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# HOMOGENIZATION OF LINEARIZED ELASTICITY SYSTEMS WITH TRACTION CONDITION IN PERFORATED DOMAINS

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Abstract. In this paper, we study the asymptotic behavior of the linearized elasticity system with nonhomogeneous traction condition in perforated domains. To do that, we use the $H_e^0$-convergence introduced by M. El Hajji in [4] which generalizes - in the case of the linearized elasticity system - the notion of $H^0$-convergence introduced by M. Briane, A. Damlamian and P. Donato in [1]. We give then some examples to illustrate this result.

## 1. Introduction

The notion of $H_e^0$-convergence was introduced by M. El Hajji in [4] for the study of the asymptotic behavior of the linearized elasticity system with homogeneous traction condition in perforated domains. It translates the notion of $H^0$-convergence introduced by M. Briane, A. Damlamian and P. Donato in [1] for the study of the diffusion system problem with homogeneous Neumann condition in perforated domains which generalizes in the case of perforated domains the $H$-convergence introduced by F. Murat and L. Tartar in [13], and the $G$-convergence for the symmetric operator introduced by S. Spagnolo in [14].

This paper is devoted to giving an application of the $H_e^0$- convergence to study the asymptotic behavior of the linearized elasticity system with nonhomogeneous traction condition in perforated domains by using the convergence of a distribution defined from data on the boundaries of the holes. This result is the analogue for the linearized elasticity of Theorem 1 given by P. Donato and M. El Hajji in [3] as an application of the $H^0$-convergence to the study of the nonhomogeneous Neumann problem.

In Section 2, we recall the definition of $H_e^0$-convergence introducing a definition of e-admissible set similar to that given by M. Briane, A. Damlamian and P. Donato in [1] for the $H^0$-convergence and by F. Murat and L. Tartar in [13] for the $H$-convergence. In Section 3, we introduce the linearized elasticity problem and we give the main result. We establish then the proof making use of some preliminary results. In Section 4, we give some applications of this result - first for the case of periodic perforated domains by holes of size $r_\varepsilon = \varepsilon$ where we use the results given by F. Lene in [11] and D. Cioranescu and P. Donato in [2]. Then we apply the results of section 3 and those of C. Georgelin in [8] and S. Kaizu in [10] when

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$r_\varepsilon \lll \varepsilon$. Finally, we apply section 3 to the case of a perforated domain with double periodicity (introduced by T. Levy in [12]) using the results given by M. El Hajji in [9] and P. Donato and M. El Hajji in [3].

## 2. Recall of $H_e^0$-convergence

In this section, we recall the definition of $H_e^0$-convergence introduced in [4]. First let us introduce the following notations.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $\varepsilon$ the general term of a positive sequence, and $c$ different positive constants independent of $\varepsilon$. We introduce the following sets:

$$\begin{aligned}
\mathcal{M}_s &= \{\text{symmetric linear operators } l : \mathbb{R}^N \to \mathbb{R}^{N^2}\},\\
\mathcal{L}(\mathcal{M}_s) &= \{\text{linear operators } p : \mathcal{M}_s \to \mathcal{M}_s\},\\
\mathcal{L}_s(\mathcal{M}_s) &= \{\text{symmetric operators } p \in \mathcal{L}(\mathcal{M}_s)\},\\
M_e(\alpha,\beta;\Omega) =& \big\{ A \in L^\infty(\Omega, \mathcal{L}_s(\mathcal{M}_s)), A(x)\xi\cdot\xi \geq \alpha|\xi|^2,\\
&\quad A^{-1}(x)\xi\cdot\xi \geq \beta|\xi|^2,\ \forall \xi \in \mathcal{L}_s(\mathcal{M}_s),\ x \text{ a.e. } \in \Omega\big\}.
\end{aligned}$$

In what follows, we use the Einstein summation convention, that is, we sum over repeated indices. We denote by $e(\cdot)$ the symmetric tensor of elasticity defined by

$$e(u) = (e_{ij}(u))_{ij} \quad \text{where} \quad e_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\}.$$

We denote by $S_\varepsilon$ a compact subset of $\Omega$. We denote the perforated domain by $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$. We denote by $\chi_\varepsilon$ the characteristic function of $\Omega_\varepsilon$ and we set

$$V_\varepsilon = \left\{ v \in [H^1(\Omega_\varepsilon)]^N,\ v|_{\partial\Omega} = 0 \right\}, \tag{1}$$

which equipped with the $H^1$-norm forms a Hilbert space.

Definition 1 (e-admissible set). The set $S_\varepsilon$ is said to be admissible (in $\Omega$) for the linearized elasticity if

$$\text{every function in } L^\infty(\Omega) \text{ weak} \star \text{ of } \chi_\varepsilon \text{ is positive almost everywhere in } \Omega, \tag{2}$$

and for each $\varepsilon$ there is an extension operator $P_\varepsilon$ from $V_\varepsilon$ to $[H_0^1(\Omega)]^N$ and there exists a real positive $C$ such that

$$\begin{aligned}
&i) \quad P_\varepsilon \in \mathcal{L}\left(V_\varepsilon, [H_0^1(\Omega)]^N\right),\\
&ii) \quad (P_\varepsilon v)\,|_{\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon,\\
&iii) \quad \|e(P_\varepsilon v)\|_{[(L^2(\Omega)]^{N^2}} \leq C\|e(v)\|_{[(L^2(\Omega_\varepsilon)]^{N^2}}, \quad \forall v \in V_\varepsilon.
\end{aligned} \tag{3}$$

Remark 1. **1)** As an example of an e-admissible set, one can consider the case of a periodic function on a perforated domain by holes of size $\varepsilon$ or $r_\varepsilon$ (see F. Lene [11] and C. Georgelin [8]). One can consider also a perforated domain with double periodicity introduced by T. Levy in [12] (see also M. El Hajji [9]).
2). Observe that if $S_\varepsilon$ is admissible in the sense of definition 1, then we have a Korn inequality in $\Omega_\varepsilon$ independent of $\varepsilon$, i.e.,

$$\|\nabla v\|_{[L^2(\Omega_\varepsilon)]^{N^2}} \leq C(\Omega)\|e(v)\|_{[L^2(\Omega)]^{N^2}}, \quad \forall v \in V_\varepsilon.$$

Indeed, from the Korn inequality in $\Omega$ and (3 iii) one has

$$\begin{aligned}
\|\nabla v\|_{[L^2(\Omega_\varepsilon)]^{N^2}} &\leq \|\nabla(P_\varepsilon v)\|_{[L^2(\Omega)]^{N^2}}\\
&\leq c(\Omega)\|e(P_\varepsilon v)\|_{[L^2(\Omega)]^{N^2}}\\
&\leq C(\Omega)\|e(v)\|_{[L^2(\Omega)]^{N^2}}, \quad \forall v \in V_\varepsilon.
\end{aligned}$$

To give the definition of $H_e^0$-convergence, we introduce the adjoint operator $P_\varepsilon^\star$ of $P_\varepsilon$ defined from $[H^{-1}(\Omega)]^N$ to $V'_\varepsilon$ by

$$\begin{aligned}\langle P_\varepsilon^\star f, v\rangle_{V'_\varepsilon, V_\varepsilon} &= \langle f, P_\varepsilon v\rangle_{[H^{-1}(\Omega)]^N, [H_0^1(\Omega)]^N} \\ &= \sum_{i=1}^{N} \langle f_i, (P_\varepsilon v)_i\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in V'_\varepsilon.\end{aligned}$$

Definition 2. Let $A^\varepsilon \in M_e(\alpha, \beta; \Omega)$ and $S_\varepsilon$ be e-admissible in $\Omega$. One says that the pair $(A^\varepsilon, S_\varepsilon)$ $H^0$-converges to $A^0$ (in the sense of the linearized elasticity) and we denote this $(A^\varepsilon, S_\varepsilon) \rightharpoonup^{H_e^0} A^0$ if for each function $f$ in $[H^{-1}(\Omega)]^N$, the solution $u^\varepsilon$ of

$$\begin{aligned}- \operatorname{div}(A^\varepsilon e(u^\varepsilon)) &= P_\varepsilon^\star f \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon(x) e(u^\varepsilon)) \cdot n &= 0 \quad \text{on } \partial S_\varepsilon, \\ u^\varepsilon &= 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{4}$$

satisfies

$$\begin{aligned} P_\varepsilon u^\varepsilon &\rightharpoonup u \quad \text{weakly in } [H_0^1(\Omega)]^N, \\ A^\varepsilon \widetilde{e(u^\varepsilon)} &\rightharpoonup A^0 e(u) \quad \text{weakly in } \left(L^2(\Omega)\right)^{N^2},\end{aligned} \tag{5}$$

where $u$ is the solution of the problem

$$\begin{aligned} -\operatorname{div}\left(A^0 e(u)\right) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{6}$$

and $\widetilde{v}$ is the extension by zero to $\Omega$ of the function $v$ defined in $\Omega_\varepsilon$.
Remark 2. **1)** If $S_\varepsilon$ is empty, the $H_e^0$-convergence reduces to the notion of $H$-convergence in elasticity introduced by G.A. Francfort and F. Murat in [7].
2). The system (4) is equivalent to the system

$$\begin{aligned} -\frac{\partial}{\partial y_j} \sigma_{ij}^\varepsilon(u^\varepsilon) &= (P_\varepsilon^\star f)_i \quad \text{in } \Omega_\varepsilon \\ \sigma_{ij}^\varepsilon(u^\varepsilon) \cdot n_j &= 0 \quad \text{on } \partial S_\varepsilon, \\ u^\varepsilon &= 0 \quad \text{on } \partial\Omega,\end{aligned}$$

where $\sigma_{ij}^\varepsilon(u^\varepsilon) = A_{ijkh}^\varepsilon e_{kh}(u^\varepsilon)$, $A^\varepsilon = (A_{ijkh}^\varepsilon)$, and whose variational formulation is written as: Find $u^\varepsilon \in V_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} \sigma_{ij}^\varepsilon(u^\varepsilon) e_{ij}(v) dx = \langle P_\varepsilon^\star f, v\rangle_{V'_\varepsilon, V_\varepsilon}.$$

We can rewrite this problem in the form: Find $u^\varepsilon \in V_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon) e(v) dx = \langle P_\varepsilon^\star f, v\rangle_{V'_\varepsilon, V_\varepsilon}.$$

Some examples will be given in section 4, when we apply the main result of this paper.

## 3. The main result

In this section, we establish a property of the $H_e^0$-convergence, and apply it to the study of the asymptotic behavior of the linearized elasticity system with nonhomogeneous traction condition. This result is analogous to the linearized elasticity of Theorem 1 in [9] given as an application of the $H^0$-convergence to the study of the nonhomogeneous Neumann problem. We then give some examples to illustrate this result.

Let $A^\varepsilon = (a^\varepsilon_{ijkh}) \in M_e(\alpha, \beta; \Omega)$ and let $S_\varepsilon$ be e-admissible in $\Omega$ such that

$$S_\varepsilon \text{ has boundary } \partial S_\varepsilon \text{ of class } C^1. \tag{7}$$

We consider the linearized elasticity system

$$\begin{gathered} -\operatorname{div}(A^\varepsilon e(u^\varepsilon)) = 0 \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon(x)e(u^\varepsilon)) \cdot n = g^\varepsilon \quad \text{on } \partial S_\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{8}$$

where

$$g^\varepsilon \in [H^{-1/2}(\partial S_\varepsilon)]^N. \tag{9}$$

It is well known that (8) has a unique solution. Our aim is to study the asymptotic behavior of the solution $u^\varepsilon$ as $\varepsilon$ approaches zero. To do that, we introduce a vectorial distribution $\nu^\varepsilon_g$ defined in $\Omega$ by

$$\langle \nu^\varepsilon_g, \varphi\rangle_{[H^{-1}(\Omega)]^N,\, [H^1_0(\Omega)]^N} = \langle g^\varepsilon, \varphi\rangle_{[H^{-1/2}(\partial S_\varepsilon)]^N,\, [H^{1/2}(\partial S_\varepsilon)]^N}, \quad \forall \varphi \in [H^1_0(\Omega)]^N. \tag{10}$$

It is easy to check that this defines $\nu^\varepsilon_g$ as an element of $[H^{-1}(\Omega)]^N$, and if $\nu^\varepsilon_g \in [L^2(\Omega)]^N$, we deduce from the Riesz Theorem that $\nu^\varepsilon_g$ is a measure. The following theorem shows that the convergence of $u^\varepsilon$ can be deduced from the $H^0_e$-convergence of $(A^\varepsilon, S_\varepsilon)$ and the convergence of $\nu^\varepsilon_g$ in $[H^{-1}(\Omega)]^N$.

**Theorem 1.** *Let $\{u^\varepsilon\}$ be the sequence of the solutions of (8). Suppose that (7) is satisfied and that*

$$\begin{aligned} &i) \quad (A^\varepsilon, S_\varepsilon) \rightharpoonup^{H^0_e} A^0, \\ &ii) \quad \text{there exists } \nu \in [H^{-1}(\Omega)]^N \text{ such that } \nu^\varepsilon_g \to \nu \quad \text{strongly in } [H^{-1}(\Omega)]^N. \end{aligned} \tag{11}$$

*Then*

$$\begin{aligned} &i) \quad P_\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } [H^1_0(\Omega)]^N, \\ &ii) \quad A^\varepsilon \widetilde{e(u^\varepsilon)} \rightharpoonup A^0 e(u) \quad \text{weakly in } \left(L^2(\Omega)\right)^{N^2}, \end{aligned} \tag{12}$$

*where $u$ is the solution of the problem*

$$\begin{gathered} -\operatorname{div}\left(A^0 e(u)\right) = \nu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{13}$$

Proof. Observe first, by using (3 ii) and (10), that

$$\langle g^\varepsilon, v\rangle_{[H^{-1/2}(\partial S_\varepsilon)]^N,\, [H^{1/2}(\partial S_\varepsilon)]^N} = \langle \nu^\varepsilon_g, P_\varepsilon v\rangle_{[H^{-1}(\Omega)]^N,\, [H^1_0(\Omega)]^N}, \quad \forall v \in V_\varepsilon.$$

Hence, problem (8) is equivalent to the problem

$$\begin{gathered} -\operatorname{div}(A^\varepsilon e(u^\varepsilon)) = P^*_\varepsilon \nu^\varepsilon_g \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon(x)e(u^\varepsilon)) \cdot n = 0 \quad \text{on } \partial S_\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

since both of the two systems have the variational formulation: Find $u^\varepsilon \in V_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} A^\varepsilon e(v^\varepsilon) e(v)\,dx = \langle \nu^\varepsilon_g, P_\varepsilon v\rangle_{V'_\varepsilon, V_\varepsilon}, \quad \forall v \in V_\varepsilon. \tag{14}$$

Let us show that there exist $c$ independent of $\varepsilon$ such that

$$\|P_\varepsilon u^\varepsilon\|_{[H_0^1(\Omega)]^N} \le c\,. \tag{15}$$

By taking $u^\varepsilon$ as a test function in the variational formulation of (14) one obtains

$$\int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon)e(v)dx = \langle \nu_g^\varepsilon, P_\varepsilon u^\varepsilon\rangle_{V'_\varepsilon,V_\varepsilon}.$$

From (3 iii) and the fact that $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ one deduces that

$$\begin{aligned}\|e\left(P_\varepsilon u^\varepsilon\right)\|^2_{[L^2(\Omega)]^{N^2}} \le& C\int_{\Omega_\varepsilon} e\left(u^\varepsilon\right)e(u^\varepsilon)dx\\ \le&\frac{C}{\alpha}\int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon)e(u^\varepsilon)dx\\ \le& c\left\|\nu_g^\varepsilon\right\|_{[H^{-1}(\Omega)]^N}\|e\left(P_\varepsilon v^\varepsilon\right)\|_{[L^2(\Omega)]^{N^2}}\,.\end{aligned}$$

Hence (15) gives (11 ii). One may deduce (up to a subsequence) that

$$P_\varepsilon u^\varepsilon \rightharpoonup u^\star \quad \text{weakly in } [H_0^1(\Omega)]^N. \tag{16}$$

Consider now the solution $v^\varepsilon$ of the problem

$$\begin{gathered}-\operatorname{div}\left(A^\varepsilon e(v^\varepsilon)\right) = P_\varepsilon^* \nu \quad \text{in } \Omega_\varepsilon,\\ \left(A^\varepsilon(x)e(v^\varepsilon)\right)\cdot n = 0 \quad \text{on } \partial S_\varepsilon,\\ v^\varepsilon = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{17}$$

From (11 i), one deduces that

$$\begin{aligned}&i)\quad P_\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } [H_0^1\left(\Omega\right)]^N,\\ &ii)\quad A^\varepsilon \widetilde{e(v^\varepsilon)} \rightharpoonup A^0 e(v) \quad \text{weakly in } \left(L^2\left(\Omega\right)\right)^{N^2},\end{aligned} \tag{18}$$

where $v$ is the solution to (13).

On the other hand, $w^\varepsilon = u^\varepsilon - v^\varepsilon$ is the solution to

$$\begin{gathered}-\operatorname{div}\left(A^\varepsilon e(w^\varepsilon)\right) = P_\varepsilon^*\left(\nu_g^\varepsilon - \nu\right) \quad \text{in } \Omega_\varepsilon,\\ \left(A^\varepsilon(x)e(w^\varepsilon)\right)\cdot n = 0 \quad \text{on } \partial S_\varepsilon,\\ w^\varepsilon = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{19}$$

By choosing $w^\varepsilon$ as a test function in the variational formulation of (19) and (3) and the fact that $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$, one has

$$\begin{aligned}\|(P_\varepsilon w^\varepsilon)\|^2_{[L^2(\Omega)]^{N^2}} \le& C\|e(w^\varepsilon)\|^2_{[L^2(\Omega_\varepsilon)]^{N^2}}\\ \le&\frac{C}{\alpha}\int_{\Omega_\varepsilon} A^\varepsilon e(w^\varepsilon)e(w^\varepsilon)dx\\ =& c\langle \nu_g^\varepsilon - \nu, P_\varepsilon w^\varepsilon\rangle_{[H^{-1}(\Omega)]^N,[H_0^1(\Omega)]^N}.\end{aligned}$$

Since $P_\varepsilon w^\varepsilon$ is bounded in $[H_0^1(\Omega)]^N$, one deduces from (12 ii) that

$$\langle \nu_g^\varepsilon - \nu, P_\varepsilon w^\varepsilon\rangle_{[H^{-1}(\Omega)]^N,[H_0^1(\Omega)]^N} \to 0,$$

which implies that

$$P_\varepsilon w^\varepsilon \to 0 \quad \text{strongly in } [H_0^1(\Omega)]^N. \tag{20}$$

This, with (18) proves that in (16) one has $u^\star = u$.

Finally, one deduces from (20) and the fact that $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ that

$$\|A^\varepsilon \widetilde{e(w^\varepsilon)}\|_{[L^2(\Omega)]^{N^2}} \le c\,\|e(w^\varepsilon)\|_{[L^2(\Omega_\varepsilon)]^{N^2}} \le c\,\|e(P_\varepsilon w^\varepsilon)\|_{[L^2(\Omega)]^{N^2}} \to 0\,.$$

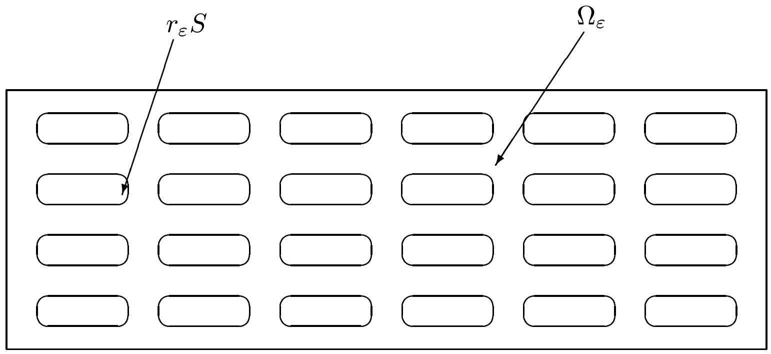


Figure 1. A periodic perforated domain

With the convergence (18 ii), it follows then that (12 ii) holds. $\diamondsuit$

Remark 3. As in the case of Theorem 1 of [9], from the linearity of the equation and the definition of the $H_e^0$- convergence, the choice of an nonhomogeneous right-hand side of the equation (8) is not restrictive.

## 4. Some applications of the main result

**The case of a periodic perforated domain.** Let $Y = [0, l_1[\times .. \times [0, l_N[$ be the representative cell, $S$ an open set of $Y$ with smooth boundary $\partial S$ such that $\overline{S} \subset Y$. Let $r_\varepsilon$ be the general term of a positive sequence which converge to zero and satisfying $r_\varepsilon \leq \varepsilon$. One denote by $\tau(r_\varepsilon \overline{S})$ the set of all the translated of $r_\varepsilon \overline{S}$ of the form $(\varepsilon k_l + r_\varepsilon \overline{S}), k \in Z^N, k_l = (k_1 l_1, .., k_N l_N)$. It represents the holes in $R^N$.

One suppose that the holes $\tau(r_\varepsilon \overline{S})$ do not intersect the boundary $\partial\Omega$. If $S_\varepsilon$ design the holes contained in $\Omega$, it follows that

$$S_\varepsilon \text{ is a finite union of the holes, i.e } S_\varepsilon = \cup_{k \in K} r_\varepsilon (k_l + \overline{S}).$$

Set $\Omega_\varepsilon = \Omega \setminus \overline{S_\varepsilon}$, by this construction, $\Omega_\varepsilon$ is a periodic perforated domain by holes of size $r_\varepsilon$ (see Figure 1)

We propose to study the asymptotic behavior of the solution $v^\varepsilon$ of the system

$$\begin{aligned} -\operatorname{div}(A^\varepsilon e(v^\varepsilon)) = 0 \quad &\text{in } \Omega_\varepsilon, \\ (A^\varepsilon(x) e(v^\varepsilon)) \cdot n = h^\varepsilon \quad &\text{on } \partial S_\varepsilon, \\ v^\varepsilon = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{21}$$

where

$$h^\varepsilon(x) = h(\frac{x}{\varepsilon}),\ h \in [L^2(\partial S)]^N \quad \text{Y-periodic.} \tag{22}$$

We suppose that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^N}{r_\varepsilon^{N-2}} = 0, \tag{23}$$

and that $A^\varepsilon = (a^\varepsilon_{ijkh})$ satisfies

$$a^\varepsilon_{ijkh}(x) = a_{ijkh}(\frac{x}{\varepsilon}),\ a_{ijkh} \in M_e(\alpha, \beta; Y^*). \tag{24}$$

In this case of a periodic perforated domain, the homogenization of system (21) has been studied by F. Lene in [11] for the case $r_\varepsilon = \varepsilon$, and C. Georgelin in [8] for the case $r_\varepsilon \lll \varepsilon$. The results obtained allow us to deduce that

$$(A^\varepsilon, S_\varepsilon) \rightharpoonup^{H_e^0} A^0, \tag{25}$$

where $A^0 = (a^0_{ijkh})$ is defined by

$$A^0_{ijkh} = \frac{1}{|Y|}\int_{Y\setminus S} A_{ijkh} e_{kh}(\chi^{kh} - P^{kh}) e_{ij}(\chi_{ij} - P^{ij}) dy, \tag{26}$$

where $P^{ij}$ is the vector all of whose components are equal to zero except the $i^{th}$ one, i.e., $(P^{ij})_k = y_j\delta_{ki}$, and for all $k, h = 1, .., N$, $\chi^{kh} \in [H^1(Y \setminus S)]^N$ Y-periodic, and is a solutin to

$$\begin{aligned} -\operatorname{div}(Ae(\chi^{kh} - P^{kh})) = 0 \quad \text{in } Y \setminus S, \\ (Ae(\chi^{kh} - P^{kh})) \cdot n = 0 \quad \text{on } \partial S, \end{aligned}$$

if $r_\varepsilon = \varepsilon$ and

$$A^0_{ijkh} = \frac{1}{|Y|}\int_{Y} a_{ijkh} e_{kh}(\chi^{kh} - P^{kh}) e_{ij}(\chi_{ij} - P^{ij}) dy, \tag{27}$$

where $P^{ij}$ is the vector all of whose components are equal to zero except the $i^{th}$ one which is equal to $y_j$, i.e., $(P^{ij})_k = y_j\delta_{ki}$, and for any $k, h = 1, .., N$, $\chi^{kh} \in [H^1(Y)]^N$ Y-periodic is a solution to

$$-\operatorname{div}(Ae(\chi^{kh} - P^{kh})) = 0 \quad \text{in } Y,$$

if $r_\varepsilon \lll \varepsilon$.

On the other hand, from the results obtain by D. Cioranescu and P. Donato in [2] for the case $r_\varepsilon = \varepsilon$, and S. Kaizu in [10] for the case $r_\varepsilon \lll \varepsilon$, we can deduce the following lemma.

**Lemma 1** ([2],[10])**.** *Let $\nu^\varepsilon_h$ be defined by (10). We suppose that (23) is satisfied and that the reference hole $S$ is star-shaped if $r_\varepsilon \lll \varepsilon$. Then*

$$\frac{\varepsilon^N}{r_\varepsilon^{N-1}}\nu^\varepsilon_h \to \nu \quad \text{in } H^{-1}(\Omega) \text{ strongly}, \tag{28}$$

*with*

$$\langle \nu, v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = I_h \int_\Omega v\, dx \quad \forall v \in H^1_0(\Omega), \tag{29}$$

*and $I_h = \frac{1}{|Y|}\int_{\partial S} h\, ds$.*

Consequently, we can apply Theorem 1 to $u^\varepsilon = \varepsilon^N v^\varepsilon / r_\varepsilon^{N-1}$ and $g^\varepsilon = \varepsilon^N h^\varepsilon / r_\varepsilon^{N-1}$ to obtain the following theorem.

**Theorem 2.** *Let $v^\varepsilon$ be a solution of (21). Suppose that (22) and (23) are satisfied and that $S$ is star-shaped if $r_\varepsilon \lll \varepsilon$. Then there exists $P_\varepsilon$ an extension operator satisfying (3) such that*

$$\begin{aligned} P_\varepsilon\Big(\frac{\varepsilon^N}{r_\varepsilon^{N-1}} v^\varepsilon\Big) \rightharpoonup v^0 \quad \text{weakly in } [H^1_0(\Omega)]^N, \\ A^\varepsilon e\Big(\frac{\varepsilon^N}{r_\varepsilon^{N-1}} \widetilde{v^\varepsilon}\Big) \rightharpoonup A^0 e(v^0) \quad \text{weakly in } \left(L^2(\Omega)\right)^{N^2}, \end{aligned}$$

*where $v^0$ is the solution to*

$$\begin{aligned} -\operatorname{div}\left(A^0 e(v^0)\right) = \nu \quad \text{in } \Omega, \\ v^0 = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{30}$$

*with $\nu$ defined by (29).*

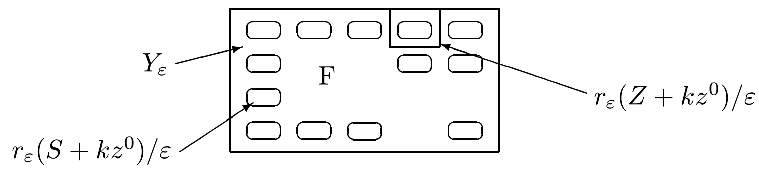


FIGURE 2. The reference cell

4.1. **The case of a perforated domain with double periodicity.** We consider the perforated domain $\Omega_\varepsilon$ defined as $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$, where $S_\varepsilon$ is a set with a double periodicity defined below. We adopt here the geometrical framework introduced in [5] and [6].

Assume that $Y$ and $Z$ are two fixed reference cells,

$$Y = ]0, y_1^o[\times ... \times]0, y_N^o[, \quad Z = ]0, z_1^o[\times ... \times]0, z_N^o[. \tag{31}$$

We set

$$y^0 = (y_1^o, .., y_N^o), \quad z^0 = (z_1^o, ..., z_N^o). \tag{32}$$

Let $F \subset Y$ and $S \subset Z$ be two closed subsets with smooth boundaries and nonempty interiors.

Suppose that $r_\varepsilon$ and $\varepsilon$ are the general term of two positive sequences such that $r_\varepsilon < \varepsilon$ and

$$\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = 0. \tag{33}$$

We assume that for each $\varepsilon > 0$ there exists a fine $K_\varepsilon \subset Z^N$, such that

$$\bigcup_{k \in K_\varepsilon} \frac{r_\varepsilon}{\varepsilon}(Z + kz^0) = Y, \tag{34}$$

and that

$$(\partial F) \cap (\bigcup_{k \in K_\varepsilon} \frac{r_\varepsilon}{\varepsilon}(S + kz^0)) = \emptyset.$$

This means that for any $\varepsilon$ the sets $Y$ and $Y \setminus F$ are exactly covered by a finite number of translated cells of $\frac{r_\varepsilon}{\varepsilon}Z$ and $\frac{r_\varepsilon}{\varepsilon}S$ respectively. Denote

$$S_Y^\varepsilon = (Y \setminus F) \cap (\bigcup_{k \in K_\varepsilon} \frac{r_\varepsilon}{\varepsilon}(S + kz^0))$$

and $Y_\varepsilon = Y \setminus S_Y^\varepsilon$. From (34) it follows that there exist a finite set $\mathcal{K}'_\varepsilon \subset Z^N$ such that

$$S_Y^\varepsilon = \bigcup_{k \in \mathcal{K}'_\varepsilon} \frac{r_\varepsilon}{\varepsilon}(S + kz^0).$$

Hence $S_Y^\varepsilon$ is a subset of $Y \setminus F$ of closed sets ("inclusions") periodically distributed with periodicity $r_\varepsilon/\varepsilon$ and of the same size as the period (see Figure 2).

We also assume that for each $\varepsilon > 0$, there exists a finite set $H_\varepsilon \subset Z^N$ such that

$$\bigcup_{h \in Z^N} \varepsilon(S_Y^\varepsilon + hy^0) \cap \Omega = \bigcup_{h \in H_\varepsilon} \varepsilon(S_Y^\varepsilon + hy^0)$$

and we set

$$S_\varepsilon \ = \ \bigcup_{h \in H_\varepsilon} \varepsilon(S_Y^\varepsilon + hy^0).$$

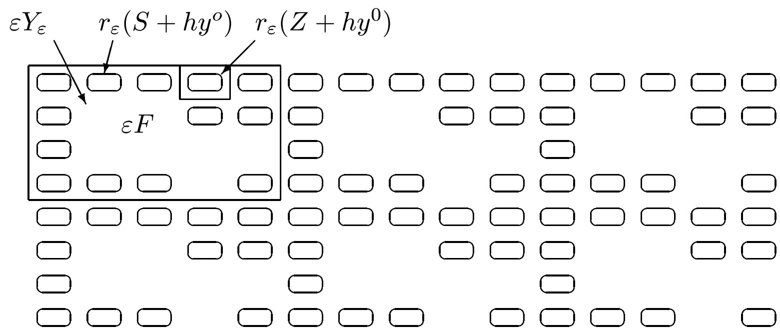


FIGURE 3. The perforated domain $\Omega_\varepsilon$

Hence, $\forall\varepsilon > 0$, $\Omega$ and $\Omega_\varepsilon$ are exactly covered by a finite number of translated cells of $\varepsilon Y_\varepsilon$ and $\varepsilon S^\varepsilon_Y$ respectively. Consequently, the structure of $\Omega_\varepsilon$ presents a double periodicity ($\varepsilon$ and $r_\varepsilon$). The zones in which the inclusions are concentrated are $\varepsilon$-periodic and of size $\varepsilon$. The inclusions in each zone are $r_\varepsilon$-periodic and of size $r_\varepsilon$ (see Figure 3).

Our aim is to apply Theorem 1 to this case of double periodicity with a matrix $A^\varepsilon = (a^\varepsilon_{ijkh})$ defined in (21) and satisfying

$$a^\varepsilon_{ijkh}(x) = a_{ijkh}(\frac{x}{\varepsilon}, \frac{x}{r_\varepsilon})$$

$$\begin{aligned} &i)\quad a_{ijkh}\quad Y\times Z-\text{periodic}\\ &ii)\quad a_{ijkh}\in L^\infty(Z, C^0(Y)) \text{ or } a_{ijkh}\in L^\infty\left(Y, C^0(Z)\right)\\ &iii)\quad a_{ijkh}=a_{ijhk}=a_{jikh}\\ &iv)\quad \exists\alpha>0 \text{ s.t. } a_{ijkh}(y,z)e_{kh}e_{ij}\ge\alpha e_{ij}e_{ij},\ \text{a.e. } (y,z)\in Y\times Z \end{aligned} \tag{35}$$

for any symmetric tensor $e_{ij}$, and $h^\varepsilon$ is defined by

$$h_\varepsilon = (\mathcal{F}^\varepsilon\circ\mathcal{Q}^\varepsilon)h, \quad h_\varepsilon\in H^{-1/2}(\partial S^\varepsilon) \tag{36}$$

where $h$ is $Z$-periodic, $h\in H^{-1/2}(\partial S)$, and

$$\langle h, 1\rangle_{H^{-1/2}(\partial S), H^{1/2}(\partial S}\neq 0. \tag{37}$$

The operator $\mathcal{Q}^\varepsilon\in\mathcal{L}\left(H^{-1/2}\left(\partial S\right), H^{-1/2}\left(\partial S^\varepsilon_Y\right)\right)$ is defined by

$$\langle\mathcal{Q}^\varepsilon z, v\rangle_{H^{-1/2}(\partial S^\varepsilon_Y),\, H^{1/2}(\partial S^\varepsilon_Y)} = \sum_{k\in\mathcal{K}'_\varepsilon}(\frac{r_\varepsilon}{\varepsilon})^{N-1}\langle z, v\circ\sigma_\varepsilon^{-1}\rangle_{H^{-1/2}(\partial S+kz^0),\, H^{1/2}(\partial S+kz^0)}, \tag{38}$$

and the operator $\mathcal{F}^\varepsilon\in\mathcal{L}(H^{-1/2}(\partial S^\varepsilon_Y), H^{-1/2}(\partial S^\varepsilon))$ is defined by

$$\langle\mathcal{F}^\varepsilon u, \phi\rangle_{H^{-1/2}(\partial S^\varepsilon),\, H^{1/2}(\partial S^\varepsilon)} = \sum_{h\in\mathcal{H}_\varepsilon}(\varepsilon)^{N-1}\langle u, \phi\circ\tau_\varepsilon^{-1}\rangle_{H^{-1/2}(\partial S^\varepsilon_Y+hy^0),\, H^{1/2}(\partial S^\varepsilon_Y+hy^0)}, \tag{39}$$

where $\sigma_\varepsilon$ and $\tau_\varepsilon$ are the homotheties

$$\sigma_\varepsilon:\quad x\longrightarrow\frac{\varepsilon}{r_\varepsilon}x, \qquad \tau_\varepsilon:\quad x\longrightarrow\frac{x}{\varepsilon}. \tag{40}$$

From the result obtained in [9], we deduce that

$$(A^\varepsilon, S_\varepsilon)\rightharpoonup^{H^0_e} A^0, \tag{41}$$

where $A^0 = (a^0_{ijkh})$ is defined as follows: set

$$d_{ijkh}(y,z) = \big(\chi_F(y) + \chi_{Y\setminus F}(y)\chi_{Z\setminus S}(z)\big)\, a_{ijkh}(y,z), \tag{42}$$

and for $l, m = 1, .., N$, let $R^{lm} = \big(R^{lm}_k\big)_{k=1,..,N}$ be the vector defined by

$$R^{lm}_k = z_m \delta_{kl}.$$

We denote by $\chi^{lm} = \chi^{lm}(y,.)$ the unique function in $[H^1(Z\setminus S)]^N$ $Z$-periodic which is a solution to

$$\begin{aligned} -\frac{\partial}{\partial z_j}[d_{ijkh} e^z_{kh}(R^{lm} - \chi^{lm})] = 0 \quad \text{in } Z\setminus S \\ d_{ijkh} e^z_{kh}(R^{lm} - \chi^{lm})\cdot n_j = 0 \quad \text{on } \partial S. \end{aligned} \tag{43}$$

We set

$$q_{ijkh} = \frac{1}{|Z|}\int_{Z\setminus S} d_{ijrs} e^z_{rs}(R^{kh} - \chi^{kh})dz.$$

Let $P^{lm} = \big(P^{lm}_k\big)_{k=1,..,N}$ be the vector defined by $P^{lm}_k = y_m\delta_{kl}$, and let $\beta^{lm}$ in $[H^1(F)]^N$ $Y$-periodic, which is a solution to

$$\begin{aligned} -\frac{\partial}{\partial y_j}[q_{ijkh} e^y_{kh}(P^{lm} - \beta^{lm})] = 0 \quad \text{in } F, \\ q_{ijkh} e^y_{kh}(P^{lm} - \beta^{lm})\cdot n_j = 0 \quad \text{on } \partial F\setminus \partial Y. \end{aligned} \tag{44}$$

We define the homogenizated coefficients by

$$a^0_{ijkh} = \frac{1}{|Y|}\int_Y q_{ijrs} e^y_{rs}(P^{kh} - \beta^{kh})dy, \tag{45}$$

where $y = (y_i)_{i=1,..,N}$ and $z = (z_i)_{i=1,..,N}$.

Observe that the coefficients $(a^0_{ijkh})$ are obtained by applying the homogenization process twice (see the classical methods of homogenization introduced by F. Murat and L. Tartar in [13] and S. Spagnolo in [14]). Indeed, first starting with the tensor $(d_{ijkh})$ and homogenizing with respect to $Z$, we obtain the tensor $(q_{ijkh})$. Then starting with $(q_{ijkh})$ and homogenizing with respect to $Y$, we obtain the tensor $(a^0_{ijkh})$.

On the other hand, using the results obtain by P. Donato and M. El Hajji in [4], we obtain

**Lemma 2.** *Let $\nu^h_\varepsilon$ be defined by (10), suppose that $\langle h,1\rangle_{H^{-1/2}(\partial S),H^{1/2}(\partial S)} \neq 0$, and that (33) is satisfied. Then*

$$r_\varepsilon \nu^\varepsilon_h \quad \to \nu \quad \textit{strongly in } H^{-1}(\Omega),$$

*where $\nu$ is given by*

$$\langle \nu,\phi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \gamma\theta I_h \int_\Omega \phi dx \quad \forall\phi\in H^1_0(\Omega), \tag{46}$$

*with*

$$\gamma = \left(\frac{|Y||Z|}{|Y-F|} - |S|\right)^{-1}, \quad I_h = \langle h,1\rangle_{H^{-1/2}(\partial S),H^{1/2}(\partial S)}, \tag{47}$$

*and $\theta$ is defined by*

$$\theta = \frac{|F|}{|Y|} + \frac{|Y\setminus F|}{|Y|}\frac{|Z\setminus S|}{|Z|}.$$

Hence, we can apply Theorem 1 to $u^\varepsilon = r_\varepsilon u^\varepsilon$ and obtain

**Theorem 3.** *Let $v^\varepsilon$ be the solution of (21). Then there exists $P_\varepsilon$ an extension operator satisfying (3) such that*

$$P_\varepsilon(r_\varepsilon v^\varepsilon) \rightharpoonup u^0 \quad \text{weakly in } [H_0^1(\Omega)]^N,$$
$$A^\varepsilon \widetilde{e(r_\varepsilon v^\varepsilon)} \rightharpoonup A^0 e(u^0) \quad \text{weakly in } \left(L^2(\Omega)\right)^{N^2},$$

*where $v^0$ is the solution of the problem*

$$\begin{aligned} -\operatorname{div}\left(A^0 e(v^0)\right) &= \nu \quad \text{in } \Omega, \\ v^0 &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{48}$$

*where $A^0 = (a^0_{ijkh})$ is given by (43)-(45), and $\nu$ defined by (46), (47).*

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