In other words $\hat{B}(\omega)z = mz$. This constant m is positive if we assume that $\sigma > 1$ and $\rho < 1$, in fact

$$m \geq \left(\frac{\sigma^{\mu-1}}{\rho^{\mu-1}} + \frac{\sigma^{\nu-1}}{\rho^{\nu-1}}\right) - \left(\frac{\sigma^{\nu-1}}{\rho^{\mu-1}} + \frac{\sigma^{\mu-1}}{\rho^{\nu-1}}\right) = \frac{(\rho^{\mu-1} - \rho^{\nu-1})(\sigma^{\nu-1} - \sigma^{\mu-1})}{\rho^{\mu+\nu-2}} > 0,$$

so that m is positive independently of the value of $\omega \in \mathbb{R}$. This implies that $\hat{B}(\omega)$ is invertible $\square$

Now we check the Palais-Smale condition for $\mathcal{F}$.

Proposition 3.2. *$\mathcal{F}$ satisfies the Palais-Smale condition.*

Proof. Let $(z_n)_{n\geq 1} \subset E$ be a sequence such that

$$|\mathcal{F}(z_n)| \leq c \quad \text{and} \quad \mathcal{F}'(z_n) \to 0. \tag{3.6}$$

Let us first prove that (3.6) implies that (z_n) is bounded. From (3.6) it follows that there exists a sequence $\varepsilon_n \to 0$ such that

$$|\mathcal{F}'(z_n)w| \leq \varepsilon_n \|w\|_E, \ \forall w \in E. \tag{3.7}$$

Let us take

$$w_n = ((w_n)_1, (w_n)_2) = \frac{\alpha\beta}{\alpha+\beta}(\frac{1}{\alpha}u_n, \frac{1}{\beta}v_n), \quad \text{where } z_n = (u_n, v_n).$$

Now, using (3.6),

$$\begin{aligned}
c + \varepsilon_n\|w_n\|_E &\geq \mathcal{F}(z_n) - \mathcal{F}'(z_n)w_n \\
&= \langle A^s u_n, A^t v_n\rangle - \int_{\partial\Omega} H(x,u_n,v_n) - \langle A^s u_n, A^t (w_n)_2\rangle \\
&- \langle A^s (w_n)_1, A^t v_n\rangle + \int_{\partial\Omega} H_u(x,u_n,v_n)(w_n)_1 + \int_{\partial\Omega} H_v(x,u_n,v_n)(w_n)_2 \\
&= \frac{\alpha\beta}{\alpha+\beta}\int_{\partial\Omega} \frac{1}{\alpha}H_u(x,u_n,v_n)u_n + \frac{1}{\beta}H_v(x,u_n,v_n)v_n - H(x,u_n,v_n) \\
&+ \left(\frac{\alpha\beta}{\alpha+\beta} - 1\right)\int_{\partial\Omega} H(x,u_n,v_n).
\end{aligned} \tag{3.8}$$

Now, by (1.12) and (1.6) we obtain

$$c(1+\|z_n\|_E) \geq \int_{\partial\Omega} H(x,u_n,v_n),$$

and then, by (1.13),

$$\int_{\partial\Omega} |u_n|^\alpha + |v_n|^\beta \leq c(1+\|u_n\|_{E^s} + \|v_n\|_{E^t}). \tag{3.9}$$

Next we consider $w = (\phi, 0)$, $\phi \in E^s$. From (3.7) we have

$$\langle A^s\phi, A^t v_n\rangle \leq \int_{\partial\Omega} |H_u(x,u_n,v_n)\phi| + \varepsilon_n\|\phi\|_{E^s}.$$

Now, by (1.11)

$$\int_{\partial\Omega} |H_u(x,u_n,v_n)\phi| \leq c\left(\int_{\partial\Omega} |u_n|^p|\phi| + |v_n|^{p\frac{q+1}{p+1}}|\phi| + |\phi|\right).$$

Using Hölder inequality the last term is bounded by

$$\|u_n\|^p_{L^\alpha(\partial\Omega)}\|\phi\|_{L^{\frac{\alpha}{\alpha-p}}(\partial\Omega)} + \|v_n\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)}\|\phi\|_{L^{\frac{\beta(p+1)}{\beta(p+1)-p(q+1)}}(\partial\Omega)} + \|\phi\|_{L^1(\partial\Omega)}.$$

Now, by Theorem 2.1, we get that the last equation is bounded by

$$\|u_n\|^p_{L^\alpha(\partial\Omega)}\|\phi\|_{E^s} + \|v_n\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)}\|\phi\|_{E^s} + \|\phi\|_{E^s}.$$

Thus,

$$|\langle A^s\phi, A^t v_n\rangle| \le c\|\phi\|_{E^s}\left(\|u_n\|^p_{L^\alpha(\partial\Omega)} + \|v_n\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)} + 1\right).$$

By duality (A^s in invertible over E^s) we get

$$\|v_n\|_{E^t} \le c\left(\|u_n\|^p_{L^\alpha(\partial\Omega)} + \|v_n\|^{p\frac{q+1}{p+1}}_{L^\beta(\partial\Omega)} + 1\right). \tag{3.10}$$

Analogously, we obtain

$$\|u_n\|_{E^s} \le c\left(\|v_n\|^q_{L^\beta\partial\Omega)} + \|u_n\|^{q\frac{p+1}{q+1}}_{L^\alpha(\partial\Omega)} + 1\right). \tag{3.11}$$

Now combining (3.9), (3.10) and (3.11), we obtain

$$\|u_n\|_{E^s} + \|v_n\|_{E^t} \le c\left(\|u_n\|^{p/\alpha}_{E^s} + \|v_n\|^{p\frac{q+1}{\beta(p+1)}}_{E^t} + \|v_n\|^{q/\beta}_{E^t} + \|u_n\|^{q\frac{p+1}{\alpha(q+1)}}_{E^s} + 1\right),$$

and as all the exponents are less than one, we get that z_n in bounded.

Now, by the compactness of $\mathcal{H}'$ and the invertibility of L we can extract a subsequence of z_n that converges in E. In fact, we can take a subsequence z_{n_j} that converges weakly in E, as $\mathcal{H}'$ is compact, it follows that $\mathcal{H}'(z_{n_j})$ converges strongly in E. Hence, using the fact that $\mathcal{F}'(z_{n_j}) \to 0$ strongly and the invertibility of L, the result follows. □

In order to apply theorem 3.1, it remains to verify (S) and (Q). The following Proposition gives (S).

Proposition 3.3. *There exists $\rho, \delta > 0$ such that $\mathcal{F}(z) \ge \delta$, $\forall z \in S$, where $S = \{B_1(u,v) \ / \ \|(u,v)\|_E = \rho,\ (u,v) \in E_+\}$.*

Proof. Let $\tilde{z} = (u,v) \in E_+$ and $z = B_1\tilde{z}$. As $(u,v) \in E_+$, then $v = A^{-t}A^s u$ and equivalently $u = A^{-s}A^t v$. We have

$$\mathcal{Q}(z) = \langle \rho^{\mu-1}A^s u, \rho^{\nu-1}A^t v\rangle = \rho^{\mu+\nu-2}\langle A^s u, A^t v\rangle,$$

We observe that, if $z = z_+ + z_-$,

$$\mathcal{Q}(z) = \frac{1}{2}\|z_+\|^2_E - \frac{1}{2}\|z_-\|^2_E,$$

hence,

$$\mathcal{Q}(z) = \frac{1}{2}\rho^{\mu+\nu-2}\|\tilde{z}\|^2_E. \tag{3.12}$$

On the other hand, (1.3)-(1.4) implies that

$$H(x,u,v) \le c(|u|^\alpha + |v|^\beta + |u|^{p+1} + |v|^{q+1}),$$

therefore

$$\begin{aligned}\mathcal{H}(z) \le\, & c\Bigg(\rho^{(\mu-1)\alpha}\int_{\partial\Omega}|u|^\alpha + \rho^{(\nu-1)\beta}\int_{\partial\Omega}|v|^\beta \\ & + \rho^{(\mu-1)(p+1)}\int_{\partial\Omega}|u|^{p+1} + \rho^{(\nu-1)(q+1)}\int_{\partial\Omega}|v|^{q+1}\Bigg).\end{aligned}$$

Now, by Theorem 2.1 we conclude that

$$\mathcal{H}(z) \le C(\rho^{(\mu-1)\alpha}\|\tilde{z}\|^\alpha_E + \rho^{(\mu-1)(p+1)}\|\tilde{z}\|^{p+1}_E + \rho^{(\nu-1)\beta}\|\tilde{z}\|^\beta_E + \rho^{(\nu-1)(q+1)}\|\tilde{z}\|^{q+1}_E). \tag{3.13}$$

Using (3.12), (3.13) and the fact that $\|\tilde{z}\|_E = \rho$, we have

$$\mathcal{F}(z) \geq \frac{1}{2}\rho^{\mu+\nu} - C(\rho^{\mu\alpha} + \rho^{\nu\beta} + \rho^{\mu(p+1)} + \rho^{\nu(q+1)}). \tag{3.14}$$

Since $\alpha \leq p+1$, $\beta \leq q+1$ and (3.2), we get $\mathcal{F}(z) \geq \frac{1}{2}\rho^{\mu+\nu} - C(\rho^{\mu\alpha} + \rho^{\nu\beta}) \geq \delta$ if ρ is small enough. □

Finally, the following Proposition gives (Q).

Proposition 3.4. *There exists constants $\sigma, M > 0$ such that $\mathcal{F}(z) \leq 0 \quad \forall z \in \partial Q$, where $Q = \{B_2(\tau z_+ + z) \ / \ 0 \leq \tau \leq \sigma, \ \|z\|_E \leq M, \ z \in E_-\}$ $z_+ = (u_+, v_+) \in E_+$ with $Au_+ = \lambda_k u_+$ for some λ_k and $\|z_+\|_E = 1$.*

Proof. For $\tau \in \mathbb{R}_+$, $z = (u, v) \in E_-$ we set, $\tilde{z} = B_2(\tau z_+ + z)$. Using the definitions of $E_\pm$ we have

$$v_+ = A^{-t}A^s u_+ \quad \text{and} \quad v = -A^{-t}A^s u,$$

therefore

$$\begin{aligned} \mathcal{Q}(\tilde{z}) &= \langle \tau\sigma^{\mu-1}A^s u_+ + \sigma^{\mu-1}A^s u, \tau\sigma^{\nu-1}A^s u_+ - \sigma^{\nu-1}A^s u\rangle \\ &= \frac{1}{2}\sigma^{\mu+\nu-2}(\tau^2 - \|z\|_E^2). \end{aligned} \tag{3.15}$$

By (1.13),

$$\int_{\partial\Omega} H(x, \tilde{z}) \geq C\left(\int_{\partial\Omega} (\sigma^{\alpha(\mu-1)}|\tau u_+ + u|^\alpha + \sigma^{\beta(\nu-1)}|\tau v_+ + v|^\beta) - |\partial\Omega|\right). \tag{3.16}$$

Now, every u can be decomposed as $u = \gamma u_+ + \hat{u}$, where $\hat{u}$ is orthogonal to u_+ in $L^2(\partial\Omega)$ and $\gamma \in \mathbb{R}$. By Hölder's inequality we have,

$$(\tau + \gamma)\int_{\partial\Omega} |u_+|^2 = \int_{\partial\Omega} (\tau u_+ + u)u_+ \leq \|\tau u_+ + u\|_{L^\alpha(\partial\Omega)}\|u_+\|_{L^{\alpha'}(\partial\Omega)},$$

hence

$$\tau + \gamma \leq C\|\tau u_+ + u\|_{L^\alpha(\partial\Omega)}. \tag{3.17}$$

Using the fact that $A^{-t}A^s u_+ = \lambda_k^{-t+s}u_+$ we have

$$\lambda_k^{-t+s}(\tau - \gamma)\int_{\partial\Omega} |u_+|^2 = \int_{\partial\Omega} (\tau v_+ + v)u_+ \leq \|\tau v_+ + v\|_{L^\beta(\partial\Omega)}\|u_+\|_{L^{\beta'}(\partial\Omega)},$$

therefore

$$\tau - \gamma \leq C\|\tau v_+ + v\|_{L^\beta(\partial\Omega)}. \tag{3.18}$$

If $\gamma \geq 0$ it follows from (3.15), (3.16) and (3.17) that

$$\mathcal{F}(\tilde{z}) \leq \frac{1}{2}\sigma^{\mu+\nu-2}\tau^2 - c\tau^\alpha\sigma^{\alpha(\mu-1)} + c|\partial\Omega|, \tag{3.19}$$

and if $\gamma < 0$, from (3.15), (3.16) and (3.18) we have

$$\mathcal{F}(\tilde{z}) \leq \frac{1}{2}\sigma^{\mu+\nu-2}\tau^2 - c\tau^\beta\sigma^{\beta(\nu-1)} + c|\partial\Omega|. \tag{3.20}$$

By the choice of μ and ν it follows from (3.19) and (3.20) that if we take $\tau = \sigma$ large we have

$$\mathcal{F}(\tilde{z}) \leq 0.$$

Now if $\|z\|_E = M$ and $0 \leq \tau \leq \sigma$ from (3.15) and (3.16) we obtain

$$\mathcal{F}(\tilde{z}) \leq \frac{1}{2}\sigma^{\mu+\nu} - \frac{1}{2}\sigma^{\mu+\nu-2}M^2 + c|\partial\Omega|,$$

then if we take M large enough we find $\mathcal{F}(\tilde{z}) \leq 0$. To finish the proof we only observe that when $\tau = 0$ we also have $\mathcal{F}(\tilde{z}) \leq 0$ by the positivity of H and (3.15). $\square$

Proof of Theorem 1.1 By Propositions 3.2, 3.3 and 3.4, $\mathcal{F}$ satisfies the hypothesis of Theorem 3.1. This provides us with a nontrivial (s,t)-weak solution of (1.1)-(1.2), that is in fact a strong solution by Theorem 2.2. $\square$

4. Positive solutions. Theorem 1.2

In this section, we will show that under the hypothesis (1.14)-(1.15), there exists a positive solution of (1.1)-(1.2). Again, we are using ideas from [6] under the functional setting of Section 2.

In order to prove Theorem 1.2, we start by redefining the Hamiltonian. Let us define $\tilde{H} : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{H}(x,u,v) = \begin{cases} H(x,u,v) & \text{if } u,v \geq 0, \\ H(x,0,v) & \text{if } u \leq 0,\ v \geq 0, \\ H(x,u,0) & \text{if } u \geq 0,\ v \leq 0, \\ 0 & \text{if } u,v \leq 0. \end{cases} \tag{4.1}$$

We observe that if (u,v) is a nontrivial strong solution of

$$\begin{aligned} \Delta u &= u \\ \Delta v &= v \quad \text{in } \Omega\,, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \frac{\partial u}{\partial \eta} &= \tilde{H}_v(x,u,v) \\ \frac{\partial v}{\partial \eta} &= \tilde{H}_u(x,u,v) \quad x \in \partial\Omega\,, \end{aligned} \tag{4.3}$$

then by the maximum principle and Hopf's Lemma we have that u and v are positive in $\overline{\Omega}$. Hence (u,v) is a strong solution of (1.1)-(1.2).

To find a nontrivial solution of (4.2)-(4.3) we want to apply the results of Section 3. By our assumption (1.15), the new Hamiltonian $\tilde{H}$ is regular. Also, it satisfies (1.3), (1.4) and (1.11), but not (1.12). So in order to adapt the proof of Theorem 1.1 to this case we observe that (1.12) was only used in the proof of the Palais-Smale condition and the condition (Q).

First, we want to prove the Palais-Smale condition for

$$\tilde{\mathcal{F}}(u,v) = \mathcal{Q}(u,v) - \int_{\partial\Omega} \tilde{H}(x,u,v).$$

Let (u_n,v_n) be a sequence in E such that

$$|\tilde{\mathcal{F}}(u_n,v_n)| \leq C \quad \text{and} \quad \tilde{\mathcal{F}}'(u_n,v_n) \to 0. \tag{4.4}$$

Mimic the proof of Proposition 3.2, we only have to show that (4.4) implies that (u_n,v_n) is bounded. Again, as in Proposition 3.2, we get

$$C(1 + \|(u_n,v_n)\|_E) \geq \int_{\partial\Omega} \tilde{H}(x,u_n,v_n),$$

using (1.12) restricted to $u, v \geq 0$. Therefore, from (1.13) we conclude

$$\int_{\partial\Omega} |u_n^+|^\alpha + |v_n^+|^\beta \leq C(1 + \|u_n\|_{E^s} + \|v_n\|_{E^t}),$$

and hence by the same arguments given in Proposition 3.2,

$$\|v_n\|_{E^t} \leq C(\|u_n^+\|^p_{L^\alpha(\partial\Omega)} + \|v_n^+\|^{p(q+1)/(p+1)}_{L^\beta(\partial\Omega)} + 1),$$

and

$$\|u_n\|_{E^s} \leq C(\|u_n^+\|^{q(p+1)/(q+1)}_{L^\alpha(\partial\Omega)} + \|v_n^+\|^q_{L^\beta(\partial\Omega)} + 1).$$

Thus

$$\begin{aligned} &\|u_n^+\|^\alpha_{L^\alpha(\partial\Omega)} + \|v_n^+\|^\beta_{L^\beta(\partial\Omega)} \\ &\quad \leq C(\|u_n^+\|^p_{L^\alpha(\partial\Omega)} + \|v_n^+\|^{p(q+1)/(p+1)}_{L^\beta(\partial\Omega)} + \|u_n^+\|^{q(p+1)/(q+1)}_{L^\alpha(\partial\Omega)} + \|v_n^+\|^q_{L^\beta(\partial\Omega)} + 1). \end{aligned}$$

By our assumptions on the exponents p, q, α and β we get that $\|u_n^+\|_{L^\alpha(\partial\Omega)}$ and $\|v_n^+\|_{L^\beta(\partial\Omega)}$ are bounded and hence $\|u_n\|_{E^s}$ and $\|v_n\|_{E^t}$ are bounded as we wanted to show. □

Now, we prove (Q). We choose Q as in Section 3 with $z_+ = (u_+, v_+) \in E^+$ such that $u_+ = \phi_1$ and $v_+ = A^{-t}A^s\phi_1 = \lambda_1^{-t+s}\phi_1$ where ϕ_1 is the first eigenfunction of A. In particular $\phi_1 > 0$ in $\overline{\Omega}$.

We observe that $(\tau\phi_1 + u)^+ = ((\tau+\gamma)\phi_1 + \hat{u})^+ \geq (\tau+\gamma)\phi_1 + \hat{u}$. Proceeding as in Proposition 3.4, we conclude the desired result (Q). □

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