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DYNAMICS OF LOGISTIC EQUATIONS WITH NON-AUTONOMOUS BOUNDED COEFFICIENTS

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Abstract. We prove that the Verhulst logistic equation with positive non-autonomous bounded coefficients has exactly one bounded solution that is positive, and that does not approach the zero-solution in the past and in the future. We also show that this solution is an attractor for all positive solutions, some of which are shown to blow-up in finite time backward. Since the zero-solution is shown to be a repeller for all solutions that remain below the aforementioned one, we obtain an attractor-repeller pair, and hence (connecting) heteroclinic orbits. The almost-periodic attractor case is also discussed. Our techniques apply to the critical threshold-level equation as well.

1. Introduction

Consider the non-autonomous logistic equation

$$\frac{du}{dt} = u(a(t) - b(t)u), \quad t \in \mathbb{R}, \tag{1}$$

where it is assumed that the carrying capacity $a : \mathbb{R} \to \mathbb{R}$ and the self-limitation coefficient $b : \mathbb{R} \to \mathbb{R}$ are continuous functions with

$$0 < \alpha \le a(t) \le A, \quad 0 < \beta \le b(t) \le B, \quad t \in \mathbb{R}, \tag{2}$$

for some positive constants α, β, A and B.

When the coefficients $a(t)$ and $b(t)$ are positive constants, Eq.(1) was introduced around 1838 by the Belgian mathematician Pierre F. Verhulst as a model for studying the dynamics of human populations with self-limitation. This nonlinear equation was proposed as an alternative to the unlimited growth model suggested earlier in that century by the British economist Thomas Malthus. It has become a classical equation in textbooks on ordinary differential equations (see e.g. Amann [1], Boyce and DiPrima [3], Hale and Koçak [9], Hirsch and Smale [11]). Due to the absence of viable census data at the time, this model was not tested and did not receive much attention for many years, until it was proven to be effective and in agreement with experimental data for populations of fruit-flies by R. Pearl in 1930, and for populations of four-beetles by G. F. Gause in 1935. Since then it

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has been used for other species, and in managerial sciences (see e.g. [3, 4]). By using the method of separation of variables and integration by partial fractions, it is easy in the constant-coefficient case to solve explicitly this equation, and completely analyze the behavior of all solutions (see e.g. [1, 3, 9, 11]). However, when the coefficients are no longer constant, the situation is different since no explicit solutions can be found in general. This situation is the subject of this paper. The time-periodic case is discussed in Hale and Koçak [9], where some of the difficulties associated with non-autonomous problems are pointed out. Let us mention that partial differential equations with logistic-type nonlinearities have also been considered recently. The reader is referred to Blat and Brown [2], Cohen and Laetsch [5], de Figueiredo [7], and Hess [10], among others, for more information. With the exception of [10] where the time-periodic problem is considered, all these papers dealt with autonomous or steady-state problems. Of course, techniques used for time-periodic problems rely heavily on the compactness of the period-interval, which implies the compactness of the associated fixed-point differential operators. This feature is clearly missing here.

In this note, we prove that the logistic equation (1) with positive non-autonomous bounded coefficients, as in (2), has exactly one bounded solution that is positive, and that does not tend to the zero-solution in the past and in the future. Positive solutions that remain above this one must blow-up in finite time backward, while negative solutions must blow-up in finite time forward. We actually obtain a quantitative estimate of the blow-up time in terms of the "initial condition" and the bounds in (2). This is accomplished in Section 2. In Section 3, we show that the unique solution obtained in Section 2 is forward-stable, and is a forward-attractor for *all* positive solutions. Hence, the zero-solution is unstable. We also show that the zero-solution is a forward-repeller (i.e. backward-attractor) for *all* solutions that remain *below* the aforementioned unique (positive) solution. In this way, we obtain an attractor-repeller pair, and so (connecting) heteroclinic orbits. This gives us a comprehensive picture of the asymptotic behavior of all solutions to Eq.(1). Our method of proof is based on uniqueness and continuation of solutions to initial-value problems, comparison techniques, maximal and minimal solutions, and ω-limit points of solutions. In Section 4, we prove that if, in addition to (2), the coefficients $a(t)$ and $b(t)$ are almost-periodic functions, then the unique bounded attractor obtained in Sections 2 and 3 is an almost-periodic solution. To show this, we use the notion of inherited separating property introduced by Amerio (see e.g. Corduneanu [6], Fink [8], Yoshizawa [14]). It should be pointed out that bounded solutions to Eq.(1) do not in general satisfy Amerio's separation condition since there is an attractor-repeller pair. However, uniqueness will imply that Amerio's separation condition is satisfied by the attractor in a small neighborhood of itself. Finally, in Section 5, we indicate how our techniques apply to the critical threshold-level equation.

Note that the nonlinearity involved in Eq.(1) is the quadratic function $f(t,u) = a(t)u - b(t)u^2$, which is a concave-down parabola for each $t \in \mathbb{R}$, with u-intercepts at $u = 0$ and $u = a(t)/b(t)$. Therefore, unlike the constant-coefficient case, the nonlinearity might have a string of non-zero u-intercepts in time. Nevertheless, the u-intercepts of the nonlinearity, and the fact that the function $f(t,u)/u$ is decreasing (in u for each $t \in \mathbb{R}$), will play a significant role in the analysis of the behavior of solutions to Eq.(1). This will be made clear in Sections 2 and 3.

2. Existence and Uniqueness

In this section we prove an existence and uniqueness result for bounded solutions to Eq.(1) that do not approach the zero-solution (in the past and in the future). It was pointed out in Hale and Koçak [9], pp. 126–128 (also see Hess [10], pp. 125–127), where the time-periodic case is discussed, that it is the uniqueness part that is the most involved to prove in the setting of non-autonomous problems. We present here a somewhat simple uniqueness argument based on the idea of the ratio of non-decaying (in the past) bounded solutions. (Our proof is even simpler in the time-periodic case, just restrict the argument to the period-interval.) The existence part follows from the notion of maximal and minimal bounded solutions, once at least one bounded (positive) solution is obtained and once it is shown that all bounded solutions are actually equi-bounded. We also prove that unbounded solutions must blow-up in finite time.

Theorem 2.1. *Suppose that the conditions in (2) are met. Then the non-autonomous logistic equation (1) has exactly one bounded solution $u : \mathbb{R} \to \mathbb{R}$ that is positive, and that does not tend to zero as $t \to \pm\infty$. Actually, $u(t)$ satisfies the inequalities*

$$\frac{\alpha}{B} \le u(t) \le \frac{A}{\beta} \quad \text{for all } t \in \mathbb{R}. \tag{3}$$

Proof. First note that the function $u \equiv 0$ is a solution of Eq.(1) on $\mathbb{R}$. Therefore, by uniqueness of solutions to initial-value problems, any non-trivial solution to Eq.(1) must be either positive or negative on its interval of definition.

Now, suppose that $u(t)$ is a non-trivial solution to Eq.(1) such that u is bounded on $\mathbb{R}$. We claim that u must satisfy the inequalities

$$0 < u(t) \le \frac{A}{\beta} \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

Indeed, suppose that $u(t_0) < 0$ for some t_0, then $u(t)$ is negative for all $t \in \mathbb{R}$ for which $u(t)$ is defined. It follows from (1) and (2) that $u(t)$ is decreasing for all such $t \in \mathbb{R}$, and that $du/dt \le \alpha u - \beta u^2$. Since $\alpha u - \beta u^2 < 0$, we derive that $(\alpha u - \beta u^2)^{-1} du/dt \ge 1$. Using partial fractions, we obtain

$$\frac{d}{dt} \ln\left(\frac{\alpha u}{\beta u - \alpha}\right) \ge \alpha \quad \text{for all such } t \in \mathbb{R}.$$

Integrating from t_0 to t, with $t_0 \le t$, and solving the inequality for $u(t)$, we get

$$u(t) \le \frac{c_0 \alpha}{c_0 \beta - \alpha e^{\alpha(t-t_0)}}, \tag{5}$$

where $c_0 = \alpha u(t_0)(\beta u(t_0) - \alpha)^{-1} > 0$. Since the right-hand side of (5) is negative for $t \ge t_0$ and has a vertical asymptote at

$$t_* = t_0 + \alpha^{-1} \ln[\alpha(\beta c_0)^{-1}] > t_0, \tag{6}$$

it follows that $u(t) \to -\infty$ as $t \to t_*^-$; that is, $u(t)$ blows up in finite time forward. This is a contradiction with the fact that $u(t)$ is bounded. Thus, $u(t)$ must be positive.

Similarly, suppose that $u(t_0) > A/\beta$ for some $t_0 \in \mathbb{R}$. Then, it follows from (1) and (2) that $u(t)$ is decreasing for all $t \le t_0$ for which $u(t)$ is defined, and that

$du/dt \leq Au - \beta u^2$. Since $Au - \beta u^2 < 0$, we derive that $(Au - \beta u^2)^{-1} du/dt \geq 1$. Using partial fractions, we obtain

$$\frac{d}{dt} \ln\left(\frac{Au}{\beta u - A}\right) \geq A \quad \text{for all such } t \in \mathbb{R}.$$

Integrating from t to t_0, with $t \leq t_0$, and solving the inequality for $u(t)$, we get

$$u(t) \geq \frac{c_0 A}{c_0\beta - Ae^{A(t_0-t)}}, \tag{7}$$

where $c_0 = Au(t_0)(\beta u(t_0) - A)^{-1} > 0$. Since the right-hand side of (7) is positive for $t \leq t_0$ and has a vertical asymptote at

$$t_* = t_0 + A^{-1} \ln[A(\beta c_0)^{-1}] < t_0, \tag{8}$$

it follows that $u(t) \to \infty$ as $t \to t_*^+$; that is, $u(t)$ blows up in finite time backward. This also is a contradiction with the fact that $u(t)$ is bounded. Thus, inequalities (4) must hold for every bounded solution to Eq.(1).

Next, we claim that there is at least one bounded (positive) solution to Eq.(1). Indeed, let $\epsilon \in \mathbb{R}$ be such that $0 < \epsilon < \alpha/B$, and consider the initial-value problem

$$\begin{cases} \dfrac{dw}{dt} &= w(a(t) - b(t)w), \quad t \in \mathbb{R}, \\ w(t_0) &= \epsilon. \end{cases} \tag{9}$$

Then the (unique) solution to Eq.(9) is defined on $\mathbb{R}$, and satisfies inequalities (4). To see this, first observe that since $w^{-1}dw/dt \geq \epsilon_2$ for all $t \leq t_0$, where $\epsilon_2 = \alpha - B\epsilon > 0$, integration yields

$$0 < w(t) \leq w(t_0)e^{\epsilon_2(t-t_0)} \quad \text{for all } t \leq t_0. \tag{10}$$

This implies that $w(t)$ can be continued indefinitely in the past, and that $w(t) \to 0$ exponentially as $t \to -\infty$. Next, we claim that $0 < w(t) \leq A/\beta$ for all $t \geq t_0$, so that $w(t)$ can be continued indefinitely in the future. Otherwise, the Intermediate Value Theorem and the fact that $w(t)$ is decreasing if $w(t) > A/\beta$ leads to a contradiction.

Now, let $I \subset \mathbb{R}$ be defined by

$$I = \{w_0 \in \mathbb{R} : \text{Eq.(1), with } u(0) = w_0, \text{ has a bounded solution}\}.$$

Set $u_0 = \sup I$, and let $u(t)$ denote the solution to Eq.(1) with initial condition $u(0) = u_0$. Then, it follows immediately from Eq.(9) that $u_0 \geq \alpha/B$. Moreover, $u_0 \leq A/\beta$. For, if not, pick $w_0 \in \mathbb{R}$ such that $A/\beta < w_0 < u_0$. Then, by (7) with $t_0 = 0$, the solution through w_0 blows up in finite time in the past. This violates the fact that u_0 is the *supremum* of initial conditions of bounded solutions to Eq.(1). A similar reasoning shows that $u(t) \leq A/\beta$ for all $t < 0$. (Otherwise, check the value of a close-by unbounded solution when it reaches the time $t = 0$, and compare it with u_0.) Therefore, It follows that $u(t) \leq A/\beta$ for all $t \in \mathbb{R}$, since otherwise (7) would again lead to a contradiction. Hence $u(t)$ satisfies the inequalities (4); i.e. $u_0 \in I$. Thus, it is the maximal bounded solution to Eq.(1).

We claim that $u(t) \geq \alpha/B$ for all $t \leq 0$. Indeed, suppose there is $t_0 < 0$ such that $u(t_0) < \alpha/B$. Pick $\epsilon \in \mathbb{R}$ such that $u(t_0) < \epsilon < \alpha/B$. Then, the solution to Eq.(9) is bounded on $\mathbb{R}$, with $w(0) > u_0$ by uniqueness of solution to initial-value problems. This contradicts the definition of u_0. Therefore, $u(t) \geq \alpha/B$ for all $t \in \mathbb{R}$.

Otherwise, the Intermediate Value Theorem and (10) lead to a contradiction. Thus, the maximal solution $u(t)$ satisfies the inequalities (3).

Now, we want to show uniqueness; that is, there is no other bounded solution to Eq.(1) that satisfies inequalities (3). For that purpose, let $J \subset I$ be defined by

$$J = \{w_0 \in \mathbb{R} : w_0 \in I \text{ and inequalities (3) hold.}\}$$

Set $v_0 = \inf J$, and let $v(t)$ denote the solution to (1) with $v(0) = v_0$. Note that $u_0 \in J$, and $\alpha/B \le v_0 \le u_0 \le A/\beta$. Moreover, by a reasoning similar to the one above, one can show that the minimal solution $v(t)$ also satisfies inequalities (3); i.e. $v_0 \in J$. Thus,

$$0 < \frac{\alpha}{B} \le v(t) \le u(t) \le \frac{A}{\beta} \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

We now proceed to show that $v(t) = u(t)$ for all $t \in \mathbb{R}$. Let us assume that $u(t) > v(t)$ for all $t \in \mathbb{R}$. Otherwise, uniqueness follows immediately.

By using (1) and (11), we immediately get that $v^{-1}dv/dt - u^{-1}du/dt \ge b(t)(u(t) - v(t))$ for all $t \in \mathbb{R}$. That is,

$$\frac{d}{dt} \ln\left(\frac{v}{u}\right) \ge b(t)(u(t) - v(t)) > 0 \quad \text{for all } t \in \mathbb{R}. \tag{12}$$

This implies that the function (v/u) is *increasing* on $\mathbb{R}$. Therefore,

$$\frac{v(t)}{u(t)} \le \frac{v(0)}{u(0)} \le c < 1 \quad \text{for all } t \le 0.$$

Consequently, $u(t) - v(t) \ge (1-c)u(t) \ge (1-c)\alpha/B = \delta > 0$ for all $t \le 0$. Using (12), and integrating from t to 0, with $t \le 0$, we obtain

$$\frac{v(t)}{u(t)} \le \frac{v(0)}{u(0)} e^{\beta\delta t} \quad \text{for all } t \le 0.$$

Hence, $v(t)/u(t) \to 0$ as $t \to -\infty$. This is a contradiction with the fact that, by (11), $v(t)/u(t) \ge \alpha\beta/AB > 0$ for all $t \in \mathbb{R}$. The proof is complete. $\diamondsuit$

Note that Theorem 2.1 fully answers the question posed in [12]. However, we would like to investigate further the asymptotic behavior of all other non-trivial (bounded or not) solutions of Eq.(1) relative to the unique solution obtained in Theorem 2.1. This will be taken up in the next section.

3. Attractor-Repeller Pair

In this section we shall prove that the unique bounded solution $u(t)$ obtained in Theorem 2.1 is a forward attractor for all positive solutions (bounded and unbounded), and so is forward asymptotically stable. We also show that the zero-solution to Eq.(1) is a backward (exponential) attractor for all solutions $v(t)$ with $v(t_1) < u(t_1)$ for some $t_1 \in \mathbb{R}$, and so is backward exponentially stable. Thus, the zero-solution is a forward (exponential) repeller for all solutions that remain below the attractor.

Theorem 3.1. *Suppose the conditions in (2) are met. Then, the bounded solution $u(t)$ given in Theorem 2.1 is an attractor for all positive solutions to Eq.(1). That is, if $v(t)$ is a positive solution to Eq.(1), then*

$$\lim_{t\to\infty} |u(t) - v(t)| = 0. \tag{13}$$

Proof. Let us first consider the case when $v(t_0) > u(t_0)$ for some $t_0 \in \mathbb{R}$. Then, an analysis of the proof of Theorem 2.1 shows that $\alpha/B \le u(t) < v(t) \le \max\{v(t_0), A/\beta\}$ for all $t \ge t_0$, and that $v(t)$ must be unbounded in the past, and actually blows-up in finite time backward, by an argument similar to (7) and (8).

By using (1) and (2), we get as before that

$$\frac{d}{dt}\ln\left(\frac{u}{v}\right) = b(t)(v-u) > 0 \quad \text{for all } t \ge t_0.$$

This implies that the function (u/v) is *increasing* on the interval $[t_0, \infty)$, with $0 < (u/v) < 1$ for all $t \ge t_0$. Therefore, $\lim_{t\to\infty}[u(t)/v(t)] = c$, where $c = \sup_{[t_0,\infty)}[u(t)/v(t)]$ is a constant such that $0 < c \le 1$.

Suppose $c < 1$. It follows that $v(t) - u(t) \ge (1-c)\alpha/B = \delta > 0$, since $u(t) \le cv(t)$ and $v(t) \ge \alpha/B$. Therefore,

$$\frac{d}{dt}\ln\left(\frac{u}{v}\right) \ge \beta\delta \quad \text{for all } t \ge t_0.$$

Integrating from t_0 to t, we obtain

$$1 > c \ge \frac{u(t)}{v(t)} \ge \frac{u(t_0)}{v(t_0)}e^{\beta\delta(t-t_0)}.$$

Letting $t \to \infty$, we reach a contradiction.

Thus, $c = 1$; i.e., $\lim_{t\to\infty}[u(t)/v(t)] = 1$. It follows that $[v(t) - u(t)] \to 0^+$ as $t \to \infty$, since $[v(t)/u(t)] \to 1^+$ as $t \to \infty$, and $u(t) \le A/\beta$ for all $t \ge t_0$. Hence, (13) holds.

Now, let us consider the case when $0 < v(t_0) < u(t_0)$ for some $t_0 \in \mathbb{R}$. Then, it follows that $0 < v(t) < u(t) \le A/\beta$ for all $t \ge t_0$. Observe that

$$\frac{d}{dt}\ln\left(\frac{v}{u}\right) = \frac{1}{v}\frac{dv}{dt} - \frac{1}{u}\frac{du}{dt} = b(t)[u(t) - v(t)] > 0$$

for all $t \ge t_0$. Thus, proceeding as above, it is now easy to conclude that $\lim_{t\to\infty}[u(t) - v(t)] = 0$. Once again (13) holds. The proof is complete. ◊

The next result shows that the zero-solution is a repeller (i.e., backward attractor) for all solutions that stay below the attractor $u(t)$.

Theorem 3.2. *Suppose the conditions in (2) are met. Then, the zero-solution exponentially repels all solutions $v(t)$ such that $v(t_1) < u(t_1)$ for some $t_1 \in \mathbb{R}$. That is, if $v(t)$ is a solution to Eq.(1) with $v(t_1) < u(t_1)$ for some $t_1 \in \mathbb{R}$, then $\lim_{t\to-\infty} v(t) = 0$ exponentially.*

Proof. Let us first consider the case when $0 < v(t_1) < u(t_1)$ for some $t_1 \in \mathbb{R}$. Then, an analysis of the proof of Theorem 2.1 shows that $0 < v(t) < u(t)$ for all $t \in \mathbb{R}$, and that there is $t_0 \le t_1$ such that $v(t_0) < \alpha/B$. Consequently, using (9) and (10), we deduce that $v(t) \to 0$ exponentially as $t \to -\infty$.

Now, consider the case when $v(t_1) < 0$ for some $t_1 \in \mathbb{R}$. Then, $v(t) < 0$ for all t in its maximal interval of definition. Moreover, it follows from the argument leading up to (5) and (6) that $v(t)$ must be decreasing, and that it blows-up in finite time forward. By using (1) and (2), we get $v^{-1}dv/dt \ge \alpha$ for $t \le t_1$; i.e., $\frac{d}{dt}\ln v \ge \alpha$ for $t \le t_1$. Integrating from t to t_1, we obtain

$$v(t_1)e^{\alpha(t-t_1)} \le v(t) < 0.$$

This implies that $v(t)$ can be continued indefinitely in the past, and that $v(t) \to 0$ exponentially as $t \to -\infty$. The proof is complete. $\diamondsuit$

4. Almost Periodic Attractor

In this section we show that if in addition to (2) the coefficients $a(t)$ and $b(t)$ are (uniformly) almost-periodic functions, then the unique bounded attractor obtained in Sections 2 and 3 is an almost-periodic solution. To accomplish this, we will show that the attractor has an inherited separating property, a notion introduced by Amerio (see e.g. [6, 8, 13, 14]). It should be pointed out that bounded solutions to Eq.(1) do not in general satisfy Amerio's separation condition since there is an attractor-repeller pair. However, uniqueness obtained in Section 2 shows that Amerio's separation condition is satisfied by the attractor in a small neighborhood of itself. The definitions of the terms used in this section may be found in [6, 8, 13, 14].

Theorem 4.1. *Suppose that, in addition to (2), the functions $a(t)$ and $b(t)$ are almost periodic. Then, the unique solution $u(t)$ obtained in Theorem 2.1 is an almost periodic function. Thus, the attractor is almost periodic.*

Proof. Let $f(t,u) = a(t)u - b(t)u^2$ be the nonlinearity given in Eq.(1), and let $\mathcal{H}(f)$ denote the hull of f (see e.g. [8, 14]). Note that each $g \in \mathcal{H}(f)$ is of the form $g(t,v) = a^*(t)v - b^*(t)v^2$, where $a^* \in \mathcal{H}(a)$ and $b^* \in \mathcal{H}(b)$. Since $a^*(t)$ and $b^*(t)$ satisfy conditions (2) as limits of translates of $a(t)$ and $b(t)$, it follows from Theorem 2.1 that each equation

$$\frac{dv}{dt} = v(a^*(t) - b^*(t)v), \quad t \in \mathbb{R},$$

has a unique bounded solution $v(t)$ satisfying (3). Therefore, uniqueness of bounded solutions in the compact interval $K = [(\alpha/B) - \delta, (A/\beta) + \delta]$, where $0 \leq \delta << \alpha/B$, is inherited by each equation with nonlinearity $g \in \mathcal{H}(f)$ ([8, 14]). It follows that the (unique) solution $u(t) \in K$ (for all $t \in \mathbb{R}$) satisfies an inherited separation condition in K in the sense of Amerio. Thus, $u(t)$ must be almost periodic (see e.g. Theorem 10.1 in Fink [8], p. 170, or Corollary 17.1 in Yoshizawa [14], p. 192). The proof is complete. $\diamondsuit$

5. Critical Threshold-level Equation

In this section we will show that the above analysis applies to the equation

$$\frac{du}{dt} = -u(a(t) - b(t)u), \quad t \in \mathbb{R}, \tag{14}$$

where it is assumed that the coefficients $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ satisfy the conditions in Section 1. Of course, in this case the nonlinearity of interest $g(t,u) = -a(t)u + b(t)u^2$ is concave-up (in u for each t).

Eq.(14) occurs for instance in fluid mechanics where it describes the evolution of a small disturbance in a *laminar* (or smooth) fluid flow. If the disturbance is below a certain threshold, it is damped out and the laminar fluid flow persists. However, if the disturbance is above the threshold, then it grows larger and the laminar flow breaks up into a turbulent one (see e.g. Boyce and DiPrima [3] for more information).

Now, let us perform the change of variables $s(t) = -t$, which reverses the time-direction. Setting $w(t) = u(s(t))$ and using the Chain Rule for differentiation, Eq.(14) becomes

$$\frac{dw}{dt} = w(\tilde{a}(t) - \tilde{b}(t)w), \quad t \in \mathbb{R}, \tag{15}$$

where $\tilde{a}(t) = a(-t)$, and $\tilde{b}(t) = b(-t)$. Note that Eq.(15) is similar to the logistic equation (1), and that the coefficients $\tilde{a}(t)$ and $\tilde{b}(t)$ satisfy (2). Thus, all the results in Sections 2–4 apply to (15), and yield the following conclusions for Eq.(14).

Theorem 5.1. *The critical threshold-level equation (14) has exactly one solution* $u : \mathbb{R} \to \mathbb{R}$ *such that*

$$0 < \frac{\alpha}{B} \le u(t) \le \frac{A}{\beta} \quad \text{for all } t \in \mathbb{R}.$$

- *The solution $u(t)$ is a repeller (i.e., backward attractor) for all other positive solutions. Thus, $u(t)$ is unstable.*
- *The zero-solution is an attractor for all solutions that remain below $u(t)$. Thus, the zero-solution is exponentially asymptotically stable.*
- *Positive solutions above $u(t)$ blow-up in finite time forward.*
- *Negative solutions blow-up in finite time backward.*
- *The solution $u(t)$ is almost periodic if $a(t)$ and $b(t)$ are almost periodic.*

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